

# Scattering Amplitudes for Binary Black holes

Fei Teng



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第十四届新物理研讨会

# Outline

Introduction: amplitudes meet GW

Classical amplitudes and EFT matching

Spinning black holes and neutron stars

Observable based formalism (KMOC)

Outlook

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# Gravitational wave: new window to probe our Universe

## New physics!

- ▶ Probe dynamics of black holes
- ▶ Test general relativity
- ▶ Black hole formation
- ▶ Early universe

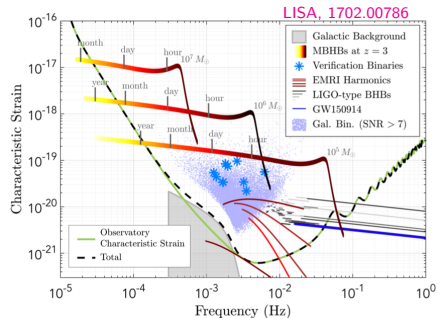
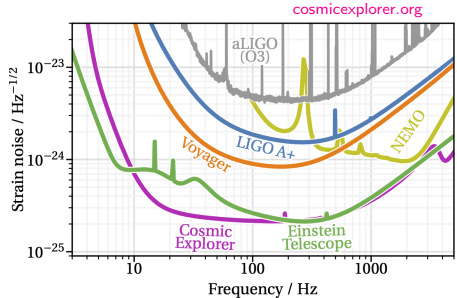
## Future ground based observatories

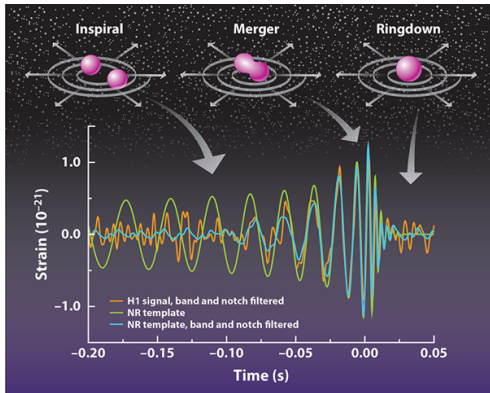
- ▶ Advanced LIGO
- ▶ Einstein Telescope
- ▶ Cosmic Explorer

## Future space based observatories

- ▶ LISA
- ▶ TaiJi
- ▶ TianQin

Require accurate theoretical prediction





Accurate theoretical prediction of the GW production puts challenges on the understanding of its source

# How to organize perturbations?

- ▶ Post-Newtonian (PN) expansion

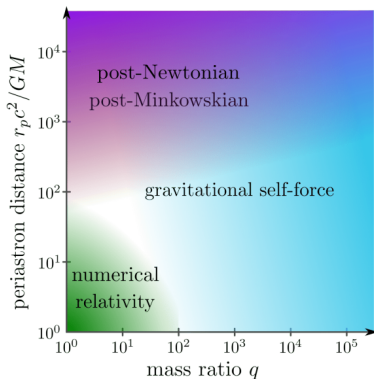
$$v^2 \sim \frac{Gm}{r} \ll 1$$

- ▶ Post-Minkowskian (PM) expansion

$$\frac{Gm}{r} \ll v^2 \sim 1$$

- ▶ Self-force expansion

$$\frac{Gm}{r} \sim v^2 \sim 1, \quad \frac{m_1}{m_2} \ll 1$$



Khalil, Buonanno, Steinhoff, Vines, 2204.05047

Amplitude-based methods naturally lead to PM expansion

PM expansion is relevant to bound orbits with large eccentricity and scattering process

# Why don't we just use numerical relativity?

Numerical relativity provides the most accurate observables (waveform, scattering angle, etc) for binaries with mass ratio  $q \sim 1$

- ▶ Works for the entire binary merger process
- ▶ Numerical error under very good control (can be considered as the TRUTH)
- ▶ Computationally expensive: need days of computation time on a super-computer to produce one waveform template
- ▶ Matched filtering analysis requires a dense sampling of the parameter space
- ▶ In light of future observatories, we need  $\sim 10^6$  waveform templates

We need analytic perturbative computations to efficiently generate observables

# Why amplitudeologists are working on GW?

It all starts with an enticing invitation



PHYSICAL REVIEW D **97**, 044038 (2018)

## High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour<sup>\*</sup>

*Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France*



(Received 29 October 2017; published 26 February 2018)

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

DOI: [10.1103/PhysRevD.97.044038](https://doi.org/10.1103/PhysRevD.97.044038)

# Latest progress

[ — GR method — ]

|      |     | 1687            | 1938  | 1973  | 2001  | 2019  | in progress |          |             |
|------|-----|-----------------|-------|-------|-------|-------|-------------|----------|-------------|
|      |     | 0PN             | 1PN   | 2PN   | 3PN   | 4PN   | 5PN         | 6PN      |             |
| 1979 | 1PM | 1               | $v^2$ | $v^4$ | $v^6$ | $v^8$ | $v^{10}$    | $v^{12}$ | $\dots G^1$ |
| 2018 | 2PM |                 | 1     | $v^2$ | $v^4$ | $v^6$ | $v^8$       | $v^{10}$ | $\dots G^2$ |
| 2019 | 3PM |                 |       | 1     | $v^2$ | $v^4$ | $v^6$       | $v^8$    | $\dots G^3$ |
| 2021 | 4PM |                 |       |       | 1     | $v^2$ | $v^4$       | $v^6$    | $\dots G^4$ |
| 2024 | 5PM | 2SF in progress |       |       |       | 1     | $v^2$       | $v^4$    | $\dots G^5$ |
|      | 6PM |                 |       |       |       |       | 1           | $v^2$    | $\dots G^6$ |

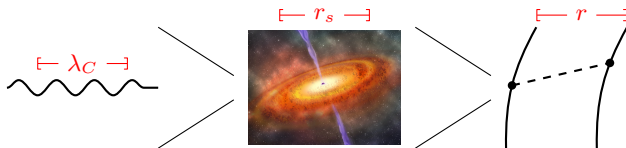
↑  
amplitude  
method  
↓

Notably, the 5PM 1SF result is obtained through an improved world-line formalism incorporated with amplitude ingredients

Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch, 2403.07781

Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch, 2411.11846

# From quantum to classical two-body problem



$$\lambda_C \ll r_s \implies Gm^2 \gg 1$$

Classical contribution is encoded in every loop order

$$r_s \ll r \implies Gmq \ll 1$$

$Gmq$  is the small parameter in the classical perturbative expansion

$$\lambda_C \ll r \implies q/m \ll 1$$

Expand in small momentum transfer and keep only the leading order  
(Bohr's correspondence principle)

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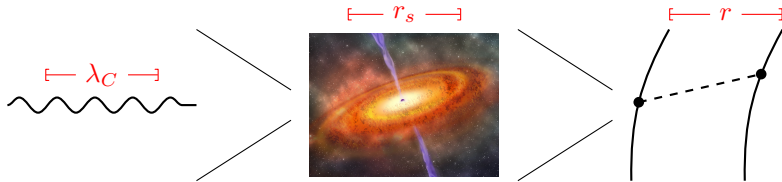
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# EFT matching using amplitudes Cheung, Rothstein, Solon, 1808.02489



Full theory: 
$$S_{\text{full}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{2} \int d^4x \sum_{i=1,2} (g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - m_i^2 \phi_i^2) + \mathcal{O}(R^2 \phi^2)$$

Implemented by method of regions  
Beneke, Smirnov, hep-ph/9711391

Classical limit  $(q, \ell, G) \rightarrow (\hbar q, \hbar \ell, \hbar^{-1} G)$

Integrate out soft gravitons

Observables  $\xleftarrow{\text{Eikonal formula}} \mathcal{M}_{\text{QFT}} = \mathcal{M}_{\text{EFT}}$

$V_{\text{PM}}$  given by an ansatz  
Solve  $V_{\text{PM}}$  by matching amplitudes

EOM

Effective theory: 
$$S_{\text{eff}} = \int dt \left[ m_1 \sqrt{1 - \mathbf{v}_1^2} + m_2 \sqrt{1 - \mathbf{v}_2^2} - V_{\text{PM}} \right]$$

# EFT matching Cheung, Rothstein, Solon, 1808.02489

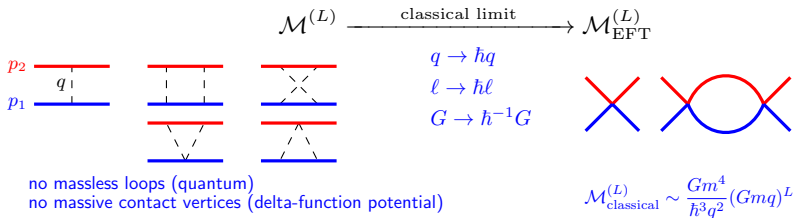
- Full theory: Schwarzschild black hole  $\implies$  scalar field  $\phi$

$$\mathcal{L} = \sqrt{-g} \left[ -\frac{1}{16\pi G} R + \frac{1}{2} \sum_{i=1,2} (g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - m_i^2 \phi_i^2) \right] + \mathcal{O}(R^2 \phi^2)$$

- Effective theory: potential  $V(\mathbf{k}, \mathbf{k}')$  given by an ansatz

$$L = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[ \sum_{i=1,2} a_i^\dagger(-\mathbf{k}) \left( i\partial_t + \sqrt{\mathbf{k}^2 + m_i^2} \right) a_i(\mathbf{k}) - \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}') a_1^\dagger(\mathbf{k}') a_1(\mathbf{k}) a_2^\dagger(-\mathbf{k}') a_2(\mathbf{k}) \right]$$

- Solve the EFT potential by matching the full theory and EFT amplitudes order-by-order in  $G$  in the classical limit



# Classical amplitudes

## Tree level

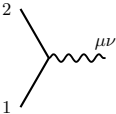
Starting from the Lagrangian and  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

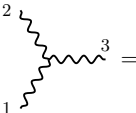
$$\mathcal{L} = \sqrt{-g} \left[ -\frac{2}{\kappa^2} R + \frac{1}{2} \sum_{i=1,2} (g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - m_i^2 \phi_i^2) \right] \quad \kappa^2 = 32\pi G$$

We derive the Feynman rules in the de Donder gauge


$$= \frac{i}{p^2 - m^2}$$


$$= \frac{i}{2q^2} \left[ \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{2}{d-2} \eta_{\mu\nu} \eta_{\rho\sigma} \right]$$


$$= \kappa p_1^{(\mu} p_2^{\nu)} - \frac{\kappa}{2} \eta^{\mu\nu} (p_1 \cdot p_2 + m^2)$$


$$=$$

(144 terms)

