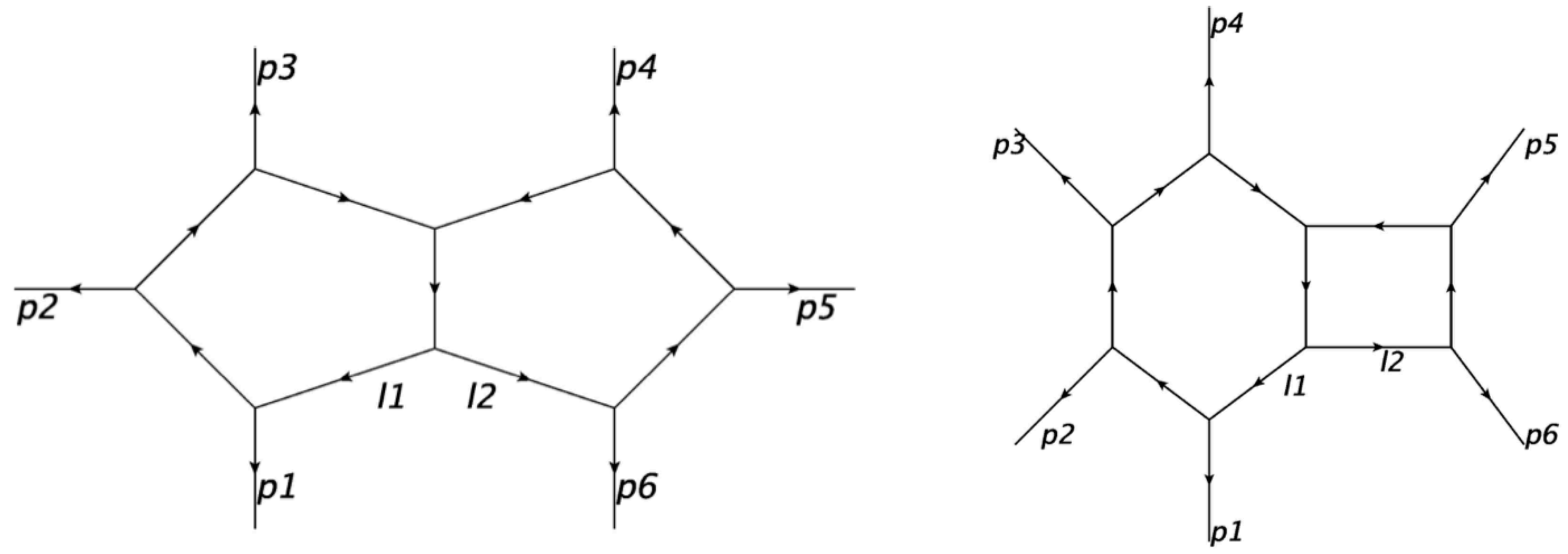
Analytic Feynman Integrals
Scattering Amplitudes
and
Wilson Loop Evaluation

