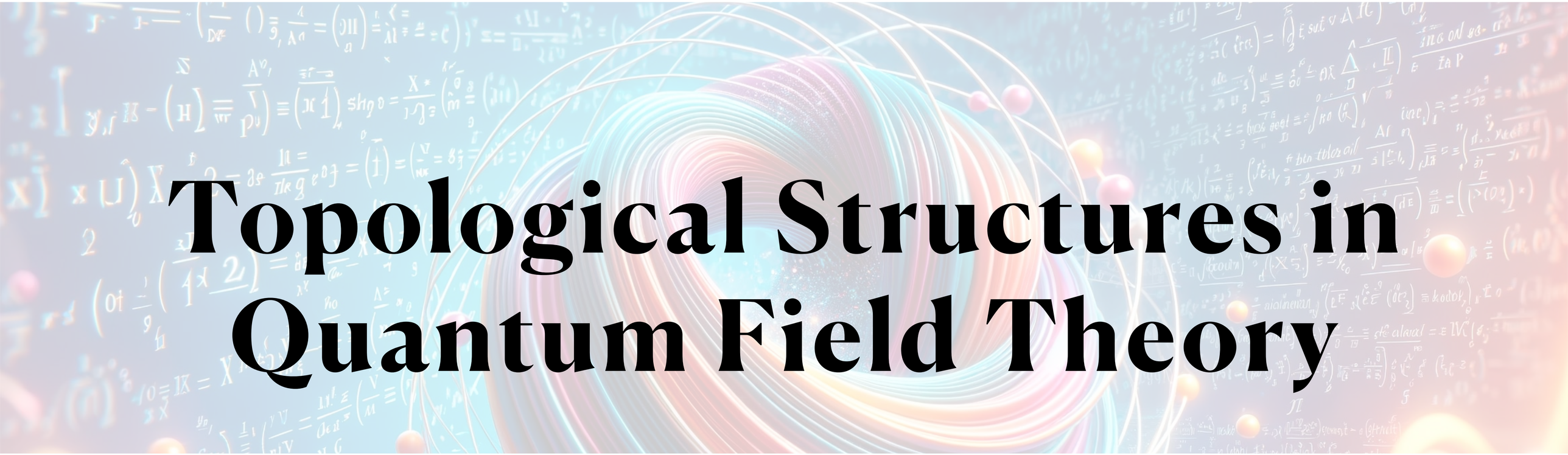


Topological Structures in Quantum Field Theory

A Short Review “from” Pedestrian

Hao Zhang
Theoretical Physics Division, Institute of High Energy Physics, Chinese Academy of Sciences
For “第十四届新物理研讨会”，Jinan, Shandong, Jul 21st, 2025

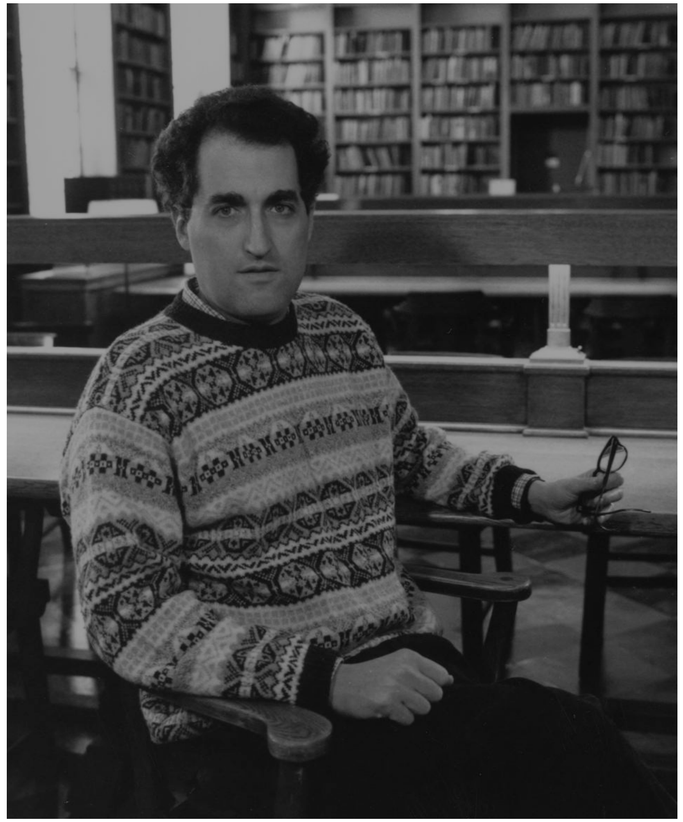
Topological structures in QFT

vs.

Topological QFT

Topological QFT

- The concept of topological QFT, to my best knowledge, was becoming well known and popular since late 1980s.



Commun. Math. Phys. 117, 353–386 (1988)

Topological Quantum Field Theory

Edward Witten*

School of Natural Sciences, Institute for Advanced Study, Olden Lan

Abstract. A twisted version of four dimensional super
is formulated. The model, which refines a nonrelativis
appears to underlie many recent developments in topo
manifolds; the Donaldson polynomial invariants of
Floer groups of three manifolds appear naturally. T
interesting from a physical viewpoint; it is in a sens
quantum field theory, albeit one in which general covariance is unbroken, there
are no gravitons, and the only excitations are topological.

Commun. Math. Phys. 121, 351–399 (1989)

Communications in Mathematical Physics

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Quantum Field Theory and the Jones Polynomial*

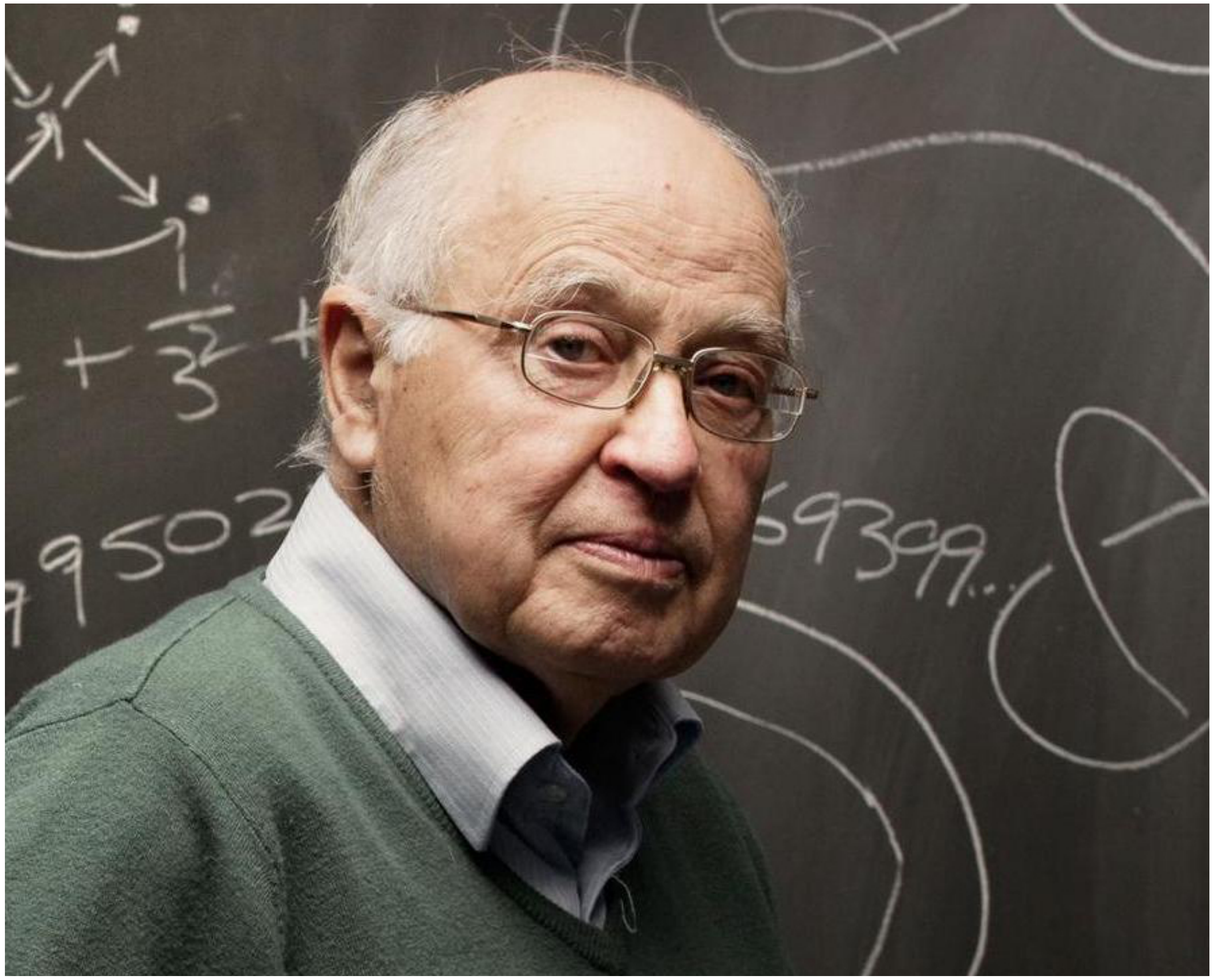
Edward Witten**

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

Topological QFT

- Two questions from Sir. Atiyah (1988):
    1. Giving a physical interpretation to Donaldson theory;
    2. Finding an intrinsically three dimensional definition of the Jones polynomial of knot theory.



Topological QFT

- The topological QFT, whose correlation functions and partition functions depend only on the topology of the background manifold, but not the metric.
- They are useful for computing and understanding the topological invariant of topological spaces.
- A very famous example: 2+1D Chern-Simons theory.

$$\mathscr{L}_{\text{Cl.}} = \frac{k}{8\pi}\int_{\mathscr{M}} \text{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

Topological QFT

- (One of) A modern definition of TQFT:

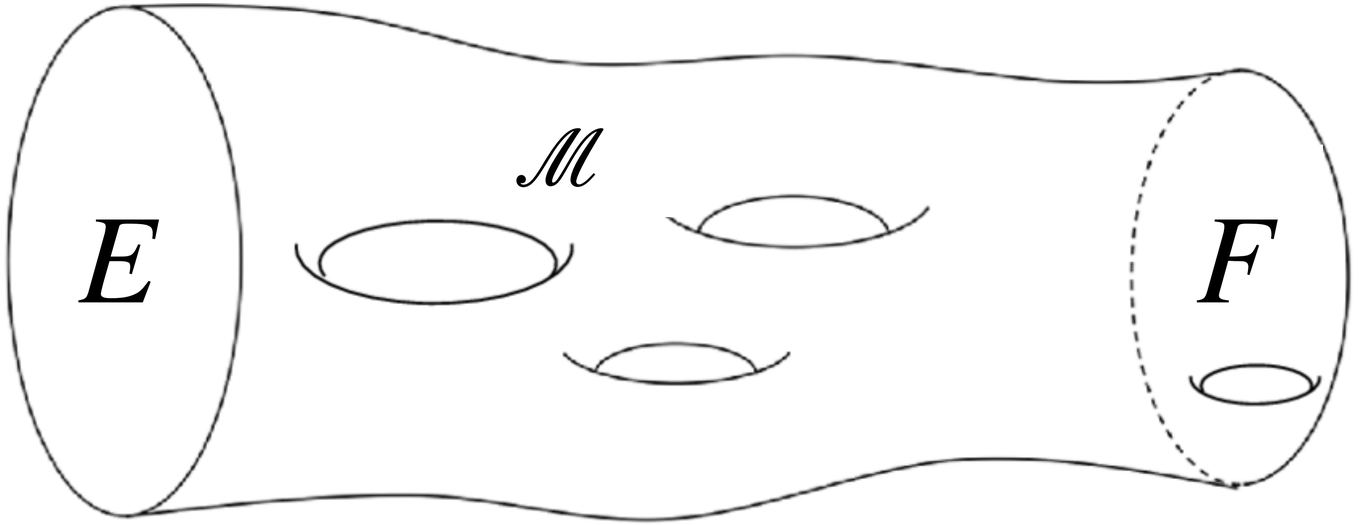
An n-dimensional oriented TQFT is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{K}}$$

- $\mathrm{Vect}_{\mathbb{K}}$: a category whose objects are $\mathbb{K}$-linear space, and the morphisms are $\mathbb{K}$ -linear transformation.
- Bord_n: a category whose objects are oriented closed (n-1)-dimensional real manifolds (spacial slice of n-dimensional spacetime), and the morphisms from the object E to the object F are the equivalence class of a bordism from E to F.

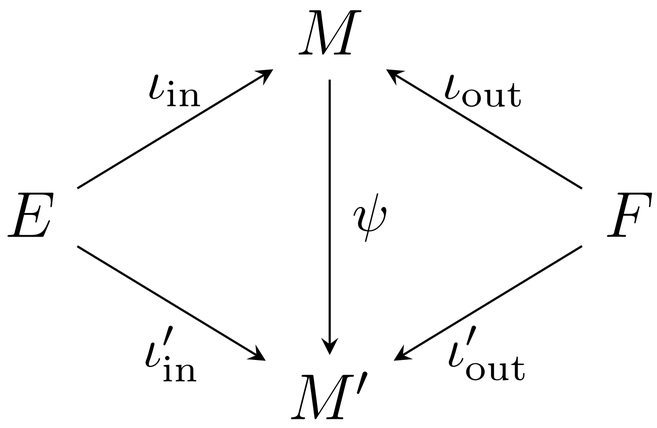
Topological QFT

- A bordism $E \to F$: an oriented compact n-dimensional manifold $\mathscr{M}$ with boundary, together with smooth maps $\iota_{\text{in}} : E \to \mathscr{M} \leftarrow F : \iota_{\text{out}}$ with image in $\partial\mathscr{M}$ such that $\bar{\iota}_{\text{in}} \sqcup \iota_{\text{out}} : \bar{E} \sqcup F \to \partial\mathscr{M}$ is an orientation-preserving diffeomorphism, where $\bar{E}$ denotes E with the opposite orientation.



Topological QFT

- Two bordism $\mathscr{M}$ and $\mathscr{M}'$ are equivalent iff there exists orientation-preserving diffeomorphism $\psi : \mathscr{M} \to \mathscr{M}'$ s.t.



Topological structures in QFT

- Being different from TQFT, topological “structures” in QFT are the topological non-trivial configurations in common QFT.

