Holographic Correlators in a Nutshell

袁 野

浙江大学 ZHEJIANG UNIVERSITY

第十四届新物理研讨会

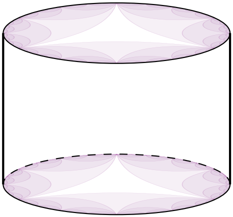
山东大学 2025.07.16

AdS/CFT correspondence

Holographic description of quantum gravity in asymptotic AdS.

- **Top-down -** Models realized in string theory / M-theory.

  [Maldacena, '97][Gubser, Klebanov, Polyakov,'98][Witten, '98] ...



  - $AdS_5 \times S^5$ ($\mathcal{N} = 4$ SYM), $AdS_4 \times S^7$ (ABJM), etc.

- **Bottom-up -** What CFTs have a weakly-coupled Einstein gravity dual? [Heemskerk et al, '09][Camanho et al, '14] ...
  - Large N – weakly coupled.
  - Large mass gap – for the EH action to exist at low energy.

What to do with holography?

- [Bulk] Properties of quantum gravity?
  [Boundary] Get help from ordinary quantum fields.
  The most useful UV-complete description for quantum gravity so far.

- [Boundary] Properties of strongly-coupled CFTs?
  [Bulk] Intuitions from perturbative expansion.
  Tools to probe physics in "non-perturbative" region of QFT.

Local operators & their bulk dual

- CFT: $\underbrace{\text{local operators}}_{\mathcal{O}_{\Delta,\ell,r}(x)} + \underbrace{\text{OPE coefficients}}_{C_{\mathcal{O}\mathcal{O}'\mathcal{O}''}} + \cdots$

  Δ: conformal dimension, ℓ: spin, ($\tau = \Delta - \ell$: twist)
  r: internal quantum numbers.
- Spectrum

$$\Delta \bullet\!\!\longrightarrow$$

$O(N^0)$ light particles — $O(N^1)$ — $O(N^2)$ back-reaction to geometry

- Light particles
  - Quantization of low-energy effective theory.
  - Scalar: $m^2_{\text{AdS}} R^2_{\text{AdS}} = \Delta(\Delta - d)$
- Need (extended) susy for the mass not to renormalize.
  - R-symmetry $\Leftrightarrow$ AdS $\times$ **S**.

Protected operators

- **Example:** $\mathcal{N}=4$ SYM contains six scalars ϕ^I. Single trace operators (with an R-symmetry polarization y_I)

$$\mathcal{O}_p(x) = y_{I_1} y_{I_2} \cdots y_{I_p}\ \mathrm{tr}\big(\phi^{I_1}(x)\phi^{I_2}(x)\cdots\phi^{I_p}(x)\big)$$

  - $y \cdot y = 0$; traceless symmetric.
  - Preserve half susy (i.e., half-BPS). $\Delta = p$, $\ell = 0$.
  - When $N \to \infty$, it corresponds to a scalar particle in AdS. $\mathcal{O}_2$: super-primary of the stress-tensor multiplet (supergraviton).
- Intuition from the perturbative bulk:

state	1-particle	2-particle	3-particle	$\cdots$
operator	$\mathcal{O}$	$[\mathcal{O}\partial^{\#}\mathcal{O}]$	$[\mathcal{O}\partial^{\#}\mathcal{O}\partial^{\#}\mathcal{O}]$	$\cdots$

  Multi-trace operator: generically not protected.
- Abundant degeneracy, e.g.,

$$[\mathcal{O}_p\partial\mathcal{O}_2], \quad [\mathcal{O}_{p-1}\partial\mathcal{O}_3], \quad [\mathcal{O}_{p-2}\partial\mathcal{O}_4], \quad \ldots$$

  $\tau = p + 2 + O(N^{-2})$, $\ell = 1$, and the same R structure..

from now on, we consider

single-trace half-BPS operators

$$\langle \mathcal{O}_{p_1}(x_1)\mathcal{O}_{p_2}(x_2)\cdots\mathcal{O}_{p_n}(x_n)\rangle$$

general structure of holographic correlators

Superconformal constraints

Four points

- Kinematics ($x_{ij}^2 = (x_i - x_j)^2$, $y_{ij} = yi \cdot y_j$)

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad \sigma = \frac{y_{12} y_{34}}{y_{13} y_{24}}, \quad \tau = \frac{y_{14} y_{23}}{y_{13} y_{24}},$$
$$U = z\bar{z}, \quad V = (1-z)(1-\bar{z}), \quad \sigma = \alpha\bar{\alpha}, \quad \tau = (1-\alpha)(1-\bar{\alpha}).$$

- Solution to superconformal Ward identity

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \#\big(\mathcal{G}_{\text{free}} + \underbrace{\frac{(z-\alpha)(z-\bar{\alpha}(\bar{z}-\alpha)(\bar{z}-\bar{\alpha}))}{(z\bar{z}\alpha\bar{\alpha})^2}}_{\mathcal{I}} \mathcal{H}\big)$$

  $\mathcal{H}$: reduced correlator. [Dolan, Gallot, Sokatchev, '04]

- Similar decomposition in other models.

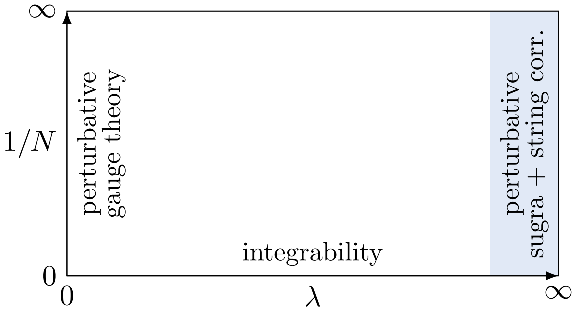
Higher points

- Can classify various R invariants. [c.f., Heslop, '22]
- No canonical decomposition.

Parameter regions

- Parameters

$$\lambda \propto \frac{R_{\text{AdS}}^2}{l_{\text{string}}^2}, \qquad \frac{\lambda}{N} \propto g_{\text{string}}.$$



- Region of interest in this talk: $1 \ll \lambda < N$.