New Physics Workshop
Zhangqiu
2025.07.15

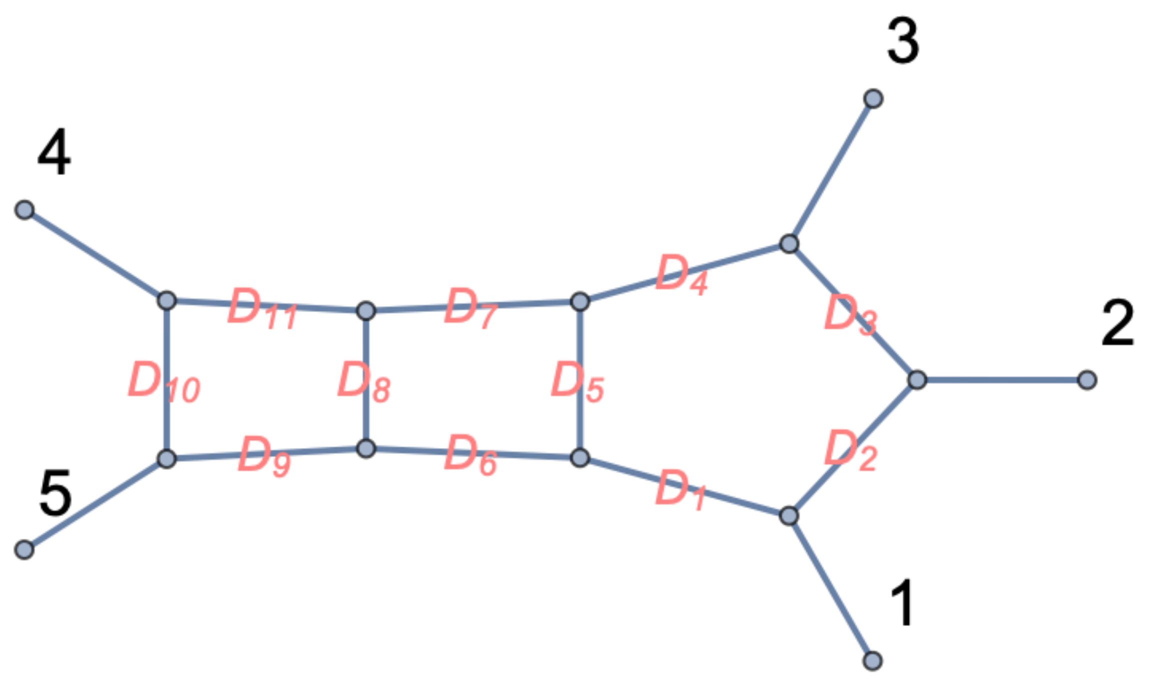
Yang Zhang
University of Science and Technology of China

Based on

Feynman integral Integral evaluation



analytic computation of all 2loop 6point massless planar integrals is done



The first analytic computation of 3loop 5-point Feynman integral family

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2501.01847 accepted in PRL

Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697 (PRD Editors' Suggestion)

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Based on package development

“**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals”
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

“Performing integration-by-parts reductions using NeatIBP 1.1 + Kira”
Wu, Boehm, Ma, Usovitsch, Xu, YZ, *arXiv: 2502.20778*

Based on perturbative QCD computations

"Two-loop amplitudes for $O(\alpha_s^2)$ corrections to $W\gamma\gamma$ production at the LHC"
Badger, Hartanto, Wu, YZ, Zoia, JHEP12(2024) 221

"Full-colour double-virtual amplitudes for associated production of a Higgs boson with a bottom-quark pair at the LHC"
Badger, Hartanto, Poncelet, Wu, YZ, Zoia, JHEP03(2025) 066

Based on Wilson loop evaluation

"Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 super Yang-Mills theory"

