Feynman Integrals and Intersection Theory

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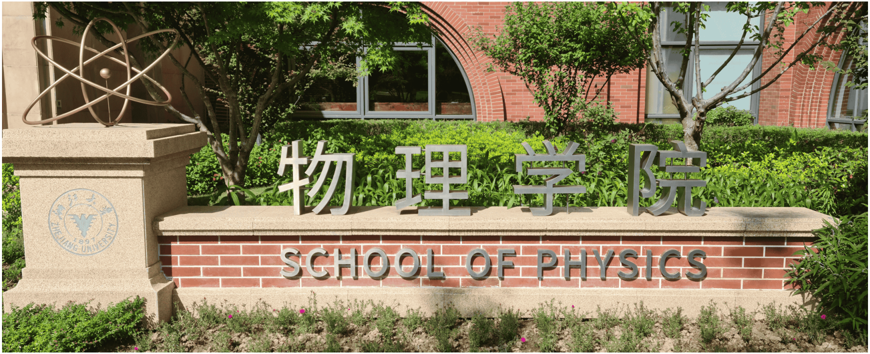
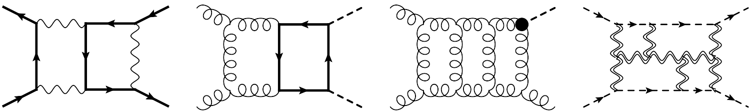


Introduction

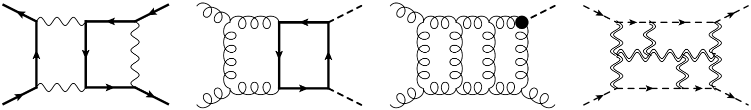


Let's say we want to compute a state-of-the-art scattering amplitude

1) Write down all Feynman diagrams
2) Perform Dirac (gamma-matrix) algebra and Lorentz algebra
3) Express in terms of scalar Feynman integrals

$$I_{a_1\cdots a_P;\cdots a_n} = \int \frac{\mathrm{d}^d k_1}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) D_2^{a_2}(k) \cdots D_P^{a_P}(k)}$$

The Ds are propagators, e.g. of the form $D_i = (k+p)^2 - m^2$,
$d = 4 - 2\epsilon$ is the space-time dimensionality,
k and p are d-dimensional momenta (internal and external),
$N(k) = \prod_{i=P+1}^{n} D_i^{-a_i}(k)$ is a numerator function,
P is the number of propagators,
L and E are the numbers of loops and (independent) legs,
$n = L(L+1)/2 + EL$ is the number of independent scalar products,
a_i are integer powers.



For state-of-the art scattering amplitude calculations

$\mathcal{O}(10\,000)$ Feynman diagrams $\rightarrow$ $\mathcal{O}(100\,000)$ Feynman integrals

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Linear relations bring this down to $\mathcal{O}(300)$ *master integrals*

Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = 0$$

Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.

IBPs implemented in some extremely optimized public codes.

The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Subsectors are sub-spaces

Not all vector spaces are *inner product spaces*

$$\begin{aligned} \langle v| &= \sum_i \langle v v_j^* \rangle \big(\boldsymbol{C}^{-1}\big)_{ji} \langle v_i| \qquad \text{with} \qquad \boldsymbol{C}_{ij} = \langle v_i v_j^* \rangle \\ &= \sum_i c_i \, \langle v_i| \qquad \qquad \big(c_i = \langle v v_i^* \rangle \ \text{ if } \ \boldsymbol{C}_{ij} = \delta_{ij}\big) \end{aligned}$$

If only there were a way to define an inner product for Feynman integrals...

The Baikov representation of Feynman integrals

$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u \, \phi = \langle \phi | \mathcal{C}]_\omega$$

$u = \mathcal{B}^\gamma$ is a multivalued function of $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

$\omega = \mathrm{d}\log(u)$ is *the twist*

$\langle \phi | \mathcal{C}]_\omega$ is a pairing of a *twisted cycle* ($\mathcal{C}$) and a *twisted cocycle* (ϕ)
(equivalence classes of contours and integrands respectively)

[P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, JHEP **1902** (2019) 139]

dim of the set of ϕs, is the number of master integrals.