Boundary view: (super-) block decomposition

- Ordinary conformal block expansion ($\tau = \Delta - \ell$)

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\tau,\ell,r} a_{\tau,\ell,r} g_{\tau,\ell}(z,\bar{z}).$$

- Super conformal block expansion

$$\mathcal{G}_{2222} = ((\text{protected) short}) + \sum_{\tau,\ell,m,n} \underbrace{\mathcal{A}_{\tau,\ell,m,n}}_{\text{(unprotected) long}} \quad .$$

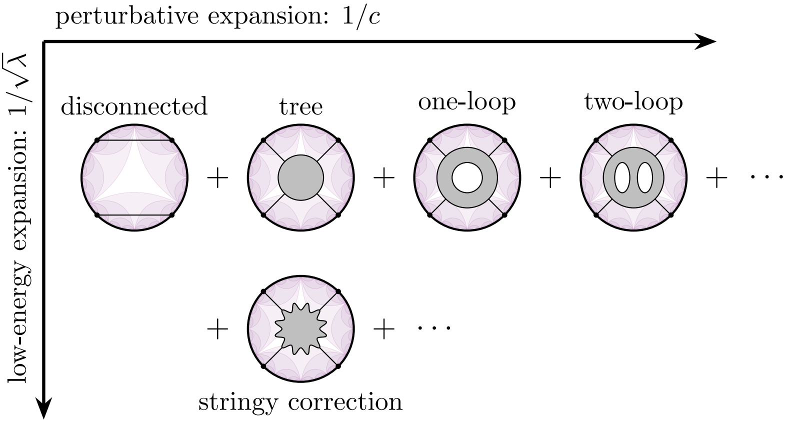
$\mathcal{G}_{\text{free}}$		$\mathcal{H}$
short		long

- $\mathcal{H}$ is bosonic

$$\mathcal{H} = \sum_{\tau,\ell} a_{\tau,\ell} g_{\tau+4,\ell}(z,\bar{z})$$

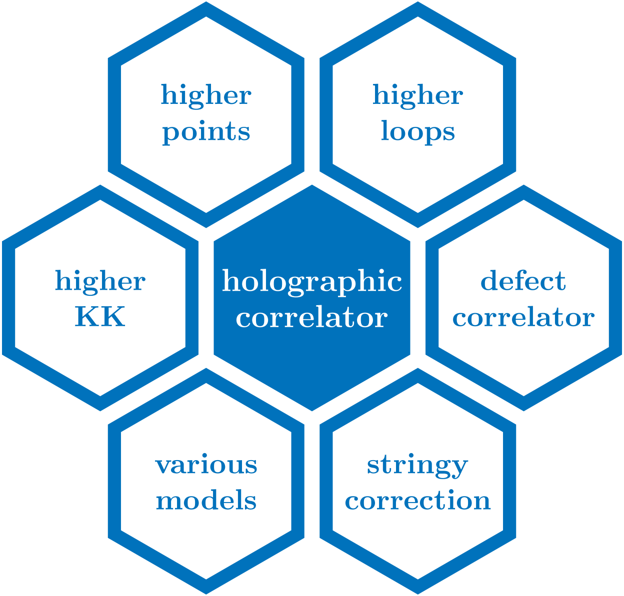
Since $\mathcal{A}_{\tau,\ell,m,n} \propto g_{\tau+4,\ell}$, the above "spectrum" directly reflects th spectrum of long operators.

Bulk view: low energy & perturbative expansion



central charge: $c = \dfrac{N^2 - 1}{2}$

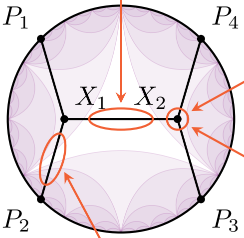
Main topics of interests



traditional computations

Traditional method: Witten diagrams

$$\Pi_{bb}(X_1, X_2) \propto \zeta^{-\Delta}\,{}_2F_1\Big(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}, 2\Delta - d + 1, -\frac{4}{\zeta}\Big)$$



read from the (effective Lagrangian)

$$\int_{\mathrm{AdS}} \mathrm{d}X$$

$$\Pi_{b\partial}(X, P) \propto \frac{1}{(-2X \cdot P)^{\Delta}}$$

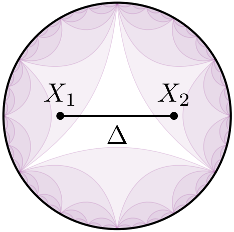
embedding formalism

bulk: $X^2 = -1$, $X = \frac{1}{z}(1, z^2 + x^2, x^\mu)$.

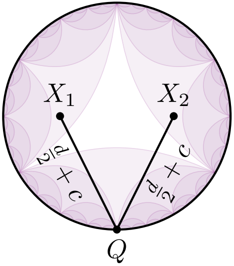
boundary: $P^2 = 0$, $P = (1, x^2, x^\mu)$.

$\zeta = (X_1 - X_2)^2 / R^2_{\mathrm{AdS}}$.

Shadow and split representation

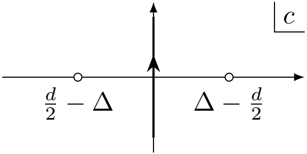


$$= \int_{-i\infty}^{+i\infty} [\mathrm{d}c] \int_{\partial \mathrm{AdS}} \mathrm{d}Q$$



- c: spectral variable [Penedones, '11]

$$[\mathrm{d}c] \propto \frac{\mathrm{d}c}{((\Delta-\frac{d}{2})^2-c^2)\Gamma(c)\Gamma(-c)}$$

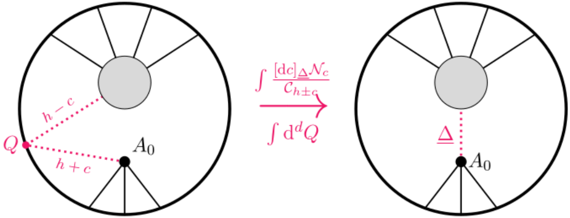


- Similar representation also holds for spinning particles. [Costa, Goncalves, Penedones, '14].

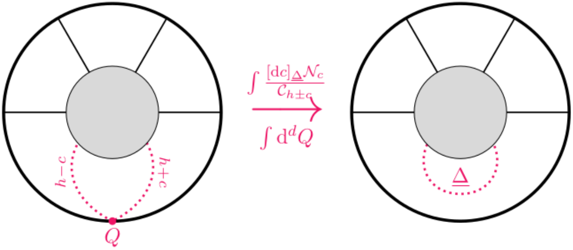
Shadow and split representation

Application 1: [EYY, '18]

- Vertex insertion

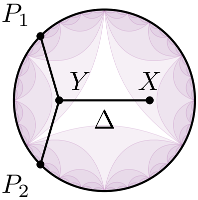


- Loop formation



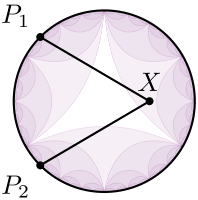
Shadow and split representation

Application 2: $(\Delta_{12} = \Delta_1 + \Delta_2,\ \delta_{12} = \Delta_1 - \Delta_2,\ h = d/2)$



$$= \int [\mathrm{d}c]\mathrm{d}Q \underbrace{\frac{1}{P_{12}^{\frac{\Delta_{12}-h-c}{2}} (Q \cdot P_1)^{\frac{\delta_{12}+h+c}{2}} (Q \cdot P_2)^{\frac{\delta_{21}+h+c}{2}}}}_{\langle O_1 O_2 O \rangle} \underbrace{\frac{1}{(Q \cdot X)^{h-c}}}_{\Pi_{b\partial}(X,Q)}$$

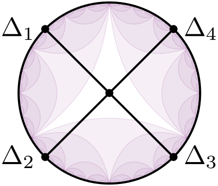
$$= \sum$$



- ☐ Truncate when $\Delta_{12} - \tau \in 2\mathbb{Z}_+$. [D'Hoker, Freedman, Rastelli, '99]
- ☐ Exchange diagram $\Rightarrow$ sum over contact diagrams.