Carrôlo, Chicherin, Henn, Yang, YZ, accepted in JHEP

Outline

Introduction

Analytic Feynman integral

Case 1: **2loop 6point Feynman integrals**

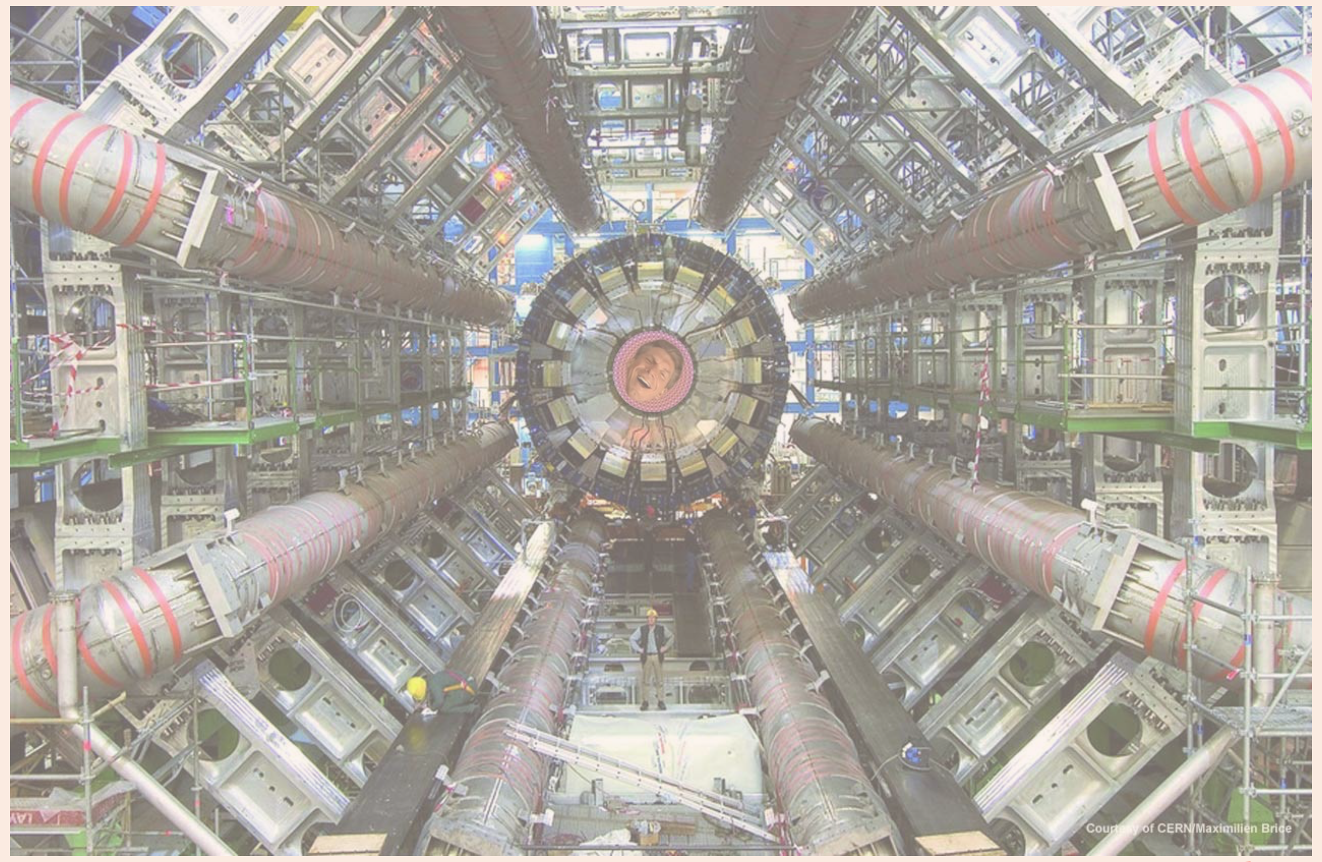
Case 2: 3loop 5point Feynman integrals

Analytic Scattering amplitudes

Wilson Loop bootstrap

Summary and Outlook

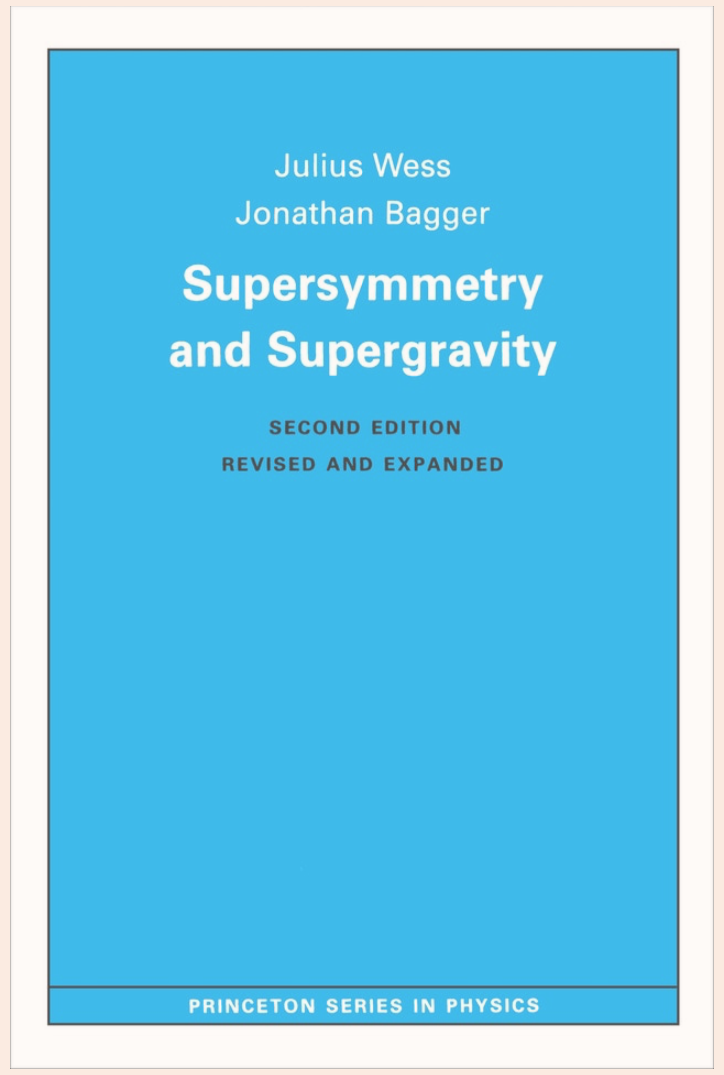
Introduction



Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

Feynman integrals, scattering amplitudes



Formal theory

N=8 supergravity UV finiteness

Gravitational wave template computations

for instance,
Driesse, Jakobsen , Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607

Why *analytic*?

- Numeric method slow or not available yet for some multi-loop multi-leg Feynman integral

  3loop 5point Feynman integrals (AMFlow not finished, pySecdec slow and inaccurate, trying AmpRed by Wen Chen)

- Theoretical aspects of quantum field theory

  for examples: 2loop N=4 SYM theory spacelike splitting amplitude

  Henn, Ma, Xu, Yan, *YZ*, Zhu, arXiv 2406.14604

- Quantum field theory computation of gravitational wave

  analytic continuation/ Fourier transform is sometimes needed

Two approaches in perturbative QFT

Feynman diagram based

Straightforward, Brute force

Heavy computation (IBP reduction)

works for all perturbative QFTs

Bootstrap

obtain the result for the Ansatz directly

easy computation

previously, usually works for highly symmetric QFTs

with the information of
the Feynman integral function space

The situation may be different soon …

Analytic Feynman integrals

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2501.01847 PRL accepted
JHEP 08(2024) 027, arXiv:2501.01847
Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
arXiv:2411.18697 PRD editor's suggestion

Goal of analyticity

Feynman integral

arguments related to
Letters, algebraic function of kinematics

