Effective Field Theory and Scattering Amplitude

Ming-Lei Xiao 肖明磊

Sun Yat-Sen University 中山大学

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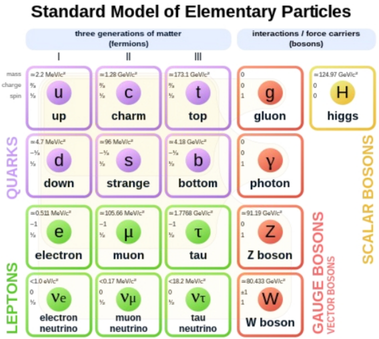
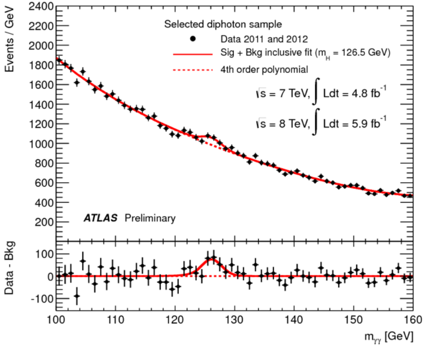
The Standard Model

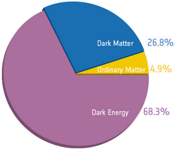
The Standard Model has been winning since its birth . . .

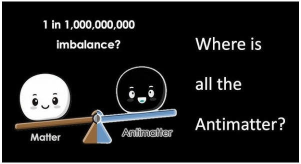


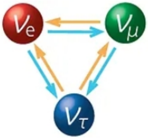


Physics Beyond the Standard Model

However, there are lots of indirect evidences of BSM:



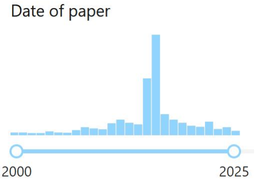




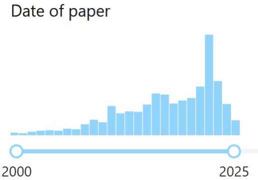
Search for Anomaly

Lack of New Physics signals and the Ambulance-Chasing game:

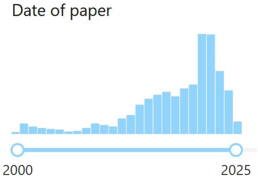
- The 750 GeV diphoton anomaly



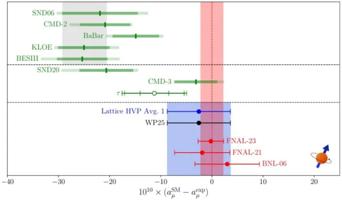
- W boson mass anomaly



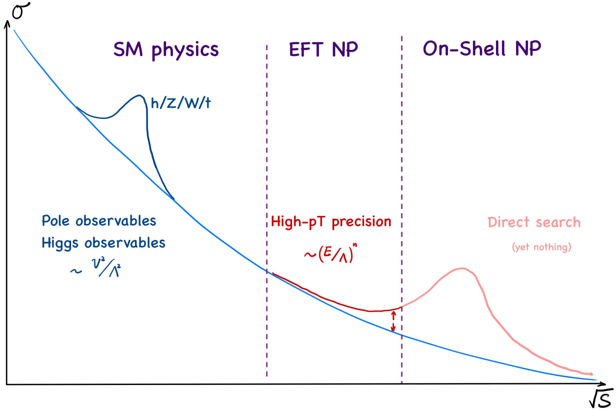
- Muon $g-2$ anomaly



Muon Theory Initiative [2505.21476]

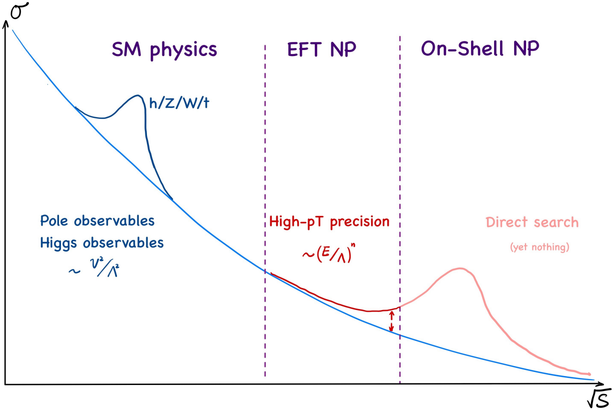


The Approach of Effective Field Theory



Paradigm shift	Experimentalist	Theorist
On-Shell New Physics	bump hunting	model building
EFT New Physics	precision measurement	effective operators

The Approach of Effective Field Theory



$$\mathcal{L}_{\mathrm{EFT}}(\Lambda) = \mathcal{L}_{\mathrm{SM}} + \sum_{d>4} \frac{c_i}{\Lambda^{d-4}} \mathcal{O}_i^{(d)}$$

Outline

Outline

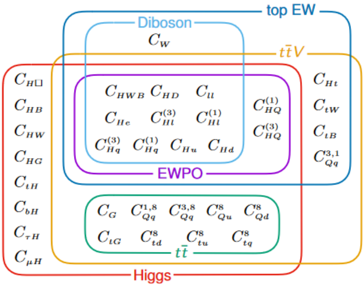
1. Effective Operators in the On-Shell Way
2. Construction of Operator Basis
3. Partial Wave Amplitudes
4. Summary

Higher Dimensional Operators

Dim-6 operators are the main concerns in the new physics searches,

$$\mathcal{L}_{\text{Warsaw}} = \sum_{i=1}^{63} \frac{c_i}{\Lambda^2} \mathcal{O}_i$$

[Grzadkowski,*et al.*, 2010]

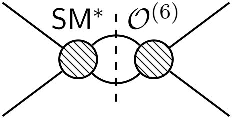


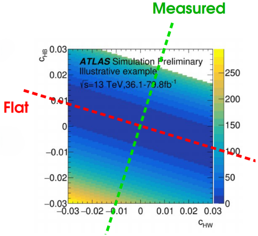
[Ellis,*et al.*, 2021]

Higher Dimensional Operators

Leading contribution beyond dim-6:

- absent at dim-6: *e.g.* nTGC [Ellis, He, Xiao, 2020]
- non-interference [Degrande, Li, 2023]





- flat directions [Boughezal, Petriello, Wiegand, 2021]
- loop-level generation [Guedes, Olgoso, Santiago, 2023]

$$F_{\mu\nu}F^{\nu}{}_{\rho}F^{\rho\mu}\ , \quad (H^\dagger H)F_{\mu\nu}F^{\mu\nu}\ , \quad B_{\mu\nu}(\bar{L}\sigma^{\mu\nu}e)H$$

Bases of Effective Operators

The EFT analysis should be based on a complete and independent set of effective operators: the operator basis

- The first such dim-6 basis, Warsaw basis, was constructed in 2010 (the last update of the paper is actually 2017).
- Historically, there were bases of operators invented for convenient characterization of certain class of processes or UV models: SILH basis, Higgs basis,. . . but they are not complete and does not consistently apply to theory outside the range of its presumption.
- After the Warsaw basis, lots of work on higher dimensional operator basis
  - dimension-7 Liao,Ma,2016
  - dimension-8 Murphy,2020; Li,Shu,Ren,Xiao,Yu,Zheng,2020
  - dimension-9 Li,Ren,Xiao,Yu,Zheng,2020; Liao,Ma,2020

Bases of Effective Operators

The EFT analysis should be based on a complete and independent set of effective operators: the operator basis

Due to a number of redundancy relations, the choice of operator basis is arbitrary

- Equation of Motion (EOM)
- Covariant Derivative Commutator (CDC), Bianchi Identity, . . .
- Integration by Part (IBP of Lagrangian)
- All sorts of group identities (Lorentz, gauge)

Bases of Effective Operators

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In the presence of repeated fields and number of flavors, additional redundancy due to permutation symmetry (flavor symmetry).

