Analytic Structures of Feynman Integrals

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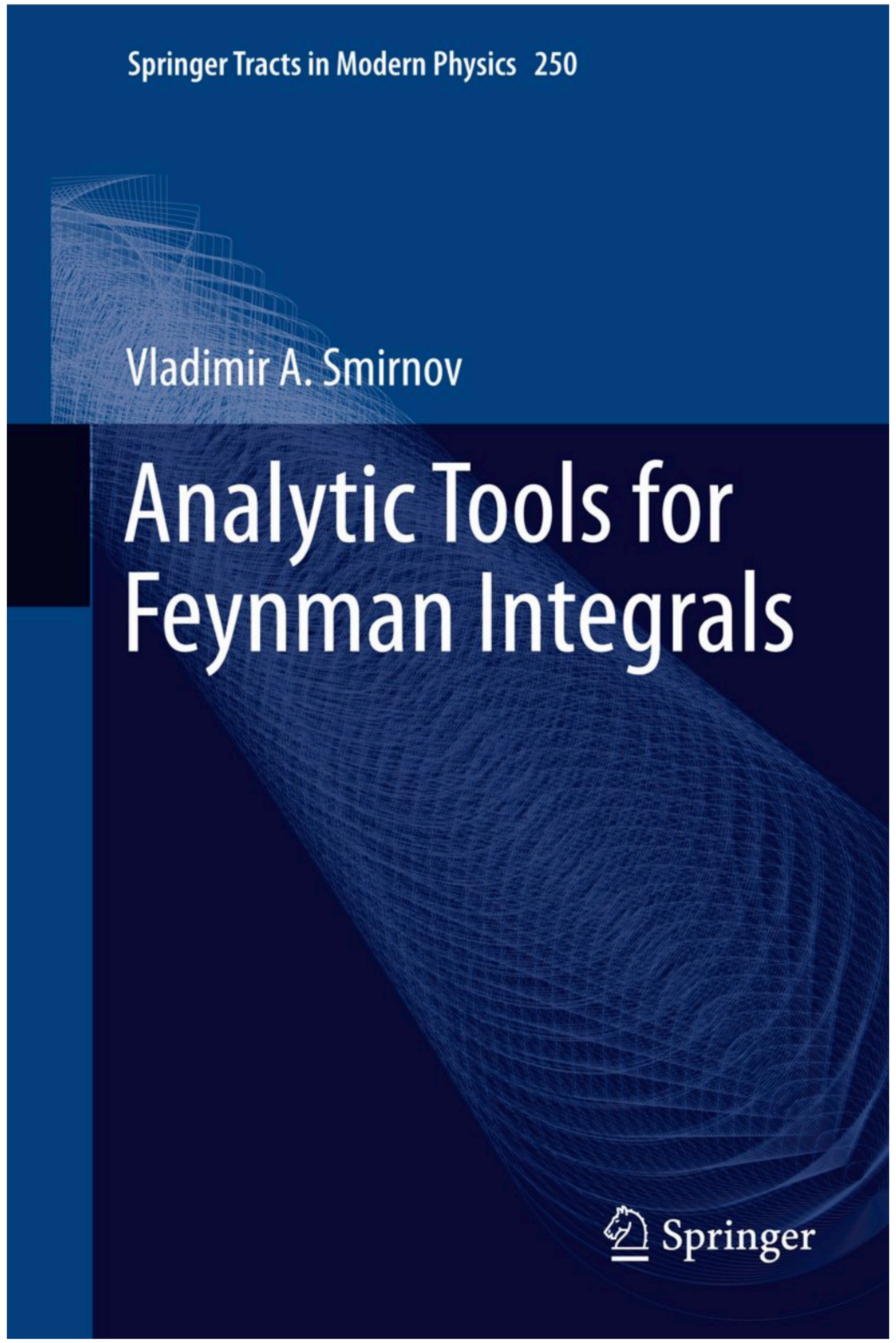
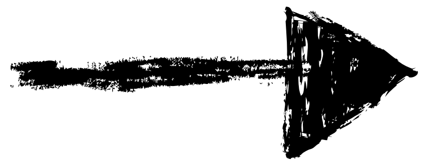
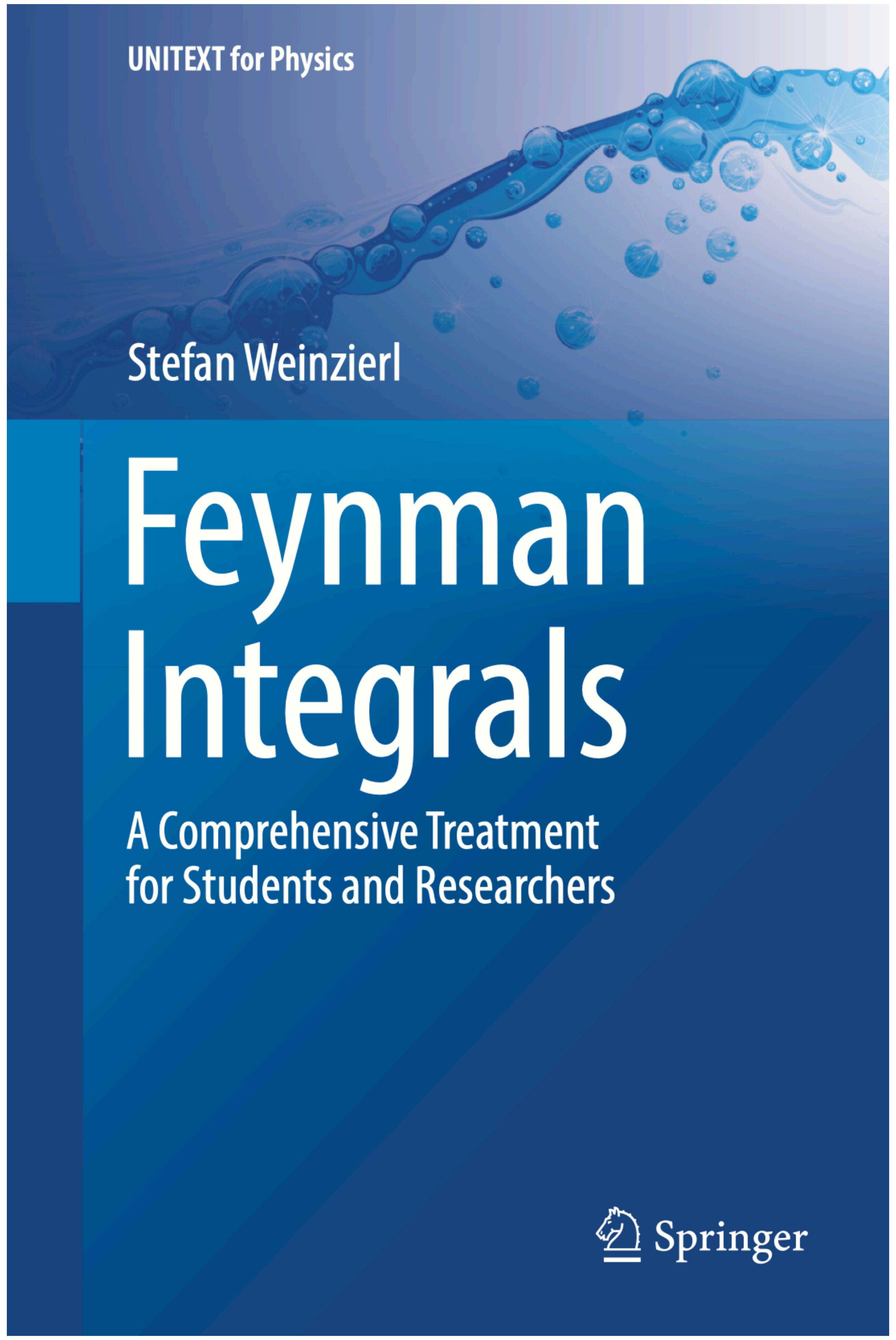
The Chinese University of Hong Kong, Shenzhen

2025.7.17 @ Zhangqiu, Jinan (济南章丘)

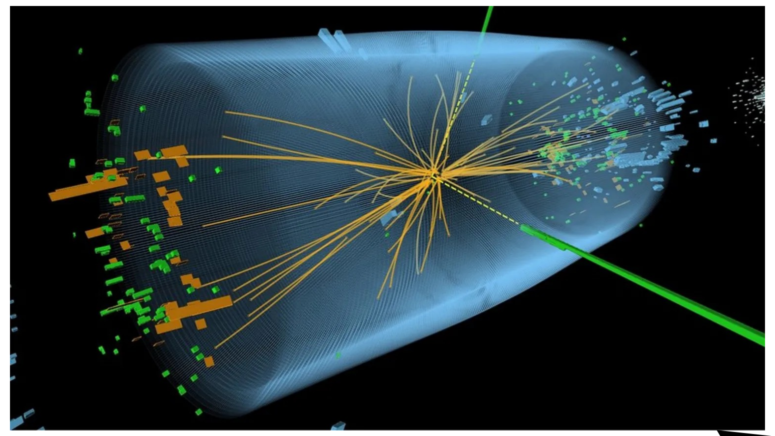
Reference

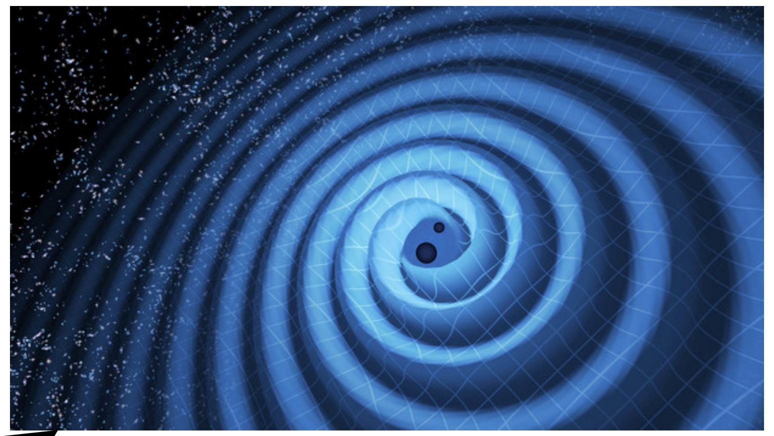

Springer Tracts in Modern Physics 250
Vladimir A. Smirnov
Analytic Tools for Feynman Integrals
Springer




UNITEXT for Physics
Stefan Weinzierl
Feynman Integrals
A Comprehensive Treatment for Students and Researchers
Springer

Motivation



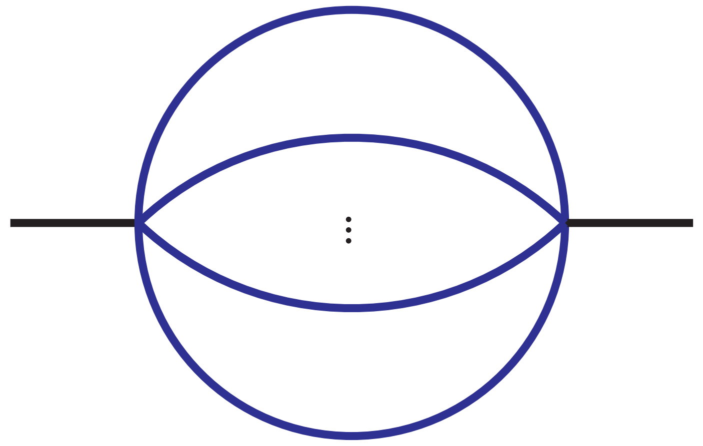


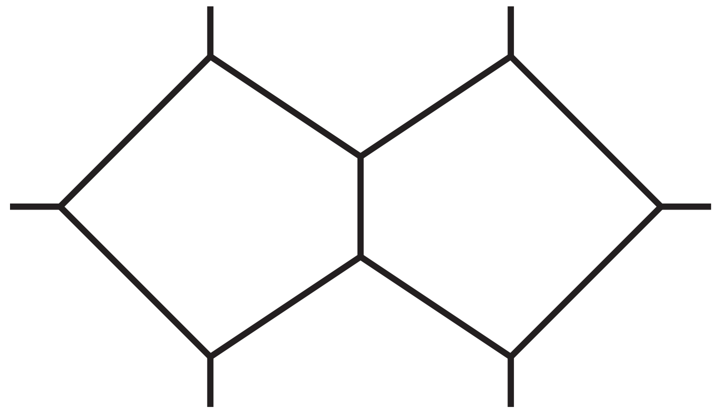
precision prediction

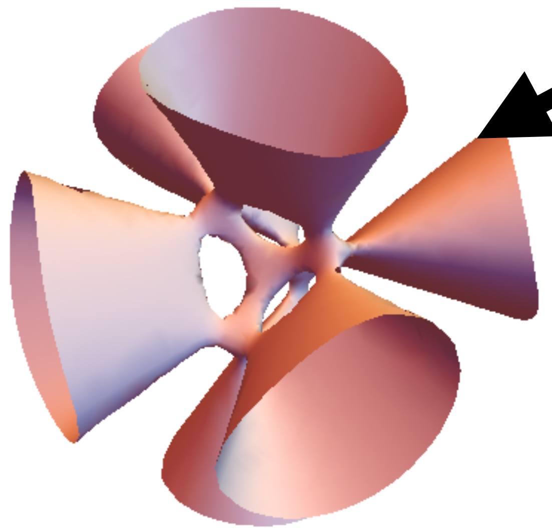
[Wen, Yang, Hjalte's talks]

$$O\left(\alpha, \{x\}\right) = \sum_n c_n F_n$$

F_n: **Feynman Integrals**







formal side: $\mathcal{N} = 4$ sYM, super Gravity, string…

Motives? Number Theory…

Mathematics

What are Feynman Integrals (Mom.)