# Classical amplitudes

## Tree level

We define a set of convenient variables such that  $\bar{p}_i \cdot q = 0$  Landshoff and Polkinghorne 1969

$$\bar{p}_1 = p_1 + q/2 \quad \bar{p}_2 = p_2 - q/2 \quad \bar{m}_i^2 = \bar{p}_i^2 \quad y = \bar{p}_1 \cdot \bar{p}_2$$

## Full quantum amplitude

$$M^{(0)} \left[ \begin{array}{c} 2 \\ q \uparrow \\ 1 \end{array} \right] = - \frac{\kappa^2 \bar{m}_1^2 \bar{m}_2^2 (2y^2 - 1)}{2\hbar^3 q^2} + \frac{\kappa^2 (4\bar{m}_1^2 + 4\bar{m}_2^2 + 3\hbar^2 q^2)}{32\hbar}$$

classical

higher order in  $q$ , quantum

## Classical amplitude

$$x = \frac{1}{d-2} \text{ for Einstein gravity}$$

$$x = 0 \text{ for } \mathcal{N} = 8 \text{ sugra}$$

$$M_{\text{qft-cl}}^{(0)} = \begin{array}{c} 2 \\ q \uparrow \\ 1 \end{array} = - \frac{\kappa^2 \bar{m}_1^2 \bar{m}_2^2 (y^2 - x)}{q^2} \quad (\text{restoring } d \text{ dependence})$$

## Impact parameter space

$$\text{FT}[f(q^2)] = \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) \hat{\delta}(2\bar{p}_2 \cdot q) f(q^2) e^{iq \cdot b} \quad \hat{\delta}(x) = 2\pi \delta(x)$$

$$\delta^{(0)} = \text{FT} \left[ M_{\text{qft-cl}}^{(0)} \right] = \frac{\kappa^2 \bar{m}_1 \bar{m}_2 (y^2 - x) (-\pi b^2)^\epsilon \Gamma(-\epsilon)}{16\pi \sqrt{y^2 - 1}}$$

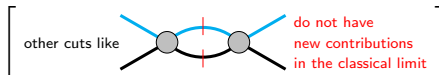
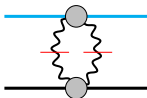
# Classical amplitudes

One loop

We use generalized unitarity to construct the integrand

Bern, Dixon, Dunbar, Kosower, hep-ph/9403226, hep-ph/9409265

spanning cuts:



We use the full quantum gravitational Compton amplitude as the building block

$$=$$

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# Classical amplitudes

## One loop

Method of regions expansion  $(q, \ell, G) \rightarrow (\hbar q, \hbar \ell, \hbar^{-1} G)$

$$\frac{1}{(p_1 + \hbar \ell)^2 - m_1^2 + i0} = \frac{1}{\hbar} \frac{1}{2\bar{p}_1 \cdot \ell + i0} + \frac{q \cdot \ell - \ell^2}{(2\bar{p}_1 \cdot \ell + i0)^2} + \dots$$

Integral family in the soft region

$$\mathcal{I}_{\pm \pm}^{a_1 a_2}[N] = \int \frac{d^d \ell}{(2\pi)^d} \frac{N(\ell)}{(2\bar{p}_1 \cdot \ell \pm i0)^{a_1} (2\bar{p}_2 \cdot \ell \pm i0)^{a_2} \ell^2 (\ell - q)^2}$$

Master integral basis under IBP reduction

$$M^{(1)} = \underbrace{\frac{ik^4 \bar{m}_1^4 \bar{m}_2^4 (2y^2 - 1)^2}{8\hbar^4} \mathcal{I}_{\square}}_{\text{super-classical } M_{\text{qft-sc}}^{(1)}} + \underbrace{\frac{3\kappa^4 (5y^2 - 1)}{32\hbar^3} \left[ \bar{m}_1^4 \bar{m}_2^2 \mathcal{I}_{\Delta} + \bar{m}_1^2 \bar{m}_2^4 \mathcal{I}_{\nabla} \right]}_{\text{classical } M_{\text{qft-cl}}^{(1)}} + \mathcal{O}(\hbar^{-2})$$

$$\mathcal{I}_{\square} = e^{\epsilon \gamma_E} \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2\bar{p}_1 \cdot \ell) \hat{\delta}(2\bar{p}_2 \cdot \ell)}{\ell^2 (\ell - q)^2} = \frac{e^{\epsilon \gamma_E}}{4\bar{m}_1 \bar{m}_2 \sqrt{y^2 - 1}} \frac{\Gamma(-\epsilon)^2 \Gamma(1 + \epsilon)}{(4\pi)^{1-\epsilon} \Gamma(-2\epsilon) (-q^2)^{1+\epsilon}}$$

$$\mathcal{I}_{\Delta} = \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2\bar{p}_1 \cdot \ell)}{\ell^2 (\ell - q)^2} = \frac{1}{16\bar{m}_1 \sqrt{-q^2}} \quad \mathcal{I}_{\nabla} = \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2\bar{p}_2 \cdot \ell)}{\ell^2 (\ell - q)^2} = \frac{1}{16\bar{m}_2 \sqrt{-q^2}}$$

# Classical amplitudes

## One loop

Leading order in  $\hbar$  expansion (super-classical):

- IR divergent, but...

$$\mathcal{I}_{\square} = \frac{1}{8\pi\bar{m}_1\bar{m}_2\sqrt{y^2-1}q^2} \left[ \frac{1}{\epsilon} - \log \frac{-q^2}{4\pi} \right] + \mathcal{O}(\epsilon)$$

- Does not contain new information at  $G^2$  order

$$\text{FT} \left[ iM_{\text{qft-sc}}^{(1)} \right] = \frac{1}{2} \left( i\delta^{(0)} \right)^2 \quad (\text{iteration of tree level})$$

Classical physics at  $G^2$  is contained in the classical amplitude

$$M_{\text{qft-cl}}^{(1)} = \frac{3\kappa^4\bar{m}_1\bar{m}_2(\bar{m}_1 + \bar{m}_2)(5y^2 - 1)}{128\sqrt{-q^2}}$$
$$\delta^{(1)} = \text{FT} \left[ M_{\text{qft-cl}}^{(1)} \right] = \frac{3\kappa^4\bar{m}_1\bar{m}_2(\bar{m}_1 + \bar{m}_2)(5y^2 - 1)}{4096\pi\sqrt{y^2-1}\sqrt{-b^2}}$$

# Matching

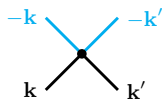
Tree level and one loop

Effective theory: potential  $V(\mathbf{k}, \mathbf{k}')$  given by an ansatz

$$L = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[ \sum_{i=1,2} a_i^\dagger(-\mathbf{k}) \left( i\partial_t + \sqrt{\mathbf{k}^2 + m_i^2} \right) a_i(\mathbf{k}) - \int \frac{d^3\mathbf{k}'}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}') a_1^\dagger(\mathbf{k}') a_1(\mathbf{k}) a_2^\dagger(-\mathbf{k}') a_2(\mathbf{k}) \right]$$

$$V(\mathbf{p}, \mathbf{p} - \mathbf{q}) = \frac{4\pi G}{\mathbf{q}^2} c_1(\mathbf{p}^2) + \frac{2\pi^2 G^2}{|\mathbf{q}|} c_2(\mathbf{p}^2) + \dots$$

Feynman rules



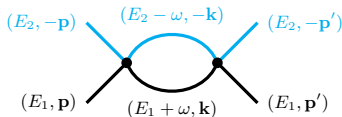
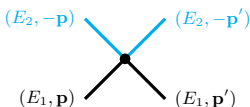
$$= -iV(\mathbf{k}, \mathbf{k}')$$

$$\frac{(E, \mathbf{k})}{\hspace{1.5cm}} = \frac{i}{E - \sqrt{\mathbf{k}^2 + m^2} + i0}$$

# Matching

Tree level and one loop

Feynman diagrams



Classical region:  $\mathbf{p}' - \mathbf{p} = \hbar \mathbf{q}$ ,  $\mathbf{k} - \mathbf{p} = \hbar \mathbf{l}$

$(E = E_1 + E_2 \text{ and } \xi = E_1 E_2 / E^2)$

$$M_{\text{EFT}}^{(0)} = -\frac{4\pi G}{\mathbf{q}^2} c_1(\mathbf{p}^2)$$

$$M_{\text{EFT}}^{(1)} = -\frac{2\pi^2 G^2}{|\mathbf{q}|} c_2(\mathbf{p}^2) + \frac{\pi^2 G^2}{E\xi|\mathbf{q}|} \left[ (1 - 3\xi) c_1^2(\mathbf{p}^2) + 4\xi^2 E^2 c_1(\mathbf{p}^2) c_1'(\mathbf{p}^2) \right]$$

$$+ \int \frac{d^{d-1} \mathbf{l}}{(2\pi)^{d-1}} \frac{32E\xi\pi^2 G^2 c_1^2(\mathbf{p}^2)}{\mathbf{l}^2 (1 + \mathbf{q})^2 (\mathbf{l}^2 + 2\mathbf{p} \cdot \mathbf{l})}$$

super-classical matches exactly the iteration  
IR divergence cancels

Solve the WCs through matching

$$M_{\text{EFT}}^{(0)} = \frac{M_{\text{qft-cl}}^{(0)}}{4E_1 E_2}$$

$$M_{\text{EFT}}^{(1)} = \frac{M_{\text{qft-sc}}^{(1)} + M_{\text{qft-cl}}^{(1)}}{4E_1 E_2}$$

$$E_1 = \sqrt{\mathbf{p}^2 + m_1^2} \quad E_2 = \sqrt{\mathbf{p}^2 + m_2^2}$$

$$\gamma = \frac{E_1 + E_2}{m_1 + m_2} \quad \xi = \frac{E_1 E_2}{(E_1 + E_2)^2}$$

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \quad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

**Hamiltonian:**  $H = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{|\mathbf{r}|^n} c_{n\text{PM}}(\mathbf{p}^2)$

$$c_{1\text{PM}} = -\frac{\nu^2(m_1 + m_2)^2}{\gamma^2 \xi} (2\sigma^2 - 1) \quad \text{Westpfahl and Goller 1979}$$

$$c_{2\text{PM}} = -\frac{\nu^2(m_1 + m_2)^3}{\gamma^2 \xi} \left[ \frac{3(5\sigma^2 - 1)}{4} - \frac{4\nu\sigma(2\sigma^2 - 1)}{\gamma \xi} + \frac{\nu^2(1 - \xi)(2\sigma^2 - 1)^2}{2\gamma^3 \xi^2} \right] \quad \text{Bel, Damour, Deruelle, Ibanez, Martin, 1981}$$