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In practice, whatever we get from matching and running should be converted to the combination of operator basis.

On-shell Basis

An easy rule to get rid of EOM (and CDC) redundancy: simply avoid the "kinetic term" of EOM ($D^2\Phi$, $\not{D}\psi$, $D^\mu F_{\mu\nu}$) and assuming symmetries among the covariant derivatives

$$D_{\mu_1} D_{\mu_2} \cdots D_{\mu_m} \Phi \simeq D_{\mu_{\sigma(1)}} D_{\mu_{\sigma(2)}} \cdots D_{\mu_{\sigma(m)}} \Phi \;, \quad \forall \sigma \in S_m$$

This is exactly a rule that the Warsaw-like operator bases follow, which can be interpreted as a simple principle:

Amplitude/Operator Correspondence

An independent set of operators is isomorphic to an independent set of on-shell local amplitudes, or their leading form factors.

$$D^2\Phi \simeq p^2 = m^2, \quad \not{D}\psi \simeq \not{p}u(p) = mu(p) \;, \quad D^\mu F_{\mu\nu} \simeq p^2\epsilon_\nu - (p\cdot\epsilon)p_\nu = m^2\epsilon$$

$$D_{\mu_1} D_{\mu_2} \cdots D_{\mu_m} \Phi \simeq p_{\mu_1} p_{\mu_2} \cdots p_{\mu_m}$$

Relation with On-shell Bootstrap

Two ways to compute the scattering amplitude

- Operators $\rightarrow$ Feynman Diagrams $\rightarrow$ Amplitudes
- On-shell amplitude seeds $\xrightarrow{\text{RR}}$ Amplitudes

A Brief Review of Recursion Relation

Momentum shift $\mathcal{A}(\hat{p}(z)) \equiv \hat{\mathcal{A}}(z)$ where $\hat{p}_\mu(z) = p_\mu + z r_\mu$

$$\mathcal{A}(p) = \hat{\mathcal{A}}(0) = \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{A}}(z)}{z} = \sum_I \frac{1}{P_I^2} \hat{\mathcal{A}}_L(z_I) \times \hat{\mathcal{A}}_R(z_I) + B_\infty$$

while the residues are determined recursively by the amplitude seeds (*e.g.* the 3-pt amplitudes in YM).

Relation with On-shell Bootstrap

A theory is said to be on-shell constructible if there is a way to do the recursion with $B_\infty = 0$.

- What is B_∞?
  A contribution to the amplitude with no poles (local).
- Why is it constructible when $B_\infty = 0$?
  The amplitude can be fully determined by the residues.
- Why is it NOT constructible when $B_\infty \neq 0$?
  There is information NOT determined by the residues
  (local contributions coming from local operators).
  - $\mathcal{A}(\phi,\phi,\phi,\phi)$ in scalar QED due to a possible operator $\lambda\phi^4$.
  - Most EFT's are not constructible.
- The missing information is encoded in the set of Wilson coefficients.

$$\text{Effective Operators} \Leftrightarrow \text{Possible Terms in } B_\infty$$

Spinor-Helicity Variables

It is convenient to construct the amplitude basis in terms of $\{|i\rangle, |i]\}$.

- Both momenta and on-shell wave functions can be expressed

$$p_i^\mu = \frac{1}{2}\langle i|\sigma^\mu|i] \ , \quad u(p_i) = \begin{pmatrix} |i\rangle \\ |i] \end{pmatrix} \ , \quad \epsilon_L^\mu(p_i) = \frac{\langle i|\sigma^\mu|r]}{\sqrt{2}[ir]}$$

- Extension to massive particles $\{|\boldsymbol{i}\rangle, |\boldsymbol{i}]\}$ Arkani-Hamed,Huang,Huang,2017

$$p_i^\mu = \frac{1}{2}\langle \boldsymbol{i}|^{[I}\sigma^\mu|\boldsymbol{i}]^{J]} \ , \quad u^I(p_i) = \begin{pmatrix} |\boldsymbol{i}\rangle^I \\ |\boldsymbol{i}]^I \end{pmatrix} \ , \quad \epsilon_{(IJ)}^\mu(p_i) = \frac{\langle \boldsymbol{i}^{(I}|\sigma^\mu|\boldsymbol{i}^{J)}]}{\sqrt{2}m}$$

- Lorentz invariant amplitudes expressed in terms of spinor brackets

$$\mathcal{M}(\langle ij\rangle, [ij]) \quad \text{independent of } |r] \text{ due to gauge invariance}$$

Examples of Amplitude Basis

- Counting helicities $\mathcal{M}(\ldots, h_i, \ldots) \sim |i\rangle^{n_i}|i]^{2h_i-n_i}$
- 3-point amplitudes (special kinematics: either $\langle ij\rangle = 0$ or $[ij] = 0$)

$$\mathcal{M}(0,0,-1) = \frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle}\ , \quad \mathcal{M}(+1,-1-1) = \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 13\rangle}$$

The denominators represent spurious poles.