$$I_{\nu_1\nu_2\cdots\nu_n}\left(D = D_{\text{int}} - 2\varepsilon;\ \{m_i^2, s_{ij}\}, \mu^2\right)$$

$$\|$$

$$e^{l\gamma_E}\left(\mu^2\right)^{|\nu|-\frac{lD}{2}}\int\frac{\mathrm{d}^D k_1}{i\pi^{D/2}}\int\frac{\mathrm{d}^D k_2}{i\pi^{D/2}}\cdots\int\frac{\mathrm{d}^D k_l}{i\pi^{D/2}}\frac{\text{Num}(\{l\})}{\left[-q_1^2+m_1^2\right]^{\nu_1}\left[-q_2^2+m_2^2\right]^{\nu_2}\cdots\left[-q_n^2+m_n^2\right]^{\nu_n}}$$

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- Dep. on dimensionless kinematics: e.g., $\mu^2 = s_{12} \rightsquigarrow \{x_i = m_i^2/s_{12}, \cdots\}$;
- Numerators will not increase the essential (analytical) complexity.

Generic Features of Feynman Integrals

$$I_{\nu_1 \nu_2 \cdots \cdot \nu_n}(\varepsilon;\ \{x_i\})$$

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$$I_{\nu_1\nu_2\cdots\nu_n}\left(\varepsilon;\ \{x_i\}\right)$$

$I_{\nu_1\nu_2\cdots\nu_n}$ is always a linear combination of a **finite** basis (master integrals):

$$I_{\nu_1\nu_2\cdots\nu_n} = \sum_{i=1}^{N_F} \text{rational}_i \times M_i,$$

e.g., $I_{21}^{\text{bubble}} = \dfrac{D-2}{4m^2(p^2-4m^2)} I_{10}^{\text{bubble}} + \dfrac{D-3}{4m^2-s} I_{11}^{\text{bubble}}$. [Wen, Yang, Hjalte's talks]

Generic Features of Feynman Integrals

$$I_{\nu_1\nu_2\cdots\nu_n}(\varepsilon;\ \{x_i\})$$

ε dependence is always meomorphic (Laurent series).

Generic Features of Feynman Integrals

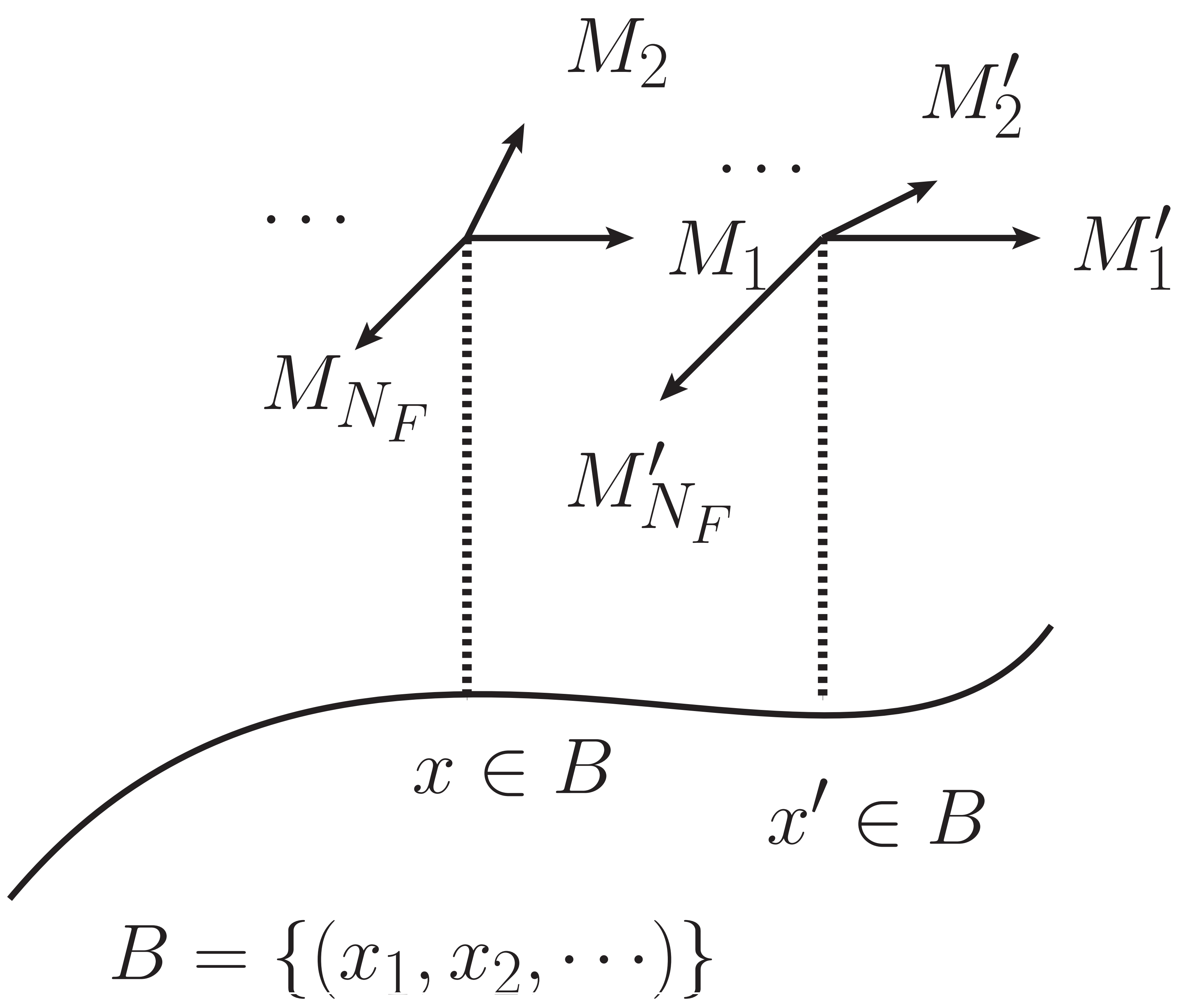
$$I_{\nu_1 \nu_2 \cdots \nu_n}\left(\varepsilon;\ \{x_i\}\right)$$

The kinematic dependence is the most non-trivial: analytic structures.

Kinematics Dependence

Kinematics vary$\longrightarrow$ natural to study differential equations of FIs (MIs).

It becomes the primary method for analytic calculation of FIs.



$$\mathrm{d}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{N_F} \end{pmatrix} = A_{N_F\times N_F}(\varepsilon, \{x\})\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{N_F} \end{pmatrix}$$

[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '00]

Canonicalization

ε-factorization:

With rotationof basis and variable change, ***ε dependence factorizes in the (Gauß Manin) connection matrix, with suitable boundary condition.***

[Henn '13]

$$\mathrm{d}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ \vdots \\ J_{N_F} \end{pmatrix} = \varepsilon\, B_{N_F\times N_F}\big(\{y(x)\}\big)\begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ \vdots \\ J_{N_F} \end{pmatrix} \longrightarrow$$

$$\vec{J} = \sum \varepsilon^n \vec{J}^{(n)}$$

$$\vec{J}^{(n+1)} = \int B_{N_F\times N_F}\vec{J}^{(n)} + \text{boundary}$$

MIs can be written as Chen's iterated integrals [Chenn '13].

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Once the ε-factorized form is derived, FIs (MIs) are viewed as solved.

Canonicalization's Another Consequence

To achieve the ε-factorized form, both rotations basis and variable changes are required.

The "mirror map": $q(x) \equiv \exp\left(2\pi i \dfrac{\psi_1(x)}{\psi_0(x)}\right)$ and its inverse $x(q)$, where $\psi_0(x)$ is holomorphic while $\psi_1(x)$ is single-logarithmic near the MUM point.

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- For FIs, the MUM point should always "exist";
- The mirror map not only exists for Calabi-Yau type FIs, but also generalizes to MPL-like FIs, e.g., [2411.07493, Wang, XW, Wang].

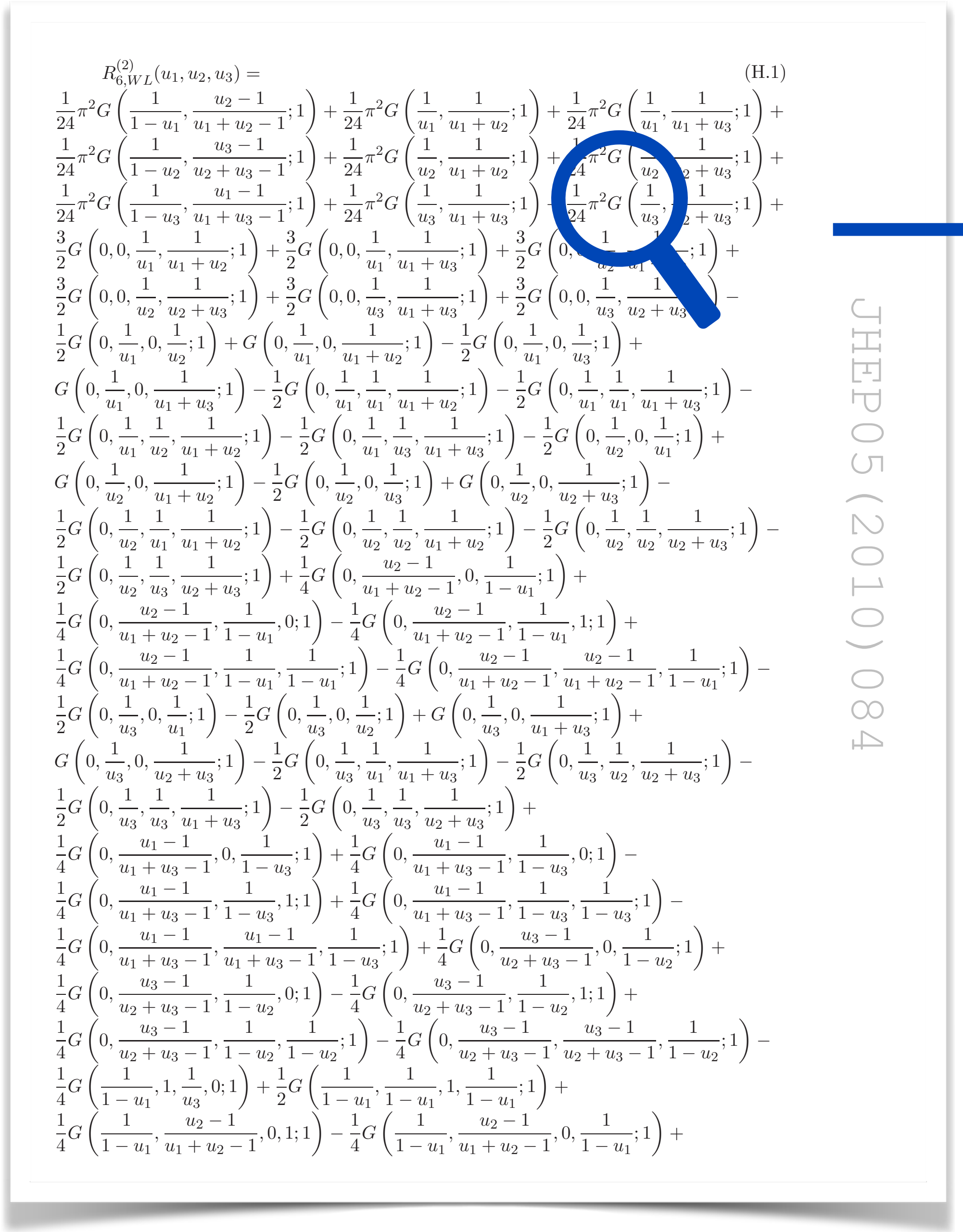
Multiple (Goncharov) Polylogarithms

For many cases, the entrys of $B_{N_F \times N_F}$ look like: $\mathrm{d}\log \eta_{ij}(x)$. In particular, the letters η_{ij} are rational functions, then we end up with MPL (GPL) [Goncharov, '98, 01'].