$$I = \sum_{i=-2L} \epsilon^i \sum_{\alpha} c_\alpha G(W_{\alpha_1}, \ldots, W_{\alpha_{2L+i}}; z)$$

Goncharov polylogarithm function

$$G(\mathbf{0}_k; z) = \frac{1}{k!}(\log z)^k, \quad G(a_1, \ldots, a_k; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots a_k; t)$$

Dimensional regularization parameter

well studied function with Hopf algebra structure

For more complicated cases, iterative integral of elliptic functions, Calabi-Yau functions can appear

Canonical Differential Equation, new insights

Better Integration-by-parts (IBP) reduction

NeatIBP, Wu, Boehm, Ma, Xu, *YZ* 2023

Comput.Phys.Commun. 295 (2024) 108999

Blade, Guan, Liu, Ma, Wu 2024

Comput.Phys.Commun. 310 (2025) 109538

Alphabet searching

Effortless, Matijasic, Miczajka to appear

https://github.com/antonela-matijasic/Effortless

BaikovLetter, Jiang, Liu, Xu, Yang, 2401.07632

PLD, Fevola, Mizera, Telen

Comput. Phys. Commun. 303 (2024) 109278

SOFIA Correia, Giroux, Mizera

2503.16601

Solving differential equation

Novel representation of one-fold integration

Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697

2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *arXiv:2501.01847*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

2loop Feynman integral: Scale frontier

2loop 5point massless	*Gehrmann, Henn, Lo Presti 2015* *Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019*	5 scales
2loop 5point one-mass	*Papadopoulos, Tommasini, Wever 2019* *Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020* *Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2023* *Jiang,Liu,Xu,Yang 2024* *Badger, Becchetti,Giraudo, Zoia 2024*	6 scales
2loop 5point two-mass	*Cordero, Figueiredo, Kraus, Page and Reina 2023* for leading-Color pp→ttH amplitudes with a light-quark loop	7 scales
2loop 6point massless	*Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2024* *for NNLO 4 jets production, 2 jets+ 2photons*	8 scales!