The IBP equation can be written

$$0 = \int_{\mathcal{C}} \mathrm{d}(u \, \xi) = \int_{\mathcal{C}} u \, \nabla_u \xi \qquad \text{where} \qquad \nabla_u := \mathrm{d} + \omega$$

Our Feynman integral: $$I = \int_{\mathcal{C}} u\,\phi = \langle \phi | \mathcal{C}]_\omega$$

A *dual* Feynman integral: $$I_{\text{dual}} = \int_{\check{\mathcal{C}}} u^{-1} \check{\phi} = [\check{\mathcal{C}} | \check{\phi} \rangle_\omega$$

The *intersection number* $\langle \phi | \check{\phi} \rangle$ is a pairing of a twisted cocycle ϕ with a *dual* twisted cocycle $\check{\phi}$

Lives up to all criteria for being a scalar product.

Why "intersection number"?

Usually intersection numbers count the intersections of curves.
Our construction generalizes that from homology to cohomology

Naive atempt: $$\langle \phi | \check{\phi} \rangle \overset{?}{:=} \int_X (u\phi)(u^{-1}\check{\phi}) = \int_X \phi\,\check{\phi}$$

This is badly defined

$$\langle \phi | \check{\phi} \rangle \;:=\; \int (u\phi)_{\text{reg}}\, (u^{-1}\check{\phi})$$

After one page of derivations; in the univariate case:

$$\Rightarrow \qquad \langle \phi | \check{\phi} \rangle \;=\; \sum_{p\in\mathcal{P}} \text{Res}_{x=p}(\psi\check{\phi}) \qquad \text{with} \qquad (d+\omega)\psi = \phi \qquad \Leftarrow$$

where

$$\omega := d\log(u) \quad \text{and} \quad \mathcal{P} \text{ is the set of zeros of } \omega$$

[Mastrolia, Mizera (2018)]

ψ can be found with a series expansion $\psi \to \psi_p = \sum \psi_{p;i}(x-p)^i$

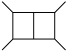
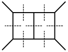
(Other options are a recursive formula, or sometimes a closed expression)

Summary:

$$I_i = \int_{\mathcal{C}} u\phi_i \;=\; \langle \phi_i | \mathcal{C}] \qquad\qquad I = \sum_i c_i I_i$$

$$c_i = \langle \phi | \check{\phi}_j \rangle (\boldsymbol{C}^{-1})_{ji} \quad \text{with} \quad \boldsymbol{C}_{ij} = \langle \phi_i | \check{\phi}_j \rangle$$

Example (double box on hepta-cut)

 $= \int \mathrm{d}^8 x \, \frac{u N(x)}{x_1 \cdots x_7} \quad \rightarrow$  $= \int u_{\text{cut}} \, \phi \,, \qquad u_{\text{cut}} = z^{d/2-3}(z+s)^{2-d/2}(z-t)^{d-5}$

We want to reduce

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower}$$

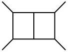
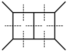
$$\varphi = z^2 \, \mathrm{d}z \,, \quad \phi_1 = 1 \, \mathrm{d}z \,, \quad \phi_2 = z \, \mathrm{d}z \,, \quad \check{\phi}_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s} \right) \mathrm{d}z \,, \quad \check{\phi}_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t} \right) \mathrm{d}z \,,$$

$$c_i = \langle \varphi | \check{\phi}_j \rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle \phi_i | \check{\phi}_j \rangle$$

We need the intersection numbers

$$\left\{ \langle \varphi | \check{\phi}_1 \rangle, \langle \varphi | \check{\phi}_2 \rangle, \langle \phi_1 | \check{\phi}_1 \rangle, \langle \phi_1 | \check{\phi}_2 \rangle, \langle \phi_2 | \check{\phi}_1 \rangle, \langle \phi_2 | \check{\phi}_2 \rangle \right\}$$

We have the univariate formula $\langle \phi | \check{\phi} \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi \check{\phi})$ with $(\mathrm{d} + \omega)\psi = \phi$

 $= \int \mathrm{d}^8 x \, \frac{u N(x)}{x_1 \cdots x_7} \quad \rightarrow$  $= \int u_{\text{cut}} \, \phi \,, \qquad u_{\text{cut}} = z^{d/2-3}(z+s)^{2-d/2}(z-t)^{d-5}$

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$$c_i = \langle \varphi | \check{\phi}_j \rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle \phi_i | \check{\phi}_j \rangle$$

$$\langle \phi | \check{\phi}_1 \rangle = \tfrac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi | \check{\phi}_2 \rangle = \tfrac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi_1 | \check{\phi}_1 \rangle = \tfrac{-s}{d-5} \,, \qquad \langle \phi_1 | \check{\phi}_2 \rangle = \tfrac{s+t}{d-5} \,,$$