$$G(;z) = 1,$$

$$G\left(x_1, x_2 \ldots, x_n; z\right) = \int_0^z \frac{\mathrm{d}z_1}{z_1 - x_1} G\left(x_2 \ldots, x_n; z_1\right);$$

$$\hookrightarrow \; G(0;z) = \ln z, \; G(0,1;z) = -\,\mathrm{Li}_2(z)\,.$$



+16 pages

for a 2-loop amplitude [Duca, Duhr, and Smirnov, 2010]

simplify

$$R_6^{(2)}\left(u_1, u_2, u_3\right) = \sum_{i=1}^{3}\left(L_4\left(x_i^+, x_i^-\right) - \frac{1}{2}\mathrm{Li}_4\left(1 - 1/u_i\right)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2\left(1 - 1/u_i\right)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

[Goncharov, Spradlin, Vergu, and Volovich, '10]

Motivation of Analytic Investigation

The magic simplification is deeply rooted in the hidden structures of MPLs: symbol letter and coaction (Hopf algebra).

Structure $\to$ Simplicity $\to$ Shortcuts

talk by Britto @ Amplitudes 2025

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The last two decades of precision predictions benefit a lot from knowledge and tools developed from the above.

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formal side: 8-loop form factor ($\mathcal{N} = 4$ sYM) [talk by Dixon @ Amplitudes 2025]

real world: 2-loop 6-point QCD + … [Yang's talk] [talk by Volovich @ Amplitudes 2025]

↑ cluster algebra

Motivation of Analytic Investigation

Structure → Simplicity → Shortcuts

talk by Britto @ Amplitudes 2025

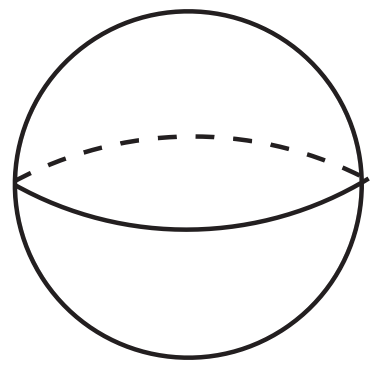
This is just the beginning of Feynman integral story.

Geometric Classification of FIs

Geometric Classification of FIs

MPL

$\Uparrow$



Riemann sphere, i.e.,
$g = 0$

last 2 decades,
most 2-loop
cases

packages for ε-factorized

form…

Geometric Classification of FIs

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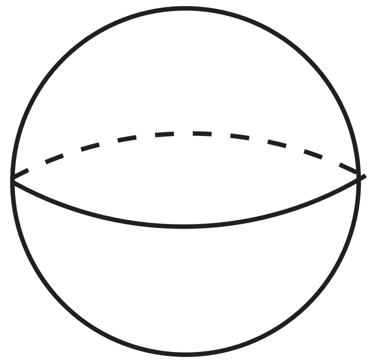

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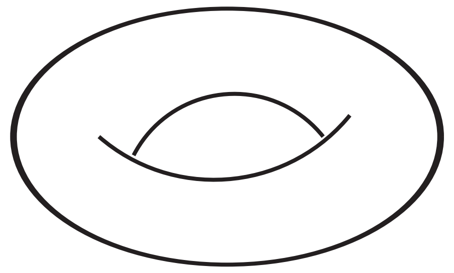

Riemann torus, i.e.,
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frontier during last
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Some 2-loop cases

some examples
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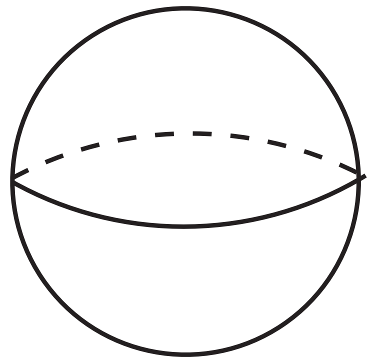

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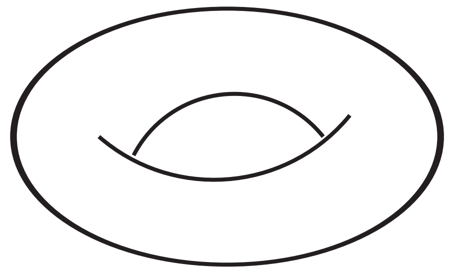

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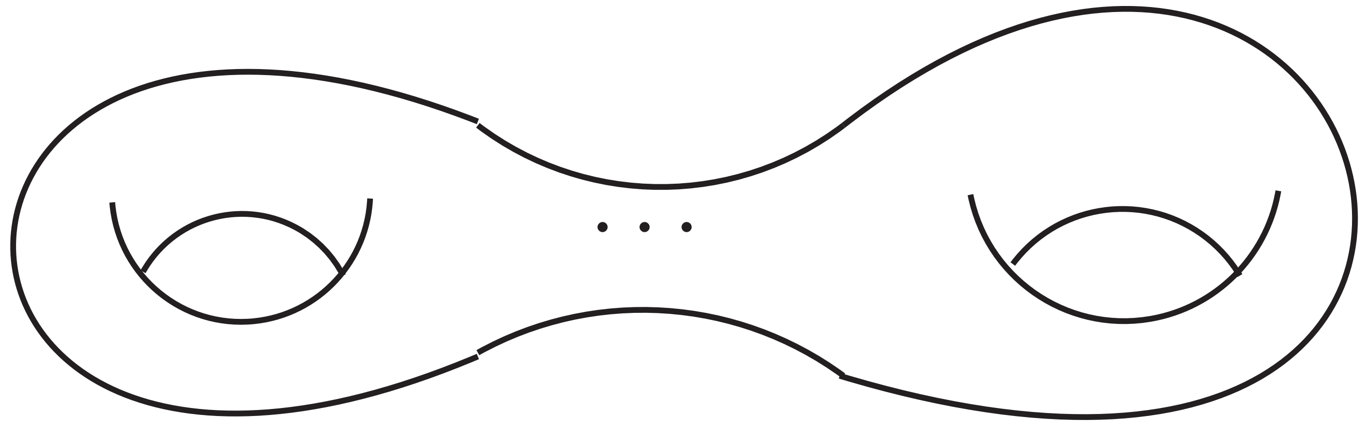
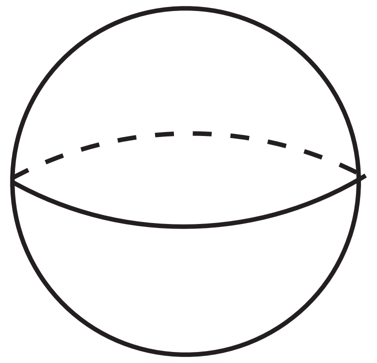


Riemann surfaces with $g \geq 2$

Geometric Classification of FIs

MPL

⇑



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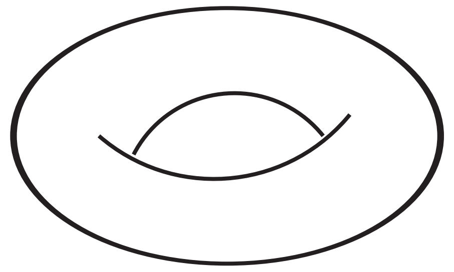

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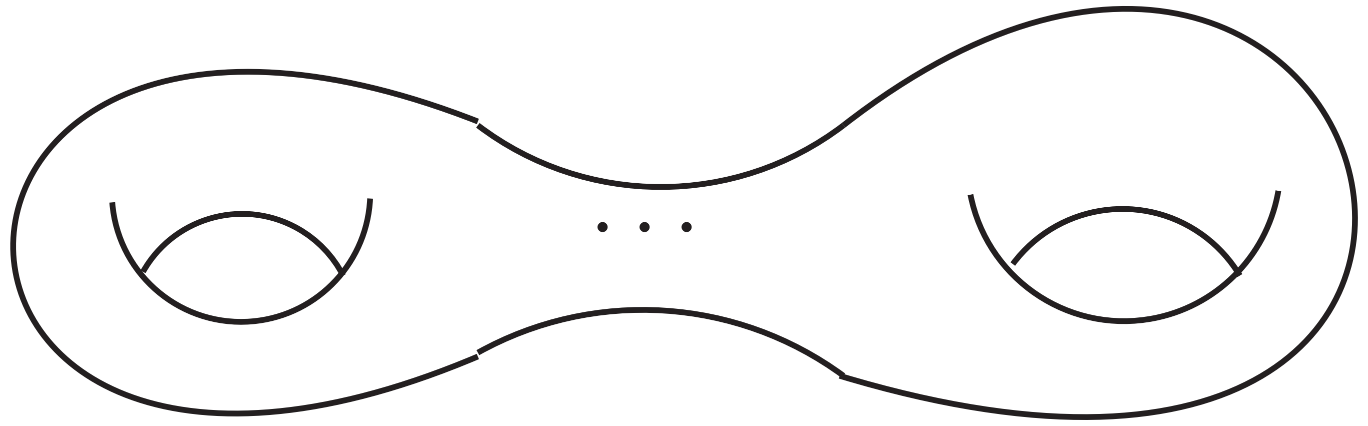
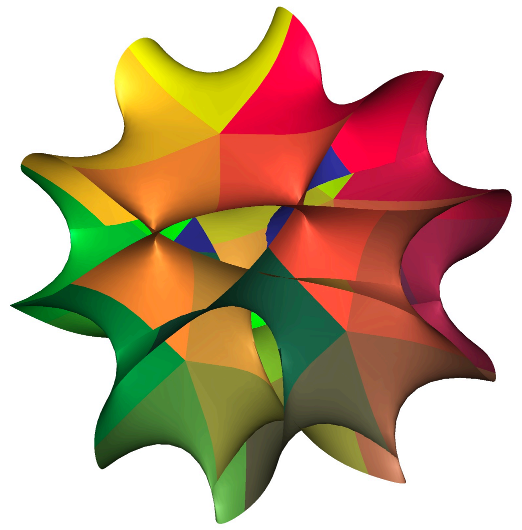
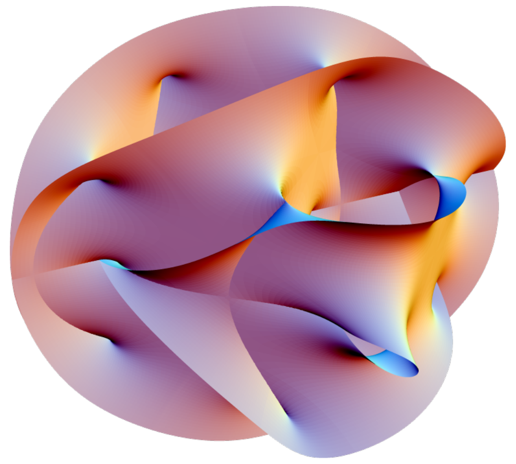