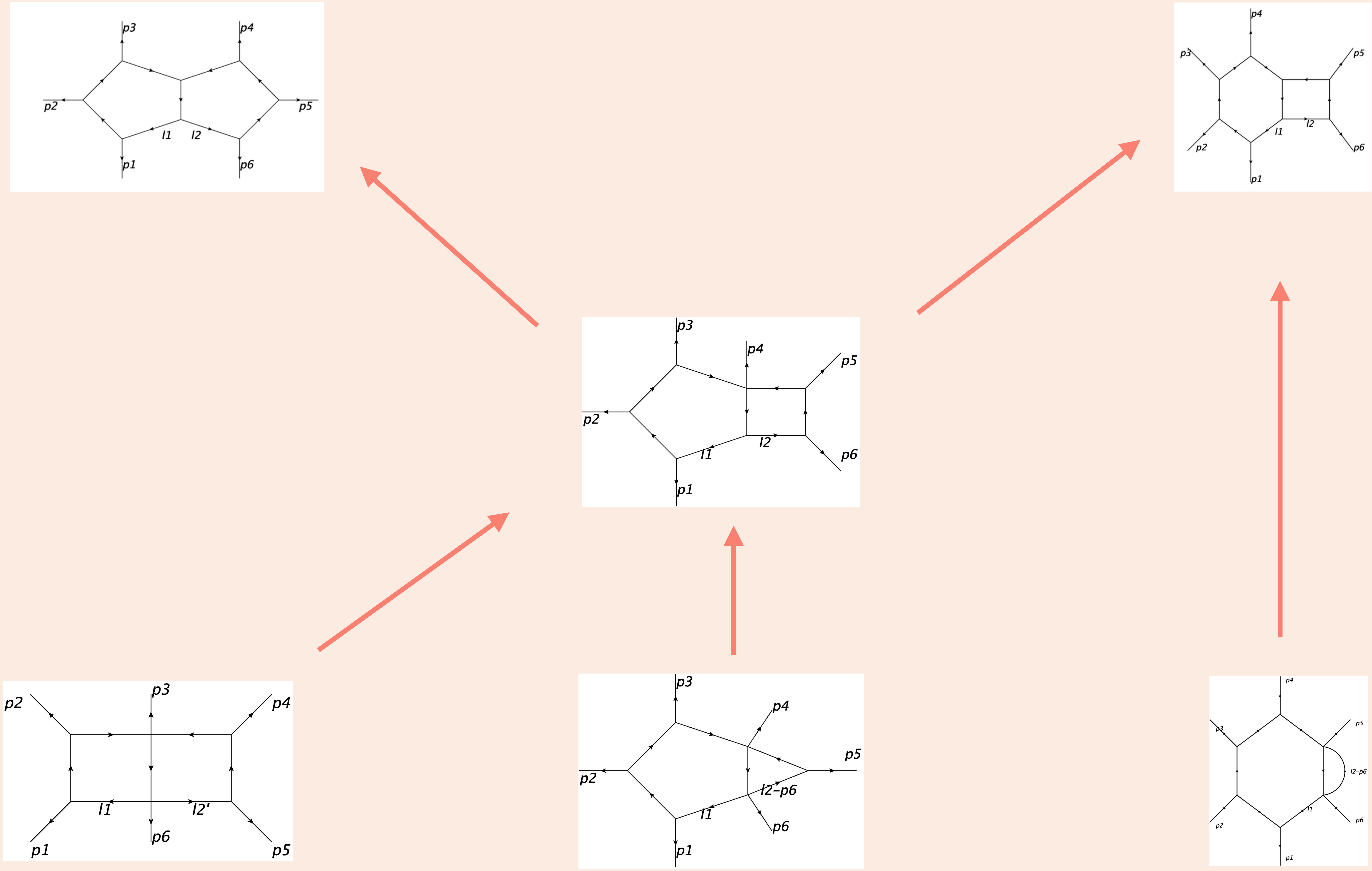
$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{345}$