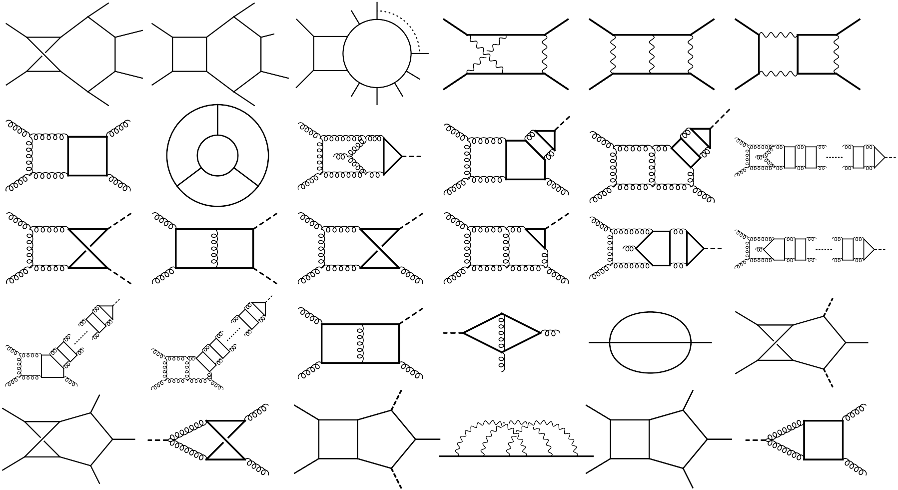
$$\langle \phi_2 | \check{\phi}_1 \rangle = \tfrac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)} \,, \qquad \langle \phi_2 | \check{\phi}_2 \rangle = \tfrac{-(3d-14)s(s+t)}{2(d-5)(d-4)} \,.$$

The result are $\quad c_0 = \dfrac{(d-4)st}{2(d-3)} \,, \quad c_1 = \dfrac{2t - 3(d-4)s}{2(d-3)} \,,$

in agreement with FIRE

Examples

On the maximal cut we did a lot of examples
[HF, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera (2019)]



Univariate: $\langle\phi|\check{\phi}\rangle = \sum_i \mathrm{Res}_{z=z_i}(\psi\check{\phi})$ with $(\partial_z + \omega)\psi = \phi$

Multivariate: The iterative approach (fibration):

[Sebastian Mizera: 1906.02099 + PhD Thesis]

[HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera: *PhysRevLett.* **123** (2019) 201602]

$$_{\mathbf{n}}\langle\phi^{(\mathbf{n})}|\check{\phi}^{(\mathbf{n})}\rangle = \sum_{p\in\mathcal{P}_n} \mathrm{Res}_{z_n=p}\Big(\psi_i^{(n)}\,{}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})}|\check{\phi}^{(\mathbf{n})}\rangle\Big)$$

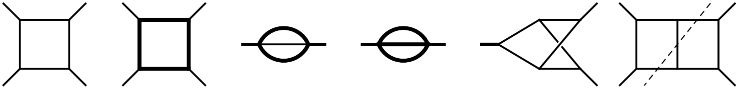
$$\big(\delta_{ij}\partial_{z_n} + \mathbf{\Omega}_{ij}^{(n)}\big)\psi_j^{(n)} = \varphi_i^{(n)}$$

$$\mathbf{\Omega}_{ij}^{(n)} = {}_{\mathbf{n-1}}\langle(\partial_{z_n}+\omega_n)e_i^{(\mathbf{n-1})}|h_k^{(\mathbf{n-1})}\rangle\big(\mathbf{C}_{(\mathbf{n-1})}^{-1}\big)_{kj}$$

$$\varphi_i^{(n)} = {}_{\mathbf{n-1}}\langle\phi^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle\big(\mathbf{C}_{(\mathbf{n-1})}^{-1}\big)_{ji}$$

$$\mathbf{C}_{ij}^{(\mathbf{n-1})} = {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})}|h_j^{(\mathbf{n-1})}\rangle$$

Examples of complete reductions with that method:



[H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, JHEP 03 (2021) **027** arXiv:2008.04823]

In twisted cohomology theory

$$I = \int_{\mathcal{C}} u\,\phi \quad \text{with all poles of } \phi \text{ being regulated by } u = \mathcal{B}^{\gamma}$$

but for uncut FIs $\phi \approx \dfrac{\mathrm{d}^n z}{z_1 \cdots z_m}$ has all poles unregulated

Solution in that paper: Introduce *regulators*

$$u \;\to\; u_{\text{reg}} = u z_1^{\rho_1} z_2^{\rho_2} \cdots z_m^{\rho_m}$$

and take the limits $\rho_i \to 0$ at the end.

This is one new scale per uncut propagator!