Topic 1: The Instanton

The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4} F_{\mu\nu,a} F^{\mu\nu}_{b} \delta^{ab}$$

The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4} F_{\mu\nu,a} F_b^{\mu\nu} \delta^{ab}$$

$$[T^a, T^b] = (-1)^{s_1} i f_c^{ab} T^c, \quad \mathrm{tr}\left(T^a T^b\right) = T_F \delta^{ab}, \quad U(\theta) = \exp\left[(-1)^{s_2} i \theta_a T^a\right]$$

$$\mathbf{A}_\mu \equiv A_{\mu,a} T^a \to U \mathbf{A}_\mu U^{-1} - (-1)^{s_3} \frac{i}{g} (\partial_\mu U) U^{-1}$$

$$\mathbf{F}_{\mu\nu} \equiv F_{\mu\nu,a} T^a = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - (-1)^{s_3} i g [\mathbf{A}_\mu, \mathbf{A}_\nu], \quad F_{\mu\nu,a} = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} + g (-1)^{s_1 + s_3} f_a^{bc} A_{\mu,b} A_{\nu,c}$$

The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- **Euclidean space: Why?**

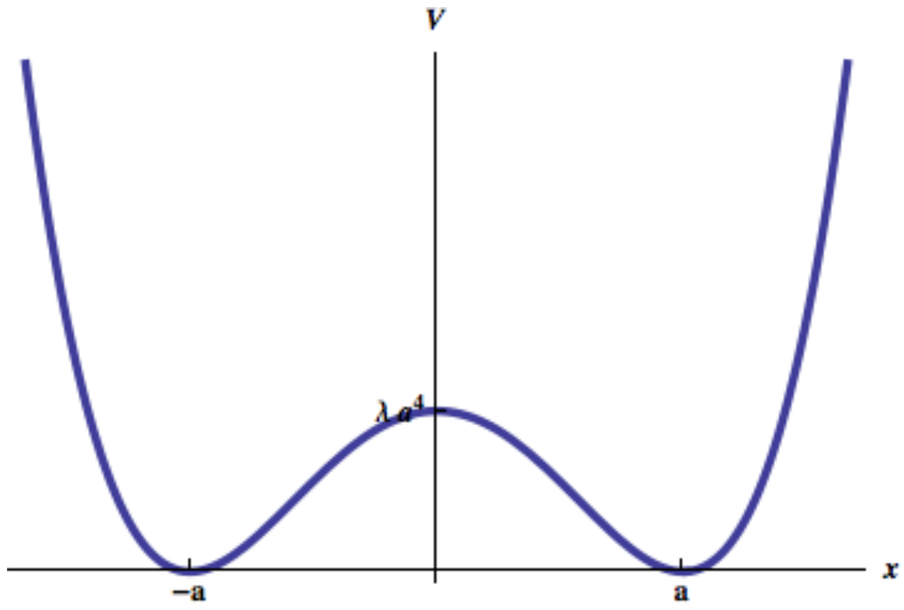
The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- **Euclidean space: Why?**

$$\left[\frac{d^2}{dx^2} - 2\lambda\left(x^2 - a^2\right)^2 + 2E\right]\psi(x) = 0$$

$$\begin{cases} E_0 = \frac{\omega}{2}\left[1 - \sqrt{\frac{2\omega^3}{\pi\lambda}}\exp\left(-\frac{\omega^3}{12\lambda}\right)\right] \\ E_1 = \frac{\omega}{2}\left[1 + \sqrt{\frac{2\omega^3}{\pi\lambda}}\exp\left(-\frac{\omega^3}{12\lambda}\right)\right] \end{cases}$$

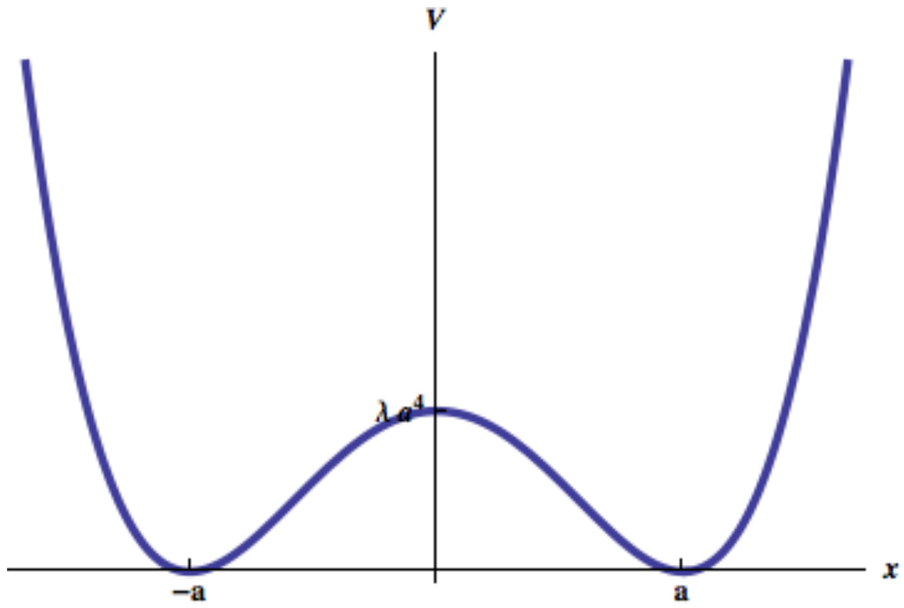
$|a\rangle$ or $|-a\rangle$?



The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- **Euclidean space: Why?**

$$\begin{aligned}
\left\langle x_f \left| e^{-T\hat{H}} \right| x_i \right\rangle &= \left\langle x_f \left| e^{-T\hat{H}} \left(\sum_{n\in\mathbb{N}} |n\rangle \langle n| \right) \right| x_i \right\rangle \\
&= \sum_{n\in\mathbb{N}} e^{-TE_n} \langle x_f \,|\, n\rangle \langle n \,|\, x_i\rangle \\
&\overset{T\to\infty}{\longrightarrow} \lim_{T\to\infty} e^{-TE_0} \langle x_f \,|\, 0\rangle \langle 0 \,|\, x_i\rangle
\end{aligned}$$



The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- **Euclidean space: Why?**

$$\begin{aligned}
\left\langle x_f \left| e^{-T\hat{H}} \right| x_i \right\rangle &= \left\langle x_f \left| e^{-T\hat{H}} \left(\sum_{n\in\mathbb{N}} |n\rangle \langle n| \right) \right| x_i \right\rangle \\
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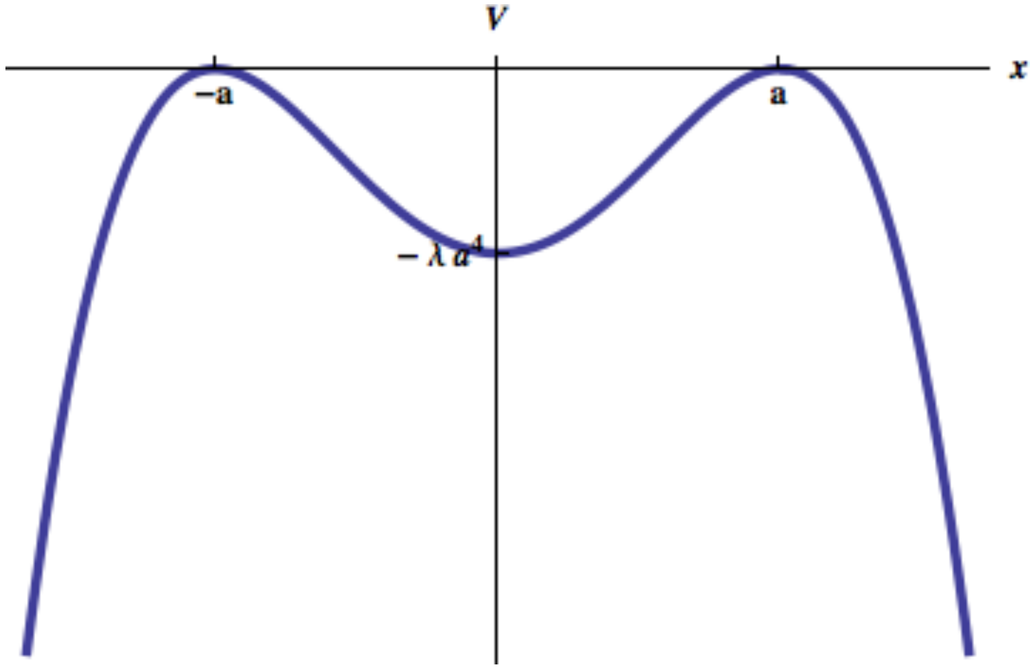
&\overset{T\to\infty}{\longrightarrow} \lim_{T\to\infty} e^{-TE_0} \langle x_f \,|\, 0\rangle \langle 0 \,|\, x_i\rangle
\end{aligned}$$



The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- From 0+1D to 3+1D?
  - From semi-classical “vacuum” to “real vacuum”
  - To get a finite transition amplitude, the action must be finite
  - No well-defined path integral!
  - No well-defined QFT (with non-trivial interaction)!
  - Mass gap?

The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- Finite action configuration — 2 possibilities

The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- Pure gauge (at infinity)

$$\lim_{\rho \to \infty} A_\mu(x) = \frac{i}{g} U(\hat{x}) \partial_\mu U^\dagger(\hat{x})$$

The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- Bogomolny bound

$$\frac{1}{2}\,\mathrm{Tr}\left[\tilde{F}_{\mu\nu} \pm F_{\mu\nu}\right]^2 = \mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}] \pm \mathrm{Tr}\left[\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}\right]$$

$$\int d^4x\,\mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}] \geq \left|\int d^4x\,\mathrm{Tr}\left[\tilde{F}^{\mu\nu}F_{\mu\nu}\right]\right|$$

$$S \geq \frac{8\pi^2|n|}{g^2}$$

The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- The solution

$$U(\hat{x}) = \frac{x_4 + i\vec{x} \cdot \vec{\sigma}}{\rho}$$

$$A_\mu(x) = \frac{i}{g} f(\rho) U(\hat{x}) \partial_\mu U^\dagger(\hat{x}) \qquad\qquad f(\rho) = \frac{\rho^2}{\rho^2 + a^2}$$

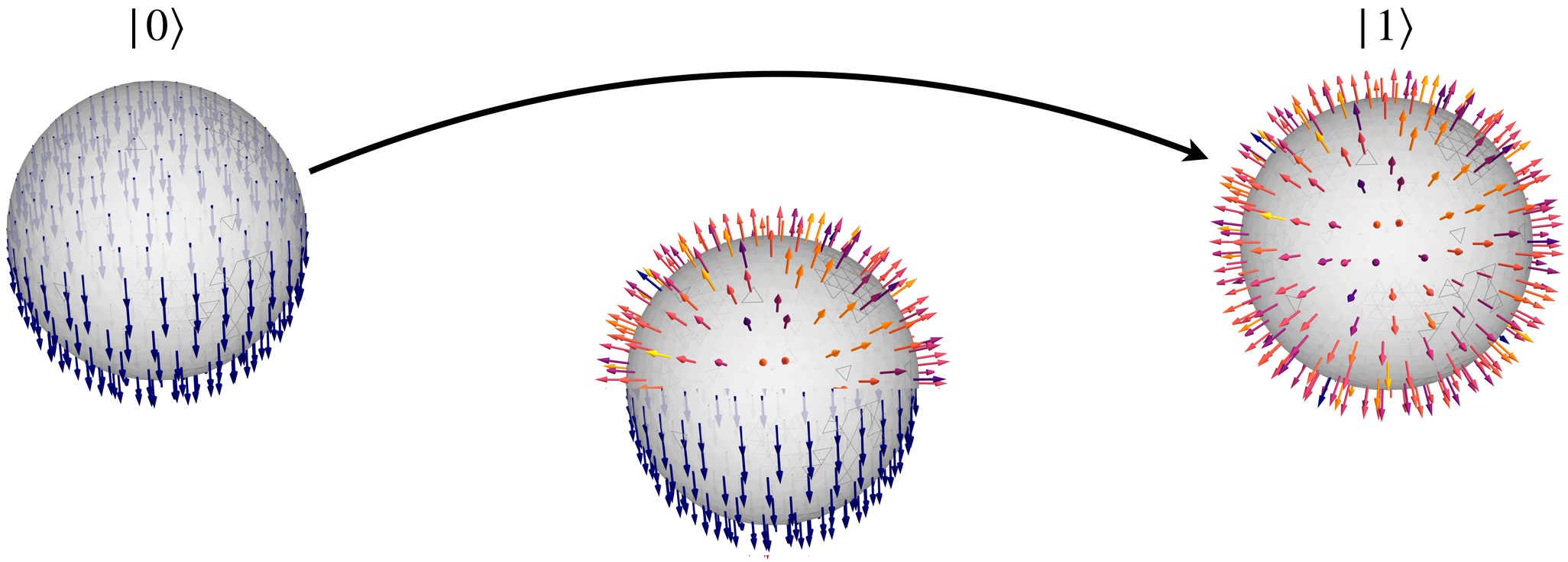
The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- Topological unequivalent vacua, the winding number.

$$n_W = \frac{1}{24\pi^2} \int d^3x \, \epsilon^{ijk} \, \mathrm{Tr}[(U^\dagger \partial_i U)(U^\dagger \partial_j U) \times (U^\dagger \partial_k U)].$$

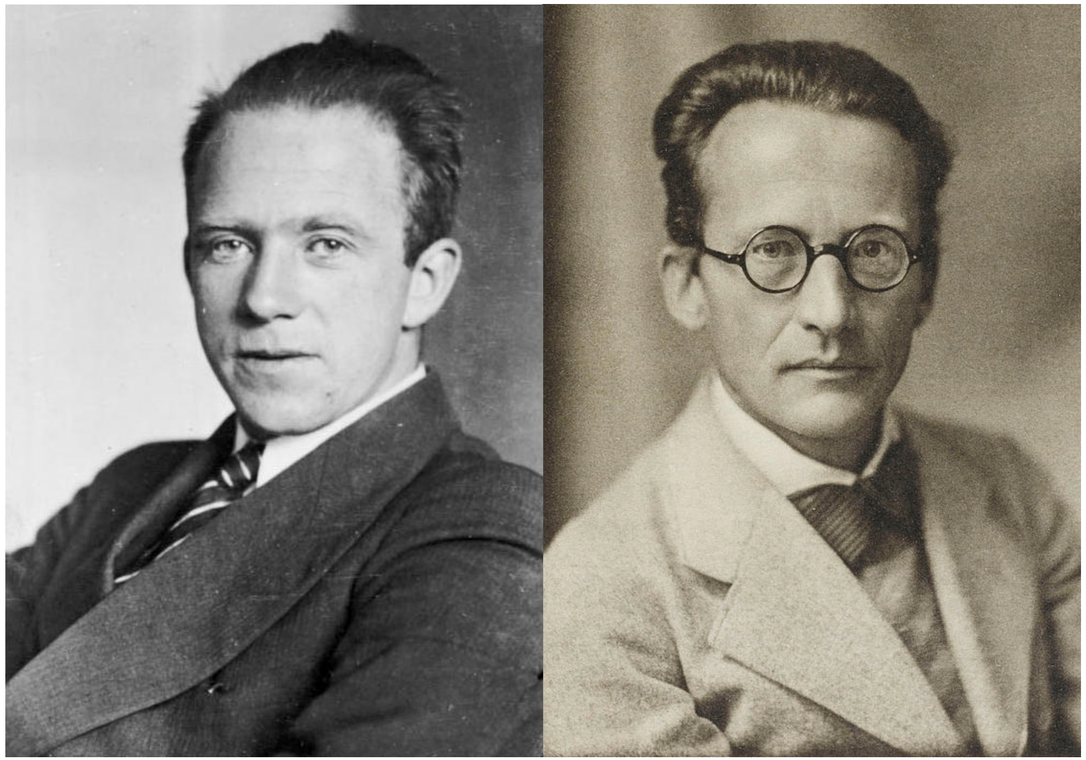
The Instanton Solution

- (Classical) Non-abelian (pure) Yang-Mills theory
- Transition induced by instanton.