- The more derivatives, the more degeneracy: *e.g.* $\psi_1\psi_2\psi_3\psi_4^\dagger D$

$$\begin{aligned}
\mathcal{M}_1 = \langle 12\rangle\langle 13\rangle[14] \quad &\simeq \quad \mathcal{O}_1 = -(D_\mu\psi_1\psi_2)(\psi_3\sigma^\mu\psi_4^\dagger)\ ,\\
\mathcal{M}_2 = \langle 12\rangle\langle 23\rangle[24] \quad &\simeq \quad \mathcal{O}_2 = -(\psi_1 D_\mu\psi_2)(\psi_3\sigma^\mu\psi_4^\dagger)\ ,\\
\mathcal{M}_3 = \langle 13\rangle\langle 23\rangle[34] \quad &\simeq \quad \mathcal{O}_3 = (\psi_1 D_\mu\psi_3)(\psi_2\sigma^\mu\psi_4^\dagger)\ .
\end{aligned}$$

Example: SMEFT at dim-6

Steps to get the operator basis [Shu,Ma,Xiao,2019]

- Enumerate all the types $((d, \{h_i\}, \{\mathbf{r}_i\}))$
  The corresponding operators are written in terms of chiral basis fields:

$$\psi = P_L \Psi\,, \quad \psi_c^\dagger = P_R \Psi\,, \quad F_L = \sigma^{\mu\nu} F_{\mu\nu}\,, \quad F_R = \bar{\sigma}^{\mu\nu} F_{\mu\nu}$$

```
In[15]:= AllTypesC[SMEFT, 6] // Catenate
         Length[%]

Out[15]= {BL^3, BL WL^2, WL^3, BL GL^2, GL^3, dc ec uc^2, ec L Q uc, dc Q^2 uc, L Q^3, BL ec H† L, BL dc H† Q,
  BL H Q uc, ec H† L WL, dc H† Q WL, H Q uc WL, dc GL H† Q, GL H Q uc, BL^2 H H†, BL H H† WL, H H† WL^2,
  GL^2 H H†, ec^2 ec†^2, ec ec† L L†, dc† ec L Q†, dc dc† L L†, L L† uc uc†, ec ec† Q Q†, L^2 L†^2,
  dc dc† ec ec†, ec ec† uc uc†, dc^2 dc†^2, dc dc† uc uc†, uc^2 uc†^2, ec Q†^2 uc, dc L† Q† uc,
  L L† Q Q†, dc dc† Q Q†, Q Q† uc uc†, Q^2 Q†^2, D ec ec† H H†, D H H† L L†, D dc H†^2 uc†,
  D dc dc† H H†, D H H† uc uc†, D H H† Q Q†, D^2 H^2 H†^2, ec H H†^2 L, dc H H†^2 Q, H^2 H† Q uc, H^3 H†^3}

Out[16]= 50
```

Example: SMEFT at dim-6

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- For each type of operators, find all the independent Lorentz structures $\mathcal{M}^{(d)}(\{h_i\})$ and gauge structures $\mathcal{T}(\{\mathbf{r}_i\})$ (invariant tensors).
  e.g. type $Q^{ai} Q^{bj} Q^{ck} L^l$:

$$\mathcal{M}_1 = \langle 12 \rangle \langle 34 \rangle\,,\ \mathcal{M}_2 = \langle 13 \rangle \langle 42 \rangle\,,\ \mathcal{M}_3 = \langle 14 \rangle \langle 23 \rangle\,,$$
$$\mathcal{T}_1 = \epsilon^{abc} \epsilon^{ij} \epsilon^{kl}\,,\ \mathcal{T}_2 = \epsilon^{abc} \epsilon^{ik} \epsilon^{lj}\,,\ \mathcal{T}_3 = \epsilon^{abc} \epsilon^{il} \epsilon^{jk}\,.$$

  Schouten identities: $\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 = 0$, $\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 = 0$.

Example: SMEFT at dim-6

Steps to get the operator basis **[Shu,Ma,Xiao,2019]**

- Enumerate all the types ($(d, \{h_i\}, \{\mathbf{r}_i\})$)
  The corresponding operators are written in terms of chiral basis fields:

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- For each type of operators, find all the independent Lorentz structures $\mathcal{M}^{(d)}(\{h_i\})$ and gauge structures $\mathcal{T}(\{\mathbf{r}_i\})$ (invariant tensors).
- In the presence of identical particles, find combinations that satisfy the Bose/Fermi-statistics.

$$n_f = 1 : \quad \mathcal{O}_{qqq} = \epsilon_{abc}\epsilon_{il}\epsilon_{jk}(Q^{ai}Q^{bj})(Q^{ck}L^l) \simeq \sum_i \mathcal{M}_i \mathcal{T}_i$$

Problems Solved and Unsolved

Solved

- All redundancy relations from EOM, CDC, Bianchi Identity are encoded in the anti-symmetric spinor brackets $\langle ii\rangle = [ii] = 0$.
- Lorentz invariance and gauge invariance are built-in.
- $D = 4$ identities (Gram determinant) are built-in.

Still problems

- IBP is equivalent to momentum conservation $\sum_i |i\rangle[i| = 0$
- Lorentz group identities are equivalent to the Schouten Identities

$$|i\rangle\langle jk\rangle + |j\rangle\langle ki\rangle + |k\rangle\langle ij\rangle = 0 \ , \quad |i][jk] + |j][ki] + |k][ij] = 0 \ .$$

- Repeated fields (flavor) is equivalent to statistics of identical particles.

Outline

Young Tensor Method

Total momentum $P^\mu = \sum_{i=1}^{N} \langle i|\sigma^\mu|i]$ is invariant under the $SU(N)$

$$|i\rangle \to \sum_j \mathcal{U}_{ij}|j\rangle\ , \quad |i] \to \sum_j \mathcal{U}^\dagger_{ij}|j]\ .$$

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The seed amplitudes $\mathcal{M}$ transform under some representation $\mathbf{R}$

$$\langle ij\rangle \to \mathcal{U}_{ik}\mathcal{U}_{jl}\langle kl\rangle \in \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\ , \quad [ij] \to \mathcal{U}^\dagger_{ik}\mathcal{U}^\dagger_{jl}[kl] \in \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\right\} N-2$$

3 invariant parameters $(d{=}N{+}n{+}\tilde{n})$

- N the number of particles
- n the number of angle brackets
- $\tilde{n}$ the number of square brackets

even the type of amplitude is not invariant!

Young Tensor Method

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e.g. $\mathcal{U}|1\rangle = |2\rangle, \mathcal{U}|2\rangle = -|1\rangle$,

$$\langle 12\rangle \to \langle 12\rangle \ , \quad \langle 13\rangle \to \langle 23\rangle \ , \quad \langle 13\rangle\langle 24\rangle \to -\langle 23\rangle\langle 14\rangle \ .$$

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Types of the same $(N, n, \tilde{n})$ may transform to each other

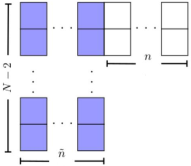
$$(4,2,0): \quad \{\psi^4,\ F_L\psi^2\phi,\ F_L^2\phi^2\}$$

$$\mathbf{R}_{N,n,\tilde{n}} \quad = \quad \mathbf{P}_{N,n,\tilde{n}} \quad \oplus \quad \underbrace{\bar{\mathbf{R}}_{N,n,\tilde{n}}}_{\text{IBP non-redundant}}$$

Young Tensor Method

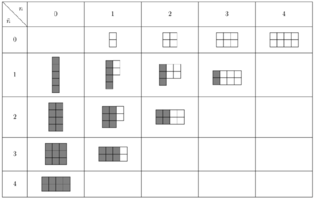
The independent set of Lorentz structures forms primary irrep. $\bar{\mathbf{R}}_{N,n,\tilde{n}}$ of $SU(N)$