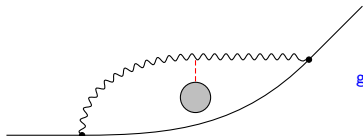
Westpfahl 1985

$$c_{3\text{PM}} = \frac{\nu^2 m^4}{\gamma^2 \xi} \left[ \frac{3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3}{12} - \frac{4\nu(3 + 12\sigma^2 - 4\sigma^4) \operatorname{arcsinh} \sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}} \right. \\ \left. - \frac{3\nu\gamma(2\sigma^2 - 1)(5\sigma^2 - 1)}{2(1 + \gamma)(1 + \sigma)} + \frac{3\nu\sigma(20\sigma^2 - 7)}{2\gamma \xi} + \frac{\nu^2(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2)(2\sigma^2 - 1)}{4\gamma^3 \xi^2} \right. \\ \left. + \frac{2\nu^3(3 - 4\xi)\sigma(2\sigma^2 - 1)^2}{\gamma^4 \xi^3} - \frac{\nu^4(1 - 2\xi)(2\sigma^2 - 1)^3}{2\gamma^6 \xi^4} \right] \quad \text{(This is what Damour was asking for)}$$

State-of-the-art:  $c_{4\text{PM}}^{\text{hyp}}$  and  $c_{5\text{PM}}^{\text{hyp 1SF}}$

- ▶  $c_{3\text{PM}}$  is not known to general relativists before computed this way
- ▶  $c_{5\text{PM}}$  for GR is obtained using the amplitude-worldline hybrid method

# Gravitational tail



gravitational tails depend on the trajectory

$$c_4^{\text{hyp}} = \frac{M^7 \nu^2}{4\xi E^2} \left[ \mathcal{M}_4^{\text{p}} + \nu \left( 4\mathcal{M}_4^{\text{t}} \log\left(\frac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right) \right] + \mathcal{D}^3 \left[ \frac{E^3 \xi^3}{3} c_1^4 \right] + \mathcal{D}^2 \left[ \left( \frac{E^3 \xi^3}{p^2} + \frac{E\xi(3\xi-1)}{2} \right) c_1^4 - 2E^2 \xi^2 c_1^2 c_2 \right] \\ + \left( \mathcal{D} + \frac{1}{p^2} \right) \left[ E\xi(2c_1 c_3 + c_2^2) + \left( \frac{4\xi-1}{4E} + \frac{2E^3 \xi^3}{p^4} + \frac{E\xi(3\xi-1)}{p^2} \right) c_1^4 + \left( (1-3\xi) - \frac{4E^2 \xi^2}{p^2} \right) c_1^2 c_2 \right], \quad (8)$$

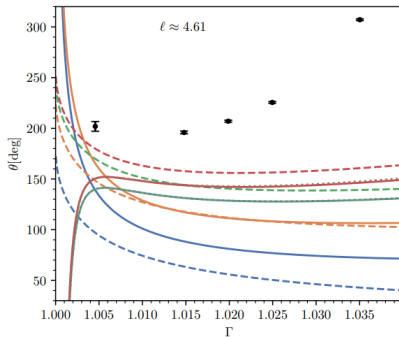
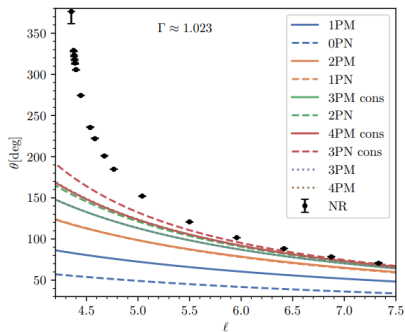
where  $\mathcal{D} = \frac{d}{dp^2}$

Bern, Parra-Martinez, Roiban, Ruf, Solon, Shen, Zeng, 2112.10750

# Comparison with numerical relativity

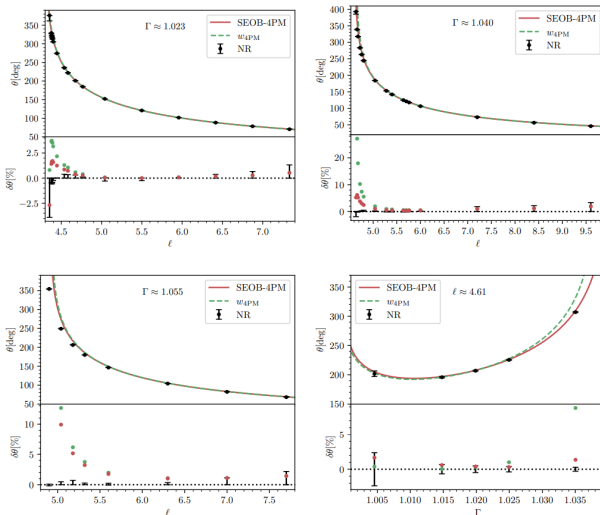
Fixed-order perturbative computation only works well for weak couplings

Buonanno, Jakobsen, Mogull, 2402.12342



# Effective-one-body (EOB) resummation is necessary for strong couplings

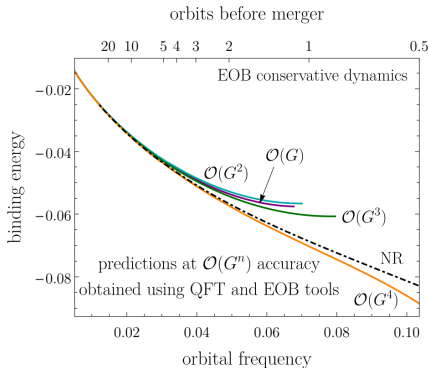
Buonanno, Damour, gr-qc/9811091 Buonanno, Jakobsen, Mogull, 2402.12342



# Bound orbits

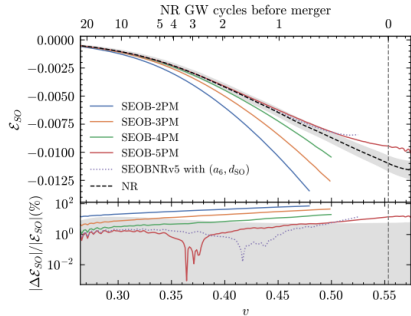
## PM-informed two-body effective Hamiltonian

- ▶ Local contribution: direct analytic continuation
- ▶ Nonlocal (tail) contribution: supplement PN results from GR computations



Khalil, Buonanno, Steinhoff, Vines, 2205.05047

Snowmass white paper, 2204.05194



Buonanno, Mogull, Patil, Pompili, 2405.19181

We have explicitly computed the impact-parameter space amplitudes up to one loop

$$\delta^{(0)} = \text{FT} \left[ M_{\text{qft-cl}}^{(0)} \right] = \frac{\kappa^2 \tilde{m}_1 \tilde{m}_2 (y^2 - 1/2) (-\pi b^2)^\epsilon \Gamma(-\epsilon)}{16\pi \sqrt{y^2 - 1}} = \frac{\kappa^2 \tilde{m}_1 \tilde{m}_2 (y^2 - 1/2)}{16\pi \sqrt{y^2 - 1}} \left[ -\frac{1}{\epsilon} - \log(-\pi b^2) - \gamma_E \right]$$

$$\delta^{(1)} = \text{FT} \left[ M_{\text{qft-cl}}^{(1)} \right] = \frac{3\kappa^4 \tilde{m}_1 \tilde{m}_2 (\tilde{m}_1 + \tilde{m}_2) (5y^2 - 1)}{4096\pi \sqrt{y^2 - 1} \sqrt{-b^2}} \quad \text{FT} \left[ iM_{\text{qft-sc}}^{(1)} \right] = \frac{1}{2} \left( i\delta^{(0)} \right)^2$$

## Eikonalization conjecture

$$\text{FT} [iM(q)] = (1 + i\Delta) e^{i\delta/\hbar} - 1$$

The conjecture is explicitly verified up to two-loop level

Di Vecchia, Heissenberg, Russo, Veneziano, 2101.05772

- $\delta = \delta^{(0)} + \delta^{(1)} + \dots$  encodes all the classical physics (generating function for classical conservative observables)

$$\text{scattering angle: } \theta = \frac{\partial \delta}{\partial J} \quad \text{time delay: } T = \frac{\partial \delta}{\partial E}$$

- $\Delta = \Delta^{(1)} + \Delta^{(2)} + \dots$  is the quantum reminder

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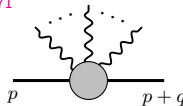
Outlook

Black holes and neutron stars can carry spin

How to incorporate spin into the EFT?

# On-shell description of spin

Bern, Luna, Roiban, Shen, Zeng, 2005.03071



- On-shell spin- $s$  states are **symmetric traceless and transverse**

$$\varepsilon_{a_1 a_2 \dots a_s} = \varepsilon_{(a_1 a_2 \dots a_s)} \quad p^{a_1} \varepsilon_{a_1 a_2 \dots a_s} = \eta^{a_1 a_2} \varepsilon_{a_1 a_2 \dots a_s} = 0$$

- Classical limit  $\implies$  **spin coherent state**  $\varepsilon_{a_1 a_2 \dots a_s}^s = \varepsilon_{a_1}^+ \varepsilon_{a_2}^+ \dots \varepsilon_{a_s}^+$  with large  $s$

$$\begin{aligned} \varepsilon_p^s \cdot M^{ab} \cdot \varepsilon_{p+q}^s &\sim S^{ab} & (M^{ab})_{c(s)}^{d(s)} &= -2is \delta_{(c_1}^{[a} \eta^{b](d_1} \delta_{c_2}^{d_2} \dots \delta_{c_s}^{d_s)} \\ \varepsilon_p^s \cdot \{M^{ab} M^{cd}\} \cdot \varepsilon_{p+q}^s &\sim S^{ab} S^{cd} & S^{ab} &= (1/m) \varepsilon^{abcd} p_c S_d \end{aligned}$$

- The spin tensor satisfy **covariant spin supplementary condition (SSC)**

$$S^{ab} p_b = 0 \quad (S^{ab} \text{ is boosted from rest frame } S^{ij})$$

- Transversality** and **covariant SSC** are related

- Spin magnitude is conserved:  $S^{ab} S_{ab} \sim S^a S_a \sim \mathbf{S}^2 = \text{const}$