We want to get rid of the regulators. Use *Relative* (twisted) cohomology:

$$I = \int_{\mathcal{C}} u\,\phi \quad \text{with} \quad \phi = \frac{\mathrm{d}^n x}{x_1 \cdots x_m} \; : \quad \text{Work } \textit{relative} \text{ to } \bigcup_i (x_i = 0)$$

Forms and contours live in a space defined *modulo* a different space
[Matsumoto (2018)], [Caron-Huot and Pokraka (2021, 2022)]
[G. Brunello, V. Chestnov, G. Crisanti, HF, M.K. Mandal, P. Mastrolia (2024)]

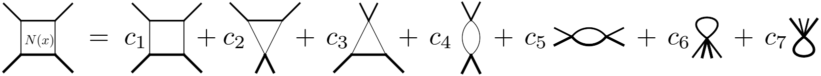
In practice this allows for a new kind of dual forms $\delta_{x_i, x_j, \ldots}$
Delta-forms extract contributions from relative boundaries

$${}_1\langle \phi | \delta_x \rangle := \mathrm{Res}_{x=0}(\phi)$$

$${}_n\langle \phi | \xi \delta_{x_{m+1} \ldots x_n} \rangle := {}_m\langle \mathrm{Res}_{x_{m+1}=0,\ldots}(\phi) | \xi \rangle$$

The delta-forms act as cutting-operators

Example: ($e\mu \to e\mu$ at one loop)



basis:

$$\phi_1 = \frac{1}{x_1x_2x_3x_4}\,,\ \phi_2 = \frac{1}{x_1x_2x_4}\,,\ \phi_3 = \frac{1}{x_2x_3x_4}\,,\ \phi_4 = \frac{1}{x_2x_4}\,,\ \phi_5 = \frac{1}{x_1x_3}\,,\ \phi_6 = \frac{1}{x_1}\,,\ \phi_7 = \frac{1}{x_3}\,.$$

dual basis:

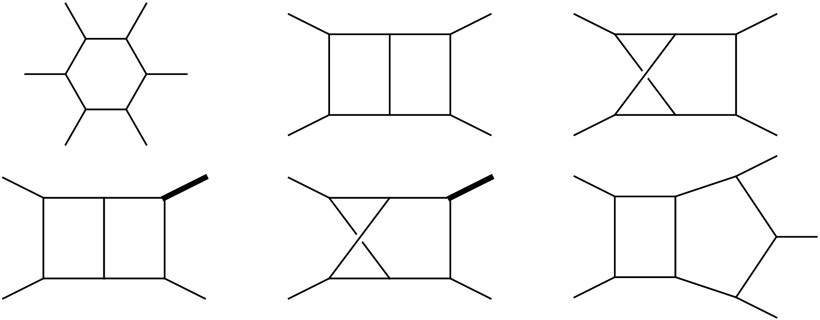
$$\check{\phi}_1 = \delta_{x_1x_2x_3x_4}\,,\ \check{\phi}_2 = \delta_{x_1x_2x_4}\,,\ \check{\phi}_3 = \delta_{x_2x_3x_4}\,,\ \check{\phi}_4 = \delta_{x_2x_4}\,,\ \check{\phi}_5 = \delta_{x_1x_3}\,,\ \check{\phi}_6 = \delta_{x_1}\,,\ \check{\phi}_7 = \delta_{x_3}\,.$$

The intersection numbers become easier to compute, many are zero, and $\boldsymbol{C}$ becomes block-triangular.

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_{1,1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{C}_{2,1} & \boldsymbol{C}_{2,2} & \ddots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{n,1} & \boldsymbol{C}_{n,2} & \cdots & \boldsymbol{C}_{n,n} \end{bmatrix}$$

No regulators needed!

Examples of complete reductions
with the relative cohomolgy approach:



[G. Brunello, V. Chestnov, G. Crisanti, HF, M.K. Mandal, P. Mastrolia (2024)]
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Moving towards the state of the art

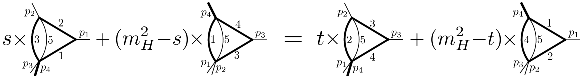
$$I_i = \int_{\mathcal{C}} u\phi_i \; = \; \langle \phi_i | \mathcal{C}] \qquad\qquad I = \sum_i c_i I_i$$

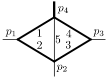
$$c_i = \langle \phi | \check{\phi}_j \rangle (\boldsymbol{C}^{-1})_{ji} \quad \text{with} \quad \boldsymbol{C}_{ij} = \langle \phi_i | \check{\phi}_j \rangle$$

To apply this you need to know the dimensionality of H^n
(i.e. the number of master integrals)

To apply the relative cohomology and the delta-forms
you additionally need to know how many there are in each sector.

magic relations:



comes from doing IBPs in 

If we pick the basis $\phi_1 = \frac{1}{x_1 x_2 x_3 x_5}$, $\phi_2 = \frac{1}{x_1 x_3 x_4 x_5}$, $\phi_3 = \frac{1}{x_2 x_3 x_4 x_5}$

what are the correct duals?