[Henning, Melia, 2019]



$\bar{n}$ \ n	0	1	2	3	4
0	ϕ^8	$\psi^2\phi^5$	$\psi^4\phi^2$, $F_{\rm L}\psi^2\phi^3$, $F_{\rm L}^2\phi^4$	$F_{\rm L}\psi^4$, $F_{\rm L}^2\psi^2\phi$, $F_{\rm L}^3\phi^2$	$F_{\rm L}^4$
1	$\psi^{\dagger 2}\phi^5$	$\psi^{\dagger 2}\psi^2\phi^2$, $\psi^\dagger\psi\phi^4 D$, $\phi^6 D^2$	$F_{\rm L}\psi^{\dagger 2}\psi^2$, $F_{\rm L}^2\psi^{\dagger 2}\phi$, $\psi^\dagger\psi^3\phi D$, $F_{\rm L}\psi^\dagger\psi\phi^2 D$, $\psi^2\phi^3 D^2$, $F_{\rm L}\phi^4 D^2$	$F_{\rm L}^2\psi^\dagger\psi D$, $\psi^4 D^2$, $F_{\rm L}\psi^2\phi D^2$, $F_{\rm L}^2\phi^2 D^2$	
2	$\psi^{\dagger 4}\phi^2$, $F_{\rm R}\psi^{\dagger 2}\phi^3$, $F_{\rm R}^2\phi^4$	$F_{\rm R}\psi^{\dagger 2}\psi^2$, $F_{\rm R}^2\psi^2\phi$, $\psi^{\dagger 3}\psi\phi D$, $F_{\rm R}\psi^\dagger\psi\phi^2 D$, $\psi^{\dagger 2}\phi^3 D^2$, $F_{\rm R}\phi^4 D^2$	$F_{\rm R}^2 F_{\rm L}^2$, $F_{\rm R}F_{\rm L}\psi^\dagger\psi D$, $\psi^{\dagger 2}\psi^2 D^2$, $F_{\rm R}\psi^2\phi D^2$, $F_{\rm L}\psi^{\dagger 2}\phi D^2$, $F_{\rm R}F_{\rm L}\phi^2 D^2$, $\phi^4 D^4$, $\psi^\dagger\psi\phi^2 D^3$		
3	$F_{\rm R}\psi^{\dagger 4}$, $F_{\rm R}^2\psi^{\dagger 2}\phi$, $F_{\rm R}^3\phi^2$	$F_{\rm R}^2\psi^\dagger\psi D$, $\psi^{\dagger 4} D^2$, $F_{\rm R}\psi^{\dagger 2}\phi D^2$, $F_{\rm R}^2\phi^2 D^2$			
4	$F_{\rm R}^4$				

dim-8 $\Longleftrightarrow$



[Li, Ren, Xiao, Yu, Zheng, 2020-2022]

Young Tensor Method

Amplitude Basis $\Longleftrightarrow$ Semi-Standard Young Tableau of $\bar{\mathbf{R}}_{N,n,\tilde{n}}$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & 4 \\ \hline 4 & 4 \\ \cline{1-2}\end{array} \simeq [35]^2\langle 24\rangle\langle 34\rangle \simeq \phi_1(\psi_2\sigma^{\mu\nu}\not{D}\bar{\sigma}_{\rho\lambda}\psi_3^\dagger)F_{L4,\mu\nu}F_{R5}^{\rho\lambda}$$

A given set of numbers to fill in the Young diagram corresponds to a given type of operators, *e.g.* $\phi_1\psi_2\psi_3^\dagger F_{L4}F_{R5}D$:

$\{1,1,2,2,2,3,4,4,4,4\}$ with basis of SSYT:

$$\left\{ \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 4 & 4 \\ \hline 4 & 4 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & 4 \\ \hline 4 & 4 \\ \cline{1-2}\end{array} \right\}_{\text{y-basis}}$$

$$\#i = \tilde{n} - 2h_i$$

[Li,Shu,Ren,Xiao,Yu,Zheng,2020]

Amplitude Reduction

How to reduce a given Lorentz structure to the amplitude basis?

In mathematics, the reduction of Young tableau is called "straightening algorithm", in which the basic relation applied (Garnir relations) actually correspond to the amplitude relations of IBP and Schouten Identities.

Step 1 Remove all the p_1 (pairs of $|1\rangle$ and $|1]$);

$$p_1 = -\sum_{i\neq 1} p_i$$

Step 2 Remove as many p_2 and p_3 as possible without generating p_1

$$e.g. \qquad \langle 12\rangle[2j] = -\sum_i \langle 1i\rangle[ij]$$

Step 3 Apply Schouten Identities to the Lorentz contractions

$$\langle il\rangle\langle jk\rangle = \langle ik\rangle\langle jl\rangle - \langle ij\rangle\langle kl\rangle \ , \quad [il][jk] = [ik][jl] - [ij][kl] \ .$$

[Li, Ren, Xiao, Yu, Zheng, 2201.04639;

in progress for massive amplitudes and the full operator reduction.]

Amplitude Reduction

How to reduce a given Lorentz structure to the amplitude basis?

In mathematics, the reduction of Young tableau is called "straightening algorithm", in which the basic relation applied (Garnir relations) actually correspond to the amplitude relations of IBP and Schouten Identities.

$$[35][15]\langle 14\rangle\langle 24\rangle = -[35][25]\langle 24\rangle^2 - [35]^2\langle 24\rangle\langle 34\rangle$$
$$= \mathcal{M}_1 - \mathcal{M}_2 \simeq \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

in general: $\mathcal{M} = \sum_{i=1}^{\mathcal{N}(\text{type})} c_i \mathcal{M}_i^{(y)}$

Obtain the coordinate of any given Lorentz structure under the y-basis.

Permutation Symmetry

- Bose/Fermi statistics enforces permutation symmetry among identical particles.
- Although $\{\mathcal{M}_i^{(\mathrm{y})}\}$ (all particles distinguishable by the labels) are independent, their symmetrizations are usually not independent (matrix $\mathcal{K}^\lambda$ usually not full-rank):

$$\mathcal{Y}^\lambda \circ \mathcal{M}_i \overset{\text{reduce}}{=} \sum_j \mathcal{K}^\lambda_{ij} \mathcal{M}_j \ , \quad \lambda \vdash m$$

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- By finding linearly-independent rows of $\mathcal{K}^\lambda$, we can obtain the independent Lorentz structures in any representation λ of the permutation group S_m for m identical particles.
- In practice, we need to put gauge structure $\mathcal{T}$ together and perform the symmetrization.

Permutation Symmetry

- We can perform symmetrization only for the exactly identical particles, or for the particles from the same flavor multiplet.
  e.g. type $QQQL$ for $n_f = 3$ SMEFT, $m = 3$ for the flavor multiplet Q_p.