Contact diagrams and $\bar{D}$ functions

- Contact diagrams



$$= \# \, \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}$$

- Simplest case

$$\bar{D}_{1111} \equiv \mathcal{W}_2(z,\bar{z}) = \frac{2}{z-\bar{z}}\Big(\mathrm{Li}_2(z) - \mathrm{Li}(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\Big).$$

- It is known that all $\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}$ satisfy differential relations among themselves. [Arutyunov et al, '02]
- Example: $\langle 2222 \rangle$ at tree level

$$\mathcal{H}_{2222} = (U^{-1})(V^{-1})\big(-\mathcal{D}_U\mathcal{D}_V(1+\mathcal{D}_U+\mathcal{D}_V)\big)\mathcal{W}_2.$$

Mellin amplitudes

- **Def** (for scalar particles) [Mack, '09] [Penedones, '10]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{-i\infty}^{+i\infty} [\mathrm{d}\gamma]\, M(\gamma) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{P_{ij}^{\gamma_{ij}}}.$$

  Mellin variables γ_{ij}, constrained by

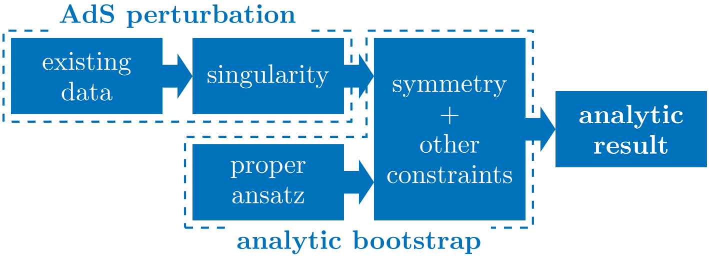
$$\gamma_{ij} = \gamma_{ji}, \qquad \prod_{j \neq i} \gamma_{ij} = \Delta_i.$$

  Analogy: $\gamma_{ij} \mapsto$ "$k_i \cdot k_j$", $\Delta_i \mapsto$ "m_i^2".
  Mellin-Mandelstam variables, i.e., $s_{ij} = \Delta_i + \Delta_j - 2\gamma_{ij}$.
- Location of poles $\Leftrightarrow$ twist $(\Delta - \ell)$ of operators in OPE.
- Factorize on poles like amplitudes. [Fitzpatrick et al, '11]
- Contact diagrams

$$M(\gamma) = \text{constant}.$$

analytic bootstrap

General strategy in analytic bootstrap



- **Symmetry**
  Crossing symmetry, superconformal symmetry, hidden symmetry, ...
- **Other physical constraints**
  Chiral twist, Drukker-Plefka twist, flat space limit, ...
- **Validity of the result** confirmed by
  cross-check + over-redundant system of equations + stability under extension of ansatz

Implementation

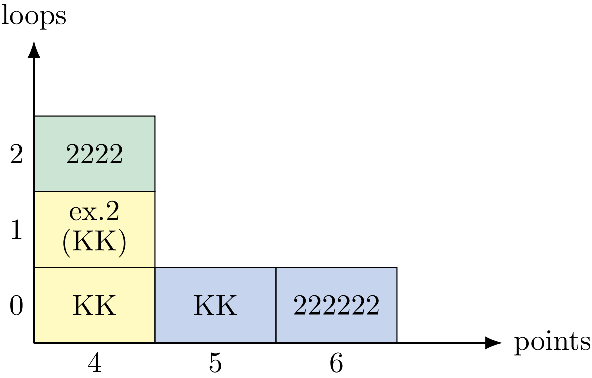
Position space

- Ansatz option 1: build upon Witten diagrams.
  Often useful at tree level.
- Ansatz option 2: build upon elementary function basis. [c.f., Yang's talk yesterday]
  Often useful at loop level.

Mellin space

- polar terms + regular terms (can be optimised).

Status of the frontier computation



- position+Mellin; position; Mellin.
- KK: all KK charge configurations (∞ correlators).
- ex.2: extremality 2, $\mathcal{E} \equiv \frac{1}{2}\sum \Delta_i - \Delta_{\max} = 2$.
  Simplest R-symmetry structure (still ∞ correlators).

Bootstrap @ tree level: four points

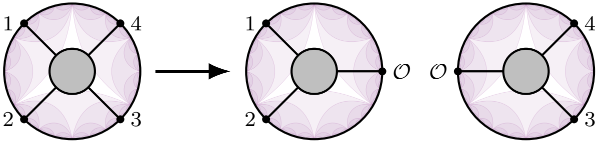
Selection rule

- Extremality (characterizing complexity in R-symmetry)

$$\mathcal{E} = \frac{1}{2}\sum_{i=1}^{n} \Delta_i - \Delta_{\max}.$$

Correlator non-vanishing only for $\mathcal{E} \geq n-2$.
(The largest dim cannot be TOO large.)

- Consider OPE (or pole in Mellin)



Sub-correlators should not vanish.

Bootstrap @ tree level: four points

Mellin space

- Example $\langle 2222 \rangle$. Mellin amplitude of $\mathcal{H}$

$$\frac{1}{(s-2)(t-2)(u-2)}\,.$$

Mellin-Mandelstam variables $s+t+u=\sum \Delta_i - 4$.

- For generic KK configurations. Ansatz [Rastelli, Zhou, '16]

$$\sum_{\substack{i+j+k=\mathcal{E}-2\\ 0\le i,j,k\le \mathcal{E}-2}} \frac{a_{ijk}\sigma^i\tau^j}{(s-s_M+2k)(t-t_M+2j)(u-u_M+2i)},$$

$s_M=\min(\Delta_{12},\Delta_{34})-2$, etc.