Riemann surfaces with $g \geq 2$

by dimension



Calabi-Yau 3 folds, K3



Calabi-Yau 4-folds

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banans, icecones…

Geometric Classification of FIs

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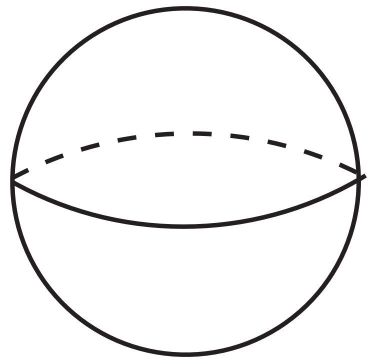

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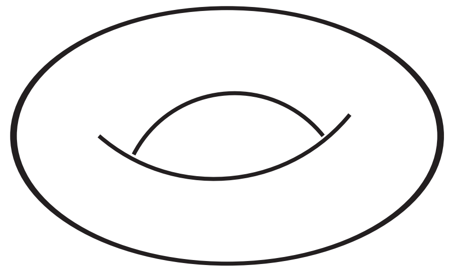

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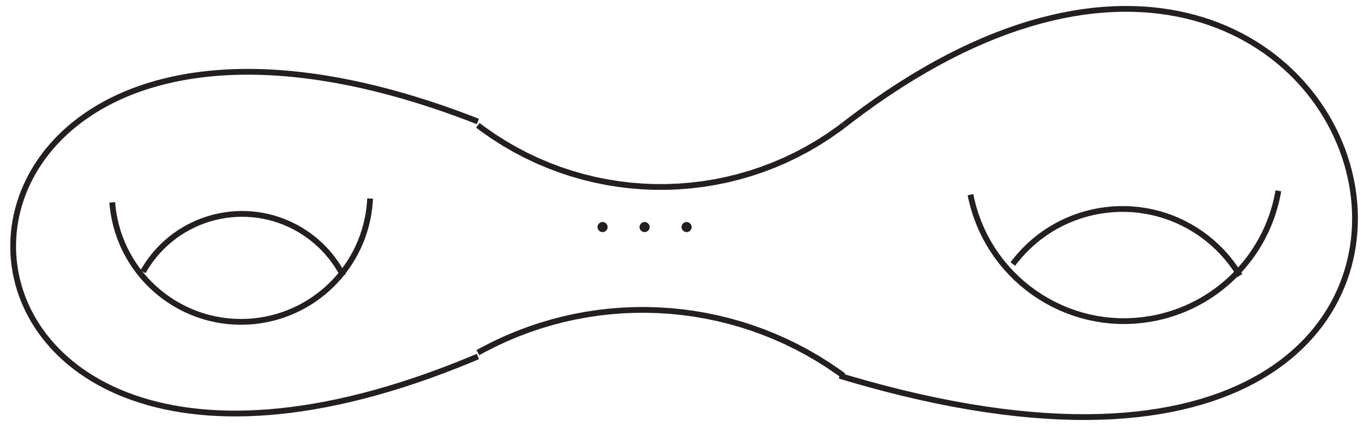

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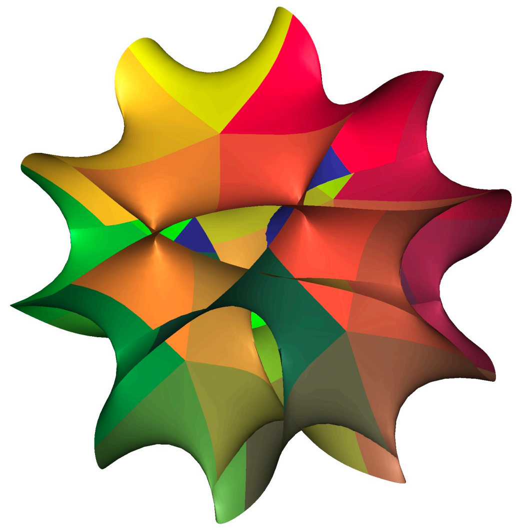

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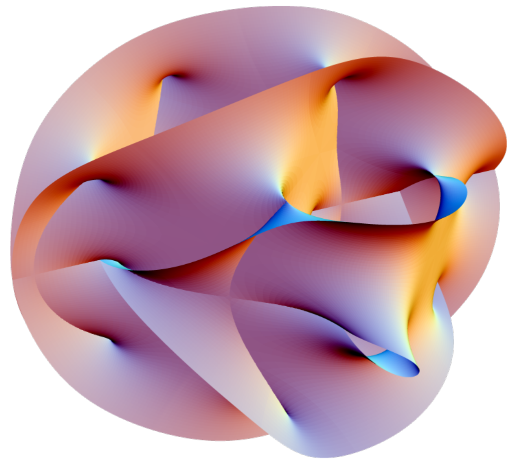

Calabi-Yau 4-folds

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[Snowmass 2021 review 2203.07088],

[a comprehensive book by Weinzierl, '22]

This Talk

Go beyond multiple polylogrithms, or equivalently, go beyond (puntured) Riemann sphere.

Structure → Simplicity → Shortcuts

→ talk by Britto @ Amplitudes 2025

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We have found a unified algorithm towards deriving ε-factorized form of any Feynman integral, inspired by Hodge theory.

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We have found a unified algorithm towards deriving ε-factorized form of any Feynman integral, inspired by Hodge theory.

Disclaimer: The algorithm reduce the complexity to a threshold, bounded by correspongding geometry, which is however inevitable!

This Talk

based on 2506.09124 + to appear by the ***ε*-collaboration**

Irís Brée

Federico Gasparotto

Antonela Matijašić

Pouria Mazloumi

Dmytro Melnichenko

Sebastian Pögel

Toni Teschke

Xing Wang

Stefan Weinziel

Konglong Wu

Xiaofeng Xu

Where are the geometric objects?

Parameter Representation of Feynman Integrals

$$\Gamma(\nu) \equiv \int_0^\infty \mathrm{d}\alpha\, \alpha^{\nu-1}\, e^{-\alpha} \rightsquigarrow \frac{1}{P^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{\nu-1}\, e^{-\alpha P}$$

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$$I_{\nu_1 \cdots \nu_n} \propto \int_{\mathbb{R}_{\geq 0}} \mathrm{d}^n\alpha \alpha^{\nu-1} \prod_{j=1}^{l} \int \frac{\mathrm{d}^D k_j}{i\pi^{D/2}} \exp\left(-\sum_{a=1}^{n} \alpha_a P_a \right)$$

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quadratic in momentua

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Schwinger rep.

$\mathcal{U}(\alpha)$: first Symanzik (homogeous) polynomial; $\mathcal{F}(\alpha;x)$: second Symanzik (homogeous) polynomial

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$$I_{\nu_1 \cdots \nu_n} = \frac{e^{l \gamma_E}}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_{\mathbb{R}_{\geq 0}} \mathrm{d}^n\alpha\, \alpha^{\nu-1} \left[\mathcal{U}(\alpha)\right]^{-\frac{D}{2}} \exp\left(-\frac{\mathcal{F}(\alpha; x)}{\mathcal{U}(\alpha)}\right)$$

Schwinger rep.

$$I_{\nu_1 \cdots \nu_n} = \frac{e^{l \gamma_E}\Gamma(\nu - lD/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_{\mathbb{R}_{\geq 0}} \mathrm{d}^n\alpha\, \alpha^{\nu-1}\, \delta\Big(1 - \sum_i \alpha_i\Big) \frac{\left[\mathcal{U}(\alpha)\right]^{\nu-(l+1)D/2}}{\left[\mathcal{F}(\alpha; x)\right]^{\nu-lD/2}}$$

Feynman rep.

$$I_{\nu_1 \cdots \nu_n} = \frac{e^{l \gamma_E}\Gamma\big(\nu - lD/2\big)}{\Gamma\big((l+1)D/2 - \nu\big)\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_{\mathbb{R}_{\geq 0}} \mathrm{d}^n\alpha\, \alpha^{\nu-1} \left[\mathcal{U}(\alpha) + \mathcal{F}(\alpha; x)\right]^{-D/2}$$

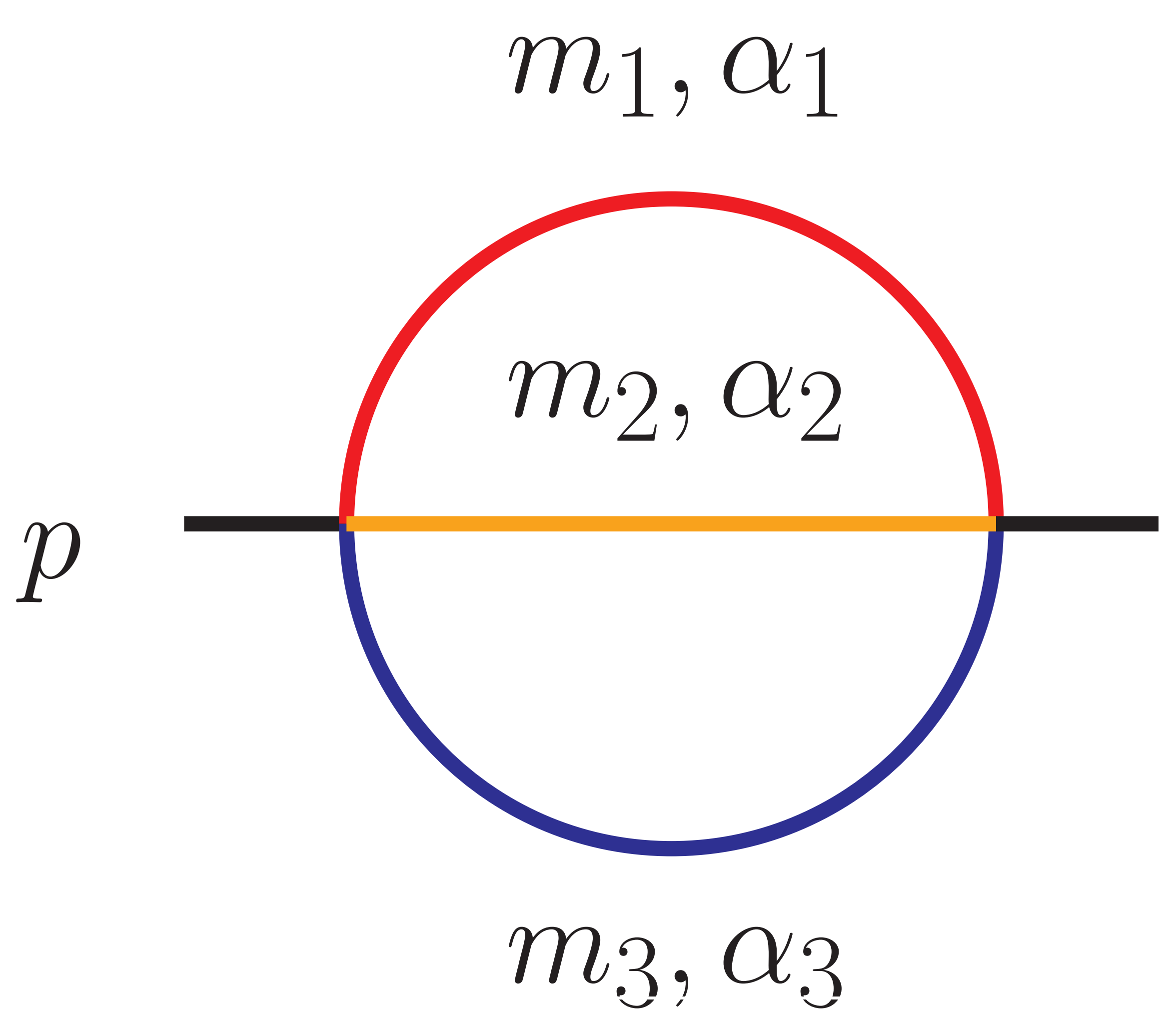
Lee-Pomeransky rep.

good for method of region [Beneke, Smirnov '91] in para. space [Smirnov; Gardi, Jones, Ma…]