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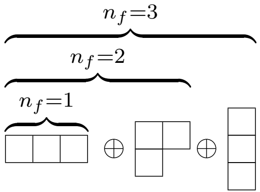
Type $QQQL$: $\{\mathcal{M}\}\otimes\{\mathcal{T}\} =$



counting irreducible tensors: $2 \times 2 = 1 + 2 + 1$

counting flavor components: $10 + 8 + 1 = 19$

Non-Linear Symmetry and Adler Zero Condition

How about Goldstone bosons with non-linear constraints? [Low,2014]

$$\mathcal{L}^{(2)} = |\partial_\mu \phi|^2 - \frac{\left|\phi^* \overleftrightarrow{\partial}_\mu \phi\right|^2}{4|\phi|^2}\left(1 - \frac{f^2}{|\phi|^2}\sin^2\frac{|\phi|}{f}\right)$$

All the vertices with $\phi^{2n+2}\partial^2/f^{2n}$ are controlled by a single parameter f

- Representation under the unbroken group H (in this case $U(1)$)
- Shift symmetry or equivalently Adler's Zero condition

$$\phi \to \phi + \epsilon \quad \Leftrightarrow \quad \lim_{p_i \to 0} \mathcal{M}(\cdots \phi_i \cdots) = 0$$

On-Shell Construction of ChPT

- It is known for more than half a century that the non-linear constraint can be reproduced in the amplitude by Adler Zero: $\lim_{p_i=0} \mathcal{M} = 0$

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Algebraic Aspects of Pionic Duality Diagrams

LEONARD SUSSKIND* AND GRAHAM FRYE
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
(Received 9 May 1969)

Certain algebraic aspects are abstracted from the duality principle and are incorporated in a simple model of pion n-point functions. An algorithm for constructing the n-point function in the tree-graph approximation is based on the duality assumption and the Adler condition which states that the amplitudes vanishes if any pion four-momentum vanishes, all others remaining on shell. The resulting amplitudes satisfy the constraints of current algebra and partial conservation of axial-vector current for $n=4$, 6, and 8, and (we conjecture) for all n. In addition, duality specifies a definite form for chiral symmetry breaking.

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$$\mathcal{L}^{(2)}=\frac{1}{2}(\partial\pi)^2+\frac{\pi^2}{2f^2}(\partial\pi)^2$$

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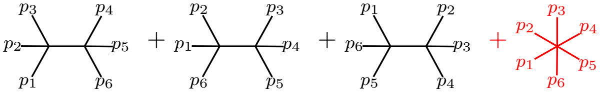
$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial\pi)^2 + \frac{\pi^2}{2f^2}(\partial\pi)^2 + \frac{\pi^4}{2f^4}(\partial\pi)^2$$

$\mathcal{M}(p_1, \ldots, p_6) =$



$$= \underbrace{\frac{1}{f^4}\left(\frac{s_{13}s_{46}}{s_{123}} + \frac{s_{24}s_{51}}{s_{234}} + \frac{s_{35}s_{62}}{s_{345}}\right)}_{\text{violate Adler Zero}} - \frac{1}{f^4}s_{135}^2 \,.$$

On-Shell Construction of ChPT

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- Extended to arbitrary n: Soft Recursion Relation **[Cheung *et al.*15']**

$$p_i \to \hat{p}_i = (1 - a_i z) p_i \ , \quad \mathcal{M}(\hat{p}_1, \ldots, \hat{p}_n) \equiv \hat{\mathcal{M}}_n(z) \ ,$$

Poles at $\hat{s}_I(z_I^{\pm}) = 0$, thus the Cauchy's theorem gives

$$\mathcal{M}(p_1, \ldots, p_n) = \hat{\mathcal{M}}_n(0) = \sum_{I,\pm} \frac{1}{s_I} \frac{\hat{\mathcal{M}}_L^{(I)}(z_I^{\pm}) \hat{\mathcal{M}}_R^{(I)}(z_I^{\pm})}{F_n(z_I^{\pm})(1 - z_I^{\pm}/z_I^{\mp})} \ ,$$

$$F_n(z) \equiv \prod_{i=1}^{n} (1 - a_i z) \quad \Rightarrow \quad B_\infty = 0 \ .$$

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- How about higher derivatives? Need new inputs (LEC)! **[Low&Yin 19']**
  - Single Trace: $\mathcal{S}_1^{(4)}(1,2,3,4)=\frac{c_1}{\Lambda^2 f^2}s_{13}^2$, $\mathcal{S}_2^{(4)}(1,2,3,4)=\frac{c_2}{\Lambda^2 f^2}s_{13}s_{23}$,
  - Double Trace: $\mathcal{S}_1^{(4)}(1,2|3,4)=\frac{d_1}{\Lambda^2 f^2}s_{13}^2$, $\mathcal{S}_2^{(4)}(1,2|3,4)=\frac{d_2}{\Lambda^2 f^2}s_{13}s_{23}$.

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(1) All soft blocks $\mathcal{S}$ satisfy Adler Zero.

(2) 1-to-1 correspond to Lagrangian terms $\mathcal{L}^{(4)}=\sum_{i=1}^4\frac{L_{4,i}}{\Lambda^2 f^2}\mathcal{O}_i$.

Lorentz Structures in ChPT

Imposing Adler's Zero condition on the combinations of y-basis

$$\left\{\lim_{p_i\to 0}\sum_j c_j \mathcal{M}_j^{(\mathrm{y})}=0\right\}_{i=1,\dots,n} \quad \Rightarrow \quad \text{soft coordinates } \mathcal{K}_i^{\alpha}=c_i^{(\alpha)}$$

Example of $n=6$ at $O(p^6)$:

Parity even	Parity odd
$\mathcal{M}_1^{\text{even}}=s_{14}s_{25}s_{36}$	$\mathcal{M}_1^{\text{odd}}=s_{12}\epsilon(3,4,5,6)$
$\mathcal{M}_2^{\text{even}}=s_{14}s_{26}s_{35}$	$\mathcal{M}_2^{\text{odd}}=s_{13}\epsilon(2,4,5,6)$
$\mathcal{M}_3^{\text{even}}=s_{15}s_{24}s_{36}$	$\mathcal{M}_3^{\text{odd}}=s_{14}\epsilon(2,3,5,6)$
$\mathcal{M}_4^{\text{even}}=s_{15}s_{26}s_{34}$	$\mathcal{M}_4^{\text{odd}}=s_{15}\epsilon(2,3,4,6)$
$\mathcal{M}_5^{\text{even}}=s_{16}s_{24}s_{35}$	$\mathcal{M}_5^{\text{odd}}=s_{23}\epsilon(1,4,5,6)$
$\mathcal{M}_6^{\text{even}}=s_{16}s_{25}s_{34}$	$\mathcal{M}_6^{\text{odd}}=s_{24}\epsilon(1,3,5,6)$
$\mathcal{M}_7^{\text{even}}=s_{13}s_{25}s_{46}$	$\mathcal{M}_7^{\text{odd}}=s_{25}\epsilon(1,3,4,6)$
$\mathcal{M}_8^{\text{even}}=s_{13}s_{26}s_{45}$	$\mathcal{M}_8^{\text{odd}}=s_{34}\epsilon(1,2,5,6)$
$\mathcal{M}_9^{\text{even}}=s_{15}s_{23}s_{46}$	$\mathcal{M}_9^{\text{odd}}=s_{35}\epsilon(1,2,4,6)$
$\mathcal{M}_{10}^{\text{even}}=s_{16}s_{23}s_{45}$	$\mathcal{M}_{10}^{\text{odd}}=s_{45}\epsilon(1,2,3,6)$
$\mathcal{M}_{11}^{\text{even}}=s_{13}s_{24}s_{56}$	
$\mathcal{M}_{12}^{\text{even}}=s_{14}s_{23}s_{56}$	
$\mathcal{M}_{13}^{\text{even}}=s_{12}s_{35}s_{46}$	
$\mathcal{M}_{14}^{\text{even}}=s_{12}s_{36}s_{45}$	
$\mathcal{M}_{15}^{\text{even}}=s_{12}s_{34}s_{56}$	

The advantage of spinor-helicity variables

The $D=4$ constraints — Schouten Identity, Gram determinant — are taken into account by the independence of the y-basis amplitudes.

Trace Structures in ChPT

Goldstones $\in$ adj. of $H = SU(N_f)$, the independent flavor structures

Group	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$	$SU(7)$	Trace
$T^{a_1a_2a_3}$	1	2	2	2	2	2	2
$T^{a_1a_2a_3a_4}$	3	8	9	9	9	9	9
$T^{a_1a_2a_3a_4a_5}$	6	32	43	44	44	44	44
$T^{a_1a_2a_3a_4a_5a_6}$	15	145	245	264	265	265	265

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$T^{a_1a_2a_3a_4a_5a_6}$	15	145	245	264	265	265	265

For each N_f, there are the Cayley-Hamilton relations:

e.g. $N_f = 3$ C-H theorem : $\quad \mathrm{tr}A^4 = \frac{1}{2}\left[\mathrm{tr}A^2\right]^2 \quad \Rightarrow$

$$\sum_{\sigma\in S_3} \mathrm{tr}\left[T^aT^{\sigma(b)}T^{\sigma(c)}T^{\sigma(d)}\right] = \mathrm{tr}\left[T^aT^b\right]\mathrm{tr}\left[T^cT^d\right] + \text{cyclic}(b,c,d)$$

Construction of Soft Blocks

The soft blocks should be totally symemtric among the flavor multiplets:

$$\mathcal{B}^{\alpha\beta} = \mathcal{Y} \circ (\mathcal{T}^{\alpha}\mathcal{M}^{\beta}) = \frac{1}{n!}\sum_{\sigma\in S_n} (\sigma \circ \mathcal{T}^{\alpha})(\sigma \circ \mathcal{M}^{\beta})$$

$$= \sum_{i=1}^{d_{\mathcal{T}}}\sum_{j=1}^{d_{\mathcal{M}}} \underbrace{\left[\frac{1}{n!}\sum_{\sigma\in S_n} c^{\alpha}{}_{i}(\sigma)\mathcal{K}^{\beta}{}_{j}(\sigma)\right]}_{\text{coordinate}} \times (\mathcal{T}_i \mathcal{M}_j^{(\mathrm{y})}),$$

$n = 6$ soft blocks:

Unbroken Group H		$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$
$O(p^6)$	P-even	3	8	13	14	15
	P-odd	0	3	4	4	4
$O(p^8)$	P-even	9	40	68	74	76
	P-odd	2	20	33	35	35

Electroweak Chiral Effective Theory

Add external sources and construct **Chiral Effective Theory (ChEFT)**

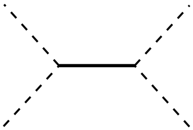
Classes	$\mathcal{N}_{\text{type}}$	$\mathcal{N}_{\text{term}}$	$\mathcal{N}_{\text{operator}}$
UhD^4	$3+6+0+0$	15	15
X^2Uh	$6+4+0+0$	10	10
$XUhD^2$	$2+6+0+0$	8	8
X^3	$4+2+0+0$	6	6
ψ^2UhD	$4+8+0+0$	13(16)	$13{n_f}^2 \quad (16{n_f}^2)$
ψ^2UhD^2	$6+10+0+0$	60(80)	$60{n_f}^2 \quad (80{n_f}^2)$
ψ^2UhX	$7+7+0+0$	22(28)	$22{n_f}^2 \quad (28{n_f}^2)$
ψ^4	$12+24+4+8$	117(160)	$\frac{1}{4}{n_f}^2(31-6n_f+335{n_f}^2) \quad ({n_f}^2(9-2n_f+125{n_f}^2))$
Total	123	261(313)	$\frac{335{n_f}^4}{4}-\frac{3{n_f}^3}{2}+\frac{411{n_f}^2}{4}+39 \quad (39+133{n_f}^2-2{n_f}^2-2{n_f}^3+125{n_f}^4)$ $\mathcal{N}_{\text{operatrs}}(n_f=1)=224(295), \quad \mathcal{N}_{\text{operatrs}}(n_f=3)=7704(11307)$

[Sun, Xiao, Yu, 2206.07722, 2210.14939]

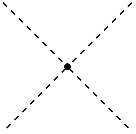
Outline

Partial Wave Amplitudes

Effective operators can be obtained by integrating out heavy resonance states



$$\sim \frac{1}{p^2 - M^2} \approx -\frac{1}{M^2} - \frac{p^2}{M^4} - \cdots \sim$$



Partial Wave Amplitudes

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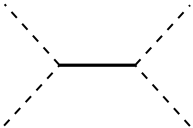

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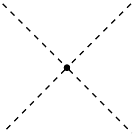


The UV couplings can be written as 1-massive-n-massless on-shell amplitudes

$$\mathcal{M}(h_1, h_2, \ldots, h_n; J) = \mathcal{C}^J(h_1, \ldots, h_n)^{\alpha_1 \ldots \alpha_{2J}} (\lambda^I_\alpha)^{\otimes 2J}$$

The C-G coefficients of n-particle state $\langle P, J, \sigma | \{\Psi(p_i, h_i)\}\rangle \sim \mathcal{C}^J(h_1, \ldots, h_n)$

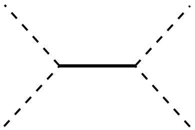
- When $n = 2$, there is a unique C-G coefficient when $|h_1 - h_2| \geq J$

$$\mathcal{C}^{J,\sigma}_{h_1,h_2} \sim \frac{[12]^{h_1+h_2-J}}{s^{(J+h_1+h_2)/2}} \left([1\boldsymbol{x}]^{J+h_1-h_2}[2\boldsymbol{x}]^{J-h_1+h_2}\right)^{\{I_1 \ldots I_{2J}\}}$$

- When $n > 2$, there are often degenerate states

Partial Wave Amplitudes

Effective operators can be obtained by integrating out heavy resonance states



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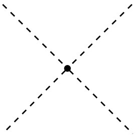

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Putting the left and right couplings together and summing over the polarizations $\{I\}$, the resulting effective local amplitude should be a partial wave amplitude with total angular momentum J **[Shu, Xiao, Zheng, 2111.08019]**

$$\mathcal{M}^J = \mathcal{C}_L^J(h_1, \ldots, h_n) \cdot \mathcal{C}_R^J(h'_1, \ldots, h'_m)$$

When $n = m = 2$, it reduces to the Wigner-d matrix $\mathcal{M}^J \sim d^J_{h_1 - h_2, h'_1 - h'_2}(\theta)$.