All planar 2loop 6point integrals

J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, 2501.01847

267
UT integrals

202
UT integrals



Momentum Twistor

external momenta $d=4$

$$p_i = x_{i+1} - x_i$$

dual coordinates

$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma} \lambda_{A,\gamma} = \mu_{A,\dot\beta}$$
$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma} \lambda_{B,\gamma} = \mu_{B,\dot\beta}$$

$$(Z_A, Z_B) \to x \qquad (Z_i, Z_{i+1}) \to x_{i+1}$$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1, \ldots 6$$

momentum twistor

$\langle Z_A Z_B Z_C Z_D \rangle$ is dual conformally invariant

$$\tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle},$$

$$(\lambda_{i,\alpha}, \tilde\lambda_{i,\dot\alpha})$$

spinor helicity

Momentum Twistor external momenta $d=4$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1,\dots 6 \qquad \tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle\langle i-1,\, i\rangle},$$

A particular parameterization Badger, Frellesvig, YZ 2013

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_2x_1} + \frac{1}{x_1} & \frac{1}{x_2x_1} + \frac{1}{x_2x_3x_1} + \frac{1}{x_1} & \frac{1}{x_2x_1} + \frac{1}{x_2x_3x_1} + \frac{1}{x_2x_3x_4x_1} + \frac{1}{x_1} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_5}{x_2} & x_6 & 1 \\ 0 & 0 & 1 & 1 & x_7 & 1 - \frac{x_8}{x_5} \end{pmatrix}$$