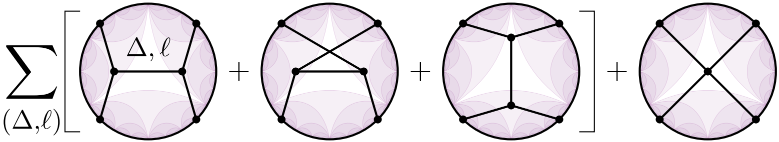
- Solution: a closed formula ($\kappa_s=|\Delta_{12}-\Delta_{34}|$, etc)

$$a_{ijk}=\frac{\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}}{i!j!k!(i+\frac{\kappa_u}{2})!(j+\frac{\kappa_t}{2})!(k+\frac{\kappa_s}{2})!}.$$

Bootstrap @ tree level: four points

Position space

- based on diagrams [Rastelli, Zhou, '17]



  In the bulk we have graviton scattering: $\ell = 0, 1, 2$.

- based on "elementary" functions

$$R_\Phi \, \mathcal{W}_2 + R_U \, \log U + R_V \, \log V + R_1 ,$$

  R: rational functions in z and $\bar{z}$.

- Also take care of structures for $\mathrm{SU}_R(4)$ R-symmetry. Allowed structures determined by spectrum.

Bootstrap @ loop level: spectrum

- Perturbative expansion of data (i labels long operators)

$$\tau_i = \tau_i^{(0)} + c^{-1}\gamma_i^{(1)} + c^{-2}\gamma_i^{(2)} + c^{-3}\gamma_i^{(3)} + \cdots\,,$$
$$a_i = a_i^{(0)} + c^{-1}a_i^{(1)} + c^{-2}a_i^{(2)} + c^{-3}a_i^{(3)} + \cdots$$

For $[\mathcal{O}\mathcal{O}]$, $a_i^{(0)}, \tau_i^{(0)}$ determined from mean field theory

$$\tau_i^{(0)} = \max(\Delta_{12}, \Delta_{34}) + 2k, \quad \text{for some } k,$$

Plug this expansion into the block expansion of $\mathcal{H}$

$$\mathcal{H}_{\text{long}} = \sum_i \left(a_i^{(0)} + c^{-1}a_i^{(1)} + \cdots\right) g_{\tau_i^{(0)} + c^{-1}\gamma_i^{(1)} + \cdots, \ell}.$$

- Note $g_{\tau,\ell}(z,\bar{z}) = (z\bar{z})^{\tau/2}(\cdots)$. The expansion in $1/c$ gives rise to powers of $\log(u)$ (s-channel). Max power at c^{-p}

$$\mathcal{H}^{(p)} \subset \frac{1}{p!\,2^p}\log(u)^p \sum_{\tau^{(0)},\ell} \langle a^{(0)}(\gamma^{(1)})^p\rangle_{\tau^{(0)},\ell}\, g_{\tau^{(0)}+4,\ell}.$$

$\langle\cdots\rangle$ due to operator degeneracy.

Bootstrap @ loop level: unitarity

Coefficients in other terms take the generic form, e.g.,

	c^{-2}	c^{-3}
$\log(u)^0$	$\langle a^{(2)} \rangle$	$\langle a^{(3)} \rangle$
$\log(u)^1$	$\langle a^{(1)}\gamma^{(1)} + a^{(0)}\gamma^{(2)} \rangle$	$\langle a^{(2)}\gamma^{(1)} + a^{(1)}\gamma^{(2)} + a^{(0)}\gamma^{(3)} \rangle$
$\log(u)^2$	$\langle a^{(0)}(\gamma^{(1)})^2 \rangle$	$\langle a^{(1)}(\gamma^{(1)})^2 + 2a^{(0)}\gamma^{(1)}\gamma^{(2)} \rangle$
$\log(u)^3$		$\langle a^{(0)}(\gamma^{(1)})^3 \rangle$

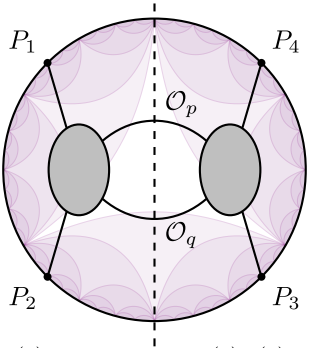
At each order c^{-p}

- The new data $a^{(k)}$ and $\gamma^{(k)}$ only show up in the $\log(u)^0$ and $\log(u)^1$ coefficients.
- $\log(u)^{k\geq 2}$ terms are in principle recursively determined by data at lower orders. In particular, leading log coefficients

$$\langle a^{(0)}(\gamma^{(1)})^k \rangle \sim \frac{\langle a^{(0)}\gamma^{(1)} \rangle^k}{\langle a^{(0)} \rangle^{k-1}}.$$

Bootstrap @ loop level: unitarity

CFT version of a "unitarity cut"



$\langle a^{(0)}\gamma^{(1)}\rangle_{p_1p_2pq}$ $\langle a^{(0)}\gamma^{(1)}\rangle_{p_3p_4pq}$

Bootstrap @ loop level: dispersion

- $\log(u)^0$ and $\log(u)^1$ terms do NOT contribute to Lorentzian singularities of the correlator.

$$\mathrm{dDisc}\,\mathcal{H}(z,\bar{z}) = \mathcal{H}(z,\bar{z}) - \frac{1}{2}\mathcal{H}^{\circlearrowleft}(z,\bar{z}) - \frac{1}{2}\mathcal{H}^{\circlearrowright}(z,\bar{z}).$$

- **Dispersion:** Lorentzian inversion formula [Caron-Huot, '17]

$$c_{\tau,\ell} =\propto \int_0^1 \frac{\mathrm{d}z}{z^2}\frac{\mathrm{d}\bar{z}}{\bar{z}^2}\left(\frac{\bar{z}-z}{\bar{z}z}\right)^2 g_{\tau+l+1,2-\tau}(z,\bar{z})\,\mathrm{dDisc}(\mathcal{H}).$$

- This integral encodes the CFT data by

$$\begin{aligned} c_{\tau,\ell} &= \sum_i \frac{a_i}{\tau-\tau_i} \\ &\supset \frac{1}{c^2}\sum_{\tau^{(0)}}\left(\frac{\langle a^{(2)}\rangle}{\tau-\tau^{(0)}} + \frac{\langle a^{(1)}\gamma^{(1)} + a^{(0)}\gamma^{(2)}\rangle}{(\tau-\tau^{(0)})^2} + \frac{\langle a^{(0)}(\gamma^{(1)})^2\rangle}{(\tau-\tau^{(0)})^3}\right) \end{aligned}$$

Bootstrap @ loop level: bootstrap

Position space

- Note at tree level, the ansatz is built from $\{\Phi, \log U, \log V, 1\}$, all are single-valued in the Euclidean signature.
- One-loop anatz

$$\mathcal{H}^{(2)} = \sum_{w=0}^{4} \sum_{i} \frac{\rho_{w,i}(z,\bar{z})}{(\bar{z}-z)^{15}} G^{\text{sv}}_{w,i}(z,\bar{z}).$$

  $G^{\text{sv}}_{w,i}$: single-valued multiple polylogarithm of weight w.
- G^{sv} can have singularities only at $z, \bar{z} = 0, 1, \infty$ $\Rightarrow$ symbol letters $\{z, \bar{z}, 1-z, 1-\bar{z}\}$.
- Highest weight ($w = 4$) and degrees in coefficient functions learned from leading log contribution.
- (one-loop) Solve by crossing and leading log contribution.