Symanzik Polynomials

$$\mathcal{U}(\alpha) = \sum_{T\text{: spanning tree}} \prod_{e \notin T} \alpha_e$$

$$\mathcal{F}(\alpha; x) = \sum_{F\text{: spanning 2-forest}} \frac{-p_F^2}{\mu^2} \prod_{e \notin F} \alpha_e + \mathcal{U} \cdot \sum_{e=1}^{n} \alpha_e \frac{m_e^2}{\mu^2}$$

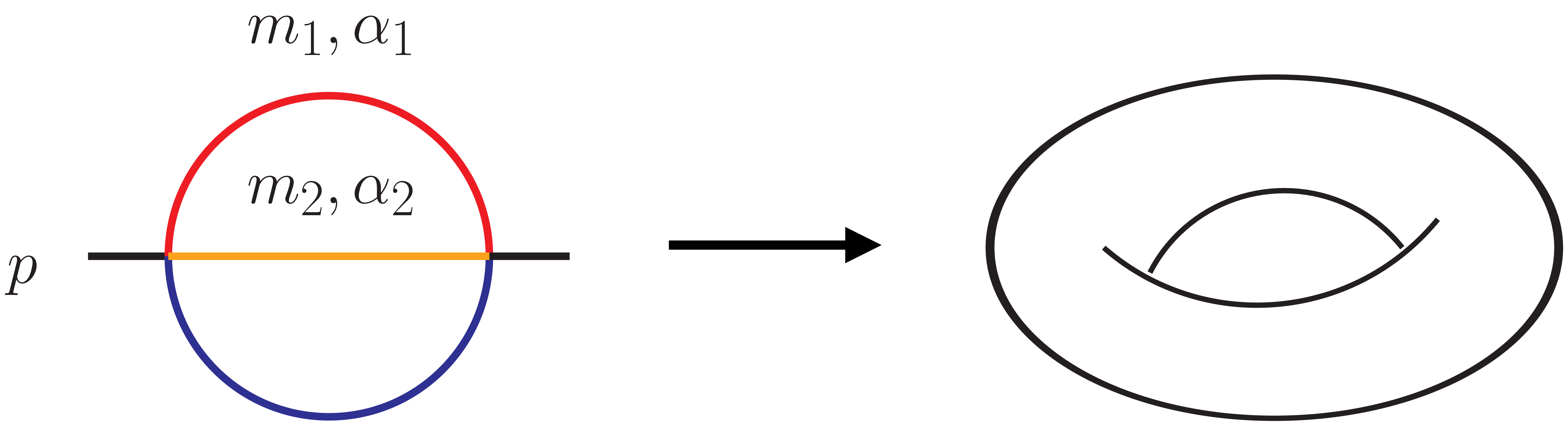


$$\mathcal{U}(\alpha) = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$$

$$\mathcal{F}(\alpha; x) = \frac{-s}{\mu^2}\alpha_1\alpha_2\alpha_3 + \mathcal{U}(\alpha)\left(\alpha_1\frac{m_1^2}{\mu^2} + \alpha_2\frac{m_2^2}{\mu^2} + \alpha_3\frac{m_3^2}{\mu^2}\right)$$

Geometry by Symanzik Polynomials

$$Y(x) = \left\{ [\alpha_1 : \alpha_2 : \alpha_3 : \cdots : \alpha_n] \,\middle|\, \mathscr{F}(\alpha; x) = 0 \right\} \subset \mathbb{CP}^{n-1}$$



torus

$$Y_{\text{sunrise}}(x_1, x_2, x_3) = \left\{ [\alpha_1 : \alpha_2 : \alpha_3] \,\middle|\, \alpha_1\alpha_2\alpha_3 = (\alpha_1 + \alpha_2 + \alpha_3) \sum_{j=1}^{3} \alpha_i x_i \right\} \subset \mathbb{CP}^2$$

Why Geometry Matters

$$I_{111}^{\text{sunrise}}\Big|_{D=2-2\varepsilon} = \frac{e^{2\gamma_E}\Gamma(3-D)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_{\mathbb{R}_{\geq 0}} \mathrm{d}^3\alpha \, \frac{\delta(1-\alpha_3)}{\left[\mathcal{U}(\alpha)\right]^{-3\varepsilon} \underline{\left[\mathcal{F}(\alpha;x)\right]}^{1+\varepsilon}}$$

The most important contribution comes from the variety (zero set, i.e., torus)!

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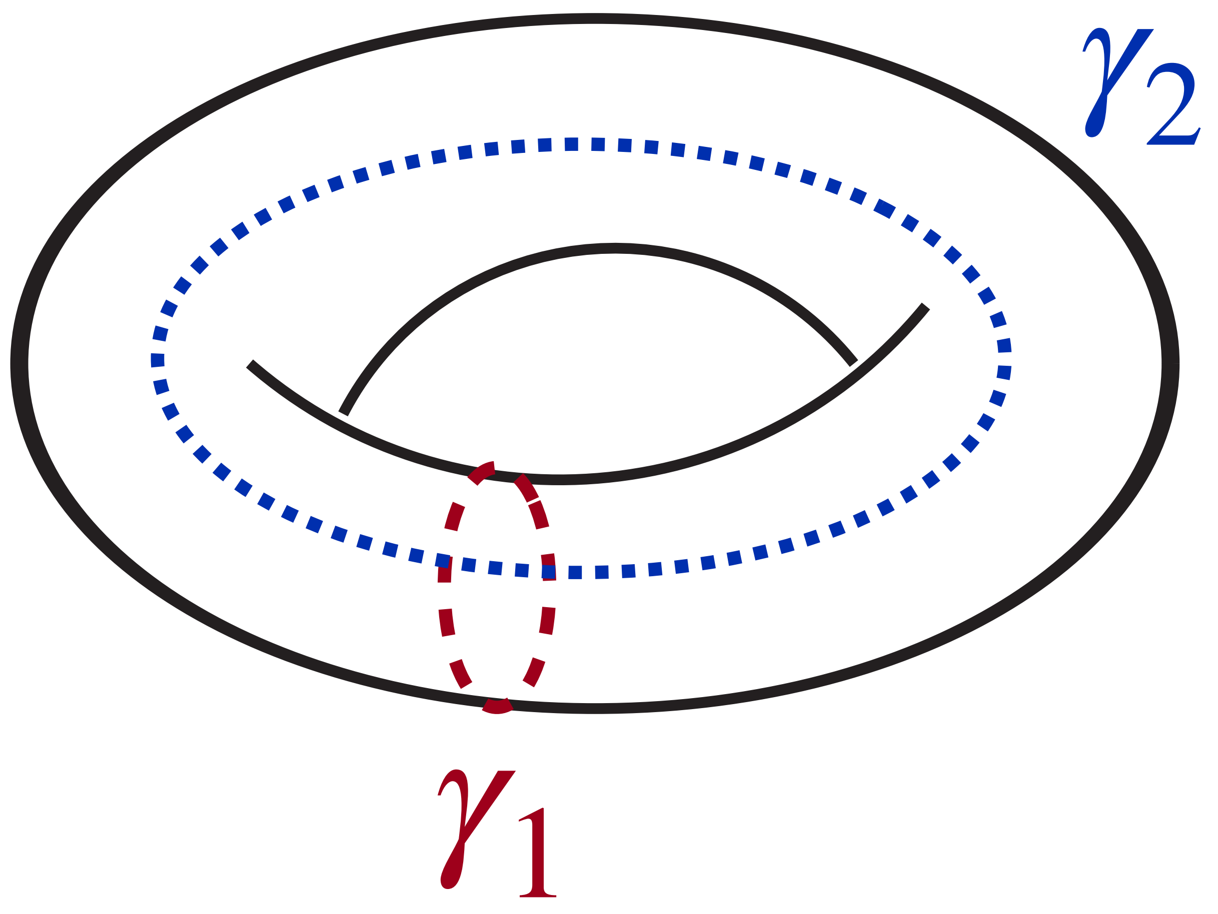

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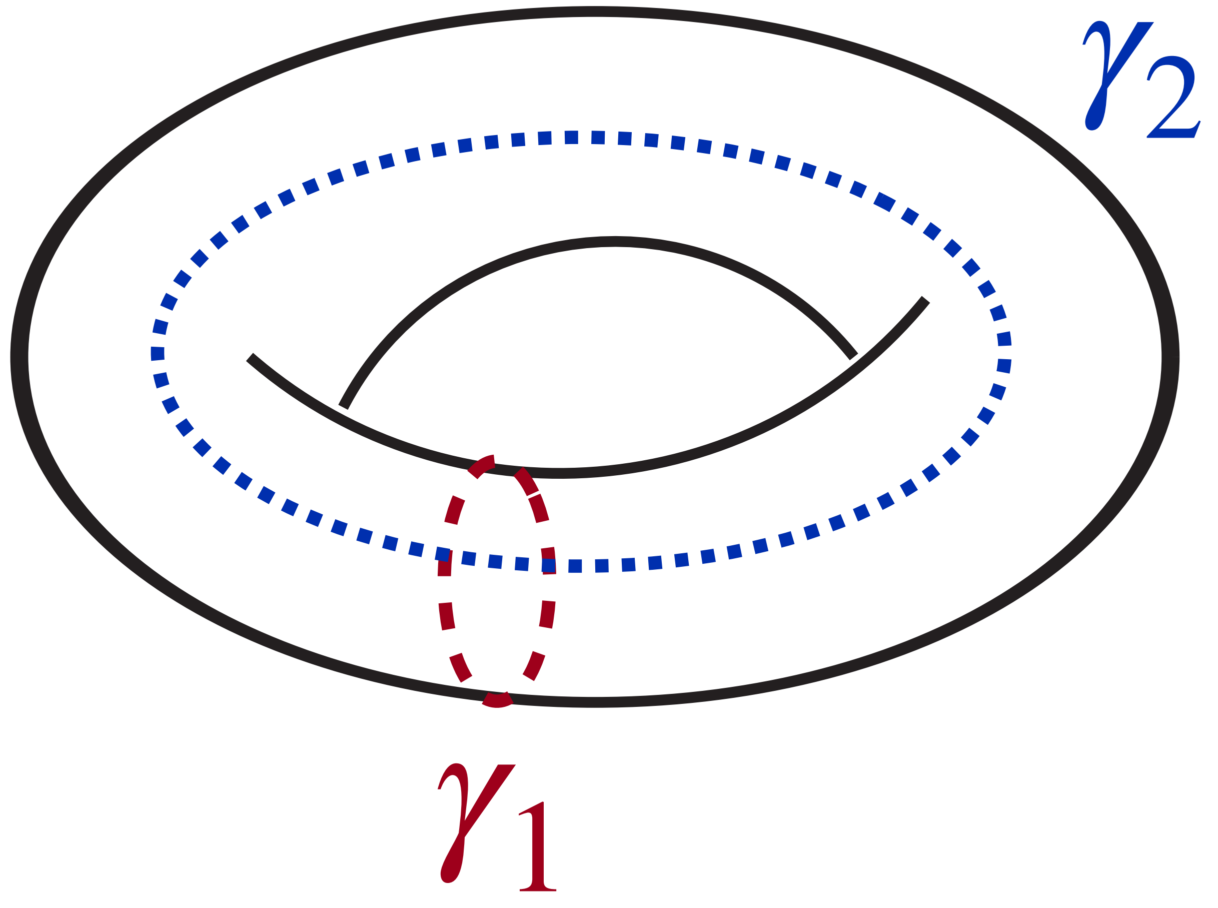

$$I_{111}^{\text{sunrise}}\Big|_{D=2} = c_1\int_{\gamma_1}\omega + c_2\int_{\gamma_2}\omega$$

$\int_{\gamma_i}\omega_j$'s are called periods of such a geometry.