# Effective field theory for higher-spin particles

Higher spin quantum field theory ( $\phi_s \equiv \phi_{a_1 a_2 \dots a_s}$ )

$$\begin{aligned}\nabla_\mu \phi_s &= \partial_\mu \phi_s + (i/2) \omega_{\mu ab} M^{ab} \phi_s \\ \mathbb{S}^a &= (-i/2m) \epsilon^{abcd} M_{cd} \nabla_b\end{aligned}$$

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2} \phi_s (\nabla^2 + m^2) \phi_s + \frac{1}{8} R_{abcd} \phi_s M^{ab} M^{cd} \phi_s - \frac{C_2}{2m^2} R_{af_1 b f_2} \nabla^a \phi_s \mathbb{S}^{(f_1} \mathbb{S}^{f_2)} \nabla^b \phi_s \\ & + \frac{D_2}{2m^2} R_{abcd} \nabla_i \phi_s \{M^{ai} M^{cd}\} \phi_s + \frac{E_2 - 2D_2}{2m^4} R_{abcd} \nabla^{(a} \nabla^{i)} \phi_s \{M^b{}_i M^d{}_j\} \nabla^{(c} \nabla^{j)} \phi_s + \mathcal{O}(M_{ab}^3)\end{aligned}$$

We prefer to use a formalism in which the classical and large spin limit is straightforward

- ▶ Contractions of  $\phi_s$  facilitated by  $M^{ab}$  only
- ▶ Propagator uniform in  $s$ :  $i\delta_{a(s)}^{b(s)}/(p^2 - m^2)$
- ▶ There are additional lower spin ( $s' < s$ ) states in the spectrum

Problematic? Not in the classical limit:

- ▶ Ghost nature easily cured by an analytic continuation on classical variables
- ▶ We get a more generic non-rigid spinning object (more internal DOFs)
- ▶ Conventional rigid spinning objects correspond to special Wilson coefficients

# Generalized spin coherent state

Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, Vines, 2308.14176

The external state now contains lower spin components. Consider the coherent sum

$$\mathcal{E}_{\mu_1 \dots \mu_s} = \varepsilon_{\mu_1 \dots \mu_s}^{(s)} + u_{(\mu_1} \varepsilon_{\mu_2 \dots \mu_s)}^{(s-1)} + \dots$$

Similar coherent sum was also considered in Aoude, Ochirov, 2108.01649, etc

Classical limit

$$\begin{aligned}\mathcal{E}_p \cdot M^{ab} \cdot \mathcal{E}_{p+q} &\sim S^{ab} & S^{ab} &= S^{ab} + (i/m)(p^a K^b - p^b K^a) \\ \mathcal{E}_p \cdot \{M^{ab} M^{cd}\} \cdot \mathcal{E}_{p+q} &\sim S^{ab} S^{cd}\end{aligned}$$

where  $K^a$  is identified as the **boost generator**, and  $S^{ab} p_b = K^a p_a = 0$

- ▶  $K^a$  emerges from the transition between spin  $s$  and lower spin states
- ▶ Consequently,  $S^{ab} S_{ab} \sim \mathbf{S}^2 - \mathbf{K}^2$  is a still constant but  $\mathbf{S}^2$  is not

# Non-minimal interactions up to $\mathcal{O}(M_{ab}^2)$

Alavertian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

$$\nabla_\mu \phi_s = \partial_\mu \phi_s + (i/2)\omega_{\mu ab} M^{ab} \phi_s$$

$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd} M_{cd} \nabla_b$$

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{1}{8}R_{abcd}\phi_s M^{ab}M^{cd}\phi_s - \frac{C_2}{2m^2}R_{af_1bf_2}\nabla^a\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2)}\nabla^b\phi_s \\ & + \frac{D_2}{2m^2}R_{abcd}\nabla_i\phi_s\{M^{ai}M^{cd}\}\phi_s + \frac{E_2 - 2D_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_iM^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s \end{aligned}$$

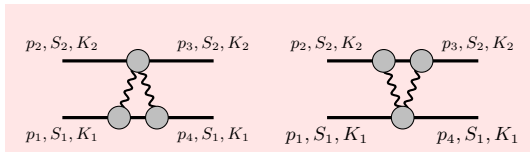
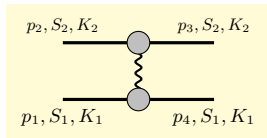
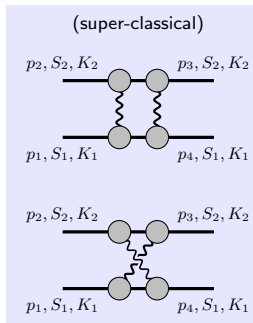
- The  $C_2$ -operator has an origin in the world-line formalism for neutron stars  
Porto, 0511061; Levi, Steinhoff, 1501.04956
- It is the only independent operator assuming that rest frame spin is the only dynamical degree of freedom
- The  $D_2$ - and  $E_2$ -operators supply additional  $\mathcal{O}(SK)$  and  $\mathcal{O}(K^2)$  interactions

$$D_2 = E_2 = 0 \quad \implies \quad \text{Conventional compact object described by } H(\mathbf{r}, \mathbf{p}, \mathbf{S})$$

$$C_2 = D_2 = E_2 = 0 \quad \implies \quad \text{Kerr black hole}$$

Generic values: generic compact object described by  $H(\mathbf{r}, \mathbf{p}, \mathbf{S}, \mathbf{K})$

# Two-body amplitudes



$$\begin{aligned}
 \mathcal{M}^{\text{body}} = & A_0 + A_1 \mathbf{L} \cdot \mathbf{S} + A_{2,1} \mathbf{S}^2 + A_{2,2} \mathbf{K}^2 + A_{2,3} \mathbf{S} \cdot \mathbf{K} + A_{2,4} (\mathbf{b} \cdot \mathbf{S})^2 \\
 & + A_{2,5} (\mathbf{p} \cdot \mathbf{S})^2 + A_{2,6} (\mathbf{b} \cdot \mathbf{K})^2 + A_{2,7} (\mathbf{p} \cdot \mathbf{K})^2 + A_{2,8} (\mathbf{b} \cdot \mathbf{S})(\mathbf{p} \cdot \mathbf{S}) \\
 & + A_{2,9} (\mathbf{L} \cdot \mathbf{S})(\mathbf{b} \cdot \mathbf{K}) + A_{2,10} (\mathbf{L} \cdot \mathbf{S})(\mathbf{p} \cdot \mathbf{K}) + A_{2,11} (\mathbf{b} \cdot \mathbf{S})(\mathbf{L} \cdot \mathbf{K}) \\
 & + A_{2,12} (\mathbf{p} \cdot \mathbf{S})(\mathbf{L} \cdot \mathbf{K}) + A_{2,13} (\mathbf{b} \cdot \mathbf{K})(\mathbf{p} \cdot \mathbf{K})
 \end{aligned}$$

# Effective Hamiltonian through matching

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, FT, 2407.10928, 2503.03739



Consider canonical spin in the COM frame

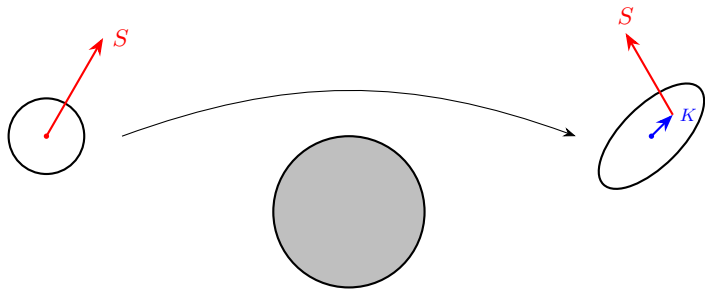
$$H = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + \sum_a \sum_{n=1}^{\infty} \left( \frac{G}{|\mathbf{r}|} \right)^n c_n^a(\mathbf{p}^2) \Sigma_a$$

where the operators  $\Sigma_a$  takes value in

$$\begin{array}{ccc} 1 & ((\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S}) / \mathbf{r}^2 & (\mathbf{r} \cdot \mathbf{K}) / \mathbf{r}^2 \\ (\mathbf{r} \cdot \mathbf{S})^2 / \mathbf{r}^4 & (\mathbf{r} \cdot \mathbf{K}) ((\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S}) / \mathbf{r}^4 & (\mathbf{r} \cdot \mathbf{K})^2 / \mathbf{r}^4 \\ \mathbf{S}^2 / \mathbf{r}^2 & (\mathbf{K} \cdot (\mathbf{p} \times \mathbf{S})) / \mathbf{r}^2 & \mathbf{K}^2 / \mathbf{r}^2 \\ (\mathbf{p} \cdot \mathbf{S})^2 / \mathbf{r}^2 & (\mathbf{r} \cdot \mathbf{S}) ((\mathbf{r} \times \mathbf{K}) \cdot \mathbf{p}) / \mathbf{r}^4 & (\mathbf{p} \cdot \mathbf{K})^2 / \mathbf{r}^2 \end{array}$$

- $c_0^a$  matches to the tree level amplitude at  $\mathcal{O}(G)$
- Iteration of  $c_0^a$  should agree exactly with the super-classical box coefficients at  $\mathcal{O}(G^2)$
- $c_1^a$  matches to the triangle coefficients at  $\mathcal{O}(G^2)$  order of  $V$
- The coefficient of  $(\mathbf{r} \cdot \mathbf{K}) / \mathbf{r}^2$  vanishes identically
- All the  $c_n^a$  coefficients are local in  $\mathbf{p}^2$

## Generic spinning body with $K$

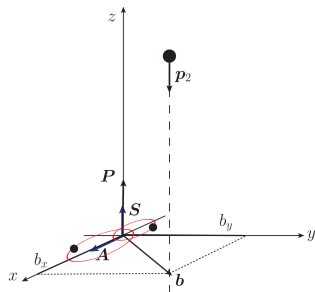


An additional conservative gapless degree of freedom

# Scattering off a Newtonian bound state

$$\begin{aligned}
 H &= \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{P}^2}{2m_1} + H_0(\mathbf{p}, \mathbf{r}) - \frac{Gm_{B1}m_2}{|\mathbf{R} + \frac{m_{B2}}{m_1}\mathbf{r}|} - \frac{Gm_{B2}m_2}{|\mathbf{R} - \frac{m_{B1}}{m_1}\mathbf{r}|} \\
 &= \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{P}^2}{2m_1} + H_0(\mathbf{p}, \mathbf{r}) - \frac{Gm_1m_2}{|\mathbf{R}|} - \frac{3G\mu_B m_2}{2|\mathbf{R}|^5} \underbrace{\left( (\mathbf{r} \cdot \mathbf{R})^2 - \frac{1}{3}|\mathbf{r}|^2|\mathbf{R}|^2 \right)}_{Q_{ij}(\mathbf{r})Q^{ij}(\mathbf{R})} + \dots
 \end{aligned}$$

where  $m_1 = m_{B1} + m_{B2}$  and  $\mu_B = m_{B1}m_{B2}/m_1$



## Scattering off a Newtonian bound state

$$\begin{aligned}\mathcal{A}_{i \rightarrow f} &= \int_{-\infty}^{+\infty} dt e^{i(E_f^B - E_i^B)t} \left\langle i \left| \frac{3G\mu_B m_2}{2|\mathbf{R}|^5} \left( (\mathbf{r} \cdot \mathbf{R})^2 - \frac{1}{3} |\mathbf{r}|^2 |\mathbf{R}|^2 \right) \right| f \right\rangle \\ &= \frac{3G\mu_B m_2 r_{\text{cl},n}^2}{2|\mathbf{b}|^2 v_0} \left[ \frac{2(\mathbf{b} \cdot \mathbf{A})^2}{|\mathbf{b}|^2} - |\mathbf{A}|^2 \right]\end{aligned}$$

- ▶ Trajectory:  $\mathbf{R} = (b_x, b_y, -v_0 t)$
- ▶ Initial and final state have the same energy; otherwise exponentially suppressed
- ▶ Use elliptical orbit coherent state with  $b v_0^2 \gg r_{\text{cl},n}$  [Bhaumik, Dutta-Roy, Ghosh, 1986](#)

$$\langle \alpha | x | \alpha \rangle = r_{\text{cl},n} \left[ \cos(2\omega_{\text{cl}} t) + \sin(2\chi) \right]$$

$$\langle \alpha | y | \alpha \rangle = r_{\text{cl},n} \sin(2\omega_{\text{cl}} t) \cos(2\chi)$$

$$\langle \alpha | z | \alpha \rangle = 0$$

- ▶ Laplace-Runge-Lenz vector  $\mathbf{A} = \sin(2\chi) \hat{\mathbf{x}}$

# Scattering off a Newtonian bound state

$$\mathcal{M}^2 \text{ body} \sim A_{2,1} \mathbf{K}^2 + A_{2,6} (\mathbf{b} \cdot \mathbf{K})^2$$

Match to the field theory amplitude:

- ▶ Spin  $\Leftrightarrow$  bound system total orbital angular momentum
- ▶ Due to the geometric configuration, the spin does not appear in  $\mathcal{A}_{i \rightarrow f}$
- ▶  $\mathbf{K}$ -vector  $\Leftrightarrow$  Laplace-Runge-Lenz vector

$$\mathbf{K} = i G m_1^2 \frac{\mu_B}{m_1} \sqrt{\frac{\mu_B}{2|\mathbf{E}_i^B|}} \mathbf{A}$$

- ▶ Wilson coefficient

$$E_2^{\text{bound 2-body}} = \frac{3|\mathbf{E}_i^B| m_1}{\mu_B^2} (m_1 r_{\text{cl},n})^2$$

$$\mathcal{L} \sim \frac{E_2}{2m^4} R_{abcd} \nabla^{(a} \nabla^{i)} \phi_s \{ M^b{}_i M^d{}_j \} \nabla^{(c} \nabla^{j)} \phi_s$$

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# KMOC formalism Kosower, Maybee, O'Connell, 1811.10950

Observable in a scattering process:

$$\begin{aligned}\Delta O &= \langle \psi_{\text{out}} | \mathbb{O} | \psi_{\text{out}} \rangle - \langle \psi_{\text{in}} | \mathbb{O} | \psi_{\text{in}} \rangle \\ &= \langle \psi_{\text{in}} | \hat{S}^\dagger \mathbb{O} \hat{S} | \psi_{\text{in}} \rangle - \langle \psi_{\text{in}} | \mathbb{O} | \psi_{\text{in}} \rangle \\ &= i \langle \psi_{\text{in}} | [\mathbb{O}, \hat{T}] | \psi_{\text{in}} \rangle + \langle \psi_{\text{in}} | \hat{T}^\dagger [\mathbb{O}, \hat{T}] | \psi_{\text{in}} \rangle\end{aligned}$$
$$\begin{aligned}\hat{S} &= 1 + i\hat{T} \\ \hat{T} - \hat{T}^\dagger &= i\hat{T}^\dagger \hat{T}\end{aligned}$$

Incoming state:  $|\psi_{\text{in}}\rangle = \int d\Phi[p_1] d\Phi[p_2] \phi(p_1) \phi(p_2) e^{ip_1 \cdot b_1 + ip_2 \cdot b_2} |p_1 p_2\rangle$

Single particle phase-space and wavefunction:

$$d\Phi[p] = \frac{d^4 p}{(2\pi)^4} 2\pi \Theta(p^0) \delta(p^2 - m^2) \quad \int d\Phi[p] |\phi(p)|^2 = 1$$

We can compute observables by dressing amplitudes and cuts with the corresponding operators

$$\begin{aligned}\Delta O &= \int d\Phi[p_1] d\Phi[p_2] d\Phi[p'_1] d\Phi[p'_2] \phi(p_1) \phi(p_2) \phi(p'_1) \phi(p'_2) e^{i(p_1 - p'_1) \cdot b_1 + i(p_2 - p'_2) \cdot b_2} \\ &\quad \times \left[ i \langle p'_1 p'_2 | [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle + \langle p'_1 p'_2 | \hat{T}^\dagger [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle \right]\end{aligned}$$

# Classical observables Kosower, Maybee, O'Connell, 1811.10950

The wave packets are highly localized, while  $q_i$  is much smaller than their spread

$$\int d\Phi[p_1]d\Phi[p_2]d\Phi[p'_1]d\Phi[p'_2]\phi(p_1)\phi(p_2)\phi^*(p'_1)\phi^*(p'_2)e^{i(p_1-p'_1)\cdot b_1+i(p_2-p'_2)\cdot b_2}$$

$$\simeq \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \quad \hat{\delta}(x) = 2\pi\delta(x)$$

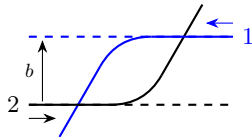
Resolution of identity in the two-massive-particle subspace

$$\mathbb{I} = \sum_X \int d\Phi[r_1]d\Phi[r_2] |r_1 r_2 X\rangle \langle r_1 r_2 X|$$

Expansion in the soft region  $(q_i, \ell, k, G) \rightarrow (\hbar q_i, \hbar \ell, \hbar k, \hbar^{-1} G)$

## Classical observables

$$\Delta O_{\text{cl}} = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \left[ i \langle p'_1 p'_2 | [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle + \langle p'_1 p'_2 | \hat{T}^\dagger [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle \right]$$



We compute the expectation value of

$$\mathbb{O} = \mathbb{P}_1^\mu \quad \mathbb{P}_1^\mu |p_1^\mu\rangle = p_1^\mu |p_1\rangle$$

Tree level impulse

$$[\Delta p_1^\mu]_{(0)} = \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} (-iq^\mu) \left[ \underbrace{-\frac{\kappa^2 \bar{m}_1^2 \bar{m}_2^2 (2y^2 - 1)}{2q^2}}_{\text{classical tree amp}} \right] = -\frac{\kappa^2 \bar{m}_1 \bar{m}_2 (2y^2 - 1)}{16\pi \sqrt{y^2 - 1}} \frac{b^\mu}{(-b^2)}$$

One loop impulse (from the S-matrix)

$$\begin{aligned} [\Delta p_1^\mu]_{(1, \perp)} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \left[ i \langle p'_1 p'_2 | [\mathbb{P}_1^\mu, \hat{T}] | p_1 p_2 \rangle \right] \\ &= \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} (-iq^\mu) \left[ M_{\text{amp-sc}}^{(1)} + M_{\text{amp-cl}}^{(1)} \right] \end{aligned}$$

We have a classically singular and IR divergent contribution  $M_{\text{qft-sc}}^{(1)}$ , but fortunately we have not done yet

$$M_{\text{amp-sc}}^{(1)} = \frac{i\kappa^4 \bar{m}_1^4 \bar{m}_2^4 (2y^2 - 1)^2}{8\hbar^4} \mathcal{I}_\square \quad M_{\text{amp-cl}}^{(1)} = \frac{3\kappa^4 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2) (5y^2 - 1)}{128 \sqrt{-q^2}}$$

# Impulse

One loop impulse (from the cut)

$$\tilde{u}_1 = \frac{y\tilde{u}_2 - \tilde{u}_1}{y^2 - 1} \text{ and } \tilde{u}_2 = \frac{y\tilde{u}_1 - \tilde{u}_2}{y^2 - 1}$$

$$\begin{aligned} [\Delta p_1^\mu]_{(1,\parallel)} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \left[ \langle p'_1 p'_2 | \hat{T}^\dagger [\mathbb{P}_1^\mu, \hat{T}] | p_1 p_2 \rangle \right] \\ &= \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} \left[ (-iq^\mu) M_{\text{cut-sc}}^{(1)} + \left( \frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1} \right) M_{\text{cut-cl}}^{(1)} \right] \end{aligned}$$

The classically singular contribution cancels exactly between the S-matrix and cut

$$M_{\text{amp-sc}}^{(1)} + M_{\text{cut-sc}}^{(1)} = 0 \quad M_{\text{cut-cl}}^{(1)} = \frac{(2y^2 - 1)^2 \bar{m}_1^3 \bar{m}_2^3}{32\pi \sqrt{y^2 - 1}} \log(-q^2)$$

Full one-loop impulse

$$\begin{aligned} [\Delta p_1^\mu]_{(1,\perp)} + [\Delta p_1^\mu]_{(1,\parallel)} &= \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} \left[ (-iq^\mu) M_{\text{amp-cl}}^{(1)} + \left( \frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1} \right) M_{\text{cut-cl}}^{(1)} \right] \\ &= - \underbrace{\frac{3\pi G^2 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2) (5y^2 - 1) b^\mu}{4\sqrt{y^2 - 1} (-b^2)^{3/2}}}_{\text{transverse}} - \underbrace{\frac{2G^2 \bar{m}_1^2 \bar{m}_2^2 (2y^2 - 1)^2}{(y^2 - 1) (-b^2)} \left[ \frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1} \right]}_{\text{longitudinal}} \end{aligned}$$

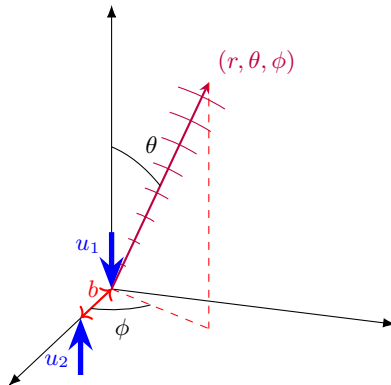
# Waveform

Herderschee, Roiban, **FT**, 2303.06112

Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**, 2402.06604

Metric perturbation:  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Waveform:  $W(T_R, \theta, \phi) = \frac{1}{4G} \lim_{r \rightarrow \infty} r \varepsilon_+^\mu \varepsilon_+^\nu h_{\mu\nu} = \frac{1}{4G} (h_+^\infty - i h_\times^\infty)$



## Operator: metric perturbation

$$\mathbb{H} = \varepsilon_+^\mu \varepsilon_+^\nu \hat{h}_{\mu\nu}(x) = \int d\Phi[k] \left[ \hat{a}_{--}(k) e^{-ik \cdot x} + \text{c.c.} \right]$$

Since there is no radiation at  $t \rightarrow -\infty$ ,

$$\Delta H(x) = \langle \psi_{\text{in}} | \hat{S}^\dagger \mathbb{H} \hat{S} | \psi_{\text{in}} \rangle = \int d\Phi[k] \left[ \tilde{J}(k) e^{-ik \cdot x} - \text{c.c.} \right] \quad \tilde{J}(k) = \langle \psi_{\text{in}} | \hat{S}^\dagger \hat{a}_{--}(k) \hat{S} | \psi_{\text{in}} \rangle$$

Integrating over  $k$  gives both retarded and advanced Green's function. We discarded the advanced one due to the boundary condition,

$$\Delta H(x) = -\frac{i}{4\pi^2} \int d^4y \frac{J(y)}{(x-y)^2 - i0} = -\frac{i}{4\pi^2} \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{k}}{(2\pi)^3} dy^0 d^3\mathbf{y} \frac{\tilde{J}(\omega, \mathbf{k}) e^{-i\omega y^0 + i\mathbf{k} \cdot \mathbf{y}}}{(x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2 - i0}$$

Assuming the spatial current  $J(y)$  is localized around  $y = 0$ ,

$$\Delta H(x) \Big|_{|\mathbf{x}| \rightarrow \infty} = \frac{1}{4\pi|\mathbf{x}|} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} W(\omega, \mathbf{n}) e^{-i\omega\tau} \quad (\tau = x^0 - |\mathbf{x}| : \text{retarded time})$$

$$\mathcal{W}(\varepsilon, \omega, \mathbf{n}, p_1, p_2, b_1, b_2) = (-i) \langle \psi_{\text{in}} | \hat{S}^\dagger \hat{a}_{--}(k) \hat{S} | \psi_{\text{in}} \rangle \Big|_{k=(\omega, \omega\mathbf{n})}^{\text{IR finite}} \quad (\text{frequency domain waveform})$$

(As we will see later, the IR divergence can be absorbed in the definition of  $\tau$ )

# Relevant matrix elements for waveform

In the space of momentum transfer:

$$(-i)\langle\psi_{\text{in}}|\hat{S}^\dagger\hat{a}(k)\hat{S}|\psi_{\text{in}}\rangle = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2p_1 \cdot q_1) \hat{\delta}(2p_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \\ \times \left[ \langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle - i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \right]$$

The matrix elements can be classified as follows:

$$\langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle \Big|_{\text{conn}} = \mathcal{M}^{\text{tree}} + \mathcal{M}^{1 \text{ loop}} \quad i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{conn}} = \mathcal{S}^{1 \text{ loop}}$$



The full connected contribution to waveform:

$$\mathcal{M}_{\text{conn}} = \mathcal{M}^{\text{tree}} + \mathcal{M}^{1 \text{ loop}} - \mathcal{S}^{1 \text{ loop}}$$

Contribution from disconnected  $T$  matrix elements:

$$\langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc}} = \mathcal{M}_{\text{const}} \quad i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc}} = \mathcal{M}_{\text{disc}}$$



# Tree-level amplitudes and LO waveform

From GR: Kovacs, Thorne, Astrophys. J. 224 62, 1978

From world line QFT: Jakobsen, Møglu, Plefka, Steinhoff, 2101.12688

From KMOC: Cristofoli, Gonzo, Kosower, O'Connell, 2107.10193

For convenience, we define the Fourier transform from  $q$ -space to  $b$ -space,

$$\text{FT}[f(q_1, q_2, k)] = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2p_1 \cdot q_1) \hat{\delta}(2p_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \hat{\delta}(q_1 + q_2 - k) f(q_1, q_2, k) \Big|_{k=(\omega, \omega \mathbf{n})}$$

LO waveform from KMOC

$$\mathcal{W}(\omega, \mathbf{n}) \Big|_{\text{LO}} = \text{FT}[\mathcal{M}^{\text{tree}}]$$

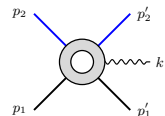
Classical tree-level amplitude Luna, Nicholson, O'Connell, White, 1711.03901

$$\begin{aligned} \mathcal{M}^{\text{tree}} &= -\kappa^2 m_1^2 m_2^2 \left[ \frac{(P_{12} \cdot \varepsilon)^2 + \sigma(Q_{12} \cdot \varepsilon)(P_{12} \cdot \varepsilon)}{q_1^2 q_2^2} + \left( \sigma^2 - \frac{1}{d-2} \right) \left( \frac{(Q_{12} \cdot \varepsilon)^2}{4q_1^2 q_2^2} - \frac{(P_{12} \cdot \varepsilon)^2}{4w_1^2 w_2^2} \right) \right] \\ &= \mathcal{M}_{d=4}^{\text{tree}} + \epsilon \mathcal{M}_{\text{extra}}^{\text{tree}} \end{aligned}$$

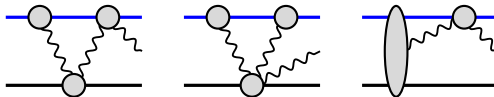
Of course, at LO the  $\mathcal{O}(\epsilon)$  term is irrelevant.

$$\begin{aligned} P_{12}^\mu &= w_1 u_2^\mu - w_2 u_1^\mu & u_i &= p_i/m_i & w_i &= u_i \cdot k \\ Q_{12}^\mu &= (q_1 - q_2)^\mu - \frac{q_1^2}{w_1} u_1^\mu + \frac{q_2^2}{w_2} u_2^\mu \end{aligned}$$

# One-loop matrix elements Herderschee, Roiban, FT, 2303.06112

$$\langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle \Big|_{1\text{-loop}} = \text{Diagram} = \mathcal{M}^{1\text{ loop}}$$


Use generalized unitarity to construct the integrand



Expand the integrand in the classical limit (method of region, Beneke, Smirnov, hep-ph/9711391)

► Matter propagator

$$\frac{1}{(p + \ell)^2 - m^2 + i0} = \frac{1}{2p \cdot \ell + \ell^2 + i0} = \frac{1}{2p \cdot \ell + i0} \sum_{n=0}^{\infty} \left( -\frac{\ell^2}{2p \cdot \ell + i0} \right)^n$$

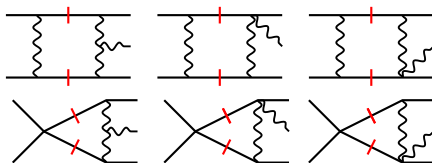
► One matter line per loop

► Matter lines do not touch before IBP

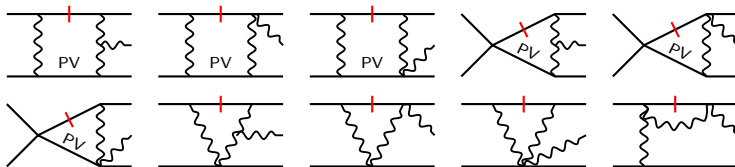
# Master integrals: amplitude Herderschee, Roiban, FT, 2303.06112

IBP automated by FIRE6: Smirnov, Chuharev, 1901.07808

Super-classical: 
$$\frac{m_1^3 m_2^3}{\hbar^4} \sum_i \alpha_i \mathcal{I}_i^{\text{cut, cut}}$$

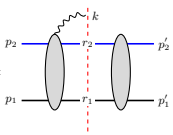


Classical: 
$$\frac{1}{\hbar^3} \sum_i (m_1^3 m_2^2 \beta_i \mathcal{I}_i^{\text{cut, pv}} + m_1^2 m_2^3 \gamma_i \mathcal{I}_i^{\text{pv, cut}}) + \frac{1}{\hbar^3} (\text{triangles and bubbles})$$



$$\text{---} \overset{\text{red tick}}{\perp} \text{---} = \hat{\delta}(2u_2 \cdot \ell) \quad \text{---} \overset{\text{PV}}{\text{---}} \text{---} = \frac{1}{2u_1 \cdot \ell + i0} + \frac{1}{2u_1 \cdot \ell - i0}$$

# Connected cut contribution $\mathcal{S}^{1 \text{ loop}}$

$$\langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{conn}}^{1 \text{ loop}} = \text{Diagram} = \sum_i \left( \frac{1}{\hbar^4} m_1^3 m_2^3 \alpha_i - \frac{1}{\hbar^3} m_1^3 m_2^2 \beta_i + \frac{1}{\hbar^3} m_1^2 m_2^3 \gamma_i \right) \mathcal{I}_i^{\text{cut, cut}}$$


- ▶ Exactly cancel the super-classical terms in the virtual amplitude
- ▶ Convert the PV to retarded [Caron-Huot, Giroux, Hannesdottir, Mizera, 2308.02125](#)

$$\text{PV} \frac{1}{2u_1 \cdot \ell} \longrightarrow \frac{1}{2u_1 \cdot \ell - i0}$$

- ▶ Classical waveform contains only imaginary IR divergences

Equivalence to world-line formalism proved by Capatti, Zeng, 2412.10864

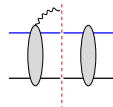
# The full connected contribution

Herderschee, Roiban, **FT**, 2303.06112  
 Brandhuber, Brown, Chen, De Angelis,  
 Gowdy, Travaglini, 2303.06111  
 Georgoudis, Heissenberg, Vazquez-Holm, 2303.07006

$$\begin{aligned}
 \mathcal{M}^{1 \text{ loop}} &= -\frac{i \kappa^2}{32\pi} (m_1 w_1 + m_2 w_2) \left[ \frac{1}{\epsilon} - \log \frac{w_1 w_2}{\mu^2} \right] \mathcal{M}_{d=4}^{\text{tree}} \\
 &\quad + \kappa^4 \left[ A_{\text{rat}}^R + \frac{A_1^R}{\sqrt{w_2^2 - q_1^2}} + \frac{A_2^R}{\sqrt{w_1^2 - q_2^2}} + \frac{A_3^R}{\sqrt{-q_1^2}} + \frac{A_4^R}{\sqrt{-q_2^2}} \right] \\
 &\quad + i \kappa^4 \left[ A_{\text{rat}}^I + A_1^I \frac{\text{arcsinh} \frac{w_2}{\sqrt{-q_1^2}}}{\sqrt{w_2^2 - q_1^2}} + A_2^I \frac{\text{arcsinh} \frac{w_1}{\sqrt{-q_2^2}}}{\sqrt{w_1^2 - q_2^2}} + A_3^I \log \frac{q_2^2}{q_1^2} + A_4^I \log \frac{w_1}{w_2} + A_5^I \frac{\text{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} \right] \\
 &= -\frac{i \kappa^2}{32\pi\epsilon} (m_1 w_1 + m_2 w_2) \mathcal{M}_{d=4}^{\text{tree}} + \widetilde{\mathcal{M}}_{1 \text{ loop}}^{\text{fin}}
 \end{aligned}$$



$$\begin{aligned}
 \mathcal{S}^{1 \text{ loop}} &= \frac{i \kappa^2 \Gamma}{64\pi} (m_1 w_1 + m_2 w_2) \left[ \frac{1}{\epsilon} - \log \frac{w_1 w_2}{(\sigma^2 - 1) \mu^2} \right] \mathcal{M}_{d=4}^{\text{tree}} \quad \Gamma = \frac{3\sigma - 2\sigma^3}{(\sigma^2 - 1)^{3/2}} \\
 &\quad + \frac{i \kappa^4}{256\pi} \frac{(2\sigma^2 - 1)^2}{(\sigma^2 - 1)^{3/2}} (m_1 w_1 + m_2 w_2) \left[ \frac{1}{\epsilon} - \log \frac{w_1 w_2}{(\sigma^2 - 1) \mu^2} \right] \frac{m_1^2 m_2^2 (u_1 \cdot f \cdot u_2)^2 (w_1^2 + \sigma w_1 w_2 + w_2^2)}{w_1^3 w_2^3} \\
 &\quad + i \kappa^4 \left[ A_{\text{rat}}^{\text{cut}} + A_1^{\text{cut}} \log \frac{w_1}{w_2} + A_2^{\text{cut}} \log \frac{w_1 w_2}{-q_1^2} + A_3^{\text{cut}} \log \frac{w_1 w_2}{-q_2^2} + A_4^{\text{cut}} \frac{\text{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} \right] \\
 &= \frac{i \kappa^2 \Gamma}{64\pi\epsilon} (m_1 w_1 + m_2 w_2) \mathcal{M}_{d=4}^{\text{tree}} + \widetilde{\mathcal{S}}_{1 \text{ loop}}^{\text{fin}} + \mathcal{S}_{1 \text{ loop}}^{\text{local}}
 \end{aligned}$$



# The full connected contribution

The IR divergence exponentiates Caron-Huot, Giroux, Hannesdottir, Mizera, 2308.02125

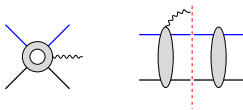
$$\mathcal{M}_{\text{conn}} = \mathcal{M}_{\text{conn}}^{\text{fin}} \exp \left[ -i\omega \frac{\kappa^2 E(1 - \Gamma/2)}{32\pi\epsilon} \right] \quad (m_1 w_1 + m_2 w_2 = \omega E)$$

such that it can be absorbed by a re-definition of retarded time

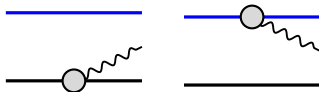
$$\tau \rightarrow \tau - \frac{\kappa^2 E(1 - \Gamma/2)}{32\pi\epsilon}$$

At one-loop level, this leads to

$$\begin{aligned} \mathcal{M}_{\text{conn}}^{\text{fin}} &= \mathcal{M}^{\text{tree}} + \mathcal{M}^{\text{1 loop}} - \mathcal{S}^{\text{1 loop}} + \frac{i\kappa^2(m_1 w_1 + m_2 w_2)(1 - \Gamma/2)}{32\pi\epsilon} \mathcal{M}^{\text{tree}} \\ &= \mathcal{M}^{\text{tree}} + \left[ \widetilde{\mathcal{M}}_{\text{1 loop}}^{\text{fin}} + \frac{i\kappa^2(m_1 w_1 + m_2 w_2)\mathcal{M}_{\text{extra}}^{\text{tree}}}{32\pi} \right] - \left[ \widetilde{\mathcal{S}}_{\text{1 loop}}^{\text{fin}} + \frac{i\kappa^2\Gamma(m_1 w_1 + m_2 w_2)\mathcal{M}_{\text{extra}}^{\text{tree}}}{64\pi} \right] \\ &= \mathcal{M}^{\text{tree}} + \mathcal{M}_{\text{1 loop}}^{\text{fin}} - \mathcal{S}_{\text{1 loop}}^{\text{fin}} \end{aligned}$$



# Disconnected amplitude contribution



They are supported only by zero-energy gravitons

$$\mathcal{W}_{\text{const}}(\omega, \mathbf{n}) = \int d\Phi_{p_1} \mathcal{M}_3(p_1, k) e^{ik \cdot b_1} + \int d\Phi_{p_2} \mathcal{M}_3(p_2, k) e^{ik \cdot b_2} = \hat{\delta}(\omega) \left[ \frac{m_1(\varepsilon \cdot u_1)^2}{u_1 \cdot n} + \frac{m_2(\varepsilon \cdot u_2)^2}{u_2 \cdot n} \right]$$

The constant background agrees with the MPM calculation

$$\mathcal{W}_{\text{const}}(\tau, \mathbf{n}) = \frac{m_1(\varepsilon \cdot u_1)^2}{u_1 \cdot n} + \frac{m_2(\varepsilon \cdot u_2)^2}{u_2 \cdot n} \quad n = (1, \mathbf{n})$$

It can be removed by a BMS supertranslation [Veneziano, Vilkovisky, 2201.11607](#)

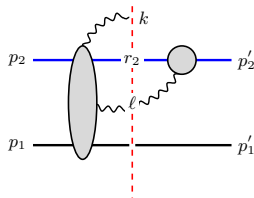
$$\delta_T \left[ \frac{h_{ab}^\infty}{r} \right] = \frac{1}{r} (2D_a D_b - \gamma_{ab} \Delta) T(n) - T(n) \partial_\tau \left[ \frac{h_{ab}^\infty}{r} \right] \quad T(n) = 2G \left[ m_1 \frac{w_1}{\omega} \log \frac{w_1}{\omega} + m_2 \frac{w_2}{\omega} \log \frac{w_2}{\omega} \right]$$

The MPM waveform is in the **intrinsic BMS frame**. If we assume  $\mathcal{M}_3 = 0$ , then we are in the **canonical BMS frame**. They are related by the above BMS supertranslation.

# Disconnected cut contribution

The  $T$  matrices in the KMOC cut can be disconnected:

$$i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc.}} \supset \int d\Phi[\ell] d\Phi[r_2]$$



- ▶ The cut graviton  $\ell$  only has support on zero energy
- ▶ We only need to consider the soft limit of the left blob

$$\mathcal{M}_6 \simeq \kappa \mathcal{M}_5 \left[ \sum_a \frac{\eta_a \varepsilon_{\mu\nu}(\ell) p_a^\mu p_a^\nu}{2\ell \cdot p_a + i0\eta_a} + \frac{\varepsilon_{\mu\nu}(\ell) k^\mu k^\nu}{2\ell \cdot k + i0} \right]$$

- ▶ The sum over massive legs only leads to zero integrals of the form

$$\int d^d \ell \delta(u_2 \cdot \ell) \delta(\ell^2) \Theta(\ell^0) \quad \int d^d \ell \delta(u_1 \cdot \ell) \delta(u_2 \cdot \ell) \delta(\ell^2) \Theta(\ell^0)$$

# Disconnected cut contribution

Contribution from the disconnected  $T$  matrix elements:

$$i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}_{hh}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc.}} = -i\kappa^2 \mathcal{M}^{\text{tree}} \left[ m_1^2 w_1^2 I(u_1, k) + m_2^2 w_2^2 I(u_2, k) \right]$$
$$I(u_1, k) = \frac{1}{m_1 \omega} \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2u_1 \cdot \ell) \hat{\delta}(\ell^2) \Theta(\ell^0)}{2\ell \cdot n}$$

This integral is **not yet properly defined** in  $d = 4$ ; further regularization is needed:

- Need to include the zero-energy graviton contribution
- We should continue  $u_1$  off-shell,  $u_1 \rightarrow \tilde{u}_1 - \beta n$ , such that

$$2u_1 \cdot \ell = (u_1 + \ell)^2 - 1 \rightarrow (\tilde{u}_1 + \ell)^2 - 1 \simeq 2u_1 \cdot \ell - 2\beta u_1 \cdot n$$

- The integral now becomes

$$I(u_1, k) = \frac{1}{m_1 \omega} \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2u_1 \cdot \ell - 2\beta u_1 \cdot n) \hat{\delta}(\ell^2) \Theta(\ell^0)}{2\ell \cdot n}$$
$$= \frac{1}{32\pi m_1 w_1} \left[ \frac{1}{\epsilon} - \log \frac{w_1^2}{\omega^2} - \log \frac{\beta^2}{\pi} \right]$$

- Divergent terms are absorbed in a further shift of retarded time

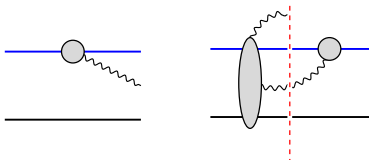
# Disconnected cut contribution [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

The disconnected  $T$  matrix elements in the cut lead to

$$\mathcal{M}_{\text{disc}} = i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc}} = -2i\omega G \left[ m_1 \frac{w_1}{\omega} \log \frac{w_1}{\omega} + m_2 \frac{w_2}{\omega} \log \frac{w_2}{\omega} \right] \mathcal{M}^{\text{tree}} = -i\omega T(n) \mathcal{M}^{\text{tree}}$$

which exactly reproduces the iteration part of the BMS supertranslation

$$\delta_T \left[ \frac{h_{ab}^\infty}{r} \right] = \frac{1}{r} (2D_a D_b - \gamma_{ab} \Delta) T(n) - T(n) \partial_\tau \left[ \frac{h_{ab}^\infty}{r} \right]$$



# Full KMOC waveform at one-loop [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

$$\mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) = \mathcal{W}_{\text{const}} + \mathcal{W}^{\text{tree}} + \text{FT} \left[ \mathcal{M}_{1\text{ loop}}^{\text{fin}} - \mathcal{S}_{1\text{ loop}}^{\text{fin}} + \mathcal{M}_{\text{disc}} \right]$$

- Fourier transform to the impact-parameter space can only be done numerically for generic kinematic setup (see Brunello, De Angelis 2403.08009 for improvements)
- Analytic computation is available under the **soft graviton limit** and/or **small relative velocity (PN) expansion**

Soft expansion (up to the first non-universal coefficient)

$$\mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) \sim \frac{\mathcal{A}}{\omega} + \mathcal{B} \log \omega + \mathcal{C} \omega (\log \omega)^2 + \mathcal{D} \omega \log \omega$$

All order conjecture in leading log: Alessio, Di Vecchia, Heissenberg, 2407.04128

PN expansion (up to 2.5PN)

$$\begin{aligned} \mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) \sim & 1 + \frac{GM^2\nu}{p_\infty} (1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \dots) \\ & + \frac{GM}{bp_\infty^2} \frac{GM^2\nu}{p_\infty} (1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \dots) \end{aligned}$$

$$M = m_1 + m_2$$

$$\nu = m_1 m_2 / M^2$$

$$p_\infty = \sqrt{\sigma^2 - 1}$$

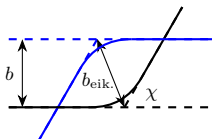
# Compare with MPM waveform [Bini, Damour, Geralico, 2309.14925]

- ▶  $W^{\text{MPM}}$  is given in terms of kinematic data in the **eikonal COM frame**, which is related to the incoming COM frame through a  $(\chi/2)$ -rotation

$$\frac{\chi_{\text{LO}}}{2} = \frac{GE}{\sqrt{-b^2}} \frac{2\sigma^2 - 1}{\sigma^2 - 1}$$

- ▶ The difference between  $|v|$ ,  $|b|$  and  $|v_{\text{eik}}|$ ,  $|b_{\text{eik}}|$  is  $\mathcal{O}(G^2)$  and thus irrelevant at one-loop. A further shift in retarded time  $\delta\tau(u_1, u_2)$  is allowed in comparison
- ▶ The KMOC waveform in the eikonal COM frame is given by

$$W_{\text{KMOC}}^{(1)}(\theta, \phi) = \mathcal{W}_{\text{KMOC}}^{(1)}(\theta, \phi) + \frac{\chi_{\text{LO}}}{2} \frac{\partial}{\partial \phi} \mathcal{W}^{\text{tree}}(\theta, \phi)$$



# Compare with MPM waveform [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

The frame rotation cancels the connected cut contribution (up to a time shift)



$$\text{FT} \left[ \mathcal{S}_{1\text{ loop}}^{\text{fin}} \right] = \frac{\chi_{\text{LO}}}{2} \frac{\partial}{\partial \phi} \mathcal{W}^{\text{tree}} - i\omega \delta\tau^{\text{cut}} \mathcal{W}^{\text{tree}}$$

This relation is verified up to  $\mathcal{O}(p_\infty^6)$  and  $\text{N}^3\text{LO}$  in soft limit, and  $\delta\tau^{\text{cut}}$  agrees with [GHR 2312.07452]

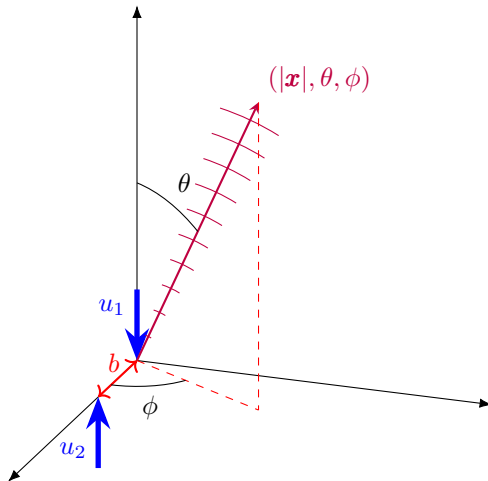
The rest agrees exactly with MPM waveform

$$\mathcal{W}^{\text{MPM}} = \mathcal{W}_{\text{const}} + \mathcal{W}^{\text{tree}} + \text{FT} \left[ \mathcal{M}_{1\text{ loop}}^{\text{fin}} + \mathcal{M}_{\text{disc}} \right] + i\omega \delta\tau^{\text{amp}} \mathcal{W}^{\text{tree}}$$

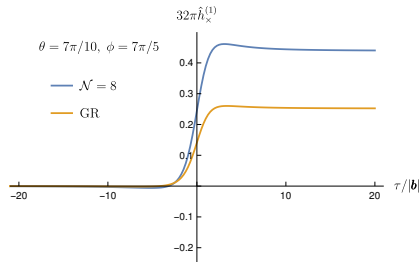
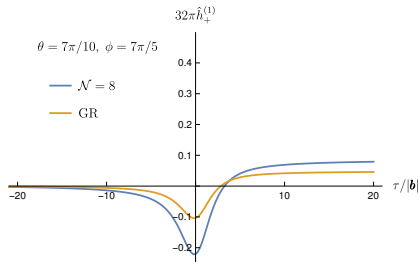
This relation is verified up to  $\mathcal{O}(p_\infty^5)$  and  $\text{N}^3\text{LO}$  in soft limit

Similar comparison up to  $\mathcal{O}(p_\infty^3)$  is done in Georgoudis, Heissenberg, Russo, 2402.06361

# Numerical results

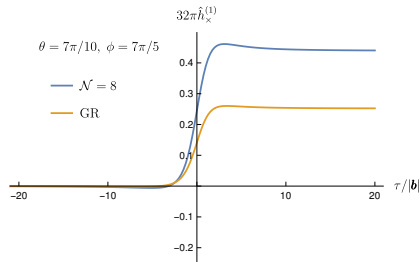
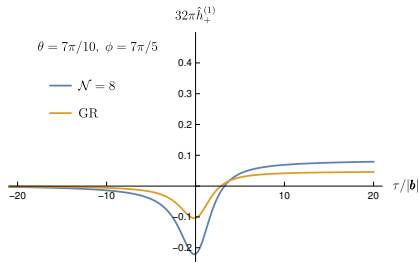


# LO waveform



Agree with Jakobsen, Mogull, Plefka, Steinhoff, 2101.12688  
From spinning binaries: De Angelis, Novichkov, Gonzo, 2309.17429  
Brandhuber, Brown, Chen, Gowdy, Travaglini, 2310.04405  
Aoude, Haddad, Heissenberg, Helset, 2310.05832

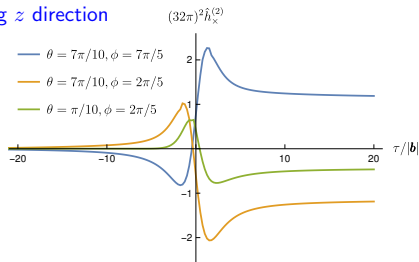
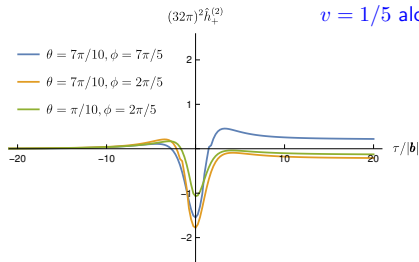
# LO waveform



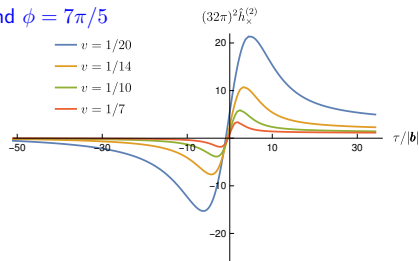
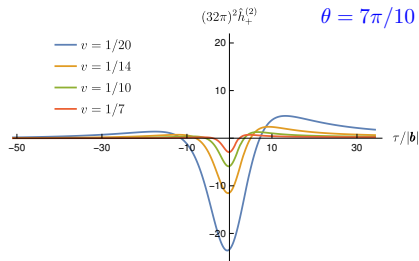
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# NLO waveform (GR)

$v = 1/5$  along  $z$  direction



$\theta = 7\pi/10$  and  $\phi = 7\pi/5$



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# Synergy between amplitude and GR

Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**, 2402.06604

One of the important takeaway messages of our new results is that, after having sorted out the subtleties that were hidden in the EFT one-loop waveform, we obtained a remarkable confirmation that the classical limit of an amplitude-based waveform does correctly incorporate the many subtle classical effects that were included in the  $O(G^2\eta^5)$ -accurate MPM waveform such as (i) radiation-reaction effects on the worldlines, (ii) high-multipolarity tail effects in the wave-zone, and (iii) cubically nonlinear multipole couplings in the exterior zone. The fact that the road leading to the present successful EFT/MPM comparison had some bumps, which taught us interesting lessons, is another example of the useful synergy between amplitude-based, and classical perturbation-theory-based, approaches to gravitational physics.

# Outlook

Explicit higher order spinning and spinless calculation

- ▶ Challenge: IBP reduction and DE for loop integrals; higher rank tensor reduction
- ▶ What do we mean by analytic computation?

Systematic inclusion of tidal operators and absorption

- ▶ Need to understand the running and mixing of WCs

How to describe generic spinning bodies?

- ▶ Perhaps relevant to future higher precision GW observations

Re-summation in observables

- ▶ SCET for gravity
- ▶ Self-force effective theory

How to directly compute bound-state observables?

- ▶ Phenomenological prescription exists and works very well in matching NR
- ▶ Remain as a theoretical challenge

Thanks for listening!