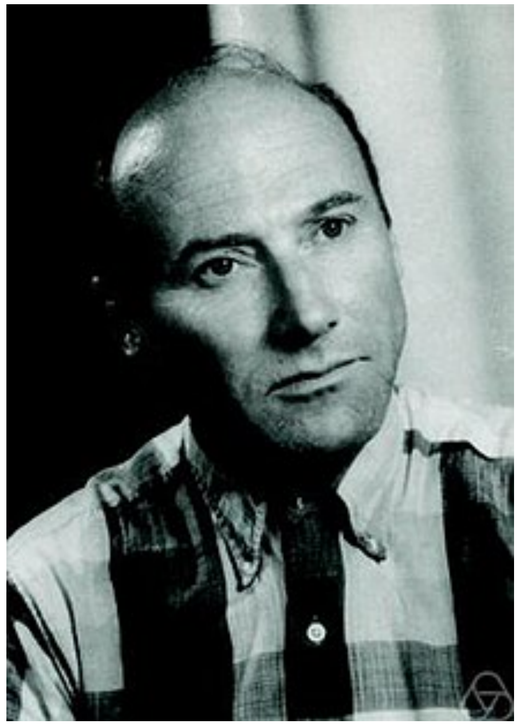
The Instanton Solution

- Non-abelian (pure) Yang-Mills theory
- Different vacuum, same representation.



The Instanton Solution

- Non-abelian (pure) Yang-Mills theory
- Different vacuum, different representations.



定理 2.18

对于 C^* 代数 $\mathfrak{A}$ 上的任意一个态 φ，都存在一个 Hilbert 空间 $\mathscr{H}_\varphi$，$\mathfrak{A}$ 在 $\mathscr{H}_\varphi$ 上的一个表示 π_φ，和 $\mathscr{H}_\varphi$ 中的一个单位矢量 Ω_φ，满足如下两个条件：

1. 对于任意的 $A \in \mathfrak{A}$

$$\varphi(A) = (\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi). \tag{2.36}$$

2. Ω_φ 是表示 π_φ 的一个**循环矢量**（*cyclic vector*），也就是说

$$\pi_\varphi(\mathfrak{A})\Omega_\varphi \equiv \{\pi_\varphi(A)\Omega_\varphi;\quad A \in \mathfrak{A}\}$$

在 $\mathscr{H}_\varphi$ 中是稠密的。

满足这两条的三元组 $(\mathscr{H}_\varphi, \pi_\varphi, \Omega_\varphi)$，作为幺正等价类是唯一的。因此，如果存在另一个由 Hilbert 空间 $\mathscr{H}'_\varphi$、$\mathfrak{A}$ 在 $\mathscr{H}'_\varphi$ 上的表示 π'_φ 和 $\mathscr{H}'_\varphi$ 中的矢量 Ω'_φ 组成的三元组，满足条件（1）和（2），则一定存在满足

$$U\pi_\varphi(A) = \pi'_\varphi(A)U \qquad （对所有的元素\ A \in \mathfrak{A}） \tag{2.37}$$

$$U\Omega_\varphi = \Omega'_\varphi$$

的从 $\mathscr{H}_\varphi$ 到 $\mathscr{H}'_\varphi$ 的幺正映射 U。

定理 2.20

下列关于 C^* 代数 $\mathfrak{A}$ 上的态 φ 和与 φ 相应的循环表示 π_φ 的条件，彼此之间是等价的。

1. φ 是纯态。
2. 表示 π_φ 是不可约的，或者说，π_ρ 的不变子空间只能是 $\mathscr{H}_\varphi$ 或 $\{0\}$。
3. $\mathscr{H}_\varphi$ 上与所有 $\pi_\varphi(A)$ $(A \in \mathfrak{A})$ 都对易的线性算子，必是某个复常数乘以恒等算子，即

$$\pi_\varphi(\mathfrak{A})' \equiv \{B \in \mathscr{B}(\mathscr{H}_\varphi);\ \forall A \in \mathfrak{A} \ \Rightarrow\ B\pi_\varphi(A) = \pi_\varphi(A)B\} = C\mathbf{1}.$$

4. 如果 $\mathscr{H}_\varphi$ 中的两个矢量 Ψ, Ψ' 表示相同的矢量态（即对于所有的 $A \in \mathfrak{A}$，我们都有 $(\Psi, \pi_\varphi(A)\Psi) = (\Psi', \pi_\varphi(A)\Psi'))$ [a]，则 Ψ' 正比于 Ψ。也就是说，存在 $\lambda \in \mathbb{C}, |\lambda| = 1$，使得 $\Psi' = \lambda\Psi$。

[a]英文原版此处遗漏后半括号。

——译者注

The Instanton Solution

- Non-abelian (pure) Yang-Mills theory
- Which vacuum?

$$\langle j|\exp(-H\tau)|i\rangle = \sum_{N_+}\sum_{N_-} \frac{\delta_{N_+-N_--j+i}}{N_+!N_-!}(K\tau e^{-S})^{N_++N_-},$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta(i-j)}$$

$$\times \exp[2K\tau\cos(\theta)\exp(-S)].$$

$$|\theta\rangle = \Sigma_n e^{in\theta}|n\rangle \qquad E(\theta) = -2K\cos(\theta)\exp(-S).$$

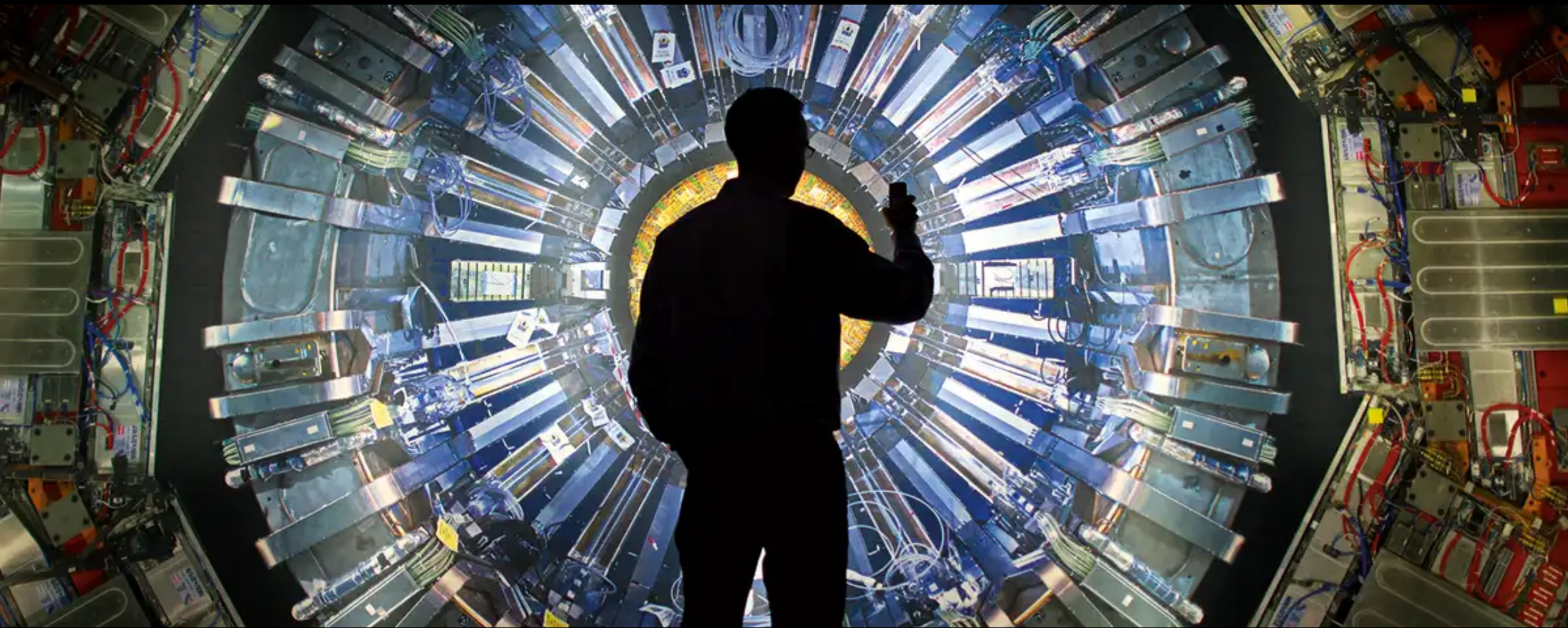
The Instanton Solution

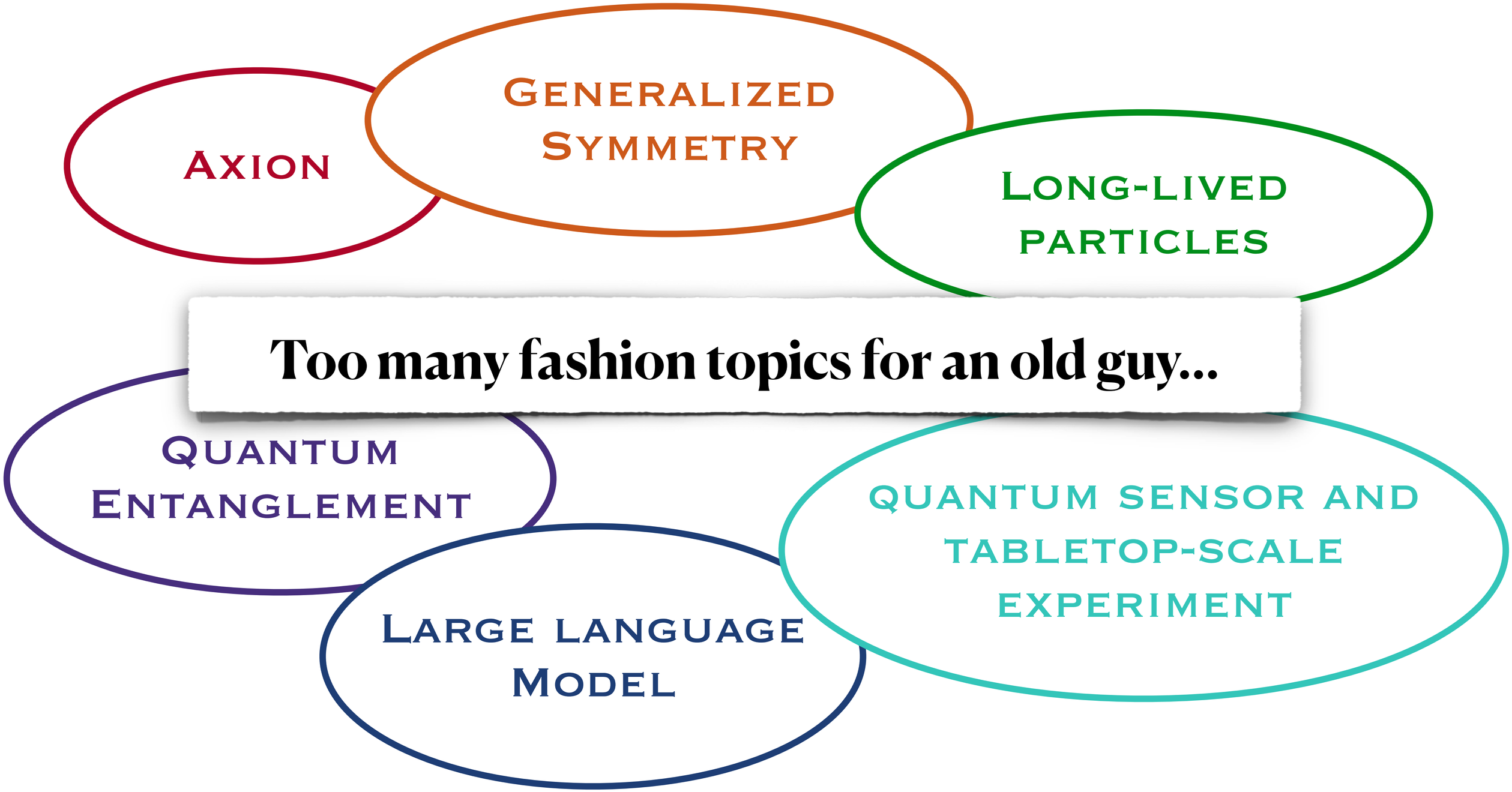
- Non-abelian (pure) Yang-Mills theory
- Farewell, instanton~

$$\mathcal{L} = \frac{i\,\theta}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^a_{\mu\nu}$$

Topic 2: The Sphaleron

As a traditional particle physics phenomenologist, what should I focus on?