Bootstrap @ loop level: bootstrap

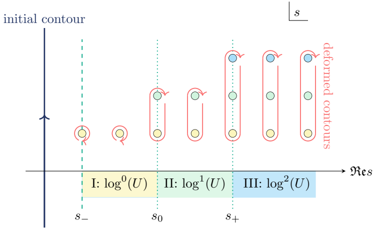
Mellin space

- Ansatz

$$\mathcal{M}^{(2)} = \mathcal{M}_{st} + \mathcal{M}_{su} + \mathcal{M}_{tu}, \quad \mathcal{M}_{st} = \sum_{s_*,t_*}^{\infty} \frac{c_{s_*,t_*}}{(s-s_*)(t-t_*)}.$$

- Pole spectrum

$$\int_{\mathcal{C}_*} \frac{\mathrm{d}s}{2\pi i} \frac{U^{\frac{s+2}{2}}}{(s-s_*)^{k+1}} h(s) = \frac{1}{2^{k+1}} U^{\frac{s_*}{2}+1} \Big(\log^k U \, h(s_*) + \cdots \Big).$$

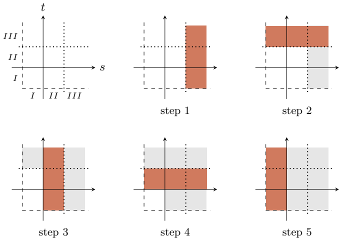


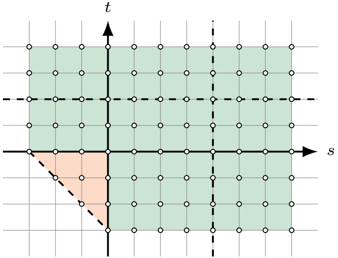
Bootstrap @ loop level: bootstrap

Mellin space

operators	data in $\mathcal{H}^{(2)}$	U expansion of $\mathcal{H}^{(2)}$	$\frac{1}{(s-s_*)(t-t_*)} \subset \widetilde{\mathcal{M}}_{st}$
long	$\langle a^{(0)}(\gamma^{(1)})^2\rangle$	$U^{\frac{\tau_0 \geq s_+}{2}+1} \log^2 U$	III: $s_* \geq s_+$
long	$\langle a^{(1)}\gamma^{(1)}\rangle$	$U^{\frac{s_0 \leq \tau_0 < s_+}{2}+1} \log U$	II: $s_0 \leq s_* < s_+$
long protected	$\langle a^{(2)}\rangle$ $\langle b^{(2)}\rangle$, $\langle c^{(2)}\rangle$	$U^{\frac{s_- \leq \tau_0 < s_0}{2}+1}$	I: $s_- \leq s_* < s_0$

- bootstrap and result [Huang, Wang, EYY, Zhou, '23]





Differential representation

$\langle 2222 \rangle$ at tree level

$$\mathcal{H}_{2222} = (U^{-1})(V^{-1})\Big(-\mathcal{D}_U\mathcal{D}_V(1+\mathcal{D}_U+\mathcal{D}_V)\Big)\mathcal{W}_2$$

Operators can be decomposed as polynomials of

$$\mathcal{D}_U \equiv U\partial_U, \quad \mathcal{D}_V \equiv V\partial_V, \quad U^{\pm 1}, \quad V^{\pm 1}.$$

position space ⟷ **Mellin space**

$\mathcal{D}_U^m\,\mathcal{H}(U,V)$	$S^m\,\mathcal{M}(S,T)$
$\mathcal{D}_V^n\,\mathcal{H}(U,V)$	$T^n\,\mathcal{M}(S,T)$
$U^a\,\mathcal{H}(U,V)$	$\Big[(-S)_a(1+S+T)_{-a}\Big]^2\mathcal{M}(S-a,T)$
$V^b\,\mathcal{H}(U,V)$	$\Big[(-T)_b(1+S+T)_{-b}\Big]^2\mathcal{M}(S,T-b)$

Differential representation

- Tree-level. Mellin amplitude of $\mathcal{W}_2$: $\mathcal{M}_2 = 1$.
- One-loop. In addition to $\mathcal{M}_2$ [Huang, Wang, EYY, '24]

$$\mathcal{M}_3(S) \equiv \xi(S) = \psi^{(0)}(-S) + \gamma_{\mathrm{E}}$$
$$\mathcal{M}_4(S,T) \equiv \Phi(S,T) = -\frac{1}{2}\Big((\xi(S)+\xi(T))^2 + \xi'(S) + \xi'(T) + \pi^2\Big)$$

they have simple residues

$$\operatorname*{Res}_{S=m} \xi(S) = 1, \qquad \operatorname*{Res}_{S=m}\operatorname*{Res}_{T=n} \Phi(S,T) = 1, \qquad m,n \in \mathbb{N}$$

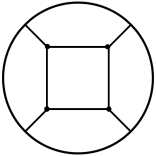
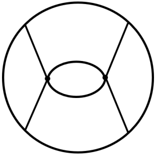
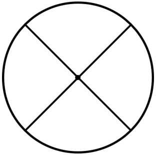
- Basis counting

# of independent SVMPLs	tree	one loop
ansatz for bootstrap	8	42
reduced correlator	4	10
seed functions	1	3

Differential representation

- In this way we worked out unified formula for all extremality 2 four-point one-loop correlators in $\mathcal{N}=4$ SYM (gravitons) and in an analog model containing gluons ($\mathrm{AdS}_5 \times \mathrm{S}^3$). [Huang, Wang, EYY, '24]
- Gluon case

$$\mathcal{H}^{(2)}_{\mathrm{YM,st}}(U,V) = \mathcal{D}_4\mathcal{W}_4 + \mathcal{D}_3\mathcal{W}_3 + \mathcal{D}_2\mathcal{W}_2$$