$$x_1 = s_{12}$$

$$x_2 = -\frac{\mathrm{Tr}_+(1234)}{2s_{12}s_{34}}$$

$$x_3 = -\frac{\mathrm{Tr}_+(1345)}{2s_{45}s_{13}}$$

$$x_4 = -\frac{\mathrm{Tr}_+(1456)}{2s_{56}s_{14}}$$

$$x_5 = \frac{s_{23}}{s_{12}}$$

$$x_6 = -\frac{\mathrm{Tr}_+(1532) + \mathrm{Tr}_+(1542)}{2s_{15}s_{12}}$$

$$x_7 = 1 + \frac{\mathrm{Tr}_+(1542) + \mathrm{Tr}_+(1543)}{2s_{15}s_{23}}$$

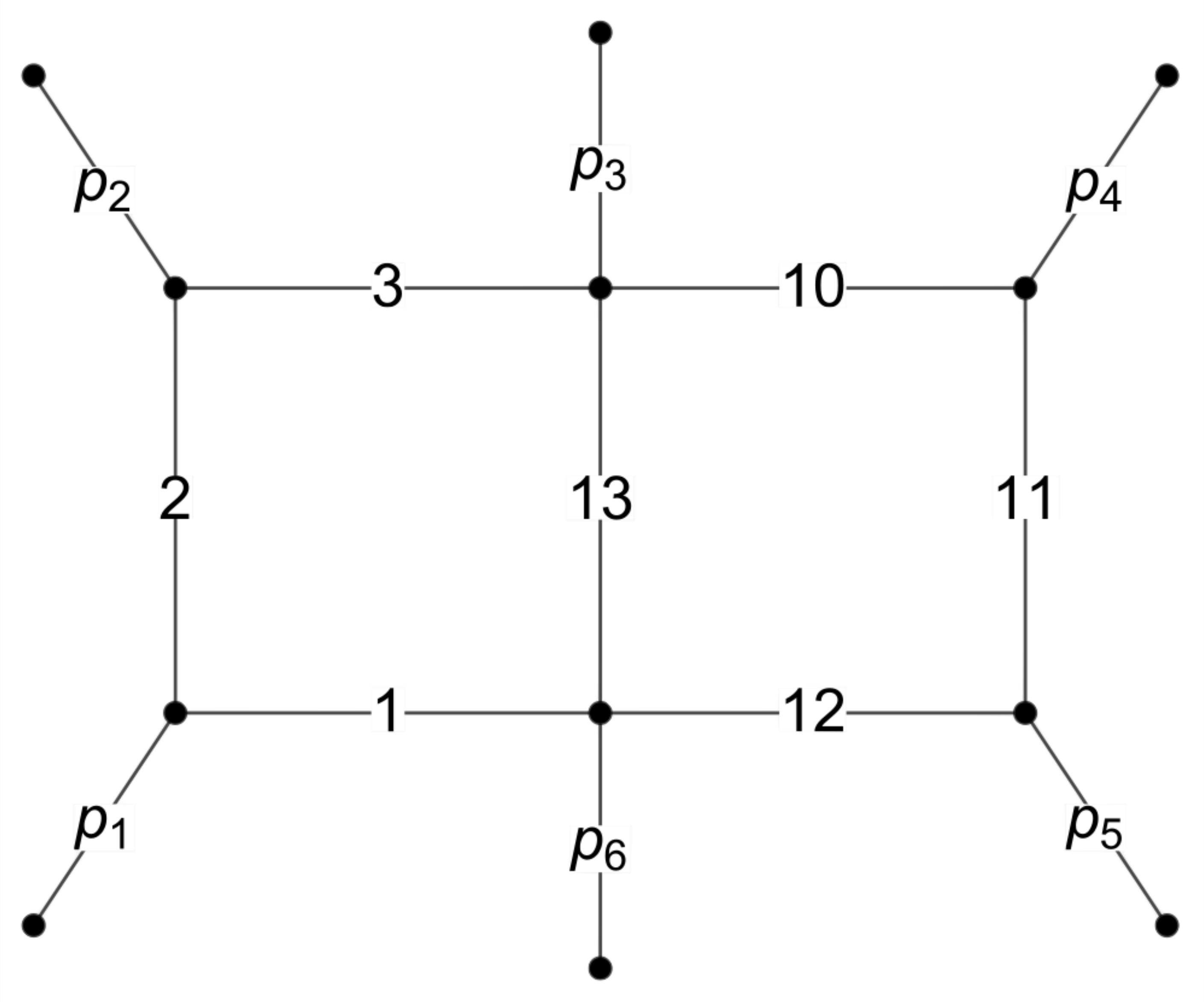
$$x_8 = \frac{s_{123}}{s_{12}}$$

Momentum parametrization rationalizes all pseudo scalars

$$\epsilon_{ijkl} \equiv 4i\epsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma, \quad \epsilon_{ijkl}^2 = G_{ijkl}$$

Uniformly transcendental (UT) basis determination

key step



$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}, \quad i = 1, \ldots, 7$$