Axion
Generalized Symmetry
Long-lived particles
Too many fashion topics for an old guy...
Quantum Entanglement
Large language Model
quantum sensor and tabletop-scale experiment

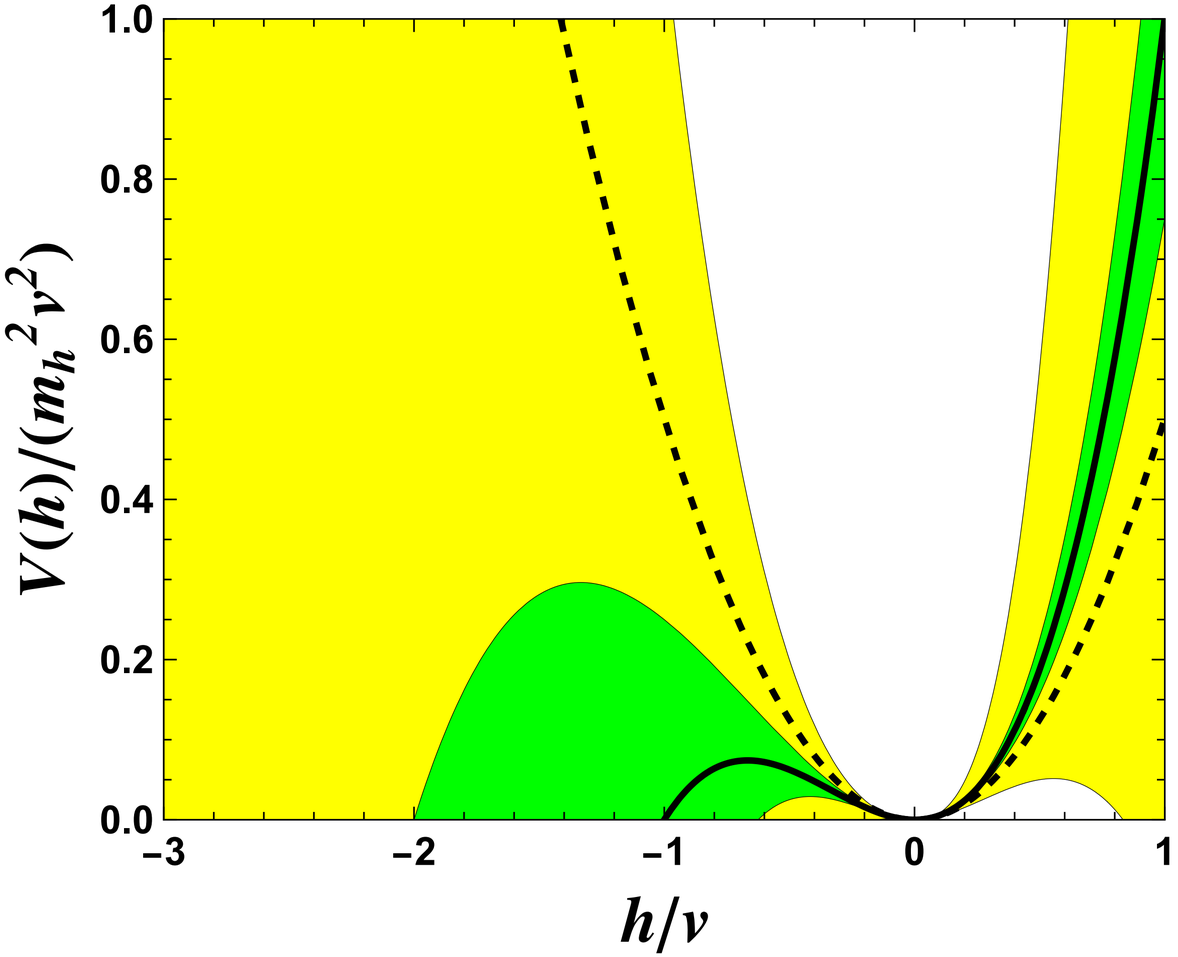
The Eternal Theme of the HL-LHC and future *pp*-collider

The Potential

- Allowed region of the cubic potential by current Higgs pair data.

$$V(h) = \frac{1}{2} m_h^2 \left(h^2 + \frac{\kappa_\lambda}{v} h^3 \right)$$

$$\kappa_\lambda \in [-1.2,\ 7.02]$$



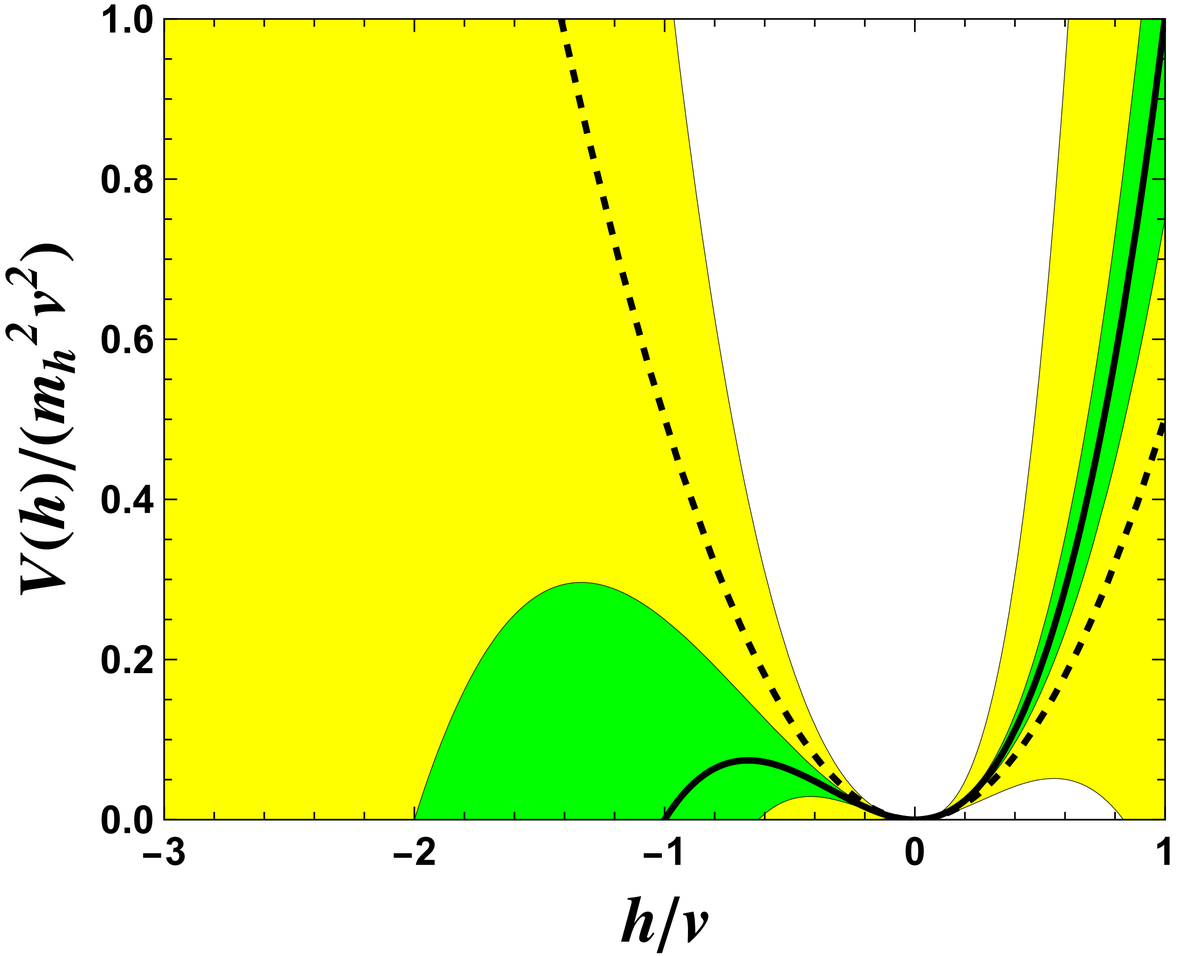
ATLAS Collaboration, Phys. Rev. Lett. 133 (2024) 101801;
CMS Collaboration, CMS PAS HIG-20-011.

The Potential

- (Expected) Allowed region of the cubic potential by HL-LHC.

$$V(h) = \frac{1}{2} m_h^2 \left(h^2 + \frac{\kappa_\lambda}{v} h^3 \right)$$

$$\kappa_\lambda \in [0.5,\ 1.6]$$



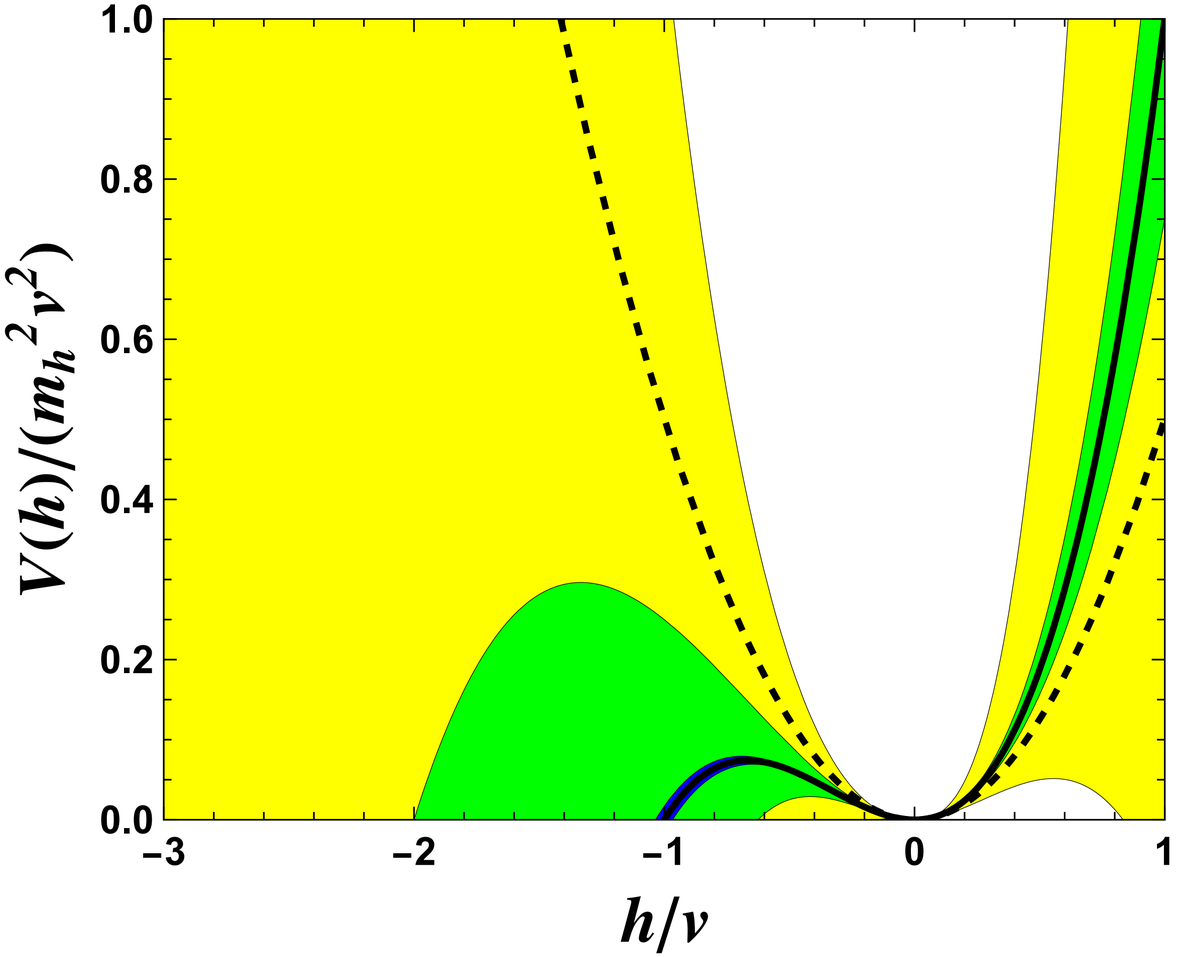
M. Mlynarikova, on behalf of the ATLAS and CMS collaborations, arXiv:2307.07772[hep-ex].

The Potential

- (Expected) Allowed region of the cubic potential by FCC-hh.

$$V(h) = \frac{1}{2} m_h^2 \left(h^2 + \frac{\kappa_\lambda}{v} h^3 \right)$$

$$\kappa_\lambda \in [0.966,\ 1.034]$$



A. Abada, et. al., FCC-hh: The Hadron Collider, Eur. Phys. J. Special Topics 228 (2019) 755.

The Potential

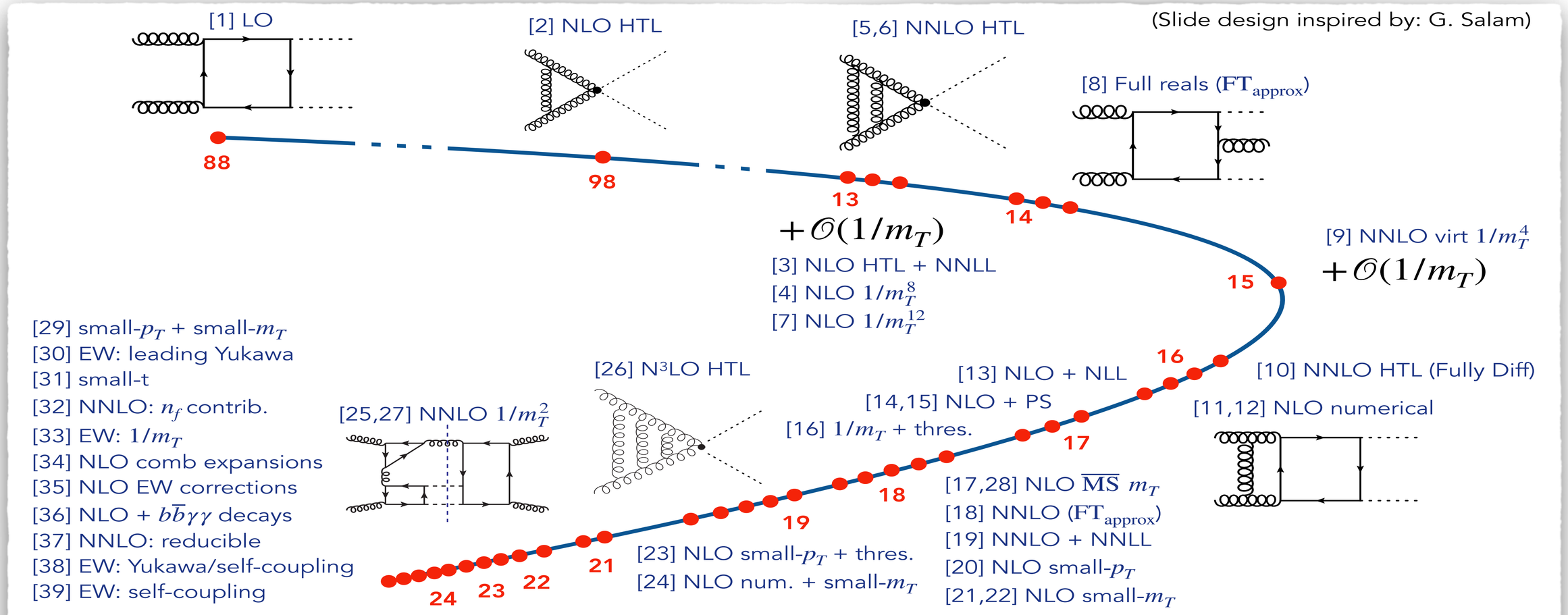
- (Expected) Allowed region of the cubic potential by FCC-hh.

When?