Operators that Produce Partial Waves

The on-shell correspondence is not only useful for basis construction

$$\text{4-fermion couplings} \begin{cases} \mathcal{O}^{(S)} = (\bar{\psi}\psi)(\bar{\chi}\chi) & \simeq \quad \mathcal{B}^{(S)} \sim d^{J=0}_{0,0}(\theta) \\ \mathcal{O}^{(V)} = (\bar{\psi}\gamma^{\mu}\psi)(\bar{\chi}\gamma_{\mu}\chi) & \simeq \quad \mathcal{B}^{(V)} \sim d^{J=1}_{1,\pm 1}(\theta) \\ \mathcal{O}^{(T)} = (\bar{\psi}\sigma^{\mu\nu}\psi)(\bar{\chi}\sigma_{\mu\nu}\chi) & \simeq \quad \mathcal{B}^{(T)} \sim d^{J=1}_{0,0}(\theta) \end{cases}$$

Operators can be classified by the angular momentum in certain channel

Partial Wave Operator Basis (J-Basis) $\simeq$ Partial Wave Amplitudes

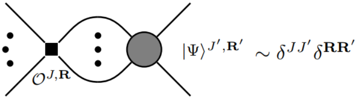
The J-basis operators can be generalized to include gauge group rep. $\mathbf{R}$

$$e.g. \qquad \mathcal{O}^{(1)/(8)}_{qu} = (\bar{q}\gamma^{\mu}T^{A}q)(\bar{u}\gamma_{\mu}T^{A}u)$$

Selection Rules

Partial wave amplitudes are powerful in the phase space integration, as the angular momentum conservation makes it block diagonal

$$\int \mathrm{dLIPS}_n\, \mathcal{C}^J(h_1,\ldots,h_n)\mathcal{C}^{J'}(h_1,\ldots,h_n)^* \sim \delta^{JJ'}$$



1. Non-interference
2. Vanishing loop diagrams
3. Vanishing ADM element

Non-Abelian

		(4,0) $V^+V^+V^-V^-$	(4,0) $V^+V^-\psi^+\bar\psi^-$	(4,0) $V^+V^-\phi\phi$	(4,0) $V^+\psi^-\psi^-\phi$	(4,2) $V^+V^+V^+V^-$	(4,2) $V^+V^+\psi^+\bar\psi^-$
(4,0)	$\psi^2\bar\psi^2$	×	0	×	0*	×	R
	$\phi^4 D^2$	×	×	0	×	×	×
	$\phi^2\psi\bar\psi D$	×	0	0	0	×	R
(4,2)	$F\psi^2\phi$	×	R	R	R	×	0
	$F^2\phi^2$	R	0	R	R	0*	0*
	ψ^4	×	0	×	0	×	0
(4,−2)	$\bar F\bar\psi^2\phi$	×	R	R	R	×	0
	$\bar F^2\phi^2$	R	0	R	R	0	0
	$\bar\psi^4$	×	0	×	R	×	0

Abelian

	(4,0) $V^+V^+V^-V^-$	(4,0) $V^+V^-\psi^+\bar\psi^-$	(4,0) $V^+V^-\phi\phi$	(4,0) $V^+\psi^-\psi^-\phi$	(4,2) $V^+V^+V^+V^-$	(4,2) $V^+V^+\psi^+\bar\psi^-$
$\psi^2\bar\psi^2$	×	0	×	0*	×	0
$\phi^4 D^2$	×	×	0	×	×	×
$\phi^2\psi\bar\psi D$	×	0	0	0	×	0
$F\psi^2\phi$	×	R	R	R	×	0
$F^2\phi^2$	R	0	R	R	0	0
ψ^4	×	0	×	0	×	0
$\bar F\bar\psi^2\phi$	×	R	R	R	×	0
$\bar F^2\phi^2$	R	0	R	R	0	0
$\bar\psi^4$	×	0	×	R	×	0

Phase Space Integration

Also a tool to compute phase space integration.

$$\int \mathrm{dLIPS}_n \, \mathcal{C}^J(h_1, \ldots, h_n) \mathcal{C}^{J'}(h_1, \ldots, h_n)^* \sim \delta^{JJ'}$$

Phase Space Integration

Diagonalize and normalize $\mathcal{C}^{J,a}$:

$$\int \mathrm{dLIPS}_n \, \mathcal{C}^{J,a}(\Phi_n)\mathcal{C}^{J,b}(\Phi_n)^* \equiv \langle \mathcal{C}^{J,a}, \mathcal{C}^{J',b} \rangle = \frac{\pi}{2(2J+1)} \delta^{JJ'} \delta^{ab}$$

Example: Wigner-D matrix $D^J_{\sigma_i \sigma_f}(\Omega) = \mathcal{C}^J_{\sigma_i}(0) \cdot \mathcal{C}^J_{\sigma_f}(\Omega)^*$

$$\begin{aligned} \frac{1}{8} \int \mathrm{d}\Omega \, D^J_{\sigma_i \sigma_f}(\Omega) D^{J'}_{\sigma_i \sigma'_f}(\Omega)^* &= \mathcal{C}^J_{\sigma_i}(0) \cdot 8 \langle \mathcal{C}^J_{\sigma'_f}, \mathcal{C}^J_{\sigma_f} \rangle \cdot \mathcal{C}^J_{\sigma_i}(0)^* \\ &= \frac{4\pi}{2J+1} \delta^{JJ'} \delta_{\sigma_f \sigma'_f} \end{aligned}$$

Partial Wave Decomposition

Partial wave decomposition:

$$\mathcal{A}(n \to m) = \sum_J \sum_{a,b} a^J_{ab} \underbrace{\mathcal{C}^{J,a}_n \cdot \mathcal{C}^{J,b*}_m}_{\mathcal{M}^{J,ab}}$$

- For local amplitudes $\mathcal{A} \in \text{span}\{\mathcal{M}^{(y)}\}$
  - Need a complete set of partial wave amplitudes.
  - They are eigenstates of angular momentum operator $(J^2?)$
    1. Lorentz invariant notion; 2. Acting on spinor variables