Univariate:

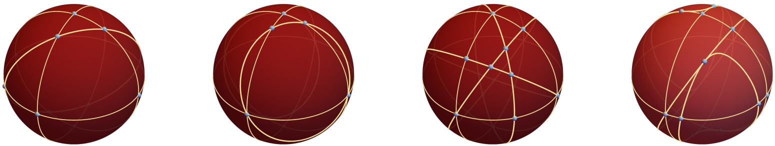
$$ {}_1\langle\phi|\check{\phi}\rangle = \sum_{p\in\mathcal{P}} \mathrm{Res}_{x=p}(\psi\check{\phi}) \quad \text{with} \quad \nabla\psi = \phi \quad \text{and} \quad \nabla = \partial_x + \omega $$

The multivariate residue approach:
[V. Chestnov, HF, F. Gasparotto, M. Mandal, P. Mastrolia (2022)]

$$ {}_n\langle\phi|\check{\phi}\rangle = \sum_{p\in\mathcal{P}} \mathrm{Res}_{p}(\psi\check{\phi}) \quad \text{with} \quad \nabla_1\nabla_2\cdots\nabla_n\psi = \phi \quad \text{and} \quad \nabla_i = \partial_{x_i} + \omega_i $$

Res_p is a *multivariate residue*

Four examples: sunrise, double-box, ${}_3F_2$, crossed-box
(all two variables, shown on $\mathbb{CP}_2$)



requires *resolution of singularities* or *blowups*
at triple-crosings (and elsewhere)

$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_0^\infty \mathrm{d}^P z \, \mathcal{G}^\beta(x) z_1^{b_1} \cdots z_P^{b_P}$$

There are parametrizations other than Baikov
such as the Lee-Pomeransky representation
(a variant of Feynman parametrization)

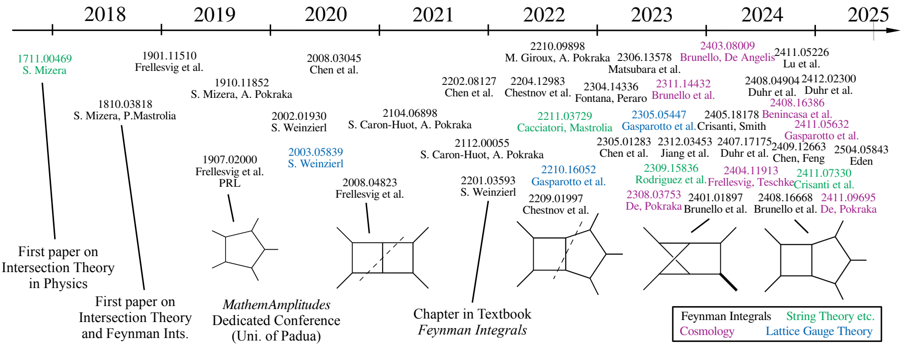
It has fewer variables, a simpler polynomial, a simpler contour...
Why not use it for intersection?

Problem: The integrals are not regulated in $z_i = 0$
Solution: Use relative cohomology again.

The delta-bases now go in the integral basis itself, and not in the dual
[M. Lu, Z. Wang, LL. Yang (2024)]

Perspectives

A timeline of intersection theory in physics



Intersection theory in physics is not just about Feynman integrals

Also path integrals, cosmological correlators, double copy integrals ...

Summary:

$$I = \sum_i c_i I_i \qquad \text{where} \qquad I_i = \int_{\mathcal{C}} u \phi_i$$

$$c_i = \langle \phi | \check{\phi}_j \rangle (\boldsymbol{C}^{-1})_{ji} \qquad \text{with} \qquad \boldsymbol{C}_{ij} := \langle \phi_i | \check{\phi}_j \rangle$$

$$\langle \phi | \check{\phi} \rangle = \sum \mathsf{Res}(\psi \check{\phi}) \qquad \text{with} \qquad (d + d\mathsf{log}(u))\psi = \phi$$

- Figure out the magic relation issue
- Better approach to the multivariate intersection number
- $\cdots$
- Make an extremely fast code

Thank you for listening!

Hjalte Frellesvig