The tentative timeline is:

- **2025**: Completion of the FCC Feasibility Study
- **2027–2028**: Decision by the CERN Member States and international partners
- **2030s**: Start of construction
- **Mid-2040s**: **FCC-ee** begins operation and runs for approximately 15 years
- **2070s**: **FCC-hh** begins operation and runs for approximately 25 years

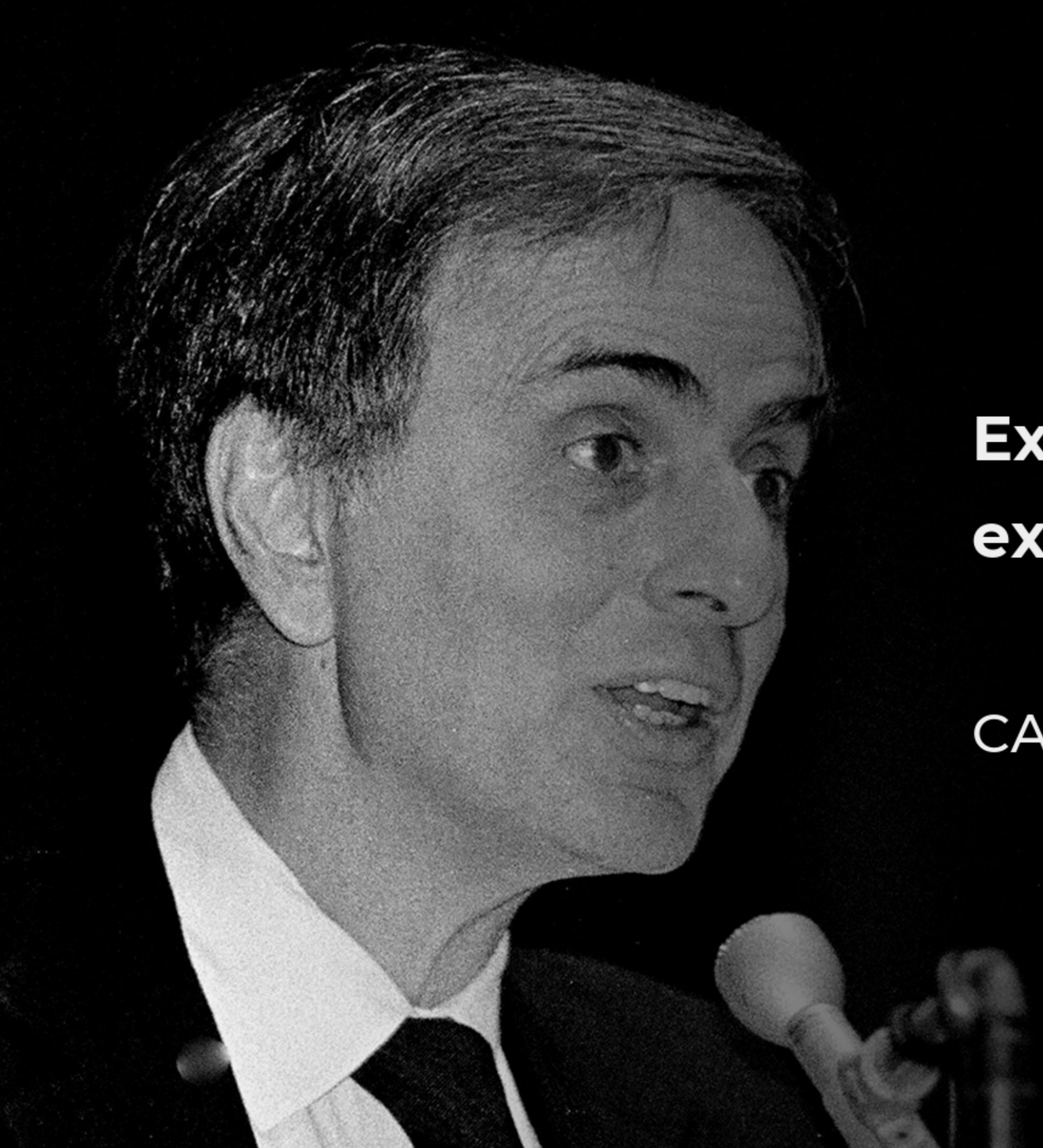
https://home.cern/science/accelerators/future-circular-collider



[1] Glover, van der Bij 88; [2] Dawson, Dittmaier, Spira 98; [3] Shao, Li, Li, Wang 13; [4] Grigo, Hoff, Melnikov, Steinhauser 13; [5] de Florian, Mazzitelli 13; [6] Grigo, Melnikov, Steinhauser 14; [7] Grigo, Hoff 14; [8] Maltoni, Vryonidou, Zaro 14; [9] Grigo, Hoff, Steinhauser 15; [10] de Florian, Grazzini, Hanga, Kallweit, Lindert, Maierhöfer, Mazzitelli, Rathlev 16; [11] Borowka, Greiner, Heinrich, SPJ, Kerner, Schlenk, Schubert, Zirke 16; [12] Borowka, Greiner, Heinrich, SPJ, Kerner, Schlenk, Zirke 16; [13] Ferrera, Pires 16; [14] Heinrich, SPJ, Kerner, Luisoni, Vryonidou 17; [15] SPJ, Kuttimalai 17; [16] Gröber, Maier, Rauh 17; [17] Baglio, Campanario, Glaus, Mühlleitner, Spira, Streicher 18; [18] Grazzini, Heinrich, SPJ, Kallweit, Kerner, Lindert, Mazzitelli 18; [19] de Florian, Mazzitelli 18; [20] Bonciani, Degrassi, Giardino, Gröber 18; [21] Davies, Mishima, Steinhauser, Wellmann 18, 18; [22] Mishima 18; [23] Gröber, Maier, Rauh 19; [24] Davies, Heinrich, SPJ, Kerner, Mishima, Steinhauser, David Wellmann 19; [25] Davies, Steinhauser 19; [26] Chen, Li, Shao, Wang 19, 19; [27] Davies, Herren, Mishima, Steinhauser 19, 21; [28] Baglio, Campanario, Glaus, Mühlleitner, Ronca, Spira 21; [29] Bellafronte, Degrassi, Giardino, Gröber, Vitti 22; [30] Davies, Mishima, Schönwald, Steinhauser, Zhang 22; [31] Davies, Mishima, Schönwald, Steinhauser 23; [32] Davies, Schönwald, Steinhauser 23; [33] Davies, Schönwald, Steinhauser, Zhang 23; [34] Bagnaschi, Degrassi, Gröber 23; [35] Bi, Huang, Huang, Ma Yu 23 [36] Li, Si, Wang, Zhang, Zhao 24; [37] Davies, Schönwald, Steinhauser, Vitti 24; [38] Heinrich, SPJ, Kerner, Stone, Vestner [39] Li, Si, Wang, Zhang, Zhao 24

From S. Jones, “Higgs production: A theory overview”, Higgs Hunting 2024.

We should not stop but work hard to understand the origin of the “mass” of the fundamental particles.



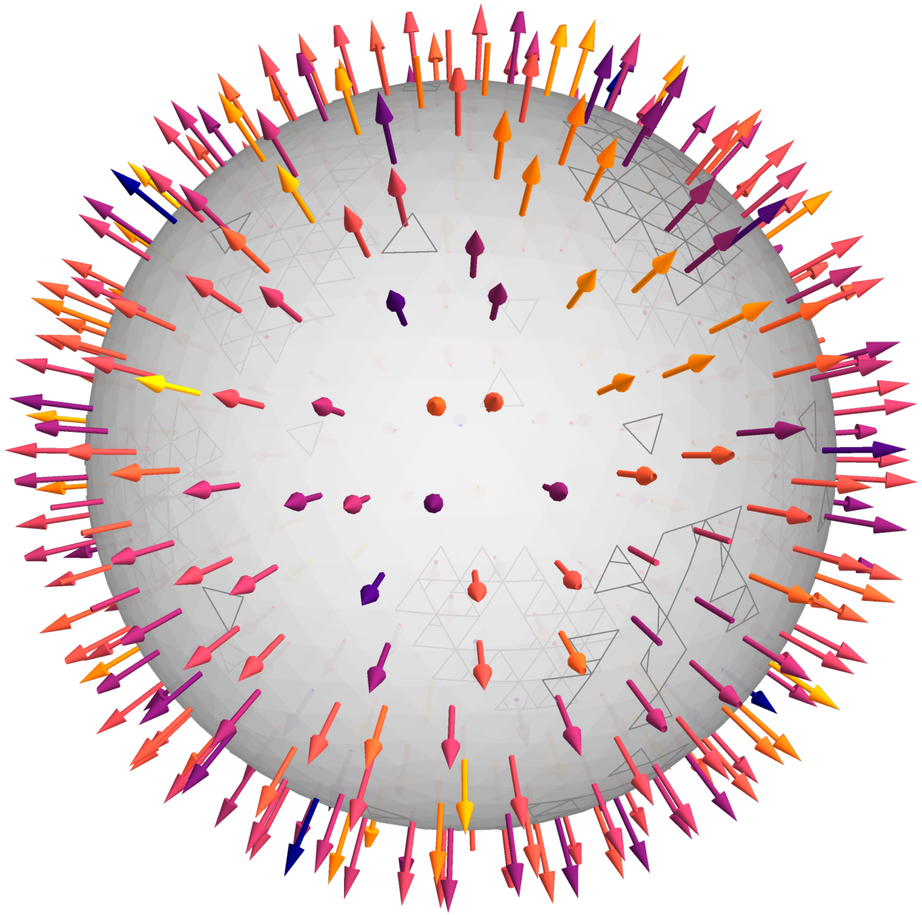
Extraordinary claims require extraordinary evidence.

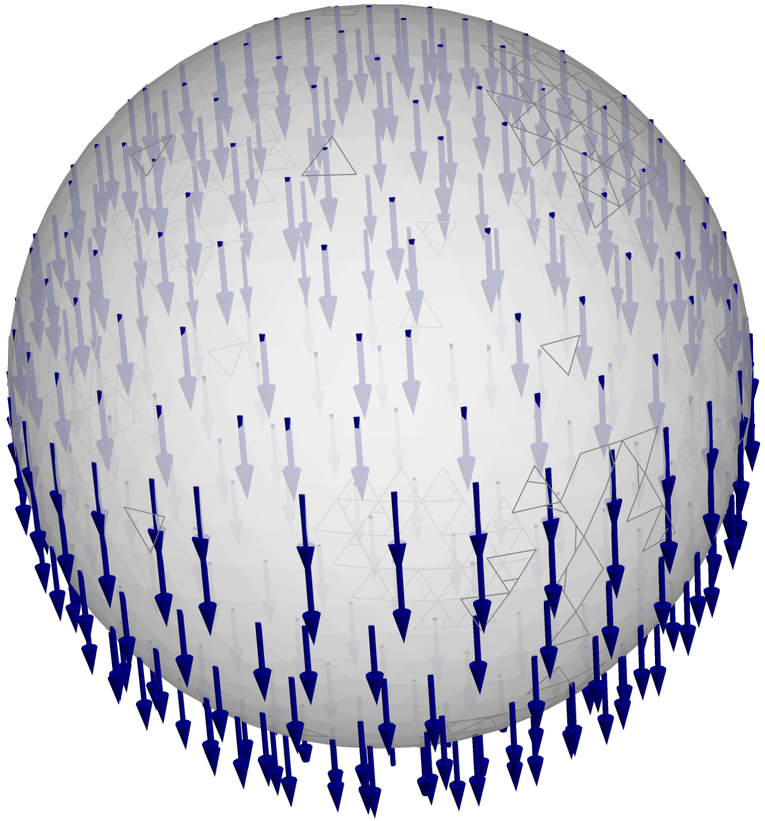
CARL SAGAN

To climb, or not to climb?

The Sphaleron, a climber

- Instanton vs sphaleron: from topological non-trivial configuration to topological non-trivial path.