This is a generic pattern, and that is why Feynman integrals also interest mathematicians

Baikov Representation of Feynman Integrals

- Treat propagators, P_i's, as integration variables: $z_i = P_i/\mu^2$; Hjalte's talk!
  In DR, one need to introduce extra variables to match #d.o.f's, leading to some non-trivial "Jacobian":

$$I_{\nu_1\cdots\nu_n} = \text{const} \times \int_{\mathscr{C}} \underbrace{\left[\mathscr{B}(z;x)\right]^{\gamma}}_{u(z)} \frac{1}{z_1^{\nu_1} z_2^{\nu_2} \cdots z_n^{\nu_n}} \mathrm{d}^n z$$

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- Baikov rep. is not for calculating FIs, but rather to study the structures therein!
  Packages.: [***Baikovletter***, Jiang, Yang; ***BaikovPackage***, Hjalte; ***SOFIA***, Correia, Giroux, Mizera]

- It translates FIs to twisted cohomology.

- This representation is perfect for cuts: just taking residues explicitly: $\text{cut}_i = \text{res}_{z_i=0}$.

From now on, we focus on the maximal cut. Since it is most relevant for analytical difficulty.

Example: Sunrise

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It has three propagators: $\{z_1, z_2, z_3\}$. The loop-by-loop Baikov rep. needs an auxiliary variable, z_4. Hence its full Baikov rep. reads:

$$I_{111}^{\text{sunrise}} \sim \int_{\mathscr{C}} u(z_1, z_2, z_3; z_4) \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4}{z_1\, z_2\, z_3}$$

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The (non-trivial) geometry is dictated by the maximal cut. So we have:

$$I_{111,\text{MC}}^{\text{sunrise}} = \frac{\pi^3 e^{2\gamma_E}}{\Gamma^2(1/2-\varepsilon)} \int_{\mathscr{C}_{\text{MC}}} \underbrace{\big[P_1(z_4)\big]^{\varepsilon} \big[P_2(z_4)\big]^{-1/2-\varepsilon} \big[P_3(z_4)\big]^{-1/2-\varepsilon}}_{u(z_4)} \mathrm{d}z_4$$

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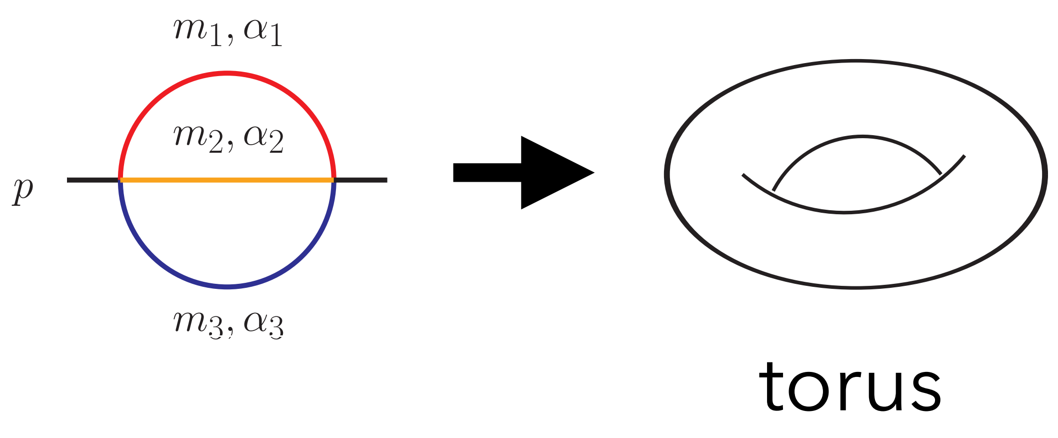


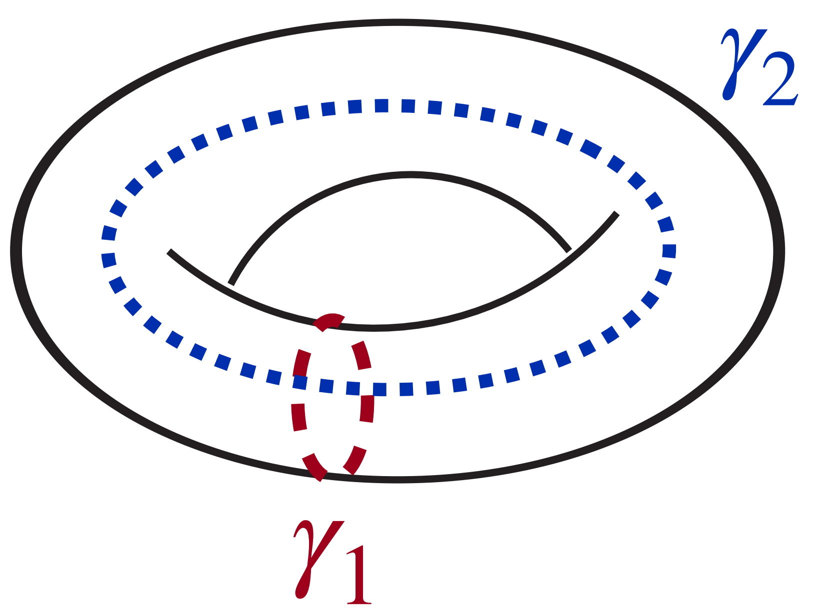
Where is (Algebraic) Geometry from Baikov?

Here, the pure geometry or the leading geometry is determined by taking $\varepsilon = 0$:

$$I^{\text{sunrise}}_{111,\text{MC}} \sim \int \frac{\mathrm{d}z_4}{\left[P_2(z_4)\,P_3(z_4)\right]^{1/2}} = \int \frac{\mathrm{d}z_4}{w_4} = \int \frac{\mathrm{d}z}{w}$$

$$\mathscr{E} = \left\{ (z_4, w_4) \in \mathbb{C}^2 \,\middle|\, w_4^2 = \prod_{i=1}^{4} \big(z_4 - a_4(x)\big) \right\}$$

$$\simeq \left\{ (z, w) \in \mathbb{C}^2 \,\middle|\, w^2 = z\,(z-1)\,\big(z - \lambda(x)\big) \right\}$$



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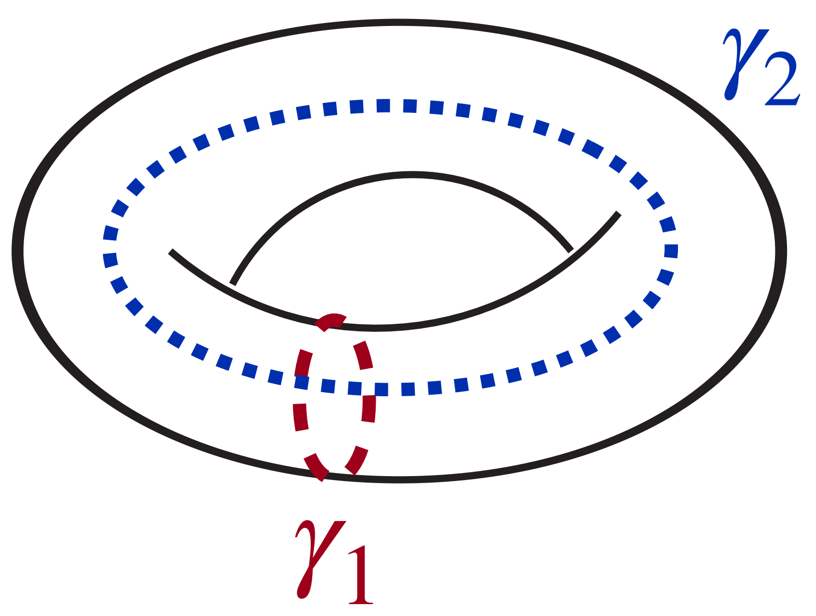

To rationlize the square root, elliptic functions are inevitable: $z = \wp(\xi), w = \wp'(\xi)$, then:

$$\left(\wp'(\xi)\right)^2 = \wp(\xi)\left(\wp(\xi) - 1\right)\left(\wp(\xi) - \lambda(x)\right)$$

$\dfrac{\mathrm{d}z}{w} = \mathrm{d}\xi$. ξ is the coordinate on the torus.

Setup

In a given sector, MIs' Baikov representations share the same twist ($b_i, b_j \in \mathbb{Z}$):

$$u(z_1, z_2, \cdots, z_{N_V}) = \prod_{i \in I_{\text{odd}}} \left[p_i(z)\right]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j \in I_{\text{even}}} \left[p_j(z)\right]^{\frac{1}{2}b_j\varepsilon}$$

- ▸ Odd polynomials $\longrightarrow$ pure geometry;
- ▸ Even polynomials $\longrightarrow$ possible punctures or marked points in the manifold;
- ▸ The Baikov rep.'s of different MIs has different rational parts:

$$M_i = C_{\text{Baikov}} \int_{\text{MC}} u(z) \underbrace{\frac{q_i(z)}{\prod_{j \in \text{all}} \left[p_j(z)\right]^{\mu_j}} \mathrm{d}z_{N_V} \wedge \cdots \wedge \mathrm{d}z_1}_{\phi_i}, \qquad \mu_j \in \mathbb{Z}$$

Setup

- Given a M_i, there is a differential form ϕ_i ($H_{\omega}^{N_V} \twoheadrightarrow V^{N_V}$);
- The revese is not quite correct: one needs to modulo IBP relations.
  $\hookrightarrow \phi_i \sim \phi_i + \nabla_u \eta$, which leads us to the twisted cohomology. [Hjalte's talk!]

- From now on, we can study the differential forms to represent the corresponding MIs;
- Besides, it is helpful to consider everything in the projective space, which takes infinite into account naturally.

$$u\left(z_1, z_2, \cdots, z_{N_V}\right) = \prod_{i \in I_{\text{odd}}} \left[p_i(z)\right]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j \in I_{\text{even}}} \left[p_j(z)\right]^{\frac{1}{2}b_j\varepsilon}$$

$$\hookrightarrow U\left(z_0, z_1, z_2, \cdots, z_{N_V}\right) = \prod_{i \in I_{\text{odd}}} \left[P_i(z_0, z)\right]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j \in I_{\text{even}}} \left[P_j(z_0, z)\right]^{\frac{1}{2}b_j\varepsilon} \cdot z_0^{\frac{1}{2}b_0\varepsilon}$$

Setup

From now on, we focus on the linear space $H_\omega^{N_V}$ (twisted cohomology) made of differential forms, classify them by some criteria and them translate back to FIs;

$$H_\omega^{N_V} = \left\{ \Psi_{\mu_0 \ldots \mu_{N_D}}[Q] = C_\varepsilon(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[Q]\, \eta \right\} \text{ modulo IBPs}$$

$$C_\varepsilon = \underbrace{\varepsilon^{-|\mu|}}_{C_{\text{clutch}}} \times \underbrace{\prod_{i \in I_{\text{odd}}} \left(-\frac{1}{2} + \frac{1}{2} b_i \varepsilon\right)_{\mu_i} \prod_{i \in I_{\text{even}}} \left(\frac{1}{2} b_i \varepsilon\right)_{\mu_i}}_{C_{\text{rel}}} \qquad (a)_n = \frac{\Gamma(a+1)}{\Gamma(a+1-n)}$$

We define two more ordering numbers on top of Laporta's algorithm: pole order o, and the number of non-zero residues r of $\Psi_{\mu_0 \cdots \mu_{N_D}}[Q]$.

$$p = N_V - o + r; \qquad q = o; \qquad w = p + q = N_V + r\,.$$

Setup

$$p = N_V - o + r; \qquad q = o; \qquad w = p + q = N_V + r\,.$$

- Within a sector, these numbers organize complexity of MIs.
- Our algorithm is we start from the highest w (weight), and then scan elements in $H_{\omega}^{N_V}$ by increasingthe pole order o. After we finish this weight, we minus the weight by 1 and repeat scanning the pole order.

In math terminlogy, this layered decomposition is called **filtration**, which is used in Hodge theory.