$\mathcal{D}_4$  $+\,\mathcal{D}_3$  $+\,\mathcal{D}_2$ 

- Graviton case: no $\mathcal{W}_3$ nor $\mathcal{W}_2$!

remarkable empirical discoveries

Hidden higher-dimensional symmetries

- Hidden higher-dim conformal symmetry was first observed in tree-level corrections to double-trace operator spectrum. [Caron-Huot, Trinh, '18]
- Useful to package all KK operators into a master operator [Caron-Huot, Coronado, '21]

$$\mathbb{O} = \sum_{p=2}^{\infty} \#_p \, \mathcal{O}_p .$$

- Generating function for reduced correlators

$$\mathbb{H} = \sum_{p,q,r,s} \#_p \#_q \#_r \#_s \mathcal{H}_{pqrs}(x_{ij}^2, y_i \cdot y_j) = \mathcal{H}_{2222}(\underbrace{x_{ij}^2 - y_i \cdot y_j}_{\text{10d distance}}).$$

- Intuition: $\mathrm{AdS}_5 \times \mathrm{S}^5$ is conformally flat.
- Similar observations in other models.

Bootstrap @ four points: higher loops

- Implications on leading log singularities [Caron-Huot, Trinh, '18]

$$\mathcal{H}^{(p)}\Big|_{\log(u)^p} = \left[\Delta^{(8)}\right]^{p-1} \mathcal{F}^{(p)}(z,\bar{z}).$$

Here ($D_z = z^2\partial_z(1-z)\partial_z$)

$$\Delta^{(8)} = \frac{z\bar{z}}{\bar{z}-z} D_z(D_z-2)D_{\bar{z}}(D_{\bar{z}}-2)\frac{\bar{z}-z}{z\bar{z}},$$

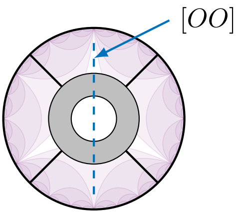
$$\mathcal{F}^{(p)}(z,\bar{z}) = \sum_{|\vec{a}|=0}^{p} \frac{\rho_{\vec{a}}(z,\bar{z})}{(\bar{z}-z)^7} \underbrace{G(\vec{a};z)}_{\text{MPL}} + (z \leftrightarrow \bar{z}).$$

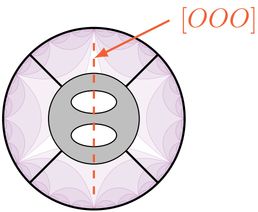
- This structure extends to the entire correlator at one loop, with a slight modification [Aprile et al, '19]

$$\mathcal{H}^{(2)} = \Delta^{(8)}\mathcal{L}^{(2)} + \frac{1}{4}\mathcal{H}^{(1)},$$

$$\mathcal{L}^{(2)} = \sum_{|\vec{a}|+|\vec{a}'|=0}^{4} \frac{\rho_{\vec{a},\vec{a}'}(z,\bar{z})}{(\bar{z}-z)^7} G(\vec{a};z)G(\vec{a}';\bar{z}).$$

Bootstrap @ four points: higher loops





$\log^2 U$

Direct unitarity recursion fails unless we know $[OOO]$ data.

Bootstrap @ four points: higher loops

Position space bootstrap for $\langle 2222 \rangle$ at two loops [Huang, EYY, '21]

- Ansatz

$$\mathcal{H}^{(3)} = \left[\Delta^{(8)}\right]^2 \mathcal{L}^{(3)} + a_2 \mathcal{H}^{(2)} + a_1 \mathcal{H}^{(1)},$$

$$\mathcal{L}^{(3)} = \sum_{w=0}^{6} \sum_{r,i} \frac{\omega_i^{w,r}(z,\bar{z})}{(\bar{z}-z)^7} G^{\mathrm{SV}}_{w,r,i}(z,\bar{z}).$$

- Result

$$\mathcal{H}^{(3)} = \left[\Delta^{(8)}\right]^2 \mathcal{L}^{(3)} + \frac{5}{4}\mathcal{H}^{(2)} - \frac{1}{16}\mathcal{H}^{(1)} + (\text{counterterms}).$$

- Also for gluon scattering in $\mathrm{AdS}_5 \times \mathrm{S}^3$. [Huang, Wang, EYY, Zhou, '23]

Bootstrap @ tree level: AdS×S Mellin formalism

- Ordinary Mellin

$$\langle p_1 p_2 \cdots p_n \rangle_{\text{int}} = \int [\mathrm{d}\gamma]\, \mathcal{M}(\gamma_{ij}; y_{ij}) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{(x_{ij}^2)^{\gamma_{ij}}}.$$

$\gamma_{ij} = \gamma_{ji}$, $\sum_{j \neq i} \gamma_{ij} = p_i$.

- Mellin on the sphere [Aprile, Vieira, '20]

$$\mathcal{M}(s, t; y_{ij}) = \sum_{n_{ij}} \mathbb{M}(\gamma_{ij}; n_{ij}) \prod_{i<j} \frac{y_{ij}^{n_{ij}}}{\Gamma(n_{ij}+1)}.$$

$n_{ij} = n_{ji}$, $\sum_{j \neq i} = p_i$.

- For $\langle \mathbb{OO} \cdots \mathbb{O} \rangle$ we simply do NOT constrain n_{ij}.

Bootstrap @ tree level: higher points

- If in case

$$\mathbb{M}(\gamma_{ij}; n_{ij}) = \mathbb{M}(\underbrace{\gamma_{ij} - n_{ij}}_{\rho_{ij}}),$$

  then we can replace $[\mathrm{d}\gamma] \mapsto [\mathrm{d}\rho]$ and the n summation yields

$$\langle \mathbb{O}\mathbb{O}\cdots\mathbb{O}\rangle_{\text{int}} = \int [\mathrm{d}\rho]\, \mathbb{M}(\rho_{ij}) \prod_{i<j} \frac{\Gamma(\rho_{ij})}{(x_{ij}^2 - y_i \cdot y_j)^{\rho_{ij}}}.$$

  $\rho_{ij} = \rho_{ji}$, $\sum_{j\neq i} \rho_{ij} = 0$.