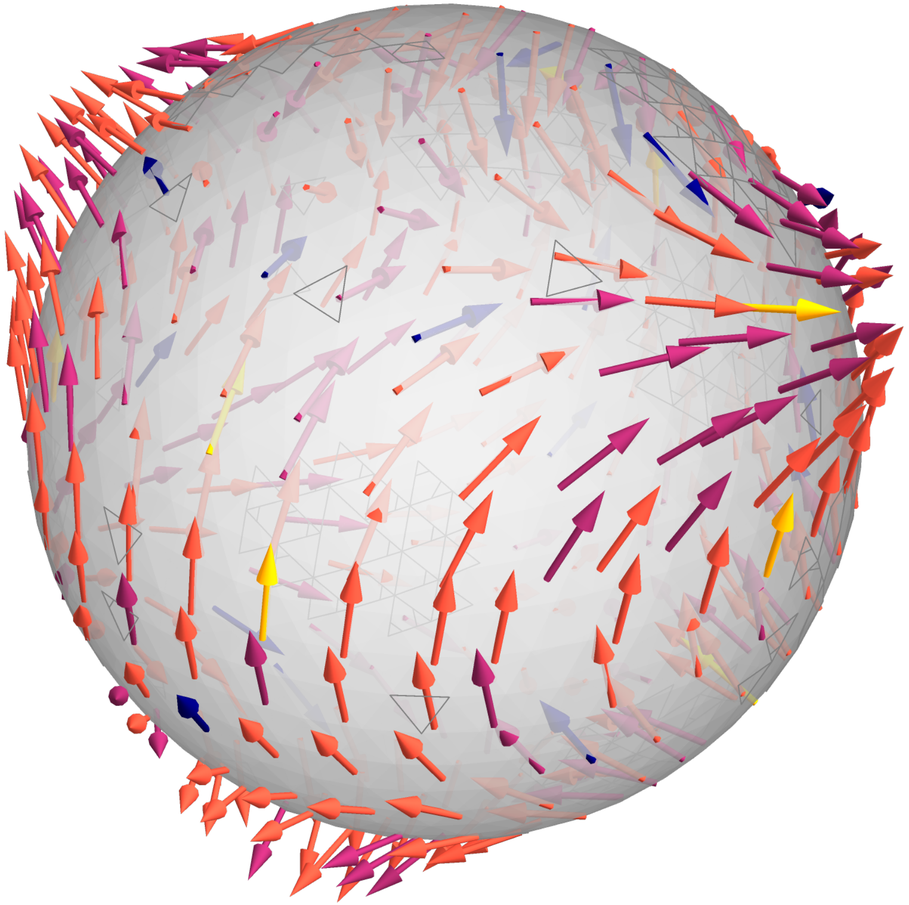
The Sphaleron Configuration

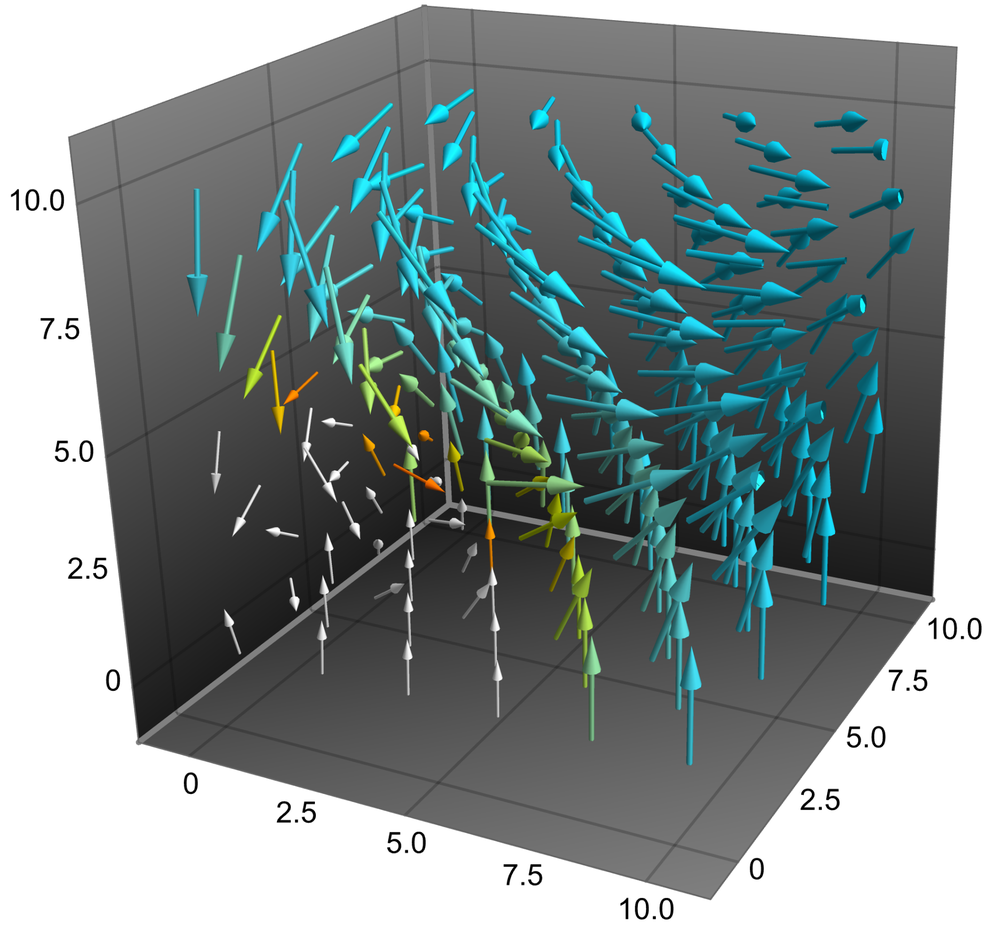
- Manton's insight

N. S. Manton, Topology in the Weinberg-Salam theory, Phys. Rev. D 28 (1983) 2019.

The Sphaleron Configuration

- A "*σφαλερος*" (means "ready to fall") with some ansatz





F. R. Klinkhamer, N. S. Manton, A saddle-point solution in the Weinberg-Salam theory, Phys. Rev. D 30 (1984) 2212.

The Sphaleron

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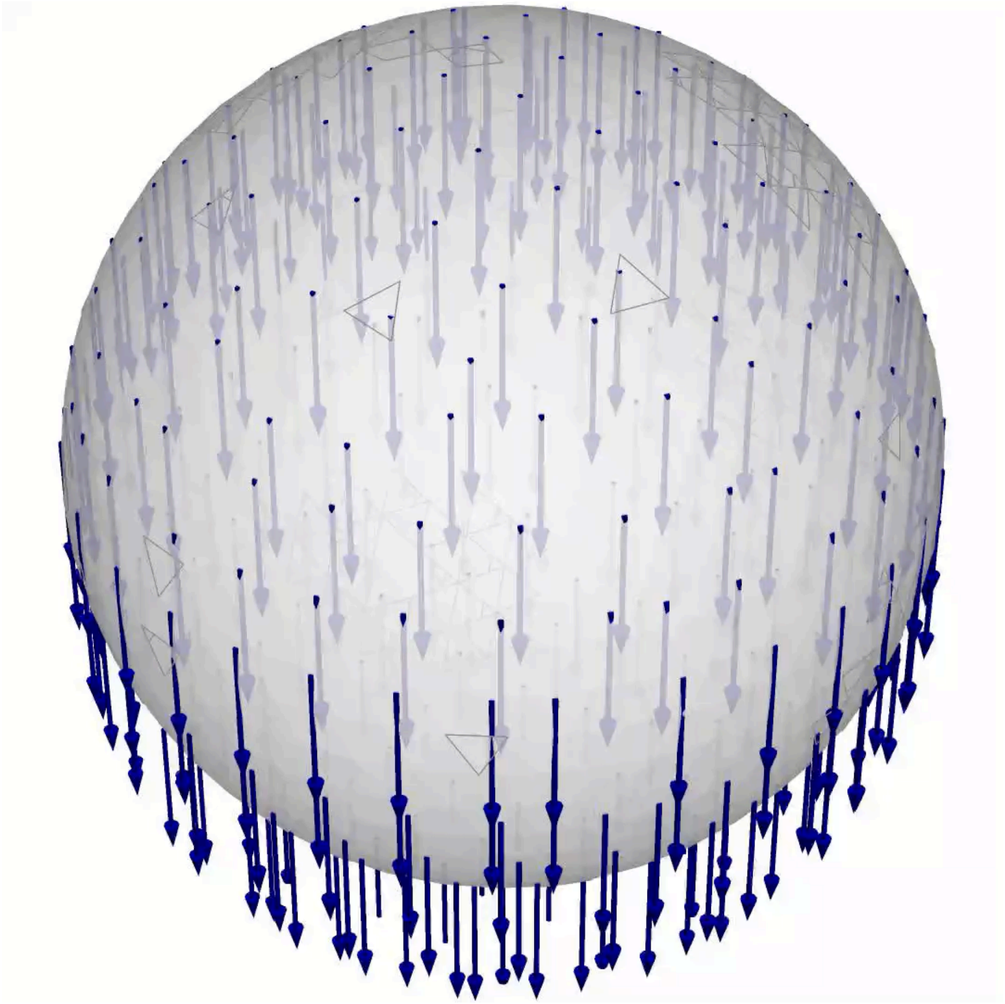
“滑子”

or

“坠子”



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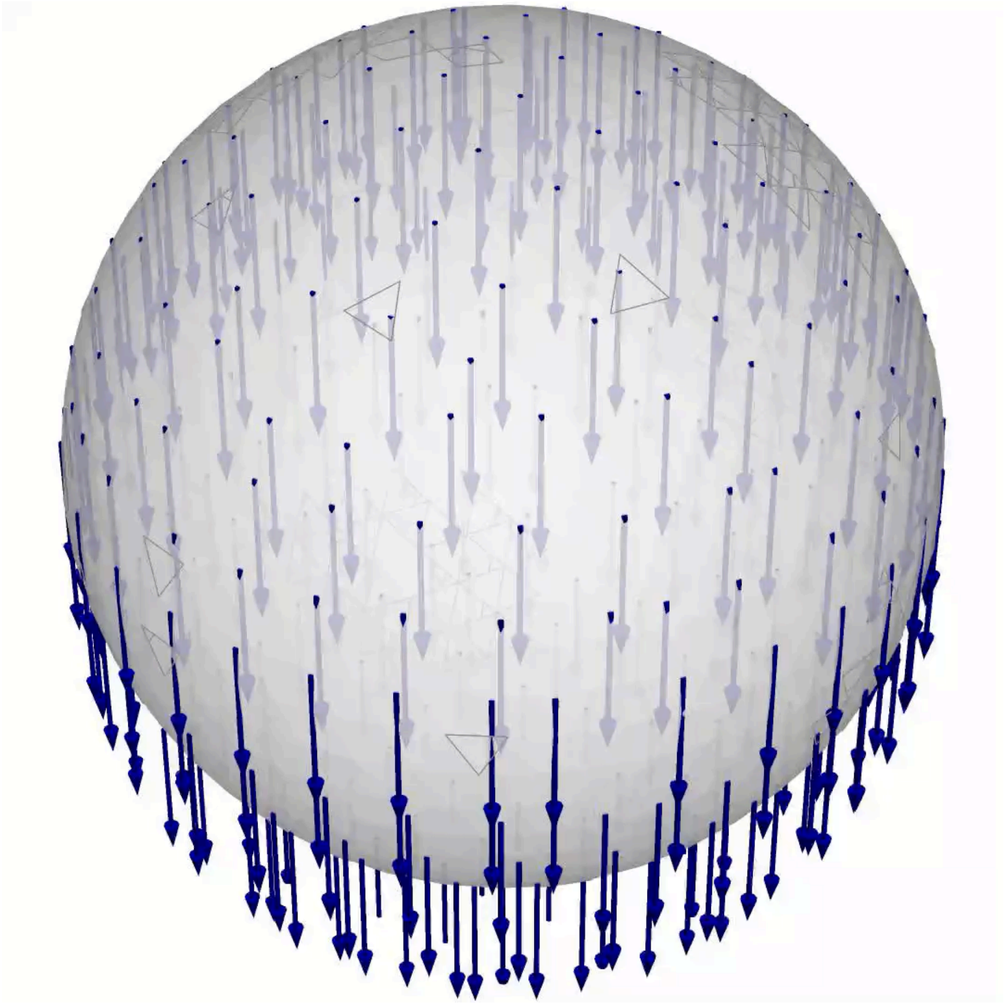

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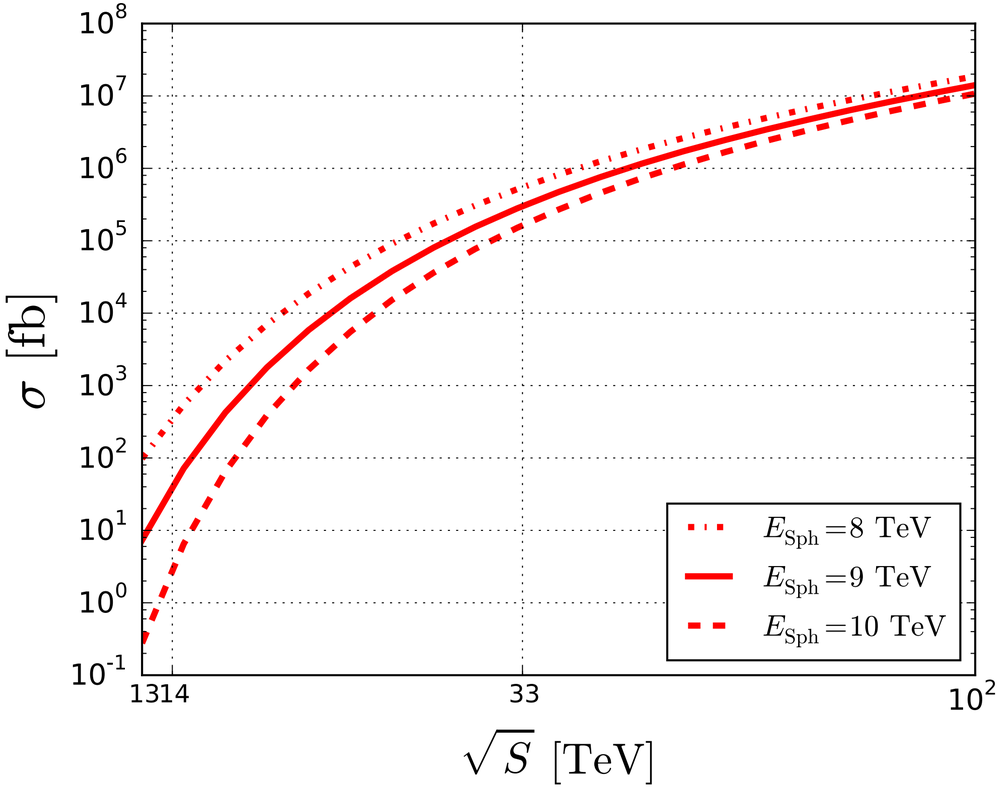
The Signal at Colliders

- It is believed that the typical signal of a sphaleron (if we can produce it) would be *B*+*L* violation final state with multi-Higgs bosons and multi-weak gauge bosons.

$$\partial_\mu J^\mu_B = \frac{1}{32\pi^2}\frac{N_B}{3}\left(g^2\mathbf{F}\wedge\mathbf{F} + g'^2 F\wedge F\right)$$

$$q_L + q_L \to (e^- ddd)(\mu^- ccc)(\tau^- bbb)uu + X$$



J. Ellis, K. Sakurai, Search for sphalerons in proton-proton collisions, JHEP 04 (2016) 086.

How many?

- An classically estimation of the “mass” of the sphaleron is ~ 9TeV (depends on the weak angle, the ansatz, ...).
- Though it seems to be too heavy to be produced at the LHC (even with the parton distribution enhancement), a 28TeV HE-LHC or 100TeV FCC-hh is energetic enough to produce it.
- It is not a easy job to predict the production and decay rate of the sphaleron precisely. Such a prediction (formula) is necessary for reading out the information of the Higgs potential hidden in the sphaleron processes.
- The effort to improve it would probably help us understand the non-perturbative properties of the quantum field theory.
- We would have 70 years to improve the prediction before our experimentalist colleagues have chance to discover it?

New Methods?

- Quantum computing and quantum simulation.

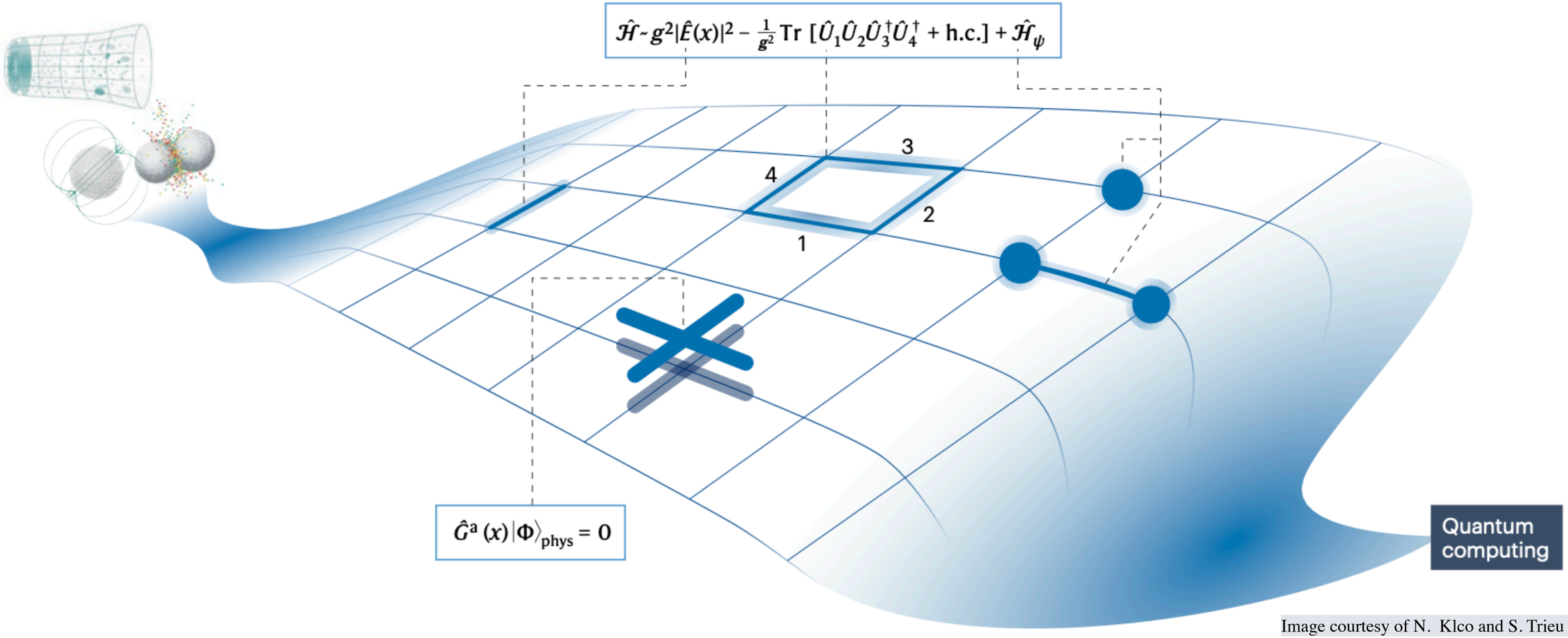


Image courtesy of N. Klco and S. Trieu

Towards to it

- The highest program of the plan: computing (or simulating) a real-time evolution with sphaleron configuration in the SM.
- But

Towards to it

- The highest program of the plan: computing (or simulating) a real-time evolution with sphaleron configuration in the SM.
- The 3+1D SU(2)×U(1) gauge theory with spontaneously symmetry breaking from a Higgs doublet and three generation chiral fermions is too complicate for a baby quantum computing.
- In this first attempt, we investigate the saddle-point configuration in a much simpler quantum field model — the 1+1D O(3) non-linear σ-model.

E. Mottola, A. Wipf, Unsuppressed fermion-number violation at high temperature: An O(3) model, Phys. Rev. D 39 (1989) 588.

The 1+1D O(3) Model

- The classical sphaleron configuration

$$S = \frac{1}{g^2}\int \mathrm{d}t\ \mathrm{d}x \left[\frac{1}{2}\partial_\mu \mathbf{n}\cdot\partial^\mu \mathbf{n} - \omega^2(1+\mathbf{n}\cdot\hat{e}_3)\right] \qquad\qquad \mathbf{n}(x)\cdot\mathbf{n}(x)\equiv 1$$

The 1+1D O(3) Model

- The classical sphaleron configuration

$$\mathbf{n}(\xi,\eta) = \begin{pmatrix} \sin\eta\sin\xi \\ \sin\eta\cos\eta(\cos\xi - 1) \\ -\sin^2\eta\cos\xi - \cos^2\eta \end{pmatrix},$$

$\eta \in (0,\pi),\ \xi \in (0,2\pi)$

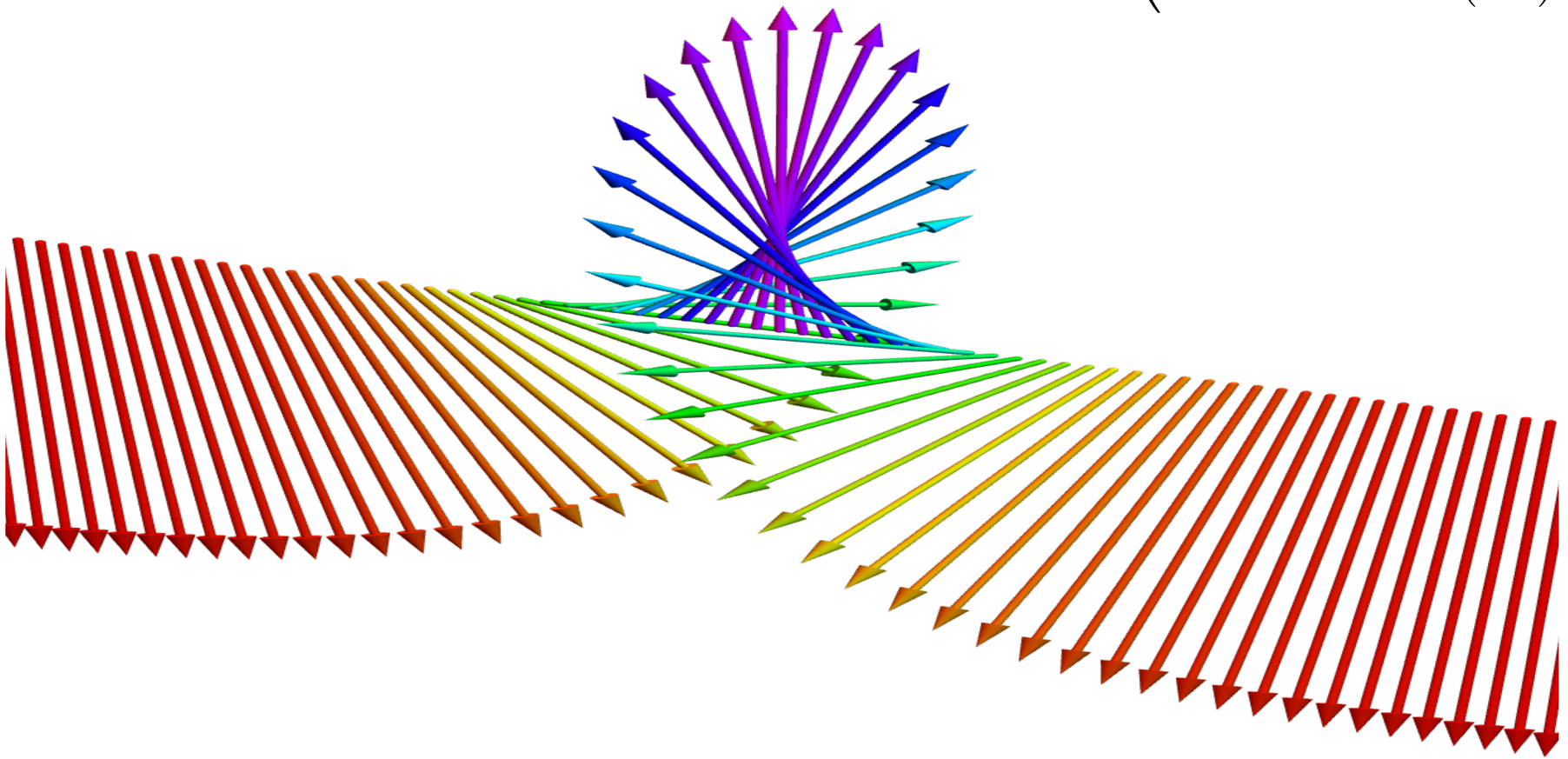
$$S = \frac{1}{g^2}\int \mathrm{d}t\ \mathrm{d}x \Big\{ \frac{1}{2}\left[(1-\cos\xi)^2 + \sin^2\xi\cos^2\eta\right]\partial_\mu\eta\partial^\mu\eta + \sin\xi\sin\eta\cos\eta\partial\eta_\mu\partial\xi^\mu + \frac{1}{2}\sin^2\eta\partial_\mu\xi\partial^\mu\xi - \omega^2\sin^2\eta(1-\cos\xi)\Big\}. \tag{3}$$

$$H = \frac{1}{g^2}\int \mathrm{d}x \left[\frac{1}{2}(1-\cos\xi)^2\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\eta}{\partial x}\sin\xi\cos\eta + \frac{\partial\xi}{\partial x}\sin\eta\right)^2 + \omega^2\sin^2\eta(1-\cos\xi)\right]. \tag{4}$$

The 1+1D O(3) Model

- The classical sphaleron configuration

$$\xi^{\mathrm{sph}}(x) = \begin{cases} 2\arcsin(\mathrm{sech}(\omega x)), & x < 0 \\ 2\pi - 2\arcsin(\mathrm{sech}(\omega x)), & x \geqslant 0 \end{cases}, \qquad \mathbf{n}(x,t) = \begin{pmatrix} -2\mathrm{sech}(\omega x)\tanh(\omega x) \\ 0 \\ -1 + 2\mathrm{sech}^2(\omega x) \end{pmatrix}.$$



The 1+1D O(3) Model

- Reconstruct the classical sphaleron configuration on a lattice

$$H_{\mathrm{tot}} = \frac{g^2}{2a}\sum_{k=-N}^{N}\mathbf{L}_k^2 + \frac{1}{ag^2}\sum_{k=-N}^{N-1}(1-\mathbf{n}_k\cdot\mathbf{n}_{k+1}) + \frac{a\omega^2}{g^2}\sum_{k=-N}^{N}(1+\mathbf{n}_k\cdot\hat{e}_3)$$

$$\mathbf{L}_k = \frac{a}{g^2}\frac{\mathrm{d}\mathbf{n}_k}{\mathrm{d}t}$$

The kinetic term

The tension term

The external field term

The 1+1D O(3) Model

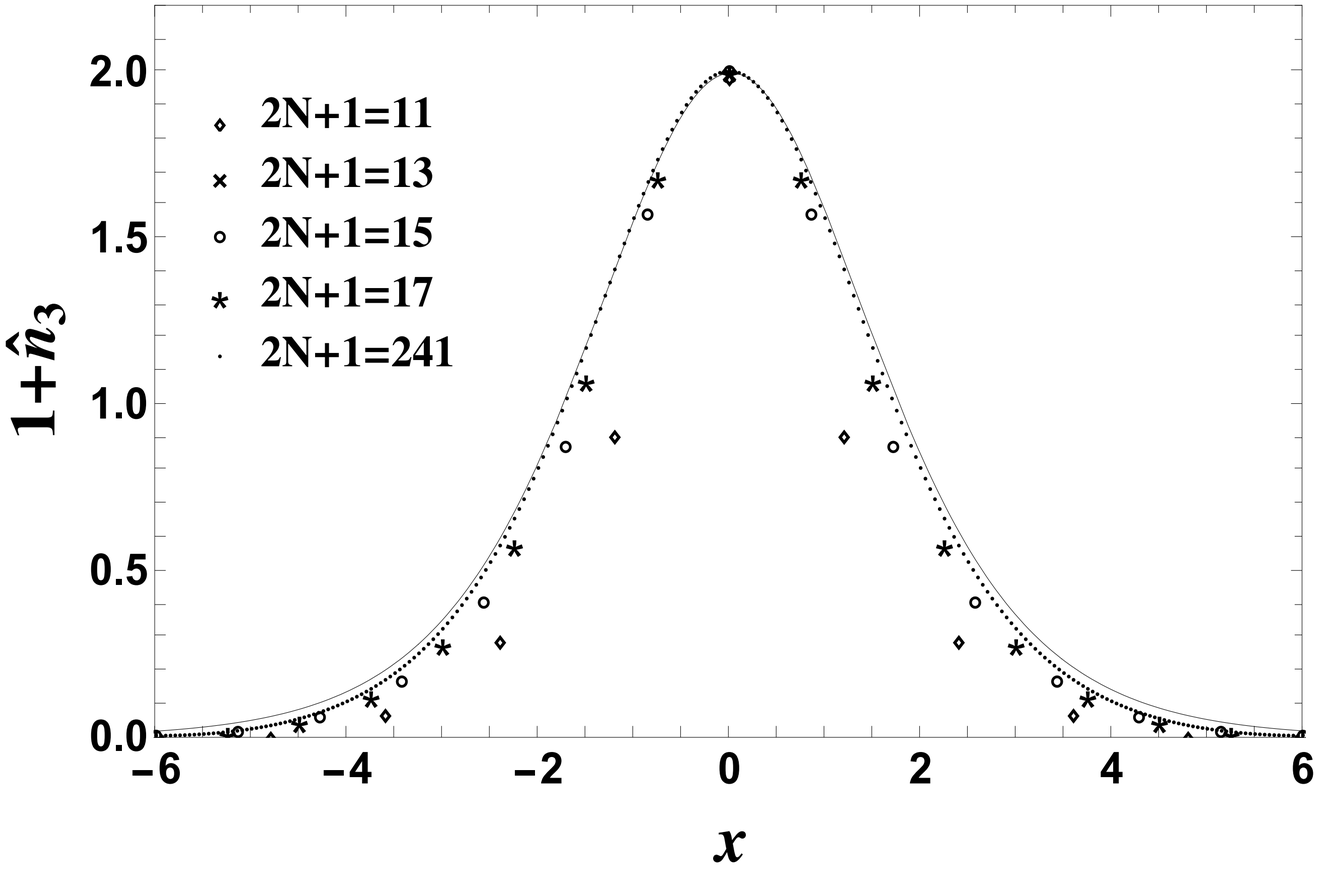
- Reconstruct the classical sphaleron configuration on a lattice

$$
\begin{aligned}
H_{\text{static}} &= \frac{1}{ag^2} \sum_{k=-N}^{N-1} (1 - \mathbf{n}_k \cdot \mathbf{n}_{k+1}) + \frac{a\omega^2}{g^2} \sum_{k=-N}^{N} (1 + \mathbf{n}_k \cdot \hat{e}_3) \\
&= \frac{\sin^2 \eta_0}{g^2} \left(\frac{2N}{L} \right) \left\{ \sum_{k=-N}^{N-1} [1 - \cos(\xi_k - \xi_{k+1})] + \left(\frac{\omega L}{2N} \right)^2 \sum_{k=-N}^{N} (1 - \cos \xi_k) \right\}
\end{aligned}
$$

The 1+1D O(3) Model

- But $\pi_1(\mathbb{Z}_{2N+1}) \neq \pi_1(S^1)$
- Boundary conditions

$$\xi_{-N} = 0, \quad \xi_0 = \pi, \quad \xi_N = 2\pi,$$
$$\text{for } \forall\ k \in [-N, N), \quad \xi_k < \xi_{k+1}.$$



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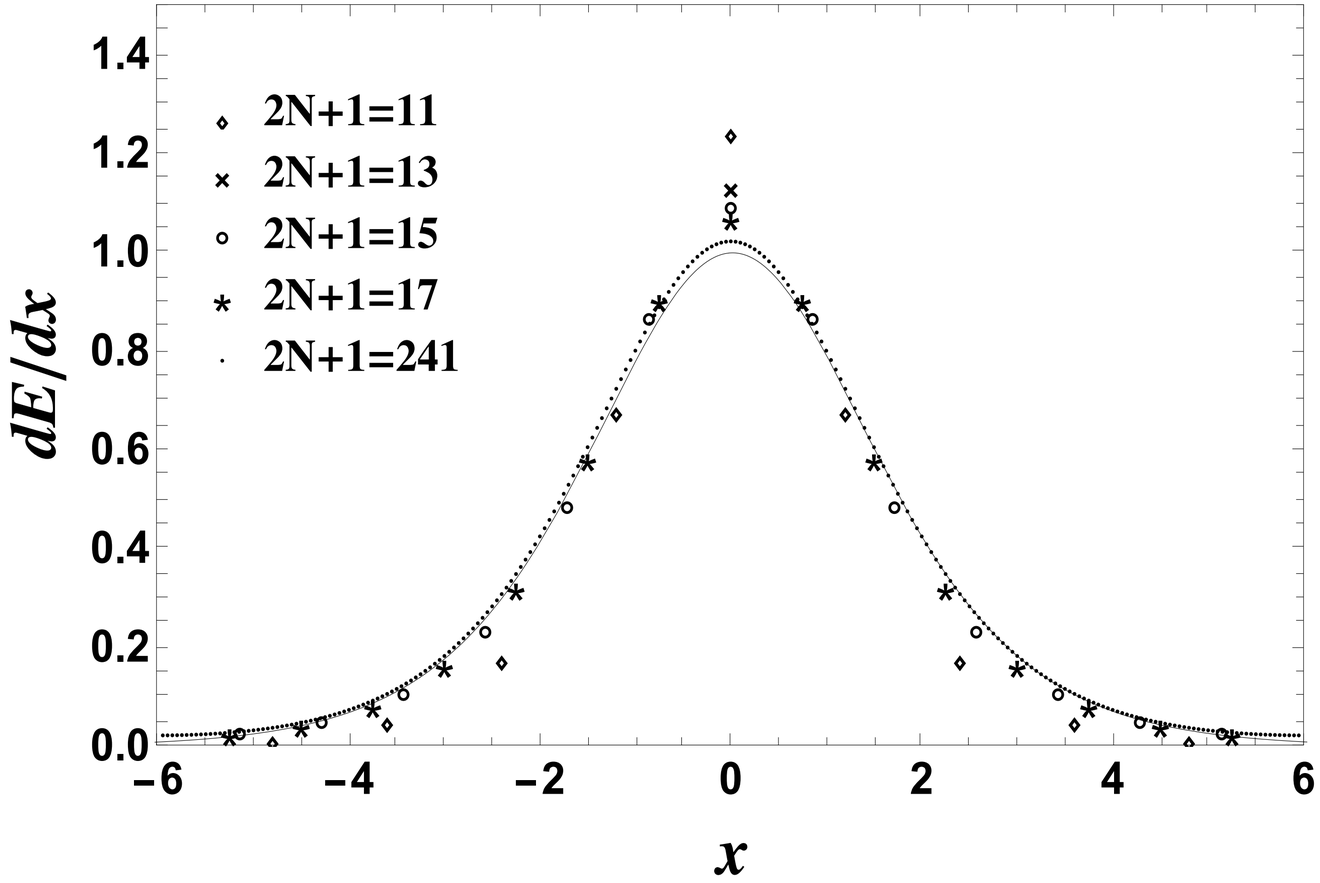

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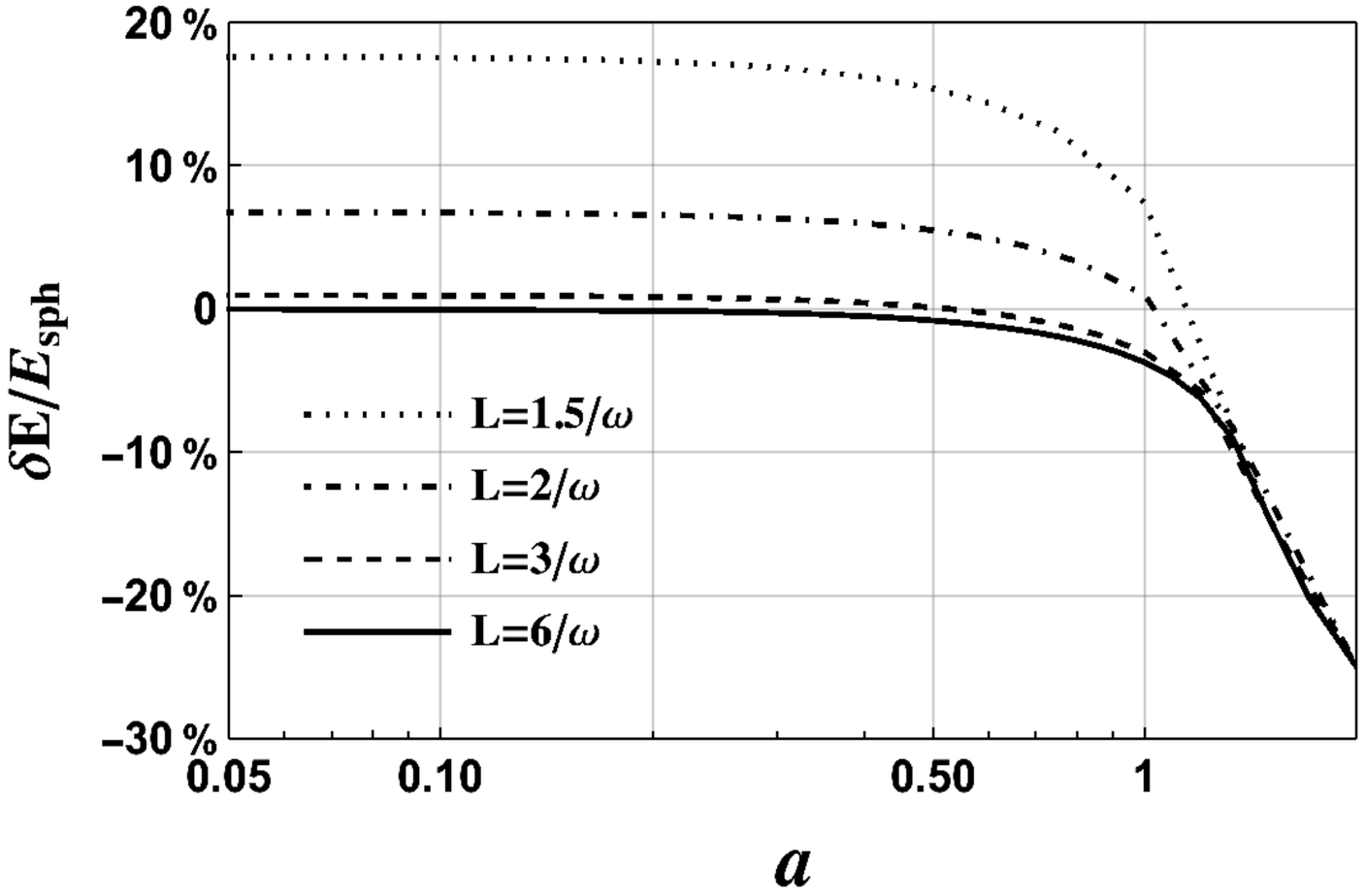
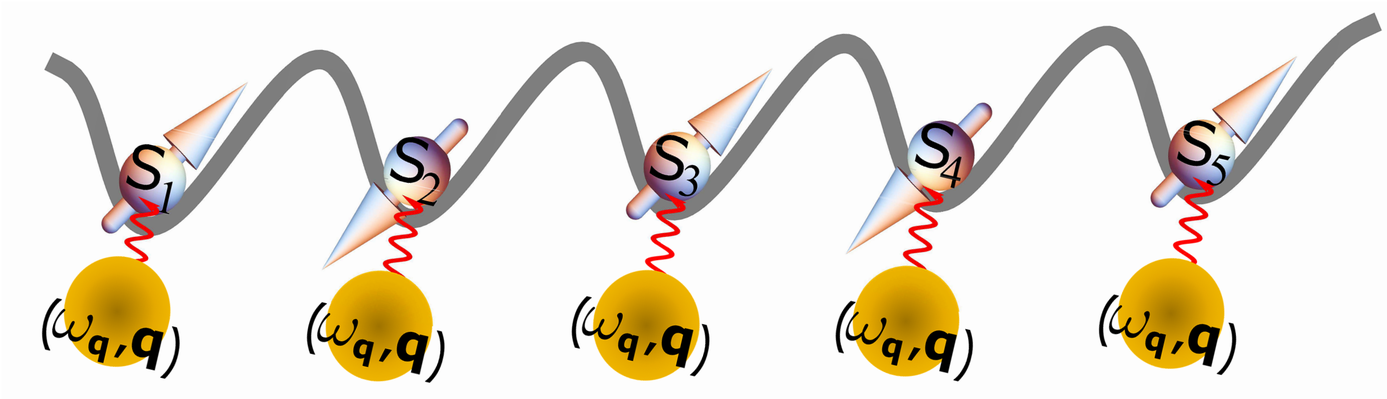


The artificial boundary condition is from the fact that the discrete LQFT losing the information of the path connecting the neighbor sites. For a more systematical treatment, see J.-Y. Chen, Instanton Density Operator in Lattice QCD from Higher Category Theory, arXiv:2406.06673[hep-lat].

The 1+1D O(3) Model

- If one freezes the kinetic mode, the classical model could be quantized as a simple Heisenberg chain.
- However, the boundary condition is different from the one used by Bethe.

$$\hat{\mathcal{H}} = -\frac{1}{ag^2}\sum_{k=-N}^{N-1}\left(\hat{\vec{\sigma}}_k \cdot \hat{\vec{\sigma}}_{k+1} - \hat{\mathbf{1}}\right) + \frac{a\omega^2}{g^2}\sum_{k=-N}^{N}\left(\hat{\mathbf{1}} + \hat{\vec{\sigma}}_k \cdot \hat{e}_3\right)$$



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$$\hat{H}_{\rm tot} = \frac{g^2}{2a}\sum_{k=-N}^{N}\hat{\mathbf{L}}_k^2 + \frac{1}{ag^2}\sum_{k=-N}^{N-1}(\mathbb{1}-\hat{\mathbf{n}}_k\cdot\hat{\mathbf{n}}_{k+1})$$
$$+\frac{a\omega^2}{g^2}\sum_{k=-N}^{N}(\mathbb{1}+\hat{\mathbf{n}}\cdot\vec{e}_z),$$

The 1+1D O(3) Model

- In principle, the evolution of the system should contain any partial wave of $\hat{\mathbf{L}}_k^2$.
- But in a real calculation, limited by the number of qubit, one has to add a "cutoff" $\ell_{\max}$ of the high (speed) partial wave: **truncation**.

$$\hat{H}_{\text{tot}} = \frac{g^2}{2a} \sum_{k=-N}^{N} \hat{\mathbf{L}}_k^2 + \frac{1}{ag^2} \sum_{k=-N}^{N-1} (\mathbb{1} - \hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}_{k+1})$$
$$+ \frac{a\omega^2}{g^2} \sum_{k=-N}^{N} (\mathbb{1} + \hat{\mathbf{n}} \cdot \vec{e}_z),$$

The 1+1D O(3) Model

- The initial state of the system is set to be the "sphaleron" such that $\langle \hat{\mathbf{n}}_k \rangle = \frac{\ell_{\max}}{\ell_{\max}+1}\mathbf{n}_k$.

$$|\mathbf{n}(\theta_k, \phi_k)\rangle = \mathcal{N} \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} Y_{lm}^*(\theta_k, \phi_k) \, |l, m\rangle$$

$$|\psi_{\mathbf{coherent}}\rangle = \bigotimes_{k=-N}^{N} |\mathbf{n}(\theta_k, \phi_k)\rangle$$

The 1+1D O(3) Model

- Trotterization

$$U(\Delta t) = e^{-i\hat{H}_K \Delta t/2}\, e^{-i\hat{H}_\omega \Delta t/2}\, e^{-i\hat{H}_J \Delta t}\, e^{-i\hat{H}_\omega \Delta t/2}\, e^{-i\hat{H}_K \Delta t/2}$$

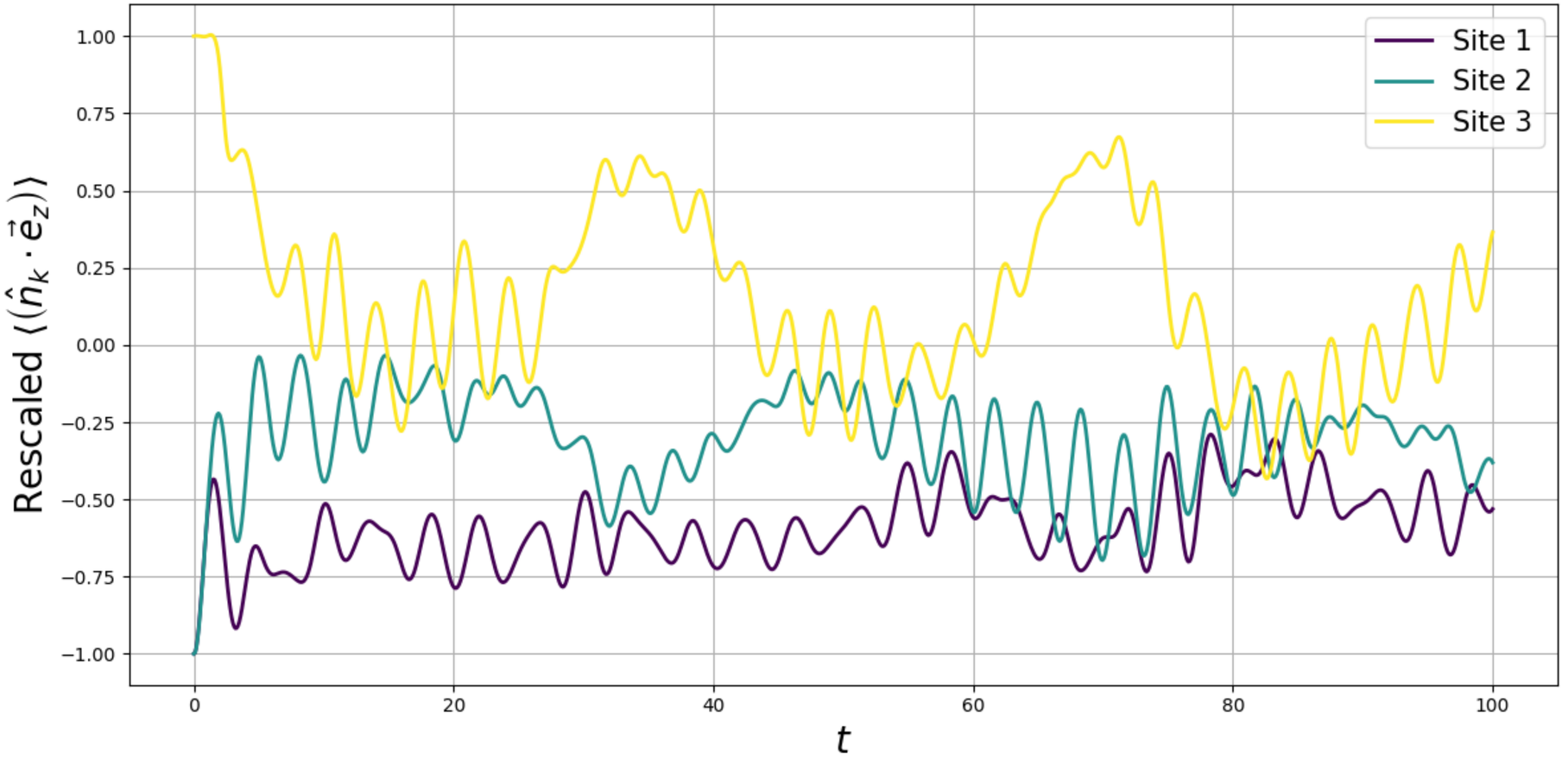
$$\hat{H}_K = \frac{g^2}{2a} \sum_k \hat{\mathbf{L}}_k^2,$$

$$\hat{H}_J = \frac{1}{ag^2} \sum_k (\mathbb{1} - \hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}_{k+1}),$$

$$\hat{H}_\omega = \frac{a\omega^2}{g^2} \sum_k (\mathbb{1} + \hat{\mathbf{n}}_k \cdot \vec{e}_z)$$

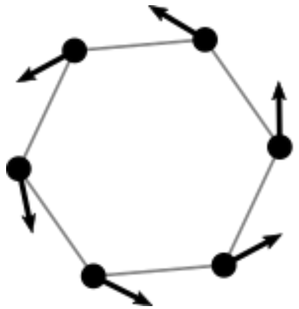
The 1+1D O(3) Model

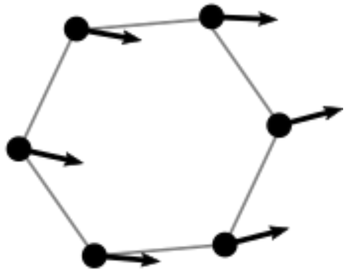
- Resource, resource, resource...



Discussions

Lattice?





How can we trust?!

- From 0+1D to 3+1D?
  - From semi-classical “vacuum” to “real vacuum”
  - To get a finite transition amplitude, the action must be finite
  - **No well-defined path integral!**
  - **No well-defined QFT (with non-trivial interaction)!**
  - **Mass gap?**