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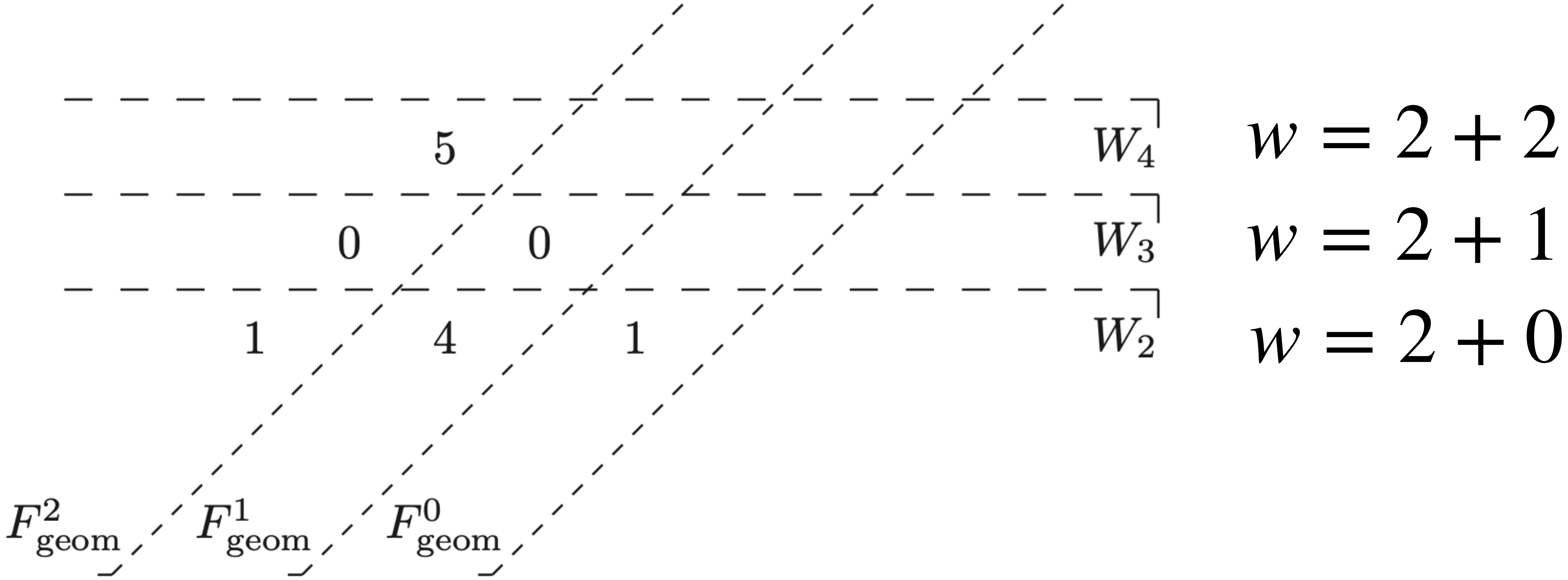

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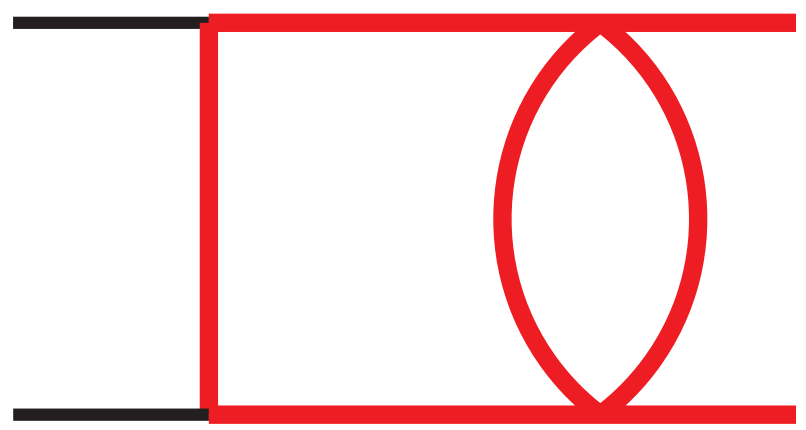
Setup

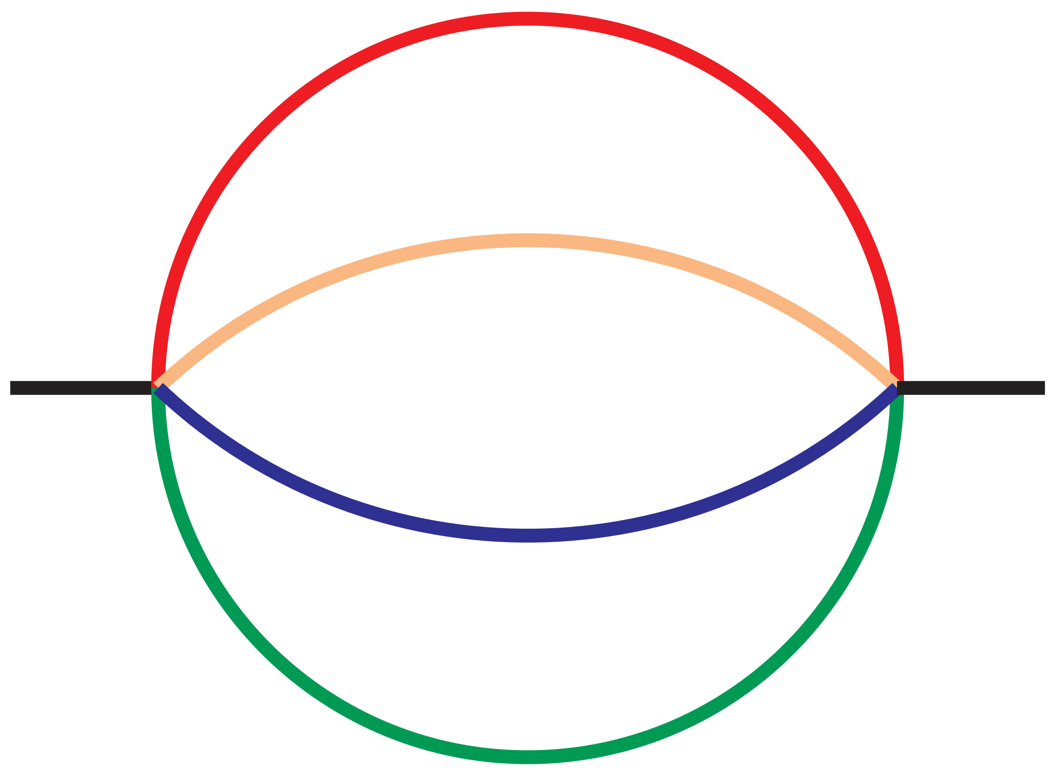
- With the filtration, we group the elements of H^{N_V} and translate back to MIs. DEQ is not ε-factorized;
- But it is in a good block lower-triangular and looks like: $A(\varepsilon; x) = \sum_{i=-N_V}^{1} \varepsilon^i A^{(i)}(x)$
- Can always remove the unwanted ε^i for $i \leq 0$, in a bottom-up way!
- Solving the constraints turns out to be equivalent to using the periods information;

- The filtration algorithm reduces the problem to the "naked" level, which is related to the geometry itself and inevitable;
- The filtration algorithm does not specify the geometry!

Two examples



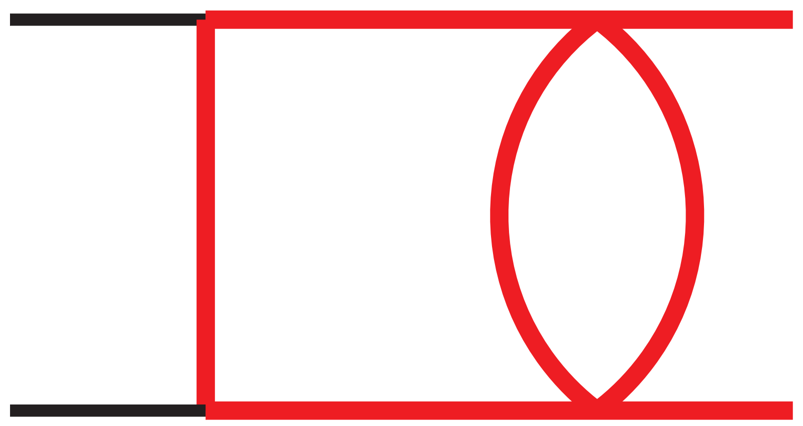
elliptic



Calabi-Yau 2-folds

lots of highly non-trivial examples worked out and in preparation…

Canonicalizing Elliptics: A New Perspective



Kira reports 3 master integrals in the top sector.

$$I_{11121,\mathrm{MC}} = C_{\mathrm{Baikov}} \int_{\mathscr{C}_{\mathrm{MC}}} \frac{\mathrm{d}z_1}{2\pi i} \left[p_1(z_1)\right]^{-\frac{1}{2}} \left[p_2(z_1)\right]^{-\frac{1}{2}-\varepsilon} \left[p_3(z_1)\right]^{-\frac{1}{2}-\varepsilon}, \quad \text{with}$$

$$p_1 = z_1 - x_2,\ p_2 = z_1 + 4 - x_2, \quad p_3 = \left(z_1 + 1\right)^2 - 4\left[x_2 + \frac{\left(1 - x_2\right)^2}{x_1}\right].$$

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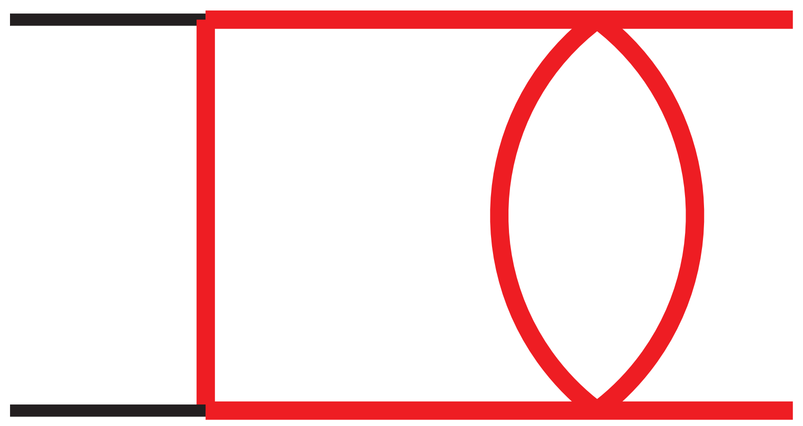

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All three polynomials are odd. After homogenization, there is one even polynomial: $P_0 = z_0$.

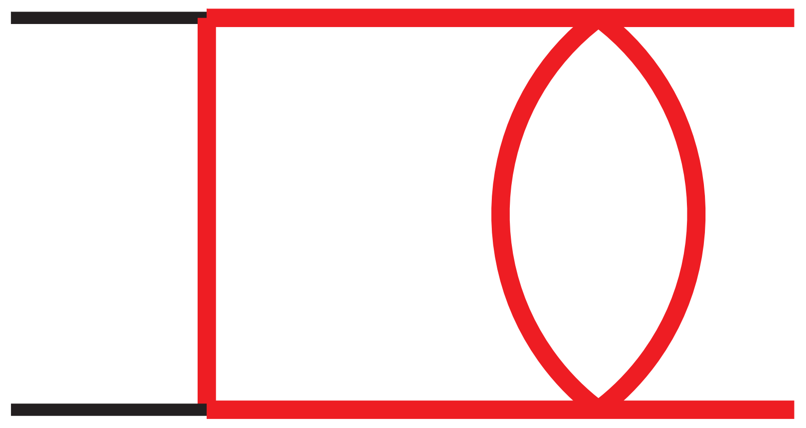
$$U\left(z_0, z_1\right) = P_0^{3\varepsilon} P_1^{-\frac{1}{2}} P_2^{-\frac{1}{2}-\varepsilon} P_3^{-\frac{1}{2}-\varepsilon}$$

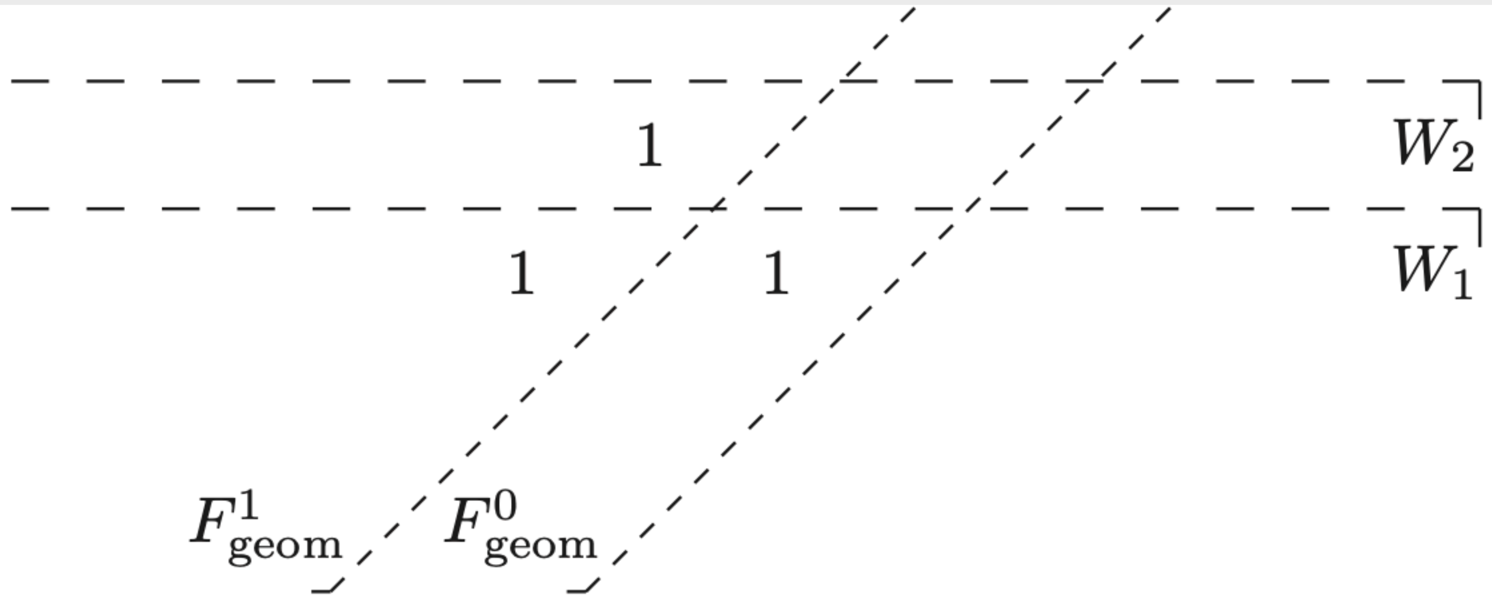
And the element in H^1_ω looks like: $\Psi_{\mu_0\ldots\mu_3}[Q] = C_{\mathrm{clutch}} C_{\mathrm{rel}}(\{\mu\})\, U(z)\, \hat{\Phi}_{\mu_0\ldots\mu_3}[Q]\, \eta$.

- U is of homogeneous degree -2, while η is of homogeneous degree $+2$. Hence $\hat{\Phi}$ should be of homegeneous degree 0.

Canonicalizing Elliptics: A New Perspective

- $w = 1 + 1 = 2$: There is only one even polynomial $P_0 = z_0$, which we can take a non-zero residue at.
  Thus, we consider $\hat{\Phi}_{1000}[z_1] = z_1/z_0$. By definition, we can read off:
  $C_{\text{clutch}} = \varepsilon^{-1}$ and $C_{\text{rel}} = 3\varepsilon$. Then $\Psi_{1000}[z_1]$ has pole order $o = 1$, and $r = 1$;
- $w = 1 + 0 = 1$: 1) The trivial one is $\hat{\Phi}_{0000}[1] = 1$, which is holomorphic. Accordingly, $C_{\text{clutch}} = C_{\text{rel}} = 1$. In this case, $\Psi_{0000}[1]$ has no pole, and hence no non-zero residue to take. We assign it pole order $o = 0$, and $r = 0$; 2) The next one should have pole order 1, but has not residue. There are some three equally good candidates. Here we choose $\hat{\Phi}_{0100}[z_0] = z_0/P_1$. Thus we have $C_{\text{clutch}} = \varepsilon^{-1}$ and $C_{\text{rel}} = -1/2$.





Canonicalizing Elliptics: A New Perspective

- Now, we can map the filtrated H^1 to the Feynman integral side:

$$M_1 = j\left(\Psi_{0000}[1]\right) = \varepsilon^3 x_1 I_{111200100}$$

$$M_2 = j\left(\Psi_{1000}[z_1]\right) = 3\varepsilon^3 x_1 I_{11120001(-1)0}$$

$$M_3 = j\left(\Psi_{0100}[z_0]\right) = -\frac{1}{2}\varepsilon^2 x_1 \left[c_1 I_{111200100} + c_2 I_{11120001(-1)0} + c_3 I_{21120001(-1)0}\right]$$

- DEQ of the above basis is good enough. Then after solving some constraints. We find

$$K_1 = \frac{\varepsilon^3 x_1}{R_{11}^{(-1)}} I_{111200100},$$

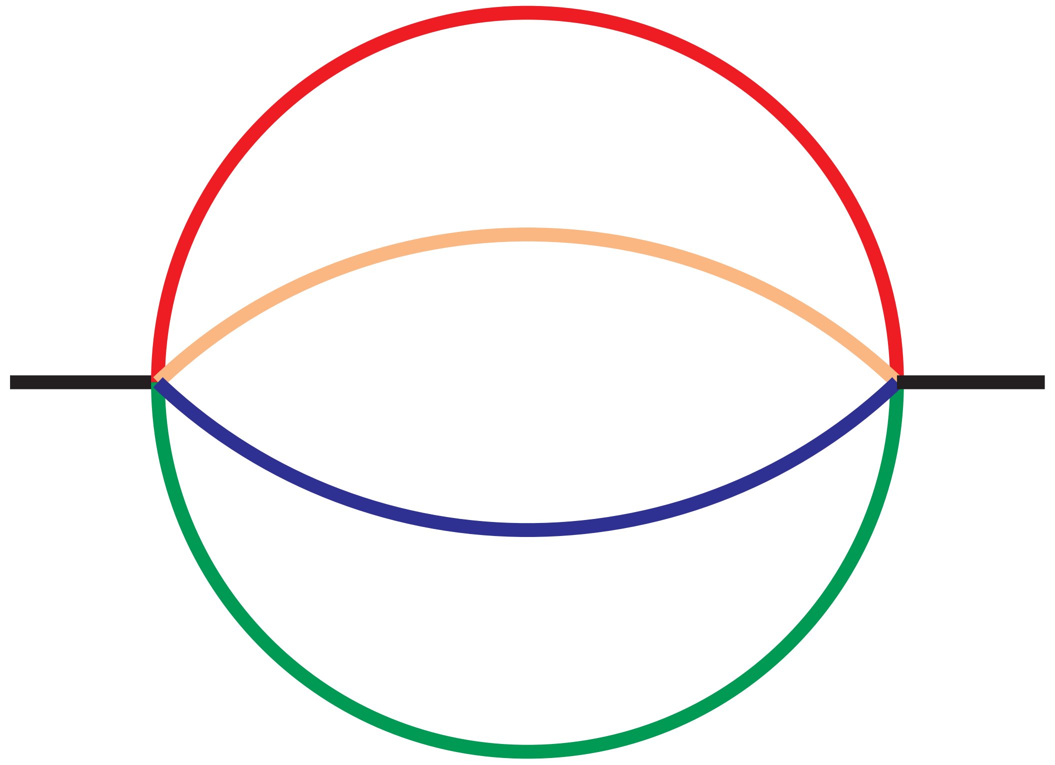
$$K_2 = 3\varepsilon^3 x_1 I_{1112001(-1)0} - R_{21}^{(0)} K_1$$

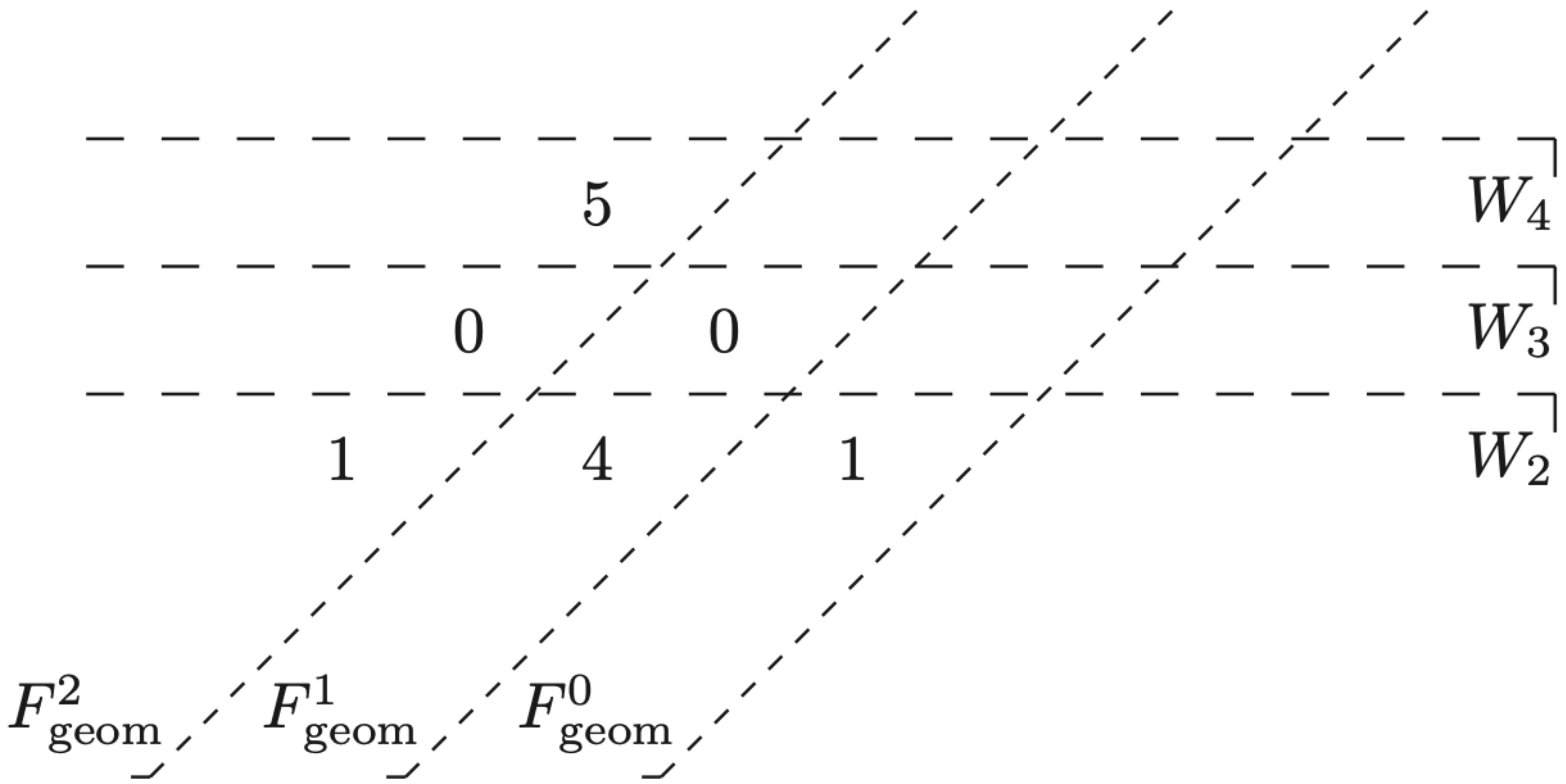
$$K_3 = \cdots$$

$\longrightarrow$ **canonical!**

K3 Example

$$u = z_5^{\varepsilon} z_6^{\varepsilon} \left[z_6^2 + 2\left(x_1 + 1\right) z_6 + \left(x_1 - 1\right)^2\right]^{-\frac{1}{2}-\varepsilon} \left[\left(z_6 - z_5\right)^2 + 2x_2\left(z_5 + z_6\right) + x_2^2\right]^{-\frac{1}{2}-\varepsilon}$$
$$\left[z_5^2 + 2\left(x_3 + x_4\right) z_5 + \left(x_4 - x_3\right)^2\right]^{-\frac{1}{2}-\varepsilon}$$


Calabi-Yau 2-folds



This algorithm brings the maximally unequal-mass banana down to its geometric essence⟶the K3 surface.

Outlook