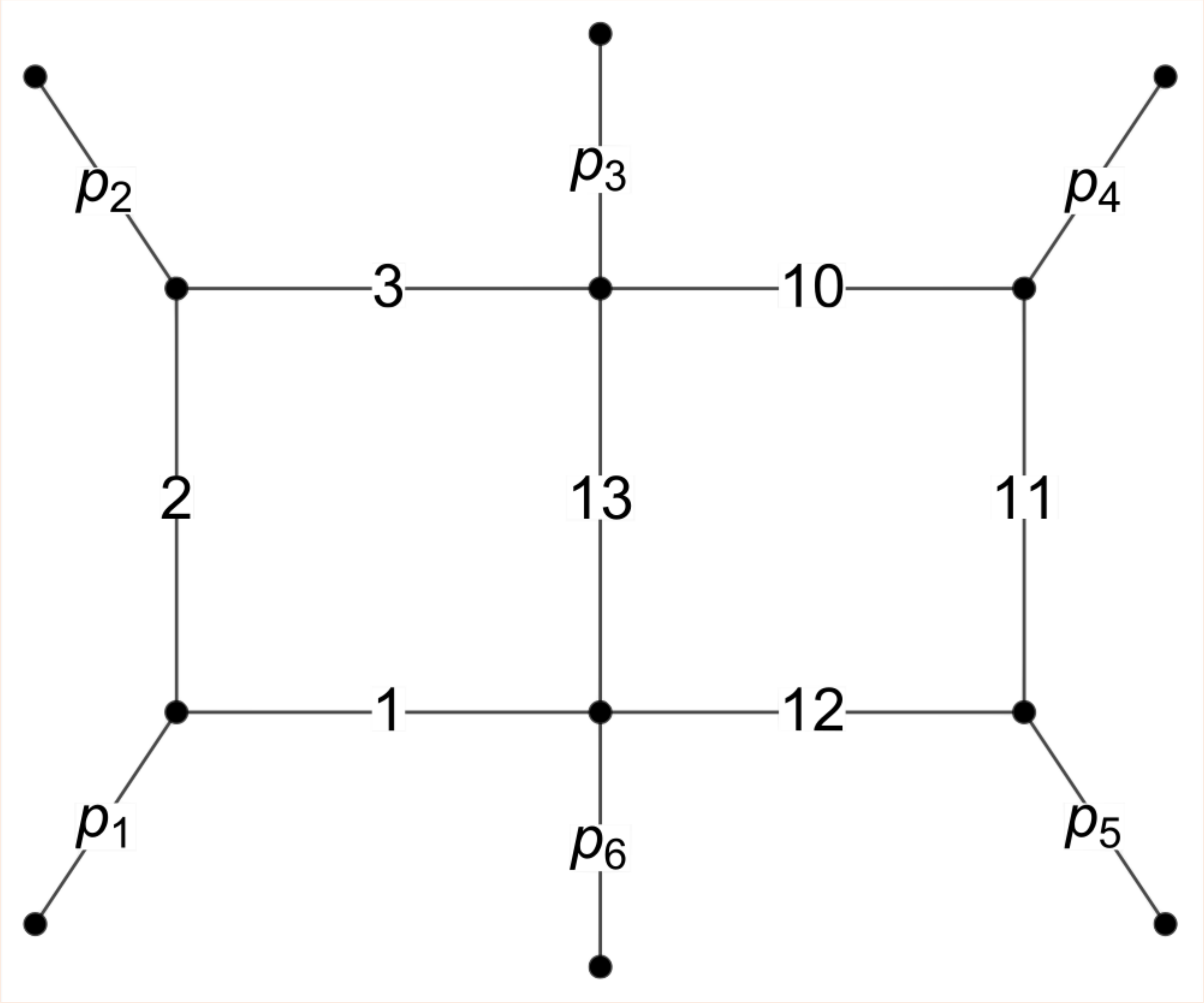
$$\begin{aligned}
N_1 &= -s_{12} s_{45} s_{156}\,, \\
N_2 &= -s_{12} s_{45} (l_1 + p_5 + p_6)^2\,, \\
N_3 &= \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix}, \\
N_4 &= \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix}, \\
N_5 &= -\frac{1}{4} \frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix}, \\
N_6 &= \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11} (s_{123} + s_{126})}{8}\,, \\
N_7 &= -\frac{1}{2\epsilon} \frac{\Delta_6}{G(1,2,4,5) D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.
\end{aligned}$$

Chiral numerator
(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)
/ Gram determinant
correspondence

$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP08(2024)027*

Chiral numerator to UT integral numerators



linear combination

$$\mathcal{N}_A = s_{45}\big(\langle 15\rangle[52] + \langle 16\rangle[62]\big) l_1 \cdot (\lambda_2 \tilde{\lambda}_1)\,,$$
$$\mathcal{N}_B = s_{45}\big([15]\langle 52\rangle + [16]\langle 62\rangle\big) l_1 \cdot (\lambda_1 \tilde{\lambda}_2)\,.$$

parity even

$$\mathcal{N}_A + \mathcal{N}_B = -\frac{1}{2} s_{12} s_{45} (l_1 + p_5 + p_6)^2 + \frac{1}{2} s_{12} s_{45} s_{156} + \ldots$$