- Four points

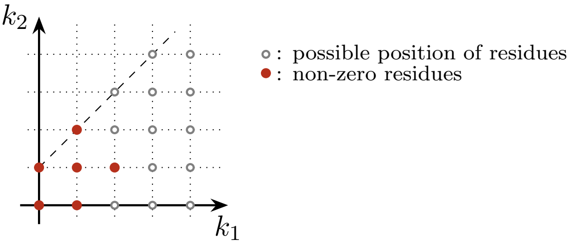
$$\mathbb{M} = \widehat{\mathcal{I}}\widetilde{\mathbb{M}}, \qquad \widetilde{\mathbb{M}} = \frac{-1}{(\rho_{12}-1)(\rho_{13}-1)(\rho_{14}-1)}.$$

Bootstrap @ tree level: higher points

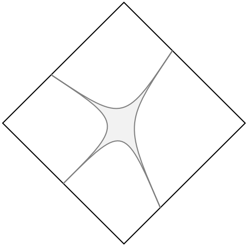
- One can design a factorization rule directly in the AdS $\times$ S Mellin space.
- Glimpse at the ansatz at five points, directly for the generating function

$$\mathbb{M}_5 = \sum_{k_1,k_2} \frac{P_{k_1,k_2}(\gamma,n)}{(\rho_{12}+k_1)(\rho_{45}+k_2)} + \sum_{k_3} \frac{P_{k_3}(\gamma,n)}{\rho_{12}+k_3} + P(n) + (\text{perm}).$$

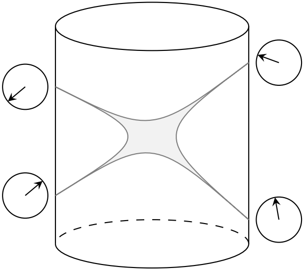
- Polar terms solved by crossing and AdS$\times$S factorization. $P(n)$ solved by Drukker-Plefka twist [Drukker, Plefka, '09].
- Glimpse at the structure of the result



Existence of a worldsheet: motivation



Virasoro-Shapiro
(all l_{s})



AdS Virasoro-Shapiro
(all l_{s} and R_{AdS})

VS + curvature correction
(all l_{s}, all KK)

Existence of a worldsheet: $\langle 2222 \rangle$

Virasoro-Shapiro ($A^{(0)}$)

$$-\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

$(S+T+U=0)$

AdS Virasoro-Shapiro

$$\langle 2222 \rangle_{\text{int}} = \int [\mathrm{d}\gamma] M(s,t) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{(x_{ij}^2)^{\gamma_{ij}}}$$

low energy

$$\frac{1}{STU} + 2\zeta_3 + 2\bar{\sigma}_2\zeta_5 + 2\bar{\sigma}_3\zeta_3^2 + \cdots$$

$(\bar{\sigma}_2 = S^2+T^2+U^2, \;\; \bar{\sigma}_3 = STU)$

$$M = M^{\text{SG}} + \lambda^{-\frac{3}{2}} M^3 + \cdots$$

Borel

$$A = A^{(0)} + \lambda^{-\frac{1}{2}} A^{(1)} + \cdots$$

Existence of a worldsheet: $\langle 2222 \rangle$

- Low-energy expansion of Mellin

$$M(s,t) = \frac{8}{(s-2)(t-2)(u-2)} + \sum_{a,b=0}^{\infty} \frac{\#_{a,b}\, \sigma_2^a \sigma_3^b}{\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} \Big(\alpha_{a,b}^{(0)} + \underbrace{\frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{a,b}^{(2)}}{\lambda} + \cdots}_{\text{curvature effects}} \Big)$$

$\sum p = 2\Sigma,\ s+t+u = 2\Sigma - 4,\ \sigma_2 = s^2+t^2+u^2,\ \sigma_3 = stu.$

- Flat space limit via Borel transformation

$$\frac{A(S,T)}{\lambda^{\frac{3}{2}}\Gamma(\Sigma-1)} = \int \frac{\mathrm{d}\beta}{2\pi i} \frac{e^{\beta}}{\beta^{\Sigma+2}}\, M\Big(\frac{2\sqrt{\lambda}S}{\beta} + \frac{2\Sigma-4}{3}, \frac{2\sqrt{\lambda}T}{\beta} + \frac{2\Sigma-4}{3}\Big)$$

$$= \frac{1}{\lambda^{\frac{3}{2}}\Gamma(\Sigma-1)} \sum_{k=0}^{\infty} \lambda^{-\frac{k}{2}} A^{(k)}(S,T)$$

Existence of a worldsheet: $\langle 2222 \rangle$

- Virasoro-Shapiro (determined by flat space result)

$$A^{(0)} = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \alpha^{(0)}_{a,b}\hat{\sigma}^{(0)}_{a,b}$$

- First curvature correction [Alday, Hansen, Silva, '22]
  (dispersive sum rules + single-valuedness)

$$A^{(1)} = -\frac{2\hat{\sigma}_2}{3\hat{\sigma}_3^2} + 2\sum_{a,b=0}^{\infty} \alpha^{(1)}_{a,b}\hat{\sigma}^{(1)}_{a,b}$$

- Result

$$A^{(0)}(S,T) = \frac{1}{\hat{\sigma}_3} + 2\zeta_3 + \hat{\sigma}_2\zeta_5 + \hat{\sigma}_3\zeta_3^2 + \cdots$$

$$\mathcal{A}^{(1)}(S,T) = -\frac{\hat{\sigma}_2}{3\hat{\sigma}_3^2} - \frac{22}{3}\hat{\sigma}_2\zeta_3^2 - \frac{537}{8}\hat{\sigma}_3\,\zeta_7 + \cdots$$

Existence of a worldsheet: $\langle 2222 \rangle$

- Worldsheet integral for Virasoro-Shapiro $A^{(0)}$

$$A^{(0)} = \int d^2z \, |z|^{-2S-2} |1-z|^{-2T-2} \, \frac{1}{U^2} \, .$$

- Worldsheet integral for $A^{(1)}$ [Alday, Hansen, Silva, '23]

$$A^{(1)}(S,T) = B^{(1)}(S,T) + B^{(1)}(U,T) + B^{(1)}(S,U),$$
$$B^{(1)}(S,T) = \int d^2z \, |z|^{-2S-2} |1-z|^{-2T-2} \, G^{(1)}(z,\bar{z}).$$

- It turns out

$$G^{(1)} = R_0(\mathcal{L}_{000} + \mathcal{L}_{111}) + R_1(\mathcal{L}_{010} + \mathcal{L}_{101}) + R_2(\mathcal{L}_{001} + \mathcal{L}_{110}) + (\text{redundancy}).$$

$$R_0 = \frac{2S}{3U}, \quad R_1 = \frac{S+5T}{6U}, \quad R_2 = \frac{2(T-S)}{3U}.$$

Existence of a worldsheet: bootstrap strategy

Use AdS×S formalism [Wang, Wu, EYY, '25]

raw data

$$\begin{tikzcd}
\boxed{\mathbb{M}(s,t;n_{ij})} \arrow[r, "\text{spherical Mellin transform}"] \arrow[d, "\text{Borel transform}"'] & \boxed{M(s,t;y_{ij})} \\
\boxed{\mathbb{A}(S,T;n_{ij})} \arrow[r] & \boxed{A(S,T;y_{ij})} \arrow[u]
\end{tikzcd}$$

ansatz

Existence of a worldsheet: ansatz

- A quick glimpse at curvature correction to sugra

$$\mathcal{A}^{(1),\mathrm{SG}} = \frac{2-\Sigma}{6\,\hat{\sigma}_3^2}\big(3(n_s S^2 + n_t T^2 + n_u U^2) - (n_s + n_t + n_u - 1)(S^2 + T^2 + U^2)\big),$$

where

$$n_s = n_{12} + n_{34}, \quad n_t = n_{14} + n_{23}, \quad n_u = n_{13} + n_{24}.$$

Hints on ingredients to appear in the ansatz.