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Poincaré Casimir: $W^2_{n\to m}\mathcal{M}^J = -P^2 J(J+1)\mathcal{M}^J$

Partial Wave Decomposition

Partial wave decomposition：

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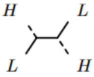
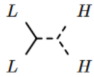
- For local amplitudes $\mathcal{A} \in \text{span}\{\mathcal{M}^{(y)}\}$
  - Need a complete set of partial wave amplitudes.
  - They are eigenstates of angular momentum operator
- For arbitrary amplitudes $\mathcal{A}$

$$\int \text{dLIPS}_n \, \mathcal{C}^{J,a*}_n \mathcal{A} \equiv \langle \mathcal{A}, \mathcal{C}^{J,a}_n \rangle = \frac{\pi}{2(2J+1)} \sum_b a^J_{ab} \mathcal{C}^{J,b}_m$$

Implication of UV Resonances

Analysing J-basis in all channels, get all tree-level UV origin:

Topology	j-basis	Quantum numbers $\{J, \mathbf{R}, Y\}$	Model
	$\mathcal{B}_{\{13\}}^{J=1/2,\mathbf{R}=1} = \mathcal{B}_1^p + \mathcal{B}_2^p.$	$\{\frac{1}{2}, 1, 0\}$	Type I
	$\mathcal{B}_{\{13\}}^{J=1/2,\mathbf{R}=3} = -\mathcal{B}_1^p + 3\mathcal{B}_2^p,$	$\{\frac{1}{2}, 3, 0\}$	Type III
	$\mathcal{B}_{\{12\}}^{J=0,\mathbf{R}=3} = -2\mathcal{B}_1^p,$	$\{0, 3, -1\}$	Type II
	$\mathcal{B}_{\{12\}}^{J=0,\mathbf{R}=1} = 2\mathcal{B}_2^p.$	$\{0, 1, -1\}$	N/A

$\Rightarrow$ Three types of seesaw models for $\mathcal{O}^{(5)} = (HL)^T \mathcal{C}(HL)$

- Completely bottom-up search
- Does NOT apply to loop-level origins

Implication of UV Resonances

47 UV resonances responsible for Dim-6 SMEFT! **[Li, Ni, Xiao, Yu, 2204.03660]**

19 scalars

Notation	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
Name	$\mathcal{S}$	$\mathcal{S}_1$	$\mathcal{S}_2$	φ	Ξ	Ξ_1	Θ_1	Θ_3
Irrep	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{1},\mathbf{1})_1$	$(\mathbf{1},\mathbf{1})_2$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$(\mathbf{1},\mathbf{3})_0$	$(\mathbf{1},\mathbf{3})_1$	$(\mathbf{1},\mathbf{4})_{\frac{1}{2}}$	$(\mathbf{1},\mathbf{4})_{\frac{3}{2}}$
Notation	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}		
Name	ω_4	ω_1	ω_2	Π_1	Π_7	ζ		
Irrep	$(\mathbf{3},\mathbf{1})_{-\frac{4}{3}}$	$(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}$	$(\mathbf{3},\mathbf{1})_{\frac{2}{3}}$	$(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$	$(\mathbf{3},\mathbf{2})_{\frac{7}{6}}$	$(\mathbf{3},\mathbf{3})_{-\frac{1}{3}}$		
Notation	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}			
Name	Ω_2	Ω_1	Ω_4	Υ_1	Φ			
Irrep	$(\mathbf{6},\mathbf{1})_{-\frac{2}{3}}$	$(\mathbf{6},\mathbf{1})_{\frac{1}{3}}$	$(\mathbf{6},\mathbf{1})_{\frac{4}{3}}$	$(\mathbf{6},\mathbf{3})_{\frac{1}{3}}$	$(\mathbf{8},\mathbf{2})_{\frac{1}{2}}$			

14 fermions

Notation	F_1	F_2	F_3	F_4	F_5	F_6	F_7
Name	N	E^c	Δ_1^c	Δ_3^c	Σ	Σ_1^c	
Irrep	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{1},\mathbf{1})_1$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$(\mathbf{1},\mathbf{2})_{\frac{3}{2}}$	$(\mathbf{1},\mathbf{3})_0$	$(\mathbf{1},\mathbf{3})_1$	$(\mathbf{1},\mathbf{4})_{\frac{1}{2}}$
Notation	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
Name	D	U	Q_5	Q_1	Q_7	T_1	T_2
Irrep	$(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}$	$(\mathbf{3},\mathbf{1})_{\frac{2}{3}}$	$(\mathbf{3},\mathbf{2})_{-\frac{5}{6}}$	$(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$	$(\mathbf{3},\mathbf{2})_{\frac{7}{6}}$	$(\mathbf{3},\mathbf{3})_{-\frac{1}{3}}$	$(\mathbf{3},\mathbf{3})_{\frac{2}{3}}$

14 vectors

Notation	V_1	V_2	V_3	V_4	V_5	V_6	V_7
Name	$\mathcal{B}$	$\mathcal{B}_1$	$\mathcal{L}_3^\dagger$	$\mathcal{W}$	$\mathcal{U}_2$	$\mathcal{U}_5$	$\mathcal{Q}_5$
Irrep	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{1},\mathbf{1})_1$	$(\mathbf{1},\mathbf{2})_{\frac{3}{2}}$	$(\mathbf{1},\mathbf{3})_0$	$(\mathbf{3},\mathbf{1})_{\frac{2}{3}}$	$(\mathbf{3},\mathbf{1})_{\frac{5}{3}}$	$(\mathbf{3},\mathbf{2})_{-\frac{5}{6}}$
Notation	V_8	V_9	V_{10}	V_{11}	V_{12}	V_{13}	V_{14}
Name	$\mathcal{Q}_1$	$\mathcal{X}$	$\mathcal{Y}_1^\dagger$	$\mathcal{Y}_5^\dagger$	$\mathcal{G}$	$\mathcal{G}_1$	$\mathcal{H}$
Irrep	$(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$	$(\mathbf{3},\mathbf{3})_{\frac{2}{3}}$	$(\mathbf{6},\mathbf{2})_{-\frac{1}{6}}$	$(\mathbf{6},\mathbf{2})_{\frac{5}{6}}$	$(\mathbf{8},\mathbf{1})_0$	$(\mathbf{8},\mathbf{1})_1$	$(\mathbf{8},\mathbf{3})_0$

Outline

Summary and Outlook

- The significance of Effective Field Theory is emphasized.
- The on-shell classification of effective operators: pheno impact?
- Young Tensor Method: a systematic method to construct the operator basis in various EFT.
- Generalization of partial wave amplitudes and operators, and its application to calculations.
- With amplitude basis, can we perform bootstrap for general EFT?

Thank you for your attention!