Scattering Amplitudes for Binary Black holes

Fei Teng

FUDAN UNIVERSITY
1905

July 24, 2025

第十四届新物理研讨会

Outline

Introduction: amplitudes meet GW

Classical amplitudes and EFT matching

Spinning black holes and neutron stars

Observable based formalism (KMOC)

Outlook

Table of Contents

Gravitational wave: new window to probe our Universe

New physics!

- Probe dynamics of black holes
- Test general relativity
- Black hole formation
- Early universe

Future ground based observatories

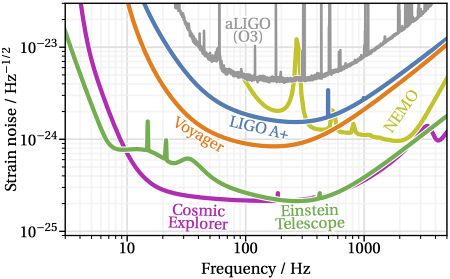
- Advanced LIGO
- Einstein Telescope
- Cosmic Explorer

Future space based observatories

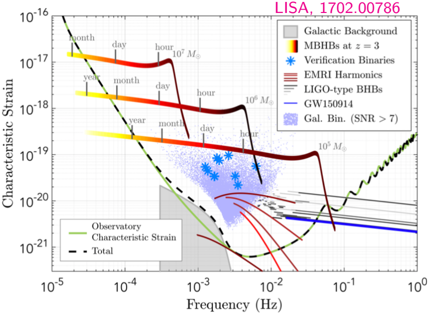
- LISA
- TaiJi
- TianQin

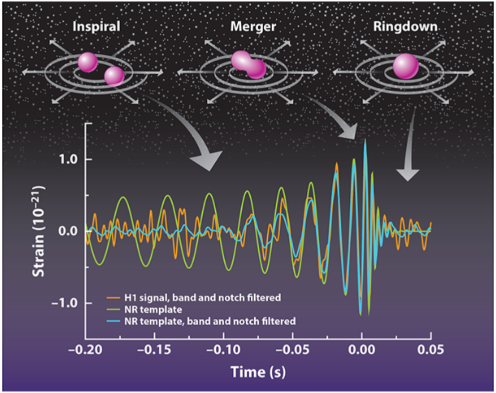
Require accurate theoretical prediction

cosmicexplorer.org



LISA, 1702.00786





Accurate theoretical prediction of the GW production puts challenges on the understanding of its source

How to organize perturbations?

- Post-Newtonian (PN) expansion

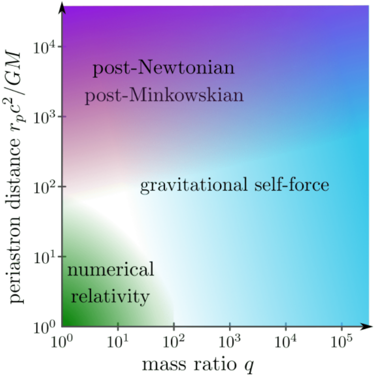
$$v^2 \sim \frac{Gm}{r} \ll 1$$

- Post-Minkowskian (PM) expansion

$$\frac{Gm}{r} \ll v^2 \sim 1$$

- Self-force expansion

$$\frac{Gm}{r} \sim v^2 \sim 1\,, \quad \frac{m_1}{m_2} \ll 1$$



Khalil, Buonanno, Steinhoff, Vines, 2204.05047

Amplitude-based methods naturally lead to PM expansion

PM expansion is relevant to bound orbits with large eccentricity and scattering process

Why don't we just use numerical relativity?

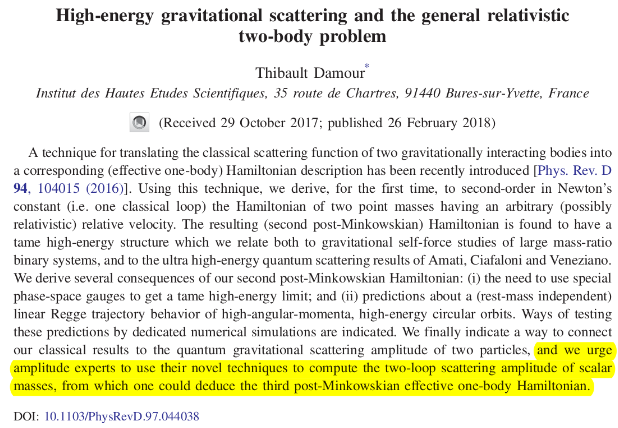
Numerical relativity provides the most accurate observables (waveform, scattering angle, etc) for binaries with mass ratio $q \sim 1$

- Works for the entire binary merger process
- Numerical error under very good control (can be considered as the TRUTH)
- Computationally expansive: need days of computation time on a super-computer to produce one waveform template
- Matched filtering analysis requires a dense sampling of the parameter space
- In light of future observatories, we need $\sim 10^6$ waveform templates

We need analytic perturbative computations to efficiently generate observables

Why amplitudeologists are working on GW?

It all starts with an enticing invitation



PHYSICAL REVIEW D **97,** 044038 (2018)

High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour[*]
Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

(Received 29 October 2017; published 26 February 2018)

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

DOI: 10.1103/PhysRevD.97.044038

Latest progress

GR method

in progress

amplitude method

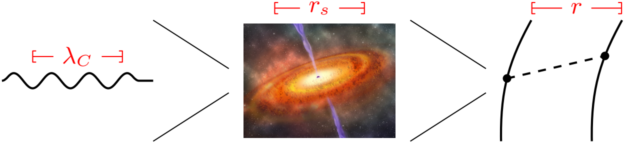
		1687	1938	1973	2001	2019				
		0PN	1PN	2PN	3PN	4PN	5PN	6PN		
1979	1PM	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	$\cdots$	G^1
2018	2PM		1	v^2	v^4	v^6	v^8	v^{10}	$\cdots$	G^2
2019	3PM			1	v^2	v^4	v^6	v^8	$\cdots$	G^3
2021	4PM				1	v^2	v^4	v^6	$\cdots$	G^4
2024	5PM	2SF in progress				1	v^2	v^4	$\cdots$	G^5
	6PM						1	v^2	$\cdots$	G^6

Notably, the 5PM 1SF result is obtained through an improved world-line formalism incorporated with amplitude ingredients

Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch, 2403.07781

Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch, 2411.11846

From quantum to classical two-body problem



$\lambda_C \ll r_s \implies Gm^2 \gg 1$

Classical contribution is encoded in every loop order

$r_s \ll r \implies Gmq \ll 1$

Gmq is the small parameter in the classical perturbative expansion

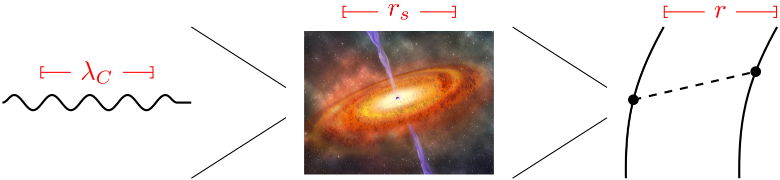
$\lambda_C \ll r \implies q/m \ll 1$

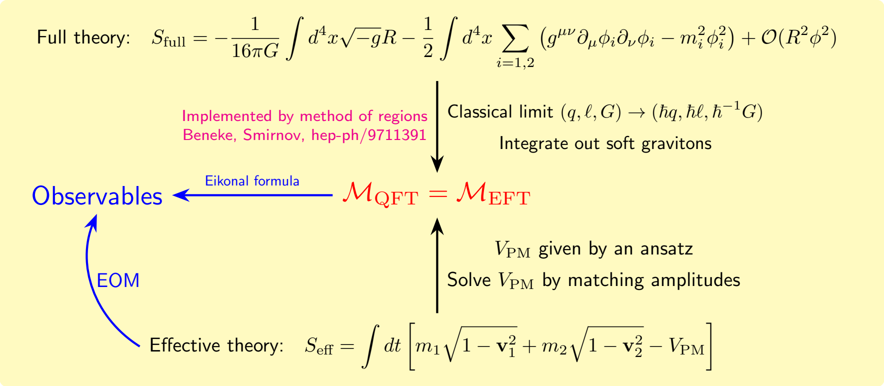
Expand in small momentum transfer and keep only the leading order
(Bohr's correspondence principle)

Table of Contents

EFT matching using amplitudes

Cheung, Rothstein, Solon, 1808.02489





Full theory: $S_{\rm full} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{2}\int d^4x \sum_{i=1,2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2\right) + \mathcal{O}(R^2\phi^2)$

Implemented by method of regions
Beneke, Smirnov, hep-ph/9711391

Classical limit $(q,\ell,G)\to(\hbar q,\hbar\ell,\hbar^{-1}G)$

Integrate out soft gravitons

Observables

Eikonal formula

$$\mathcal{M}_{\rm QFT} = \mathcal{M}_{\rm EFT}$$

EOM

$V_{\rm PM}$ given by an ansatz

Solve $V_{\rm PM}$ by matching amplitudes

Effective theory: $S_{\rm eff} = \int dt\left[m_1\sqrt{1-\mathbf{v}_1^2} + m_2\sqrt{1-\mathbf{v}_2^2} - V_{\rm PM}\right]$

EFT matching

Cheung, Rothstein, Solon, 1808.02489

- Full theory: Schwarzschild black hole $\Longrightarrow$ scalar field ϕ

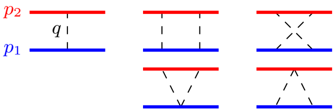
$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{16\pi G}R + \frac{1}{2}\sum_{i=1,2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2\right)\right] + \mathcal{O}(R^2\phi^2)$$

- Effective theory: potential $V(\mathbf{k},\mathbf{k}')$ given by an ansatz

$$L = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\sum_{i=1,2} a_i^\dagger(-\mathbf{k})\left(i\partial_t + \sqrt{\mathbf{k}^2 + m_i^2}\right)a_i(\mathbf{k}) - \int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} V(\mathbf{k},\mathbf{k}') a_1^\dagger(\mathbf{k}') a_1(\mathbf{k}) a_2^\dagger(-\mathbf{k}') a_2(\mathbf{k})\right]$$

- Solve the EFT potential by matching the full theory and EFT amplitudes order-by-order in G in the classical limit

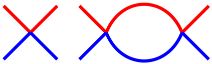
$$\mathcal{M}^{(L)} \xrightarrow{\text{classical limit}} \mathcal{M}^{(L)}_{\text{EFT}}$$



$q \to \hbar q$

$\ell \to \hbar \ell$

$G \to \hbar^{-1} G$



no massless loops (quantum)
no massive contact vertices (delta-function potential)

$$\mathcal{M}^{(L)}_{\text{classical}} \sim \frac{Gm^4}{\hbar^3 q^2}(Gmq)^L$$

Classical amplitudes

Tree level

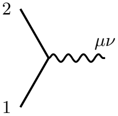
Starting from the Lagrangian and $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

$$\mathcal{L} = \sqrt{-g}\left[-\frac{2}{\kappa^2}R + \frac{1}{2}\sum_{i=1,2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - m_i^2\phi_i^2\right)\right] \qquad \kappa^2 = 32\pi G$$

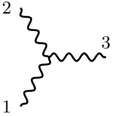
We derive the Feynman rules in the de Donder gauge

$$= \frac{i}{p^2 - m^2}$$

$$= \frac{i}{2q^2}\left[\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right]$$



$$= \kappa p_1^{(\mu}p_2^{\nu)} - \frac{\kappa}{2}\eta^{\mu\nu}(p_1\cdot p_2 + m^2)$$



$=$

(144 terms)

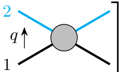
Classical amplitudes

Tree level

We define a set of convenient variables such that $\bar{p}_i \cdot q = 0$ Landshoff and Polkinghorne 1969

$$\bar{p}_1 = p_1 + q/2 \qquad \bar{p}_2 = p_2 - q/2 \qquad \bar{m}_i^2 = \bar{p}_i^2 \qquad y = \bar{p}_1 \cdot \bar{p}_2$$

Full quantum amplitude

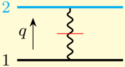


$$M^{(0)}\left[\qquad\right] = -\underset{\text{classical}}{\frac{\kappa^2 \bar{m}_1^2 \bar{m}_2^2 (2y^2 - 1)}{2\hbar^3 q^2}} + \underset{\text{higher order in } q\text{, quantum}}{\frac{\kappa^2 (4\bar{m}_1^2 + 4\bar{m}_2^2 + 3\hbar^2 q^2)}{32\hbar}}$$

Classical amplitude

$\mathsf{x} = \dfrac{1}{d-2}$ for Einstein gravity

$\mathsf{x} = 0$ for $\mathcal{N} = 8$ sugra



$$M^{(0)}_{\text{qft-cl}} = \qquad = -\frac{\kappa^2 \bar{m}_1^2 \bar{m}_2^2 (y^2 - \mathsf{x})}{q^2}$$

(restoring d dependence)

Impact parameter space

$\text{FT}[f(q^2)] = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q)\hat{\delta}(2\bar{p}_2 \cdot q) f(q^2) e^{iq\cdot b} \qquad \hat{\delta}(x) = 2\pi\delta(x)$

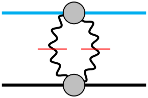
$$\delta^{(0)} = \text{FT}\left[M^{(0)}_{\text{qft-cl}}\right] = \frac{\kappa^2 \bar{m}_1 \bar{m}_2 (y^2 - \mathsf{x})(-\pi b^2)^{\epsilon}\Gamma(-\epsilon)}{16\pi\sqrt{y^2 - 1}}$$

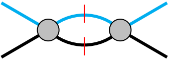
Classical amplitudes

One loop

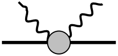
We use generalized unitarity to construct the integrand

Bern, Dixon, Dunbar, Kosower, hep-ph/9403226, hep-ph/9409265

spanning cuts: 

[other cuts like  do not have new contributions in the classical limit]

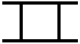
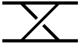
We use the full quantum gravitational Compton amplitude as the building block



$$
\begin{aligned}
= \; & -\frac{\mathrm{dot}[e[2],p[1]]^2\,\mathrm{dot}[e[3],p[1]]^2}{\mathrm{l}[p[1]+q[2]]}-\frac{2\,\mathrm{dot}[e[2],p[1]]^2\,\mathrm{dot}[e[3],p[1]]\times\mathrm{dot}[e[3],q[2]]}{\mathrm{l}[p[1]+q[2]]}-\frac{\mathrm{dot}[e[2],p[1]]^2\,\mathrm{dot}[e[3],q[2]]^2}{\mathrm{l}[p[1]+q[2]]}+\\
& \frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],p[1]]\times\mathrm{dot}[e[3],p[1]]\times\mathrm{dot}[p[1],q[2]]}{\mathrm{l}[p[1]+q[2]]}+\frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],p[1]]\times\mathrm{dot}[e[3],q[2]]\times\mathrm{dot}[p[1],q[2]]}{\mathrm{l}[p[1]+q[2]]}-\\
& \frac{\mathrm{dot}[e[2],e[3]]^2\,\mathrm{dot}[p[1],q[2]]^2}{4\,\mathrm{l}[p[1]+q[2]]}-\frac{\mathrm{dot}[e[2],p[1]]^2\,\mathrm{dot}[e[3],p[1]]^2}{\mathrm{l}[p[1]+q[3]]}-\frac{2\,\mathrm{dot}[e[2],p[1]]\times\mathrm{dot}[e[2],q[3]]\,\mathrm{dot}[e[3],p[1]]^2}{\mathrm{l}[p[1]+q[3]]}-\frac{\mathrm{dot}[e[2],q[3]]^2\,\mathrm{dot}[e[3],p[1]]^2}{\mathrm{l}[p[1]+q[3]]}+\\
& \frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],p[1]]\times\mathrm{dot}[e[3],p[1]]\times\mathrm{dot}[p[1],q[3]]}{\mathrm{l}[p[1]+q[3]]}+\frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],q[3]]\times\mathrm{dot}[e[3],p[1]]\times\mathrm{dot}[p[1],q[3]]}{\mathrm{l}[p[1]+q[3]]}-\\
& \frac{\mathrm{dot}[e[2],e[3]]^2\,\mathrm{dot}[p[1],q[3]]^2}{4\,\mathrm{l}[p[1]+q[3]]}-\frac{\mathrm{dot}[e[2],q[3]]^2\,\mathrm{dot}[e[3],p[1]]^2}{\mathrm{l}[q[2]+q[3]]}+\frac{2\,\mathrm{dot}[e[2],p[1]]\times\mathrm{dot}[e[2],q[3]]\times\mathrm{dot}[e[3],p[1]]\times\mathrm{dot}[e[3],q[2]]}{\mathrm{l}[q[2]+q[3]]}-\\
& \frac{\mathrm{dot}[e[2],p[1]]^2\,\mathrm{dot}[e[3],q[2]]^2}{\mathrm{l}[q[2]+q[3]]}-\frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],q[3]]\times\mathrm{dot}[e[3],p[1]]\times\mathrm{dot}[p[1],q[2]]}{\mathrm{l}[q[2]+q[3]]}+\\
& \frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],p[1]]\times\mathrm{dot}[e[3],q[2]]\times\mathrm{dot}[p[1],q[2]]}{\mathrm{l}[q[2]+q[3]]}-\frac{\mathrm{dot}[e[2],e[3]]^2\,\mathrm{dot}[p[1],q[2]]^2}{4\,\mathrm{l}[q[2]+q[3]]}+\frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],q[3]]\times\mathrm{dot}[e[3],p[1]]\times\mathrm{dot}[p[1],q[3]]}{\mathrm{l}[q[2]+q[3]]}-\\
& \frac{\mathrm{dot}[e[2],e[3]]\times\mathrm{dot}[e[2],p[1]]\times\mathrm{dot}[e[3],q[2]]\times\mathrm{dot}[p[1],q[3]]}{\mathrm{l}[q[2]+q[3]]}+\frac{\mathrm{dot}[e[2],e[3]]^2\,\mathrm{dot}[p[1],q[2]]\times\mathrm{dot}[p[1],q[3]]}{2\,\mathrm{l}[q[2]+q[3]]}-\frac{\mathrm{dot}[e[2],e[3]]^2\,\mathrm{dot}[p[1],q[3]]^2}{4\,\mathrm{l}[q[2]+q[3]]}
\end{aligned}
$$

(22 terms)

The part of quantum integrand that contains the classical contribution

$\mathcal{I}_4^{\text{quantum}} =$  $+$  $+$  $+$ 

Classical amplitudes

One loop

Method of regions expansion $(q, \ell, G) \to (\hbar q, \hbar \ell, \hbar^{-1} G)$

$$\frac{1}{(p_1+\hbar\ell)^2 - m_1^2 + i0} = \frac{1}{\hbar}\frac{1}{2\bar{p}_1\cdot\ell + i0} + \frac{q\cdot\ell - \ell^2}{(2\bar{p}_1\cdot\ell + i0)^2} + \dots$$

Integral family in the soft region

$$\mathcal{I}^{a_1\, a_2}_{\pm\, \pm}[N] = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{N(\ell)}{(2\bar{p}_1\cdot\ell \pm i0)^{a_1}(2\bar{p}_2\cdot\ell \pm i0)^{a_2}\ell^2(\ell-q)^2}$$

Master integral basis under IBP reduction

$$M^{(1)} = \underset{\text{super-classical } M^{(1)}_{\text{qft-sc}}}{\frac{i\kappa^4\bar{m}_1^4\bar{m}_2^4(2y^2-1)^2}{8\hbar^4}\mathcal{I}_{\square}} + \underset{\text{classical } M^{(1)}_{\text{qft-cl}}}{\frac{3\kappa^4(5y^2-1)}{32\hbar^3}\left[\bar{m}_1^4\bar{m}_2^2\mathcal{I}_{\Delta} + \bar{m}_1^2\bar{m}_2^4\mathcal{I}_{\nabla}\right]} + \mathcal{O}(\hbar^{-2})$$

$$\mathcal{I}_{\square} = e^{\epsilon\gamma_{\mathrm{E}}}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\hat{\delta}(2\bar{p}_1\cdot\ell)\hat{\delta}(2\bar{p}_2\cdot\ell)}{\ell^2(\ell-q)^2} = \frac{e^{\epsilon\gamma_{\mathrm{E}}}}{4\bar{m}_1\bar{m}_2\sqrt{y^2-1}}\frac{\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{(4\pi)^{1-\epsilon}\Gamma(-2\epsilon)(-q^2)^{1+\epsilon}}$$

$$\mathcal{I}_{\Delta} = \int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\hat{\delta}(2\bar{p}_1\cdot\ell)}{\ell^2(\ell-q)^2} = \frac{1}{16\bar{m}_1\sqrt{-q^2}} \qquad \mathcal{I}_{\nabla} = \int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\hat{\delta}(2\bar{p}_2\cdot\ell)}{\ell^2(\ell-q)^2} = \frac{1}{16\bar{m}_2\sqrt{-q^2}}$$

Classical amplitudes

One loop

Leading order in $\hbar$ expansion (super-classical):

- IR divergent, but...

$$\mathcal{I}_{\square} = \frac{1}{8\pi \bar{m}_1 \bar{m}_2 \sqrt{y^2-1}\, q^2} \left[\frac{1}{\epsilon} - \log \frac{-q^2}{4\pi} \right] + \mathcal{O}(\epsilon)$$

- Does not contain new information at G^2 order

$$\mathrm{FT}\left[iM^{(1)}_{\text{qft-sc}} \right] = \frac{1}{2} \left(i\delta^{(0)} \right)^2 \qquad \text{(iteration of tree level)}$$

Classical physics at G^2 is contained in the classical amplitude

$$M^{(1)}_{\text{qft-cl}} = \frac{3\kappa^4 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2)(5y^2-1)}{128\sqrt{-q^2}}$$

$$\delta^{(1)} = \mathrm{FT}\left[M^{(1)}_{\text{qft-cl}} \right] = \frac{3\kappa^4 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2)(5y^2-1)}{4096\pi \sqrt{y^2-1}\sqrt{-b^2}}$$

Matching

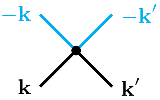
Tree level and one loop

Effective theory: potential $V(\mathbf{k},\mathbf{k}')$ given by an ansatz

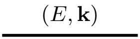
$$L = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\sum_{i=1,2} a_i^\dagger(-\mathbf{k})\Big(i\partial_t + \sqrt{\mathbf{k}^2 + m_i^2}\Big)a_i(\mathbf{k}) - \int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} V(\mathbf{k},\mathbf{k}')a_1^\dagger(\mathbf{k}')a_1(\mathbf{k})a_2^\dagger(-\mathbf{k}')a_2(\mathbf{k})\right]$$

$$V(\mathbf{p},\mathbf{p}-\mathbf{q}) = \frac{4\pi G}{\mathbf{q}^2}c_1(\mathbf{p}^2) + \frac{2\pi^2 G^2}{|\mathbf{q}|}c_2(\mathbf{p}^2) + \dots$$

Feynman rules



$$= -iV(\mathbf{k},\mathbf{k}')$$

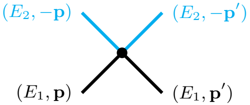


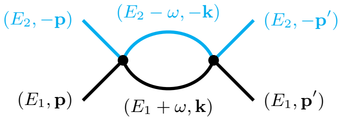
$$= \frac{i}{E - \sqrt{\mathbf{k}^2 + m^2} + i0}$$

Matching

Tree level and one loop

Feynman diagrams





Classical region: $\mathbf{p}' - \mathbf{p} = \hbar\mathbf{q}\,, \mathbf{k} - \mathbf{p} = \hbar\mathbf{l}$

($E = E_1 + E_2$ and $\xi = E_1 E_2 / E^2$)

$$M^{(0)}_{\text{EFT}} = -\frac{4\pi G}{\mathbf{q}^2} c_1(\mathbf{p}^2)$$

$$M^{(1)}_{\text{EFT}} = -\frac{2\pi^2 G^2}{|\mathbf{q}|} c_2(\mathbf{p}^2) + \frac{\pi^2 G^2}{E\xi|\mathbf{q}|}\Big[(1-3\xi)c_1^2(\mathbf{p}^2) + 4\xi^2 E^2 c_1(\mathbf{p}^2) c_1'(\mathbf{p}^2)\Big]$$

$$+ \int \frac{\mathrm{d}^{d-1}\mathbf{l}}{(2\pi)^{d-1}} \frac{32 E\xi\pi^2 G^2 c_1^2(\mathbf{p}^2)}{\mathbf{l}^2(\mathbf{l}+\mathbf{q})^2(\mathbf{l}^2 + 2\mathbf{p}\cdot\mathbf{l})}$$

super-classical matches exactly the iteration
IR divergence cancels

Solve the WCs through matching

$$M^{(0)}_{\text{EFT}} = \frac{M^{(0)}_{\text{qft-cl}}}{4E_1 E_2} \qquad M^{(1)}_{\text{EFT}} = \frac{M^{(1)}_{\text{qft-sc}} + M^{(1)}_{\text{qft-cl}}}{4E_1 E_2}$$

Cheung, Rothstein, Solon, 1808.02489
Bern, Cheung, Roiban, Solon, Shen, Zeng, 1901.04424
Bern, Parra-Martinez, Roiban, Ruf, Solon, Shen, Zeng, 2112.10750
Bern, Herrmann, Roiban, Ruf, Smirnov, 2406.01554

$$E_1 = \sqrt{\mathbf{p}^2 + m_1^2} \qquad E_2 = \sqrt{\mathbf{p}^2 + m_2^2}$$
$$\gamma = \frac{E_1 + E_2}{m_1 + m_2} \qquad \xi = \frac{E_1 E_2}{(E_1 + E_2)^2}$$
$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \qquad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

Hamiltonian: $H = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{|\mathbf{r}|^n} c_{n\mathrm{PM}}(\mathbf{p}^2)$

$$c_{1\mathrm{PM}} = -\frac{\nu^2 (m_1 + m_2)^2}{\gamma^2 \xi} (2\sigma^2 - 1)$$

Westpfahl and Goller 1979

$$c_{2\mathrm{PM}} = -\frac{\nu^2 (m_1 + m_2)^3}{\gamma^2 \xi} \left[\frac{3(5\sigma^2 - 1)}{4} - \frac{4\nu\sigma(2\sigma^2 - 1)}{\gamma\xi} + \frac{\nu^2 (1 - \xi)(2\sigma^2 - 1)^2}{2\gamma^3 \xi^2} \right]$$

Bel, Damour, Deruelle, Ibanez, Martin, 1981
Westpfahl 1985

$$\begin{aligned} c_{3\mathrm{PM}} = \frac{\nu^2 m^4}{\gamma^2 \xi} \Bigg[& \frac{3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3}{12} - \frac{4\nu(3 + 12\sigma^2 - 4\sigma^4) \operatorname{arcsinh} \sqrt{\frac{\sigma - 1}{2}}}{\sqrt{\sigma^2 - 1}} \\ & - \frac{3\nu\gamma(2\sigma^2 - 1)(5\sigma^2 - 1)}{2(1 + \gamma)(1 + \sigma)} + \frac{3\nu\sigma(20\sigma^2 - 7)}{2\gamma\xi} + \frac{\nu^2(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2)(2\sigma^2 - 1)}{4\gamma^3\xi^2} \\ & + \frac{2\nu^3(3 - 4\xi)\sigma(2\sigma^2 - 1)^2}{\gamma^4 \xi^3} - \frac{\nu^4(1 - 2\xi)(2\sigma^2 - 1)^3}{2\gamma^6 \xi^4} \Bigg] \end{aligned}$$

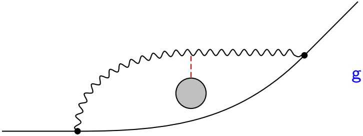
(This is what Damour was asking for)

State-of-the-art: $c_{4\mathrm{PM}}^{\mathrm{hyp}}$ and $c_{5\mathrm{PM}}^{\mathrm{hyp}\ 1\mathrm{SF}}$

- $c_{3\mathrm{PM}}$ is not known to general relativists before computed this way
- $c_{5\mathrm{PM}}$ for GR is obtained using the amplitude-worldline hybrid method

Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch, 2403.07781
Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch, 2411.11846

Gravitational tail



gravitational tails depend on the trajectory

$$\begin{aligned} c_4^{\text{hyp}} = \frac{M^7\nu^2}{4\xi E^2}\left[\mathcal{M}_4^{\text{p}} + \nu\left(4\mathcal{M}_4^{\text{t}}\log\left(\tfrac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}}\right)\right] + \mathcal{D}^3\left[\frac{E^3\xi^3}{3}c_1^4\right] + \mathcal{D}^2\left[\left(\frac{E^3\xi^3}{\boldsymbol{p}^2} + \frac{E\xi(3\xi-1)}{2}\right)c_1^4 - 2E^2\xi^2c_1^2c_2\right] \\ + \left(\mathcal{D} + \frac{1}{\boldsymbol{p}^2}\right)\left[E\xi(2c_1c_3 + c_2^2) + \left(\frac{4\xi-1}{4E} + \frac{2E^3\xi^3}{\boldsymbol{p}^4} + \frac{E\xi(3\xi-1)}{\boldsymbol{p}^2}\right)c_1^4 + \left((1-3\xi) - \frac{4E^2\xi^2}{\boldsymbol{p}^2}\right)c_1^2c_2\right], \qquad (8) \end{aligned}$$

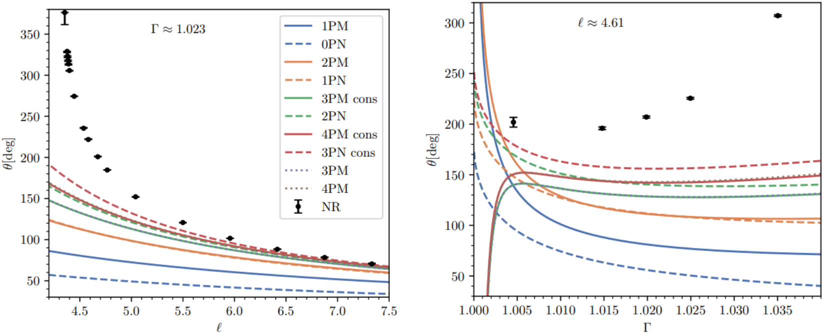
where $\mathcal{D} = \dfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{p}^2}$

Bern, Parra-Martinez, Roiban, Ruf, Solon, Shen, Zeng, 2112.10750

Comparison with numerical relativity

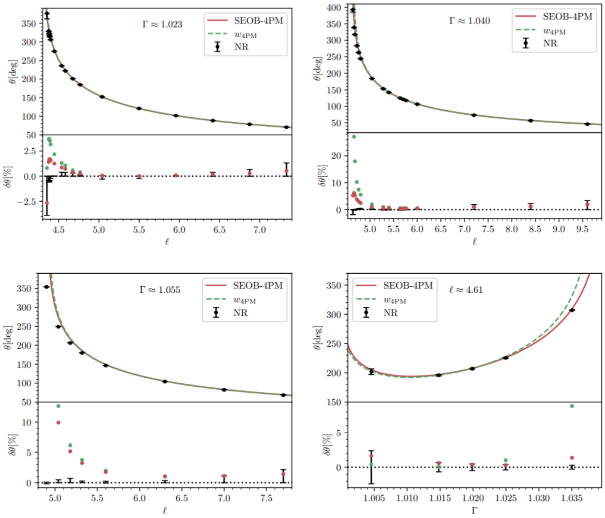
Fixed-order perturbative computation only works well for weak couplings

Buonanno, Jakobsen, Mogull, 2402.12342



Effective-one-body (EOB) resummation is necessary for strong couplings

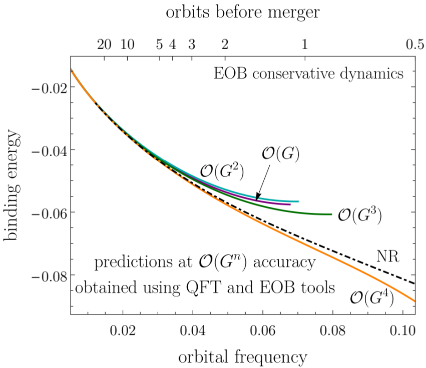
Buonanno, Damour, gr-qc/9811091 Buonanno, Jakobsen, Mogull, 2402.12342



Bound orbits

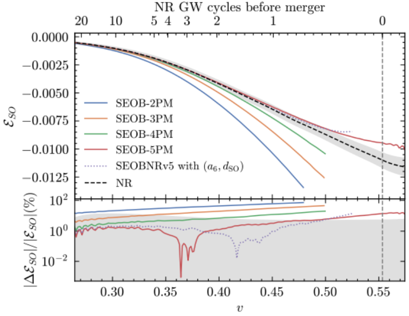
PM-informed two-body effective Hamiltonian

- Local contribution: direct analytic continuation
- Nonlocal (tail) contribution: supplement PN results from GR computations



Khalil, Buonanno, Steinhoff, Vines, 2205.05047

Snowmass white paper, 2204.05194



Buonanno, Mogull, Patil, Pompili, 2405.19181

Eikonalization Di Vecchia, Heissenberg, Russo, Veneziano, 2306.16488

We have explicitly computed the impact-parameter space amplitudes up to one loop

$$\delta^{(0)} = \mathrm{FT}\left[M^{(0)}_{\text{qft-cl}}\right] = \frac{\kappa^2 \bar{m}_1 \bar{m}_2 (y^2 - 1/2)(-\pi b^2)^\epsilon \Gamma(-\epsilon)}{16\pi\sqrt{y^2-1}} = \frac{\kappa^2 \bar{m}_1 \bar{m}_2 (y^2 - 1/2)}{16\pi\sqrt{y^2-1}}\left[-\frac{1}{\epsilon} - \log(-\pi b^2) - \gamma_{\mathrm{E}}\right]$$

$$\delta^{(1)} = \mathrm{FT}\left[M^{(1)}_{\text{qft-cl}}\right] = \frac{3\kappa^4 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2)(5y^2 - 1)}{4096\pi\sqrt{y^2-1}\sqrt{-b^2}} \qquad \mathrm{FT}\left[iM^{(1)}_{\text{qft-sc}}\right] = \frac{1}{2}\left(i\delta^{(0)}\right)^2$$

Eikonalization conjecture

$$\mathrm{FT}\left[iM(q)\right] = (1 + i\Delta)e^{i\delta/\hbar} - 1$$

The conjecture is explicitly verified up to two-loop level

Di Vecchia, Heissenberg, Russo, Veneziano, 2101.05772

- $\delta = \delta^{(0)} + \delta^{(1)} + \dots$ encodes all the classical physics (generating function for classical conservative observables)

$$\text{scattering angle:} \quad \theta = \frac{\partial \delta}{\partial J} \qquad \text{time delay:} \quad T = \frac{\partial \delta}{\partial E}$$

- $\Delta = \Delta^{(1)} + \Delta^{(2)} + \dots$ is the quantum reminder

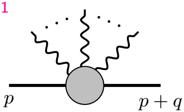
Table of Contents

Black holes and neutron stars can carry spin

How to incorporate spin into the EFT?

On-shell description of spin

Bern, Luna, Roiban, Shen, Zeng, 2005.03071



- On-shell spin-s states are symmetric traceless and transverse

$$\varepsilon_{a_1 a_2 \dots a_s} = \varepsilon_{(a_1 a_2 \dots a_s)} \qquad p^{a_1} \varepsilon_{a_1 a_2 \dots a_s} = \eta^{a_1 a_2} \varepsilon_{a_1 a_2 \dots a_s} = 0$$

- Classical limit $\Longrightarrow$ spin coherent state $\varepsilon^s_{a_1 a_2 \dots a_s} = \varepsilon^+_{a_1} \varepsilon^+_{a_2} \dots \varepsilon^+_{a_s}$ with large s

$$\varepsilon^s_p \cdot M^{ab} \cdot \varepsilon^s_{p+q} \sim S^{ab} \qquad (M^{ab})_{c(s)}{}^{d(s)} = -2is\delta^{[a}_{(c_1} \eta^{b](d_1} \delta^{d_2}_{c_2} \dots \delta^{d_s)}_{c_s)}$$

$$\varepsilon^s_p \cdot \{M^{ab} M^{cd}\} \cdot \varepsilon^s_{p+q} \sim S^{ab} S^{cd} \qquad S^{ab} = (1/m)\varepsilon^{abcd} p_c S_d$$

- The spin tensor satisfy covariant spin supplementary condition (SSC)

$$S^{ab} p_b = 0 \qquad (S^{ab} \text{ is boosted from rest frame } S^{ij})$$

- Transversality and covariant SSC are related
- Spin magnitude is conserved: $S^{ab} S_{ab} \sim S^a S_a \sim \boldsymbol{S}^2 = \mathrm{const}$

Effective field theory for higher-spin particles

$$\nabla_\mu\phi_s = \partial_\mu\phi_s + (i/2)\omega_{\mu ab}M^{ab}\phi_s$$
$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd}M_{cd}\nabla_b$$

Higher spin quantum field theory ($\phi_s \equiv \phi_{a_1a_2\ldots a_s}$)

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\phi_s(\nabla^2+m^2)\phi_s + \frac{1}{8}R_{abcd}\phi_s M^{ab}M^{cd}\phi_s - \frac{C_2}{2m^2}R_{af_1bf_2}\nabla^a\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2)}\nabla^b\phi_s\\ &+ \frac{D_2}{2m^2}R_{abcd}\nabla_i\phi_s\{M^{ai}M^{cd}\}\phi_s + \frac{E_2-2D_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_iM^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s + \mathcal{O}(M^3_{ab})\end{aligned}$$

We prefer to use a formalism in which the classical and large spin limit is straightforward

- Contractions of ϕ_s facilitated by M^{ab} only
- Propagator uniform in s: $i\delta^{b(s)}_{a(s)}/(p^2-m^2)$
- There are additional lower spin ($s' < s$) states in the spectrum

Problematic? Not in the classical limit:

- Ghost nature easily cured by an analytic continuation on classical variables
- We get a more generic non-rigid spinning object (more internal DOFs)
- Conventional rigid spinning objects correspond to special Wilson coefficients

Bern, Kosmopoulos, Luna, Roiban, **FT**, 2203.06202
Bern, Kosmopoulos, Luna, Roiban, Scheopner, **FT**, Vines, 2308.14176
Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, **FT**, 2407.10928, 2503.03739

Generalized spin coherent state

Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, Vines, 2308.14176

The external state now contains lower spin components. Consider the coherent sum

$$\mathcal{E}_{\mu_1\ldots\mu_s} = \varepsilon^{(s)}_{\mu_1\ldots\mu_s} + u_{(\mu_1}\varepsilon^{(s-1)}_{\mu_2\ldots\mu_s)} + \ldots$$

Similar coherent sum was also considered in Aoude, Ochirov, 2108.01649, etc

Classical limit

$$\mathcal{E}_p \cdot M^{ab} \cdot \mathcal{E}_{p+q} \sim \mathsf{S}^{ab} \qquad \mathsf{S}^{ab} = S^{ab} + (i/m)(p^a K^b - p^b K^b)$$

$$\mathcal{E}_p \cdot \{M^{ab} M^{cd}\} \cdot \mathcal{E}_{p+q} \sim \mathsf{S}^{ab}\mathsf{S}^{cd}$$

where K^a is identified as the boost generator, and $S^{ab}p_b = K^a p_a = 0$

- K^a emerges from the transition between spin s and lower spin states
- Consequently, $\mathsf{S}^{ab}\mathsf{S}_{ab} \sim \boldsymbol{S}^2 - \boldsymbol{K}^2$ is a still constant but $\boldsymbol{S}^2$ is not

Non-minimal interactions up to $\mathcal{O}(M_{ab}^2)$

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

$$\nabla_\mu \phi_s = \partial_\mu \phi_s + (i/2)\omega_{\mu ab} M^{ab}\phi_s$$

$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd} M_{cd} \nabla_b$$

$$\mathcal{L} = -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{1}{8} R_{abcd}\phi_s M^{ab} M^{cd}\phi_s - \frac{C_2}{2m^2} R_{af_1bf_2} \nabla^a \phi_s \mathbb{S}^{(f_1}\mathbb{S}^{f_2)} \nabla^b \phi_s$$
$$+ \frac{D_2}{2m^2} R_{abcd} \nabla_i \phi_s \{M^{ai} M^{cd}\}\phi_s + \frac{E_2 - 2D_2}{2m^4} R_{abcd} \nabla^{(a}\nabla^{i)}\phi_s \{M^b{}_i M^d{}_j\} \nabla^{(c}\nabla^{j)}\phi_s$$

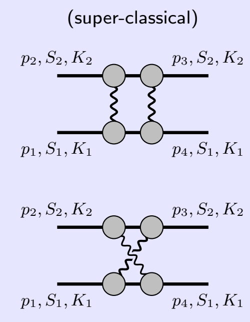
- The C_2-operator has an origin in the world-line formalism for neutron stars
  Porto, 0511061; Levi, Steinhoff, 1501.04956
- It is the only independent operator assuming that rest frame spin is the only dynamical degree of freedom
- The D_2- and E_2-operators supply additional $\mathcal{O}(SK)$ and $\mathcal{O}(K^2)$ interactions

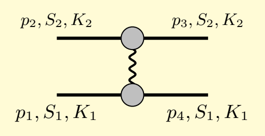
$D_2 = E_2 = 0 \quad \Longrightarrow$ Conventional compact object described by $H(\mathbf{r}, \mathbf{p}, \mathbf{S})$

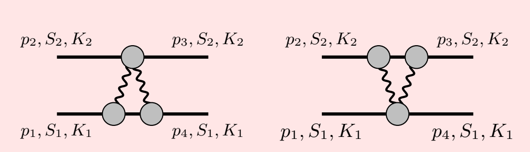
$C_2 = D_2 = E_2 = 0 \quad \Longrightarrow$ Kerr black hole

Generic values: generic compact object described by $H(\mathbf{r}, \mathbf{p}, \mathbf{S}, \mathbf{K})$

Two-body amplitudes



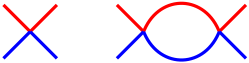




$$\begin{aligned}\mathcal{M}^{2\text{ body}} = {} & A_0 + A_1 \boldsymbol{L}\cdot\boldsymbol{S} + A_{2,1}\boldsymbol{S}^2 + A_{2,2}\boldsymbol{K}^2 + A_{2,3}\boldsymbol{S}\cdot\boldsymbol{K} + A_{2,4}(\boldsymbol{b}\cdot\boldsymbol{S})^2 \\ & + A_{2,5}(\boldsymbol{p}\cdot\boldsymbol{S})^2 + A_{2,6}(\boldsymbol{b}\cdot\boldsymbol{K})^2 + A_{2,7}(\boldsymbol{p}\cdot\boldsymbol{K})^2 + A_{2,8}(\boldsymbol{b}\cdot\boldsymbol{S})(\boldsymbol{p}\cdot\boldsymbol{S}) \\ & + A_{2,9}(\boldsymbol{L}\cdot\boldsymbol{S})(\boldsymbol{b}\cdot\boldsymbol{K}) + A_{2,10}(\boldsymbol{L}\cdot\boldsymbol{S})(\boldsymbol{p}\cdot\boldsymbol{K}) + A_{2,11}(\boldsymbol{b}\cdot\boldsymbol{S})(\boldsymbol{L}\cdot\boldsymbol{K}) \\ & + A_{2,12}(\boldsymbol{p}\cdot\boldsymbol{S})(\boldsymbol{L}\cdot\boldsymbol{K}) + A_{2,13}(\boldsymbol{b}\cdot\boldsymbol{K})(\boldsymbol{p}\cdot\boldsymbol{K})\end{aligned}$$

Effective Hamiltonian through matching

Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739



Consider canonical spin in the COM frame

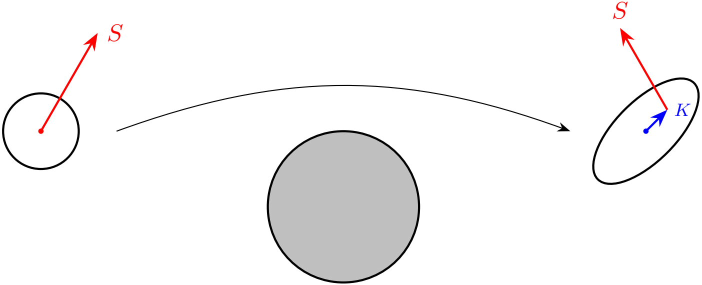
$$H=\sqrt{\mathbf{p}^2+m_1^2}+\sqrt{\mathbf{p}^2+m_2^2}+\sum_a\sum_{n=1}^{\infty}\left(\frac{G}{|\mathbf{r}|}\right)^n c_n^a(\mathbf{p}^2)\Sigma_a$$

where the operators Σ_a takes value in

$$\begin{array}{ccc}
\mathbf{1} & \left((\mathbf{r}\times\mathbf{p})\cdot\mathbf{S}\right)/\mathbf{r}^2 & \left(\mathbf{r}\cdot\mathbf{K}\right)/\mathbf{r}^2 \\
\left(\mathbf{r}\cdot\mathbf{S}\right)^2/\mathbf{r}^4 & \left(\mathbf{r}\cdot\mathbf{K}\right)\left((\mathbf{r}\times\mathbf{p})\cdot\mathbf{S}\right)/\mathbf{r}^4 & \left(\mathbf{r}\cdot\mathbf{K}\right)^2/\mathbf{r}^4 \\
\mathbf{S}^2/\mathbf{r}^2 & \left(\mathbf{K}\cdot(\mathbf{p}\times\mathbf{S})\right)/\mathbf{r}^2 & \mathbf{K}^2/\mathbf{r}^2 \\
\left(\mathbf{p}\cdot\mathbf{S}\right)^2/\mathbf{r}^2 & \left(\mathbf{r}\cdot\mathbf{S}\right)\left((\mathbf{r}\times\mathbf{K})\cdot\mathbf{p}\right)/\mathbf{r}^4 & \left(\mathbf{p}\cdot\mathbf{K}\right)^2/\mathbf{r}^2
\end{array}$$

- c_0^a matches to the tree level amplitude at $\mathcal{O}(G)$
- Iteration of c_0^a should agree exactly with the super-classical box coefficients at $\mathcal{O}(G^2)$
- c_1^a matches to the triangle coefficients at $\mathcal{O}(G^2)$ order of V
- The coefficient of $\left(\mathbf{r}\cdot\mathbf{K}\right)/\mathbf{r}^2$ vanishes identically
- All the c_n^a coefficients are local in $\mathbf{p}^2$

Generic spinning body with K

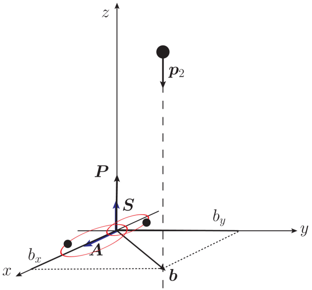


An additional conservative gapless degree of freedom

Scattering off a Newtonian bound state

$$\begin{aligned} H &= \frac{\boldsymbol{p}_2^2}{2m_2} + \frac{\boldsymbol{P}^2}{2m_1} + H_0(\boldsymbol{p},\boldsymbol{r}) - \frac{G m_{B1} m_2}{|\boldsymbol{R} + \frac{m_{B2}}{m_1}\boldsymbol{r}|} - \frac{G m_{B2} m_2}{|\boldsymbol{R} - \frac{m_{B1}}{m_1}\boldsymbol{r}|} \\ &= \frac{\boldsymbol{p}_2^2}{2m_2} + \frac{\boldsymbol{P}^2}{2m_1} + H_0(\boldsymbol{p},\boldsymbol{r}) - \frac{G m_1 m_2}{|\boldsymbol{R}|} - \frac{3G\mu_B m_2}{2|\boldsymbol{R}|^5} \underbrace{\left((\boldsymbol{r}\cdot\boldsymbol{R})^2 - \frac{1}{3}|\boldsymbol{r}|^2|\boldsymbol{R}|^2 \right)}_{Q_{ij}(\boldsymbol{r})Q^{ij}(\boldsymbol{R})} + \ldots \end{aligned}$$

where $m_1 = m_{B1} + m_{B2}$ and $\mu_B = m_{B1} m_{B2}/m_1$



Scattering off a Newtonian bound state

$$\mathcal{A}_{\mathrm{i}\to\mathrm{f}} = \int_{-\infty}^{+\infty} dt e^{i(\mathsf{E}_{\mathrm{f}}^{B} - \mathsf{E}_{\mathrm{i}}^{B})t} \left\langle \mathrm{i} \left| \frac{3G\mu_B m_2}{2|\boldsymbol{R}|^5} \left((\boldsymbol{r}\cdot\boldsymbol{R})^2 - \frac{1}{3}|\boldsymbol{r}|^2|\boldsymbol{R}|^2 \right) \right| \mathrm{f} \right\rangle$$
$$= \frac{3G\mu_B m_2 r_{\mathrm{cl},n}^2}{2|\boldsymbol{b}|^2 v_0} \left[\frac{2(\boldsymbol{b}\cdot\boldsymbol{A})^2}{|\boldsymbol{b}|^2} - |\boldsymbol{A}|^2 \right]$$

- Trajectory: $\boldsymbol{R} = (b_x, b_y, -v_0 t)$
- Initial and final state have the same energy; otherwise exponentially suppressed
- Use elliptical orbit coherent state with $bv_0^2 \gg r_{\mathrm{cl},n}$ Bhaumik, Dutta-Roy, Ghosh, 1986

$$\langle\alpha|x|\alpha\rangle = r_{\mathrm{cl},n}\Big[\cos(2\omega_{\mathrm{cl}}t) + \sin(2\chi)\Big]$$
$$\langle\alpha|y|\alpha\rangle = r_{\mathrm{cl},n}\sin(2\omega_{\mathrm{cl}}t)\cos(2\chi)$$
$$\langle\alpha|z|\alpha\rangle = 0$$

- Laplace-Runge-Lenz vector $\boldsymbol{A} = \sin(2\chi)\hat{\boldsymbol{x}}$

Scattering off a Newtonian bound state

$$\mathcal{M}^{2\text{ body}} \sim A_{2,1}\boldsymbol{K}^2 + A_{2,6}(\boldsymbol{b}\cdot\boldsymbol{K})^2$$

Match to the field theory amplitude:

- Spin ⇔ bound system total orbital angular momentum
- Due to the geometric configuration, the spin does not appear in $\mathcal{A}_{\mathrm{i}\to\mathrm{f}}$
- $\boldsymbol{K}$-vector ⇔ Laplace-Runge-Lenz vector

$$\boldsymbol{K} = i\,Gm_1^2\frac{\mu_B}{m_1}\sqrt{\frac{\mu_B}{2|\mathsf{E}_{\mathrm{i}}^B|}}\,\boldsymbol{A}$$

- Wilson coefficient

$$E_2^{\text{bound 2-body}} = \frac{3|\mathsf{E}_{\mathrm{i}}^B|m_1}{\mu_B^2}(m_1 r_{\mathrm{cl},n})^2$$

$$\mathcal{L} \sim \frac{E_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_iM^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s$$

Table of Contents

KMOC formalism Kosower, Maybee, O'Connell, 1811.10950

Observable in a scattering process:

$$\begin{aligned}\Delta O &= \langle\psi_{\text{out}}|\mathbb{O}|\psi_{\text{out}}\rangle - \langle\psi_{\text{in}}|\mathbb{O}|\psi_{\text{in}}\rangle \\ &= \langle\psi_{\text{in}}|\hat{S}^\dagger\mathbb{O}\hat{S}|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|\mathbb{O}|\psi_{\text{in}}\rangle \\ &= i\langle\psi_{\text{in}}|[\mathbb{O},\hat{T}]|\psi_{\text{in}}\rangle + \langle\psi_{\text{in}}|\hat{T}^\dagger[\mathbb{O},\hat{T}]|\psi_{\text{in}}\rangle\end{aligned}$$

$$\hat{S} = 1 + i\hat{T}$$
$$\hat{T} - \hat{T}^\dagger = i\hat{T}^\dagger\hat{T}$$

Incoming state: $|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi[p_1]\mathrm{d}\Phi[p_2]\phi(p_1)\phi(p_2)e^{ip_1\cdot b_1+ip_2\cdot b_2}|p_1p_2\rangle$

Single particle phase-space and wavefunction:

$$\mathrm{d}\Phi[p] = \frac{\mathrm{d}^4p}{(2\pi)^4}2\pi\Theta(p^0)\delta(p^2-m^2) \qquad \int \mathrm{d}\Phi[p]|\phi(p)|^2 = 1$$

We can compute observables by dressing amplitudes and cuts with the corresponding operators

$$\Delta O = \int \mathrm{d}\Phi[p_1]\mathrm{d}\Phi[p_2]\mathrm{d}\Phi[p_1']\mathrm{d}\Phi[p_2']\phi(p_1)\phi(p_2)\phi(p_1')\phi(p_2')e^{i(p_1-p_1')\cdot b_1+i(p_2-p_2')\cdot b_2}$$
$$\times\Big[i\langle p_1'p_2'|[\mathbb{O},\hat{T}]|p_1p_2\rangle + \langle p_1'p_2'|\hat{T}^\dagger[\mathbb{O},\hat{T}]|p_1p_2\rangle\Big]$$

Classical observables Kosower, Maybee, O'Connell, 1811.10950

The wave packets are highly localized, while q_i is much smaller than their spread

$$\int \mathrm{d}\Phi[p_1]\mathrm{d}\Phi[p_2]\mathrm{d}\Phi[p_1']\mathrm{d}\Phi[p_2']\phi(p_1)\phi(p_2)\phi^*(p_1')\phi^*(p_2')e^{i(p_1-p_1')\cdot b_1+i(p_2-p_2')\cdot b_2}$$

$$\simeq \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\hat{\delta}(2\bar{p}_1\cdot q_1)\hat{\delta}(2\bar{p}_2\cdot q_2)e^{iq_1\cdot b_1+iq_2\cdot b_2} \qquad \hat{\delta}(x)=2\pi\delta(x)$$

Resolution of identity in the two-massive-particle subspace

$$\mathbb{I}=\sum_X\int \mathrm{d}\Phi[r_1]\mathrm{d}\Phi[r_2]|r_1r_2X\rangle\langle r_1r_2X|$$

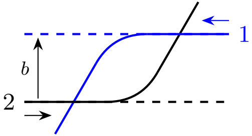
Expansion in the soft region $(q_i,\ell,k,G)\to(\hbar q_i,\hbar\ell,\hbar k,\hbar^{-1}G)$

Classical observables

$$\Delta O_{\mathrm{cl}}=\int\frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\hat{\delta}(2\bar{p}_1\cdot q_1)\hat{\delta}(2\bar{p}_2\cdot q_2)e^{iq_1\cdot b_1+iq_2\cdot b_2}\left[i\langle p_1'p_2'|[\mathbb{O},\hat{T}]|p_1p_2\rangle+\langle p_1'p_2'|\hat{T}^\dagger[\mathbb{O},\hat{T}]|p_1p_2\rangle\right]$$

Impulse

Through 3PM: Hermann, Parra-Martinez, Ruf, Zeng, 2104.03957



We compute the expectation value of

$$\mathbb{O} = \mathbb{P}_1^\mu \qquad \mathbb{P}_1^\mu |p_1^\mu\rangle = p_1^\mu |p_1\rangle$$

Tree level impulse

$$[\Delta p_1^\mu]_{(0)} = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq\cdot b}(-iq^\mu) \underbrace{\left[-\frac{\kappa^2 \bar{m}_1^2 \bar{m}_2^2 (2y^2-1)}{2q^2}\right]}_{\text{classical tree amp}} = -\frac{\kappa^2 \bar{m}_1 \bar{m}_2 (2y^2-1)}{16\pi\sqrt{y^2-1}} \frac{b^\mu}{(-b^2)}$$

One loop impulse (from the S-matrix)

$$\begin{aligned}[\Delta p_1^\mu]_{(1,\perp)} &= \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \frac{\mathrm{d}^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1\cdot b_1 + iq_2 \cdot b_2} \left[i\langle p_1' p_2' | [\mathbb{P}_1^\mu, \hat{T}] | p_1 p_2\rangle \right] \\ &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq\cdot b}(-iq^\mu) \left[M^{(1)}_{\text{amp-sc}} + M^{(1)}_{\text{amp-cl}} \right]\end{aligned}$$

We have a classically singular and IR divergent contribution $M^{(1)}_{\text{qft-sc}}$, but fortunately we have not done yet

$$M^{(1)}_{\text{amp-sc}} \frac{i\kappa^4 \bar{m}_1^4 \bar{m}_2^4 (2y^2-1)^2}{8\hbar^4} \mathcal{I}_\square \qquad M^{(1)}_{\text{amp-cl}} = \frac{3\kappa^4 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2)(5y^2-1)}{128\sqrt{-q^2}}$$

Impulse

One loop impulse (from the cut)

$\check{u}_1 = \frac{y\bar{u}_2 - \bar{u}_1}{y^2-1}$ and $\check{u}_2 = \frac{y\bar{u}_1 - \bar{u}_2}{y^2-1}$

$$
\begin{aligned}
[\Delta p_1^\mu]_{(1,\|)} &= \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\hat{\delta}(2\bar{p}_1\cdot q_1)\hat{\delta}(2\bar{p}_2\cdot q_2)e^{iq_1\cdot b_1 + iq_2\cdot b_2}\left[\langle p_1' p_2'|\hat{T}^\dagger[\mathbb{P}_1^\mu, \hat{T}]|p_1 p_2\rangle\right] \\
&= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\hat{\delta}(2\bar{p}_1\cdot q)e^{iq\cdot b}\left[(-iq^\mu)M^{(1)}_{\text{cut-sc}} + \left(\frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1}\right)M^{(1)}_{\text{cut-cl}}\right]
\end{aligned}
$$

The classically singular contribution cancels exactly between the S-matrix and cut

$$
M^{(1)}_{\text{amp-sc}} + M^{(1)}_{\text{cut-sc}} = 0 \qquad M^{(1)}_{\text{cut-cl}} = \frac{(2y^2-1)^2\bar{m}_1^3\bar{m}_2^3}{32\pi\sqrt{y^2-1}}\log(-q^2)
$$

Full one-loop impulse

$$
\begin{aligned}
[\Delta p_1^\mu]_{(1,\perp)} + [\Delta p_1^\mu]_{(1,\|)} &= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\hat{\delta}(2\bar{p}_1\cdot q)e^{iq\cdot b}\left[(-iq^\mu)M^{(1)}_{\text{amp-cl}} + \left(\frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1}\right)M^{(1)}_{\text{cut-cl}}\right] \\
&= \underbrace{-\frac{3\pi G^2\bar{m}_1\bar{m}_2(\bar{m}_1+\bar{m}_2)(5y^2-1)b^\mu}{4\sqrt{y^2-1}\,(-b^2)^{3/2}}}_{\text{transverse}} \underbrace{-\frac{2G^2\bar{m}_1^2\bar{m}_2^2(2y^2-1)^2}{(y^2-1)(-b^2)}}_{\text{longitudinal}}\left[\frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1}\right]
\end{aligned}
$$

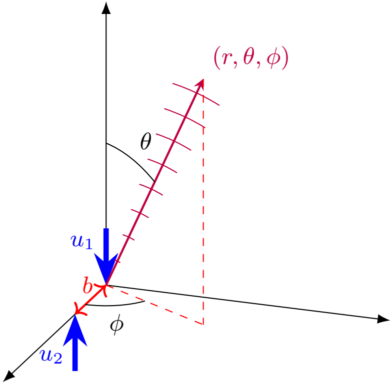
Waveform

Herderschee, Roiban, **FT**, 2303.06112

Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**, 2402.06604

Metric perturbation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Waveform: $W(T_{\mathrm{R}}, \theta, \phi) = \dfrac{1}{4G} \lim_{r\to\infty} r \varepsilon_+^\mu \varepsilon_+^\nu h_{\mu\nu} = \dfrac{1}{4G}(h_+^\infty - i h_\times^\infty)$



Waveform

Cristofoli, Gonzo, Kosower, O'Connell, 2107.10193

Operator: metric perturbation

$$\mathbb{H} = \varepsilon_+^\mu \varepsilon_+^\nu \hat{h}_{\mu\nu}(x) = \int \mathrm{d}\Phi[k] \left[\hat{a}_{--}(k)\, e^{-ik\cdot x} + \mathrm{c.c}\right]$$

Since there is no radiation at $t \to -\infty$,

$$\Delta H(x) = \langle\psi_{\rm in}|\hat{S}^\dagger \mathbb{H} \hat{S}|\psi_{\rm in}\rangle = \int \mathrm{d}\Phi[k]\Big[\tilde{J}(k) e^{-ik\cdot x} - \mathrm{c.c}\Big] \qquad \tilde{J}(k) = \langle\psi_{\rm in}|\hat{S}^\dagger \hat{a}_{--}(k)\hat{S}|\psi_{\rm in}\rangle$$

Integrating over k gives both retarded and advanced Green's function. We discarded the advanced one due to the boundary condition,

$$\Delta H(x) = -\frac{i}{4\pi^2}\int \mathrm{d}^4 y \frac{J(y)}{(x-y)^2 - i0} = -\frac{i}{4\pi^2}\int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{d}y^0 \mathrm{d}^3\boldsymbol{y} \frac{\tilde{J}(\omega,\boldsymbol{k})\, e^{-i\omega y^0 + i\boldsymbol{k}\cdot\boldsymbol{y}}}{(x^0-y^0)^2 - |\boldsymbol{x}-\boldsymbol{y}|^2 - i0}$$

Assuming the spatial current $J(y)$ is localized around $y = 0$,

$$\Delta H(x)\Big|_{|\boldsymbol{x}|\to\infty} = \frac{1}{4\pi|\boldsymbol{x}|}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}W(\omega,\boldsymbol{n})e^{-i\omega\tau} \qquad (\tau = x^0 - |\boldsymbol{x}| : \text{ retarded time})$$

$$\mathcal{W}(\varepsilon,\omega,\boldsymbol{n},p_1,p_2,b_1,b_2) = (-i)\langle\psi_{\rm in}|\hat{S}^\dagger \hat{a}_{--}(k)\hat{S}|\psi_{\rm in}\rangle\Big|_{k=(\omega,\omega\boldsymbol{n})}^{\text{IR finite}} \qquad \text{(frequency domain waveform)}$$

(As we will see later, the IR divergence can be absorbed in the definition of τ)

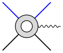
Relevant matrix elements for waveform

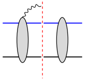
In the space of momentum transfer:

$$(-i)\langle\psi_{\rm in}|\hat{S}^\dagger \hat{a}(k)\hat{S}|\psi_{\rm in}\rangle = \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\hat{\delta}(2p_1\cdot q_1)\hat{\delta}(2p_2\cdot q_2)e^{iq_1\cdot b_1+iq_2\cdot b_2}$$
$$\times\left[\langle p_1'p_2'k|\hat{T}|p_1p_2\rangle - i\langle p_1'p_2'|\hat{T}^\dagger\hat{a}(k)\hat{T}|p_1p_2\rangle\right]$$

The matrix elements can be classified as follows:

$$\langle p_1'p_2'k|\hat{T}|p_1p_2\rangle\Big|_{\rm conn} = \mathcal{M}^{\rm tree}+\mathcal{M}^{1\ \rm loop} \qquad i\langle p_1'p_2'|\hat{T}^\dagger\hat{a}(k)\hat{T}|p_1p_2\rangle\Big|_{\rm conn} = \mathcal{S}^{1\ \rm loop}$$



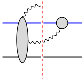


The full connected contribution to waveform:

$$\mathscr{M}_{\rm conn} = \mathcal{M}^{\rm tree}+\mathcal{M}^{1\ \rm loop}-\mathcal{S}^{1\ \rm loop}$$

Contribution from disconnected T matrix elements:

$$\langle p_1'p_2'k|\hat{T}|p_1p_2\rangle\Big|_{\rm disc} = \mathscr{M}_{\rm const} \qquad i\langle p_1'p_2'|\hat{T}^\dagger\hat{a}(k)\hat{T}|p_1p_2\rangle\Big|_{\rm disc} = \mathscr{M}_{\rm disc}$$



Tree-level amplitudes and LO waveform

From GR: Kovacs, Thorne, Astrophys. J. 224 62, 1978
From world line QFT: Jakobsen, Mogull, Plefka, Steinhoff, 2101.12688
From KMOC: Cristofoli, Gonzo, Kosower, O'Connell, 2107.10193

For convenience, we define the Fourier transform from q-space to b-space,

$$\mathrm{FT}\Big[f(q_1,q_2,k)\Big] = \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\hat{\delta}(2p_1\cdot q_1)\hat{\delta}(2p_2\cdot q_2)e^{iq_1\cdot b_1+iq_2\cdot b_2}\hat{\delta}(q_1+q_2-k)f(q_1,q_2,k)\Big|_{k=(\omega,\omega\boldsymbol{n})}$$

LO waveform from KMOC

$$\mathcal{W}(\omega,\boldsymbol{n})\Big|_{\mathrm{LO}} = \mathrm{FT}\Big[\mathcal{M}^{\mathrm{tree}}\Big]$$

Classical tree-level amplitude Luna, Nicholson, O'Connell, White, 1711.03901

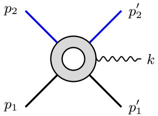
$$\mathcal{M}^{\mathrm{tree}} = -\kappa^2 m_1^2 m_2^2\left[\frac{(P_{12}\cdot\varepsilon)^2+\sigma(Q_{12}\cdot\varepsilon)(P_{12}\cdot\varepsilon)}{q_1^2 q_2^2}+\left(\sigma^2-\frac{1}{d-2}\right)\left(\frac{(Q_{12}\cdot\varepsilon)^2}{4q_1^2q_2^2}-\frac{(P_{12}\cdot\varepsilon)^2}{4w_1^2w_2^2}\right)\right]$$
$$= \mathcal{M}^{\mathrm{tree}}_{d=4}+\epsilon\mathcal{M}^{\mathrm{tree}}_{\mathrm{extra}}$$

$$P_{12}^\mu = w_1 u_2^\mu - w_2 u_1^\mu \qquad u_i = p_i/m_i \qquad w_i = u_i\cdot k$$
$$Q_{12}^\mu = (q_1-q_2)^\mu - \frac{q_1^2}{w_1}u_1^\mu + \frac{q_2^2}{w_2}u_2^\mu$$

Of course, at LO the $\mathcal{O}(\epsilon)$ term is irrelevant.

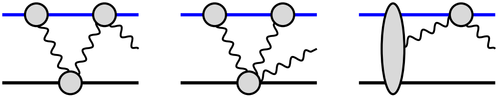
One-loop matrix elements Herderschee, Roiban, **FT**, 2303.06112

$$\langle p_1' p_2' k | \hat{T} | p_1 p_2 \rangle \Big|_{\text{1-loop}} =$$



$$= \mathcal{M}^{1\text{ loop}}$$

Use generalized unitarity to construct the integrand



Expand the integrand in the classical limit (method of region, Beneke, Smirnov, hep-ph/9711391)

- Matter propagator

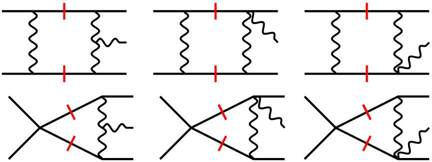
$$\frac{1}{(p+\ell)^2 - m^2 + i0} = \frac{1}{2p\cdot\ell + \ell^2 + i0} = \frac{1}{2p\cdot\ell + i0}\sum_{n=0}^{\infty}\left(-\frac{\ell^2}{2p\cdot\ell + i0}\right)^n$$

- One matter line per loop
- Matter lines do not touch before IBP

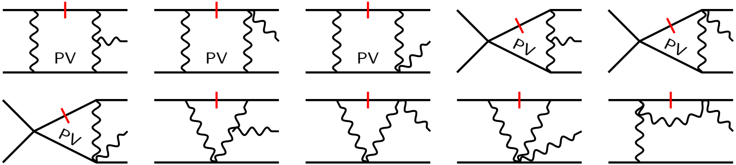
Master integrals: amplitude Herderschee, Roiban, **FT**, 2303.06112

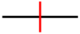
IBP automated by FIRE6: Smirnov, Chuharev, 1901.07808

Super-classical: $\frac{m_1^3 m_2^3}{\hbar^4}\sum_i \alpha_i \mathcal{I}_i^{\mathrm{cut,cut}}$



Classical: $\frac{1}{\hbar^3}\sum_i (m_1^3 m_2^2\,\beta_i\,\mathcal{I}_i^{\mathrm{cut,pv}} + m_1^2 m_2^3\,\gamma_i\,\mathcal{I}_i^{\mathrm{pv,cut}}) + \frac{1}{\hbar^3}(\text{triangles and bubbles})$

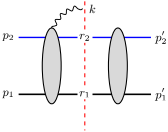


 $= \hat{\delta}(2u_2 \cdot \ell)$

$= \frac{1}{2u_1 \cdot \ell + i0} + \frac{1}{2u_1 \cdot \ell - i0}$

Connected cut contribution $\mathcal{S}^{1\text{ loop}}$

$$\langle p_1' p_2' | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{conn}}^{1\text{ loop}} =$$



$$= \sum_i \left(\frac{1}{\hbar^4} m_1^3 m_2^3 \alpha_i - \frac{1}{\hbar^3} m_1^3 m_2^2 \beta_i + \frac{1}{\hbar^3} m_1^2 m_2^3 \gamma_i \right) \mathcal{I}_i^{\text{cut,cut}}$$

- Exactly cancel the super-classical terms in the virtual amplitude
- Convert the PV to retarded Caron-Huot, Giroux, Hannesdottir, Mizera, 2308.02125

$$\text{PV}\frac{1}{2u_1 \cdot \ell} \longrightarrow \frac{1}{2u_1 \cdot \ell - i0}$$

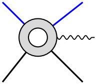
- Classical waveform contains only imaginary IR divergences

Equivalence to world-line formalism proved by Capatti, Zeng, 2412.10864

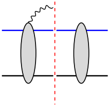
The full connected contribution

Herderschee, Roiban, **FT**, 2303.06112
Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini, 2303.06111
Georgoudis, Heissenberg, Vazquez-Holm, 2303.07006

$$\begin{aligned}\mathcal{M}^{1\text{ loop}} &= -\frac{i\,\kappa^2}{32\pi}(m_1 w_1 + m_2 w_2)\left[\frac{1}{\epsilon} - \log\frac{w_1 w_2}{\mu^2}\right]\mathcal{M}^{\text{tree}}_{d=4}\\ &\quad + \kappa^4\left[A^R_{\text{rat}} + \frac{A^R_1}{\sqrt{w_2^2 - q_1^2}} + \frac{A^R_2}{\sqrt{w_1^2 - q_2^2}} + \frac{A^R_3}{\sqrt{-q_1^2}} + \frac{A^R_4}{\sqrt{-q_2^2}}\right]\\ &\quad + i\,\kappa^4\left[A^I_{\text{rat}} + A^I_1\frac{\text{arcsinh}\,\frac{w_2}{\sqrt{-q_1^2}}}{\sqrt{w_2^2 - q_1^2}} + A^I_2\frac{\text{arcsinh}\,\frac{w_1}{\sqrt{-q_2^2}}}{\sqrt{w_1^2 - q_2^2}} + A^I_3\log\frac{q_2^2}{q_1^2} + A^I_4\log\frac{w_1}{w_2} + A^I_5\frac{\text{arccosh}\,\sigma}{(\sigma^2-1)^{3/2}}\right]\\ &= -\frac{i\,\kappa^2}{32\pi\epsilon}(m_1 w_1 + m_2 w_2)\mathcal{M}^{\text{tree}}_{d=4} + \widetilde{\mathcal{M}}^{\text{fin}}_{1\text{ loop}}\end{aligned}$$



$$\begin{aligned}\mathcal{S}^{1\text{ loop}} &= \frac{i\,\kappa^2\Gamma}{64\pi}(m_1 w_1 + m_2 w_2)\left[\frac{1}{\epsilon} - \log\frac{w_1 w_2}{(\sigma^2-1)\mu^2}\right]\mathcal{M}^{\text{tree}}_{d=4} \qquad \Gamma = \frac{3\sigma - 2\sigma^3}{(\sigma^2-1)^{3/2}}\\ &\quad + \frac{i\,\kappa^4}{256\pi}\frac{(2\sigma^2-1)^2}{(\sigma^2-1)^{3/2}}(m_1 w_1 + m_2 w_2)\left[\frac{1}{\epsilon} - \log\frac{w_1 w_2}{(\sigma^2-1)\mu^2}\right]\frac{m_1^2 m_2^2 (u_1\cdot f\cdot u_2)^2(w_1^2 + \sigma w_1 w_2 + w_2^2)}{w_1^3 w_2^3}\\ &\quad + i\,\kappa^4\left[A^{\text{cut}}_{\text{rat}} + A^{\text{cut}}_1\log\frac{w_1}{w_2} + A^{\text{cut}}_2\log\frac{w_1 w_2}{-q_1^2} + A^{\text{cut}}_3\log\frac{w_1 w_2}{-q_2^2} + A^{\text{cut}}_4\frac{\text{arccosh}\,\sigma}{(\sigma^2-1)^{3/2}}\right]\\ &= \frac{i\,\kappa^2\Gamma}{64\pi\epsilon}(m_1 w_1 + m_2 w_2)\mathcal{M}^{\text{tree}}_{d=4} + \widetilde{\mathcal{S}}^{\text{fin}}_{1\text{ loop}} + \mathcal{S}^{\text{local}}_{1\text{ loop}}\end{aligned}$$



The full connected contribution

The IR divergence exponentiates Caron-Huot, Giroux, Hannesdottir, Mizera, 2308.02125

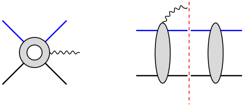
$$\mathscr{M}_{\text{conn}} = \mathscr{M}_{\text{conn}}^{\text{fin}} \exp\left[-i\omega\frac{\kappa^2 E(1-\Gamma/2)}{32\pi\epsilon}\right] \qquad (m_1 w_1 + m_2 w_2 = \omega E)$$

such that it can be absorbed by a re-definition of retarded time

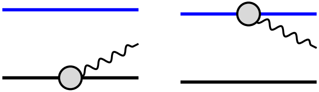
$$\tau \to \tau - \frac{\kappa^2 E(1-\Gamma/2)}{32\pi\epsilon}$$

At one-loop level, this leads to

$$\begin{aligned}
\mathscr{M}_{\text{conn}}^{\text{fin}} &= \mathcal{M}^{\text{tree}} + \mathcal{M}^{1\text{ loop}} - \mathcal{S}^{1\text{ loop}} + \frac{i\kappa^2(m_1w_1+m_2w_2)(1-\Gamma/2)}{32\pi\epsilon}\mathcal{M}^{\text{tree}} \\
&= \mathcal{M}^{\text{tree}} + \left[\widetilde{\mathcal{M}}_{1\text{ loop}}^{\text{fin}} + \frac{i\kappa^2(m_1w_1+m_2w_2)\mathcal{M}_{\text{extra}}^{\text{tree}}}{32\pi}\right] - \left[\widetilde{\mathcal{S}}_{1\text{ loop}}^{\text{fin}} + \frac{i\kappa^2\Gamma(m_1w_1+m_2w_2)\mathcal{M}_{\text{extra}}^{\text{tree}}}{64\pi}\right] \\
&= \mathcal{M}^{\text{tree}} + \mathcal{M}_{1\text{ loop}}^{\text{fin}} - \mathcal{S}_{1\text{ loop}}^{\text{fin}}
\end{aligned}$$



Disconnected amplitude contribution



They are supported only by zero-energy gravitons

$$\mathcal{W}_{\text{const}}(\omega,\boldsymbol{n}) = \int \mathrm{d}\Phi_{p_1}\mathcal{M}_3(p_1,k)e^{ik\cdot b_1} + \int \mathrm{d}\Phi_{p_2}\mathcal{M}_3(p_2,k)e^{ik\cdot b_2} = \hat{\delta}(\omega)\left[\frac{m_1(\varepsilon\cdot u_1)^2}{u_1\cdot n} + \frac{m_2(\varepsilon\cdot u_2)^2}{u_2\cdot n}\right]$$

The constant background agrees with the MPM calculation

$$\mathcal{W}_{\text{const}}(\tau,\boldsymbol{n}) = \frac{m_1(\varepsilon\cdot u_1)^2}{u_1\cdot n} + \frac{m_2(\varepsilon\cdot u_2)^2}{u_2\cdot n} \qquad n = (1,\boldsymbol{n})$$

It can be removed by a BMS supertranslation Veneziano, Vilkovisky, 2201.11607

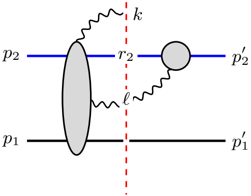
$$\delta_T\left[\frac{h^{\infty}_{ab}}{r}\right] = \frac{1}{r}(2D_aD_b - \gamma_{ab}\Delta)T(n) - T(n)\partial_\tau\left[\frac{h^{\infty}_{ab}}{r}\right] \qquad T(n) = 2G\left[m_1\frac{w_1}{\omega}\log\frac{w_1}{\omega} + m_2\frac{w_2}{\omega}\log\frac{w_2}{\omega}\right]$$

The MPM waveform is in the intrinsic BMS frame. If we assume $\mathcal{M}_3 = 0$, then we are in the canonical BMS frame. They are related by the above BMS supertranslation.

Disconnected cut contribution

The T matrices in the KMOC cut can be disconnected:

$$i\,\langle p_1' p_2' | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc.}} \supset \int \mathrm{d}\Phi[\ell] \mathrm{d}\Phi[r_2]$$



- The cut graviton ℓ only has support on zero energy
- We only need to consider the soft limit of the left blob

$$\mathcal{M}_6 \simeq \kappa\, \mathcal{M}_5 \left[\sum_a \frac{\eta_a \varepsilon_{\mu\nu}(\ell) p_a^\mu p_a^\nu}{2\ell \cdot p_a + i0\eta_a} + \frac{\varepsilon_{\mu\nu}(\ell) k^\mu k^\nu}{2\ell \cdot k + i0} \right]$$

- The sum over massive legs only leads to zero integrals of the form

$$\int \mathrm{d}^d \ell\, \delta(u_2 \cdot \ell) \delta(\ell^2) \Theta(\ell^0) \qquad \int \mathrm{d}^d \ell\, \delta(u_1 \cdot \ell) \delta(u_2 \cdot \ell) \delta(\ell^2) \Theta(\ell^0)$$

Disconnected cut contribution

Contribution from the disconnected T matrix elements:

$$i\left\langle p_1' p_2' \right| \hat{T}^{\dagger} \hat{a}_{hh}(k) \hat{T} \left| p_1 p_2 \right\rangle \Big|_{\text{disc.}} = -i\kappa^2 \mathcal{M}^{\text{tree}} \Big[m_1^2 w_1^2 I(u_1, k) + m_2^2 w_2^2 I(u_2, k) \Big]$$

$$I(u_1, k) = \frac{1}{m_1 \omega} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2u_1 \cdot \ell) \hat{\delta}(\ell^2) \Theta(\ell^0)}{2\ell \cdot n}$$

This integral is not yet properly defined in $d = 4$; further regularization is needed:

- Need to include the zero-energy graviton contribution
- We should continue u_1 off-shell, $u_1 \to \tilde{u}_1 - \beta n$, such that

$$2u_1 \cdot \ell = (u_1 + \ell)^2 - 1 \to (\tilde{u}_1 + \ell)^2 - 1 \simeq 2u_1 \cdot \ell - 2\beta u_1 \cdot n$$

- The integral now becomes

$$I(u_1, k) = \frac{1}{m_1 \omega} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2u_1 \cdot \ell - 2\beta u_1 \cdot n) \hat{\delta}(\ell^2) \Theta(\ell^0)}{2\ell \cdot n}$$
$$= \frac{1}{32\pi m_1 w_1} \left[\frac{1}{\epsilon} - \log \frac{w_1^2}{\omega^2} - \log \frac{\beta^2}{\pi} \right]$$

- Divergent terms are absorbed in a further shift of retarded time

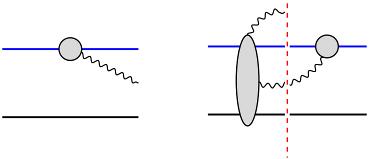
Disconnected cut contribution [DB, TD, SDA, AG, AH, RR, **FT**, 2402.06604]

The disconnected T matrix elements in the cut lead to

$$\mathscr{M}_{\rm disc} = i\langle p_1' p_2' | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\rm disc} = -2i\omega G \left[m_1 \frac{w_1}{\omega} \log \frac{w_1}{\omega} + m_2 \frac{w_2}{\omega} \log \frac{w_2}{\omega} \right] \mathcal{M}^{\rm tree} = -i\omega T(n) \mathcal{M}^{\rm tree}$$

which exactly reproduces the iteration part of the BMS supertranslation

$$\delta_T \left[\frac{h_{ab}^\infty}{r} \right] = \frac{1}{r}(2D_a D_b - \gamma_{ab}\Delta) T(n) - T(n) \partial_\tau \left[\frac{h_{ab}^\infty}{r} \right]$$



Full KMOC waveform at one-loop [DB, TD, SDA, AG, AH, RR, **FT**, 2402.06604]

$$\mathcal{W}^{\text{KMOC}}(\omega,\boldsymbol{n}) = \mathcal{W}_{\text{const}} + \mathcal{W}^{\text{tree}} + \text{FT}\Big[\mathcal{M}^{\text{fin}}_{1\ \text{loop}} - \mathcal{S}^{\text{fin}}_{1\ \text{loop}} + \mathscr{M}_{\text{disc}}\Big]$$

- Fourier transform to the impact-parameter space can only be done numerically for generic kinematic setup (see Brunello, De Angelis 2403.08009 for improvements)
- Analytic computation is available under the soft graviton limit and/or small relative velocity (PN) expansion

Soft expansion (up to the first non-universal coefficient)

$$\mathcal{W}^{\text{KMOC}}(\omega,\boldsymbol{n}) \sim \frac{\mathcal{A}}{\omega} + \mathcal{B}\log\omega + \mathcal{C}\omega(\log\omega)^2 + \mathcal{D}\omega\log\omega$$

All order conjecture in leading log: Alessio, Di Vecchia, Heissenberg, 2407.04128

PN expansion (up to 2.5PN)

$M = m_1 + m_2$

$\nu = m_1 m_2 / M^2$

$p_\infty = \sqrt{\sigma^2 - 1}$

$$\mathcal{W}^{\text{KMOC}}(\omega,\boldsymbol{n}) \sim 1 + \frac{GM^2\nu}{p_\infty}(1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \ldots)$$
$$+ \frac{GM}{bp_\infty^2}\frac{GM^2\nu}{p_\infty}(1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \ldots)$$

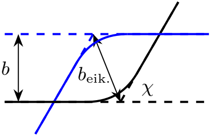
Compare with MPM waveform [Bini, Damour, Geralico, 2309.14925]

- W^{MPM} is given in terms of kinematic data in the eikonal COM frame, which is related to the incoming COM frame through a $(\chi/2)$-rotation

$$\frac{\chi_{\mathrm{LO}}}{2} = \frac{GE}{\sqrt{-b^2}} \frac{2\sigma^2 - 1}{\sigma^2 - 1}$$

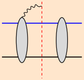
- The difference between $|v|$, $|b|$ and $|v_{\mathrm{eik}}|$, $|b_{\mathrm{eik}}|$ is $\mathcal{O}(G^2)$ and thus irrelevant at one-loop. A further shift in retarded time $\delta\tau(u_1, u_2)$ is allowed in comparison
- The KMOC waveform in the eikonal COM frame is given by

$$W^{(1)}_{\mathrm{KMOC}}(\theta,\phi) = \mathcal{W}^{(1)}_{\mathrm{KMOC}}(\theta,\phi) + \frac{\chi_{\mathrm{LO}}}{2} \frac{\partial}{\partial\phi} \mathcal{W}^{\mathrm{tree}}(\theta,\phi)$$



Compare with MPM waveform [DB, TD, SDA, AG, AH, RR, **FT**, 2402.06604]

The frame rotation cancels the connected cut contribution (up to a time shift)



$$\mathrm{FT}\Big[\mathcal{S}^{\mathrm{fin}}_{1\ \mathrm{loop}}\Big] = \frac{\chi_{\mathrm{LO}}}{2}\frac{\partial}{\partial\phi}\mathcal{W}^{\mathrm{tree}} - i\omega\delta\tau^{\mathrm{cut}}\mathcal{W}^{\mathrm{tree}}$$

This relation is verified up to $\mathcal{O}(p_\infty^6)$ and $\mathrm{N}^3\mathrm{LO}$ in soft limit, and $\delta\tau^{\mathrm{cut}}$ agrees with [GHR 2312.07452]

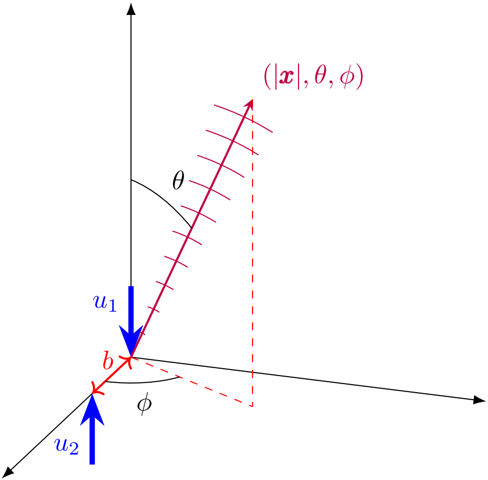
The rest agrees exactly with MPM waveform

$$W^{\mathrm{MPM}} = \mathcal{W}_{\mathrm{const}} + \mathcal{W}^{\mathrm{tree}} + \mathrm{FT}\Big[\mathcal{M}^{\mathrm{fin}}_{1\ \mathrm{loop}} + \mathscr{M}_{\mathrm{disc}}\Big] + i\omega\delta\tau^{\mathrm{amp}}\mathcal{W}^{\mathrm{tree}}$$

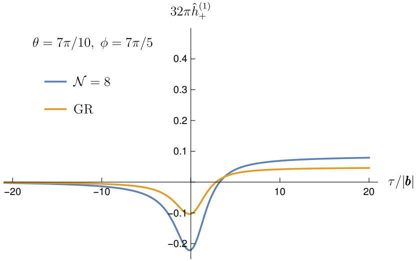
This relation is verified up to $\mathcal{O}(p_\infty^5)$ and $\mathrm{N}^3\mathrm{LO}$ in soft limit

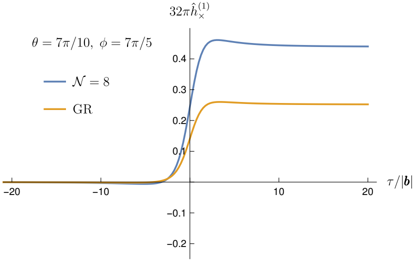
Similar comparison up to $\mathcal{O}(p_\infty^3)$ is done in Georgoudis, Heissenberg, Russo, 2402.06361

Numerical results



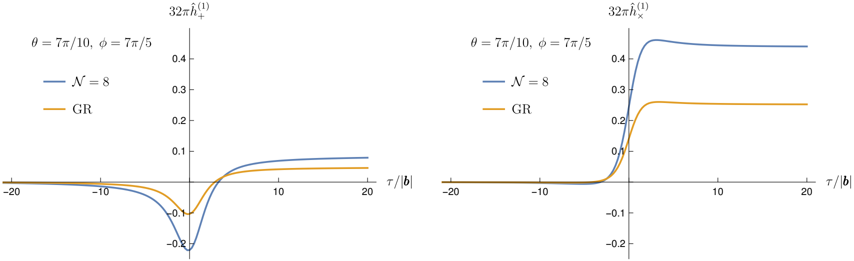
LO waveform





Agree with Jakobsen, Mogull, Plefka, Steinhoff, 2101.12688
From spinning binaries: De Angelis, Novichkov, Gonzo, 2309.17429
Brandhuber, Brown, Chen, Gowdy, Travaglini, 2310.04405
Aoude, Haddad, Heissenberg, Helset, 2310.05832

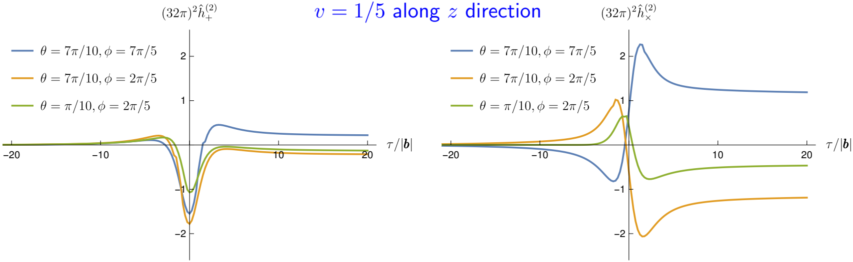
LO waveform



Agree with Jakobsen, Mogull, Plefka, Steinhoff, 2101.12688
From spinning binaries: De Angelis, Novichkov, Gonzo, 2309.17429
Brandhuber, Brown, Chen, Gowdy, Travaglini, 2310.04405
Aoude, Haddad, Heissenberg, Helset, 2310.05832

NLO waveform (GR)

$v = 1/5$ along z direction



$\theta = 7\pi/10$ and $\phi = 7\pi/5$

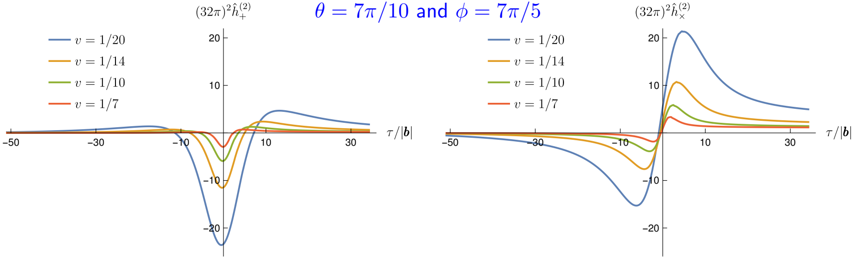
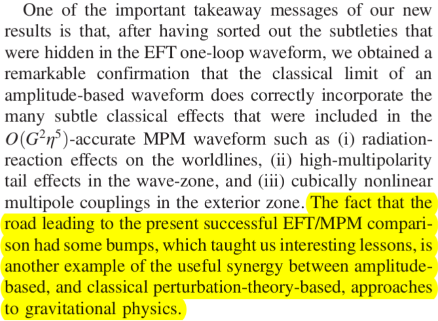


Table of Contents

Synergy between amplitude and GR

Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**, 2402.06604



One of the important takeaway messages of our new results is that, after having sorted out the subtleties that were hidden in the EFT one-loop waveform, we obtained a remarkable confirmation that the classical limit of an amplitude-based waveform does correctly incorporate the many subtle classical effects that were included in the $O(G^2\eta^5)$-accurate MPM waveform such as (i) radiation-reaction effects on the worldlines, (ii) high-multipolarity tail effects in the wave-zone, and (iii) cubically nonlinear multipole couplings in the exterior zone. The fact that the road leading to the present successful EFT/MPM comparison had some bumps, which taught us interesting lessons, is another example of the useful synergy between amplitude-based, and classical perturbation-theory-based, approaches to gravitational physics.

Outlook

Explicit higher order spinning and spinless calculation

- Challenge: IBP reduction and DE for loop integrals; higher rank tensor reduction
- What do we mean by analytic computation?

Systematic inclusion of tidal operators and absorption

- Need to understand the running and mixing of WCs

How to describe generic spinning bodies?

- Perhaps relevant to future higher precision GW observations

Re-summation in observables

- SCET for gravity
- Self-force effective theory

How to directly compute bound-state observables?

- Phenomenological prescription exists and works very well in matching NR
- Remain as a theoretical challenge

Thanks for listening!