- Ansatz in terms of worldsheet integrals

$$\mathcal{A}^{(1)}(S,T) = \mathcal{B}^{(1)}(S,T;n_s,n_t) + \mathcal{B}^{(1)}(U,T;n_u,n_t) + \mathcal{B}^{(1)}(S,U;n_s,n_u)\,,$$

$$\mathcal{B}^{(1)}(S,T;n_s,n_t) = \int d^2z\,|z|^{-2S-2}|1-z|^{-2T-2}\,\mathcal{G}^{(1)}(z,\bar{z};n_s,n_t)\,.$$

Existence of a worldsheet: ansatz

- Worldsheet integrand contains SVMPLs at weight 3, with singularities only at $z = 0, 1, \infty$

$$\mathcal{L}_i^s = (\mathcal{L}_{000}^s, \mathcal{L}_{001}^s\,, \mathcal{L}_{010}^s, \zeta_3)\ ,$$
$$\mathcal{L}_i^a = (\mathcal{L}_{000}^a, \mathcal{L}_{001}^a, \mathcal{L}_{010}^a)\ .$$

$\mathcal{L}_i^{s/a}$ is symmetric/anti-symmetric under $z \leftrightarrow 1-z$, e.g.,

$$\mathcal{L}_{000}^s = \mathcal{L}_{000} + \mathcal{L}_{111} = \frac{\log^3|z|^2}{3!} + \frac{\log^3|1-z|^2}{3!}\,.$$

- Build the integrand by linear combination

$$\mathcal{G}^{(1)}(z,\bar{z};n_s,n_t) = \sum_{i=1}^{4} \mathcal{R}_i^s \mathcal{L}_i^s + \sum_{j=1}^{3} \mathcal{R}_j^a \mathcal{L}_j^a\,.$$

The use of SVMPLs guarantees that the amplitude only contains single-valued MZVs.

Existence of a worldsheet: ansatz

- The prefactors $\mathcal{R}_i^{a/s}(S,T)$ are homogeneous functions with a universal denominator U^2 and enjoy the general structure

$$\mathcal{R}_i^{a/s}(S,T) = \frac{f_1^{a/s}\,S^2 + f_2^{a/s}\,S\,T + f_3^{a/s}\,T^2}{U^2}\,,$$

where $f_i^{a/s}$ are linear combinations of

$$\{\Sigma\,n_s, \Sigma\,n_t, \Sigma\,n_u, \Sigma\,, n_s, n_t, n_u\}$$

with unknown coefficients.

- ▷ Degree and denominator motivated by $A^{(0)}$ and $A^{(1)}_{22pp}$.
- ▷ Parameters motivated by the structure of $\mathcal{A}^{(1),\mathrm{SG}}$.

$$\mathcal{A}^{(1),\mathrm{SG}} = \frac{2-\Sigma}{6\,\hat{\sigma}_3^2}\Big(3(n_sS^2+n_tT^2+n_uU^2) - (n_s+n_t+n_u-1)(S^2+T^2+U^2)\Big),$$

Existence of a worldsheet: $\langle pqrs \rangle$ result

- The final result (apart from redundancies) only depends on five linearly independent SVMPLs

$$\mathcal{L}_{1/4} = \mathcal{L}^{s/a}_{000}\,, \quad \mathcal{L}_2 = -\mathcal{L}^{s}_{010} + 4\zeta_3\,,$$
$$\mathcal{L}_3 = \mathcal{L}^{s}_{000} - \mathcal{L}^{s}_{001} - \mathcal{L}^{s}_{010}\,,$$
$$\mathcal{L}_5 = \mathcal{L}^{a}_{000} - \mathcal{L}^{a}_{001} - \mathcal{L}^{a}_{010}\,.$$

- In terms of this basis

$$\mathcal{G}^{(1)}(z,\bar{z};n_s,n_t) = (\Sigma - 2) \times \sum_{i=1}^{5} \mathcal{L}_i \mathcal{R}_i\,,$$

$$\mathcal{R}_1 = \frac{(n_s - n_t)(S-T) + 2U}{12(S+T)}\,, \quad \mathcal{R}_2 = \frac{1}{2}\frac{1}{\Sigma - 2}\,,$$

$$\mathcal{R}_3 = \frac{n_s\, S + n_t\, T + n_u\, U}{12(S+T)}\,, \; \mathcal{R}_4 = \frac{n_s - n_t}{12}\,,$$

$$\mathcal{R}_5 = \frac{n_s S - n_t T + (n_u + 2)(T - S)}{12(S+T)}\,.$$

Topics not covered in this talk

- Other perturbation methods, e.g., large spin perturbation. [c.f., Alday, '15]
- Other versions of dispersion relations, e.g., CSDR. [c.f., Gopakumar, Sinha, Zahed, '21]
- Other AdS backgrounds, and related techniques such as MRV limit. [c.f., Alday, Zhou, '20]
- Seek for a 10d effective action for stringy corrections. [c.f., Abl, Heslop, Lipstein, '20]
- Integrated correlators. [c.f., Binder, Chester, Pufu, Wang, '19]
- correlators involving defects, giant gravitons. [c.f., Chen, Jiang, Zhou, '25]
- Differential relations among (loop-level) Witten diagrams. [c.f., Alaverdian, Herderschee, Roiban, Teng, '24]
- ...

Open problems

- Understand the spectrum of triple-trace and higher-trace operators.
- AdS version of a "Parke-Taylor" formula?
- Understand the function space in Mellin.
- Develop method for loop corrections at higher points (even one-loop five-points).
- Develop proper method at two loops.
- Explore the power of AdS$\times$S formalism.
- Understand the origin of the hidden symmetry, and to what extends it holds.
- Diagrammatic decomposition of holographic correlators.
- Correlators of other operators (e.g., semi-short operators, giant gravitons & dual giant gravitons).
- ...

Thank you very much!

Questions & comments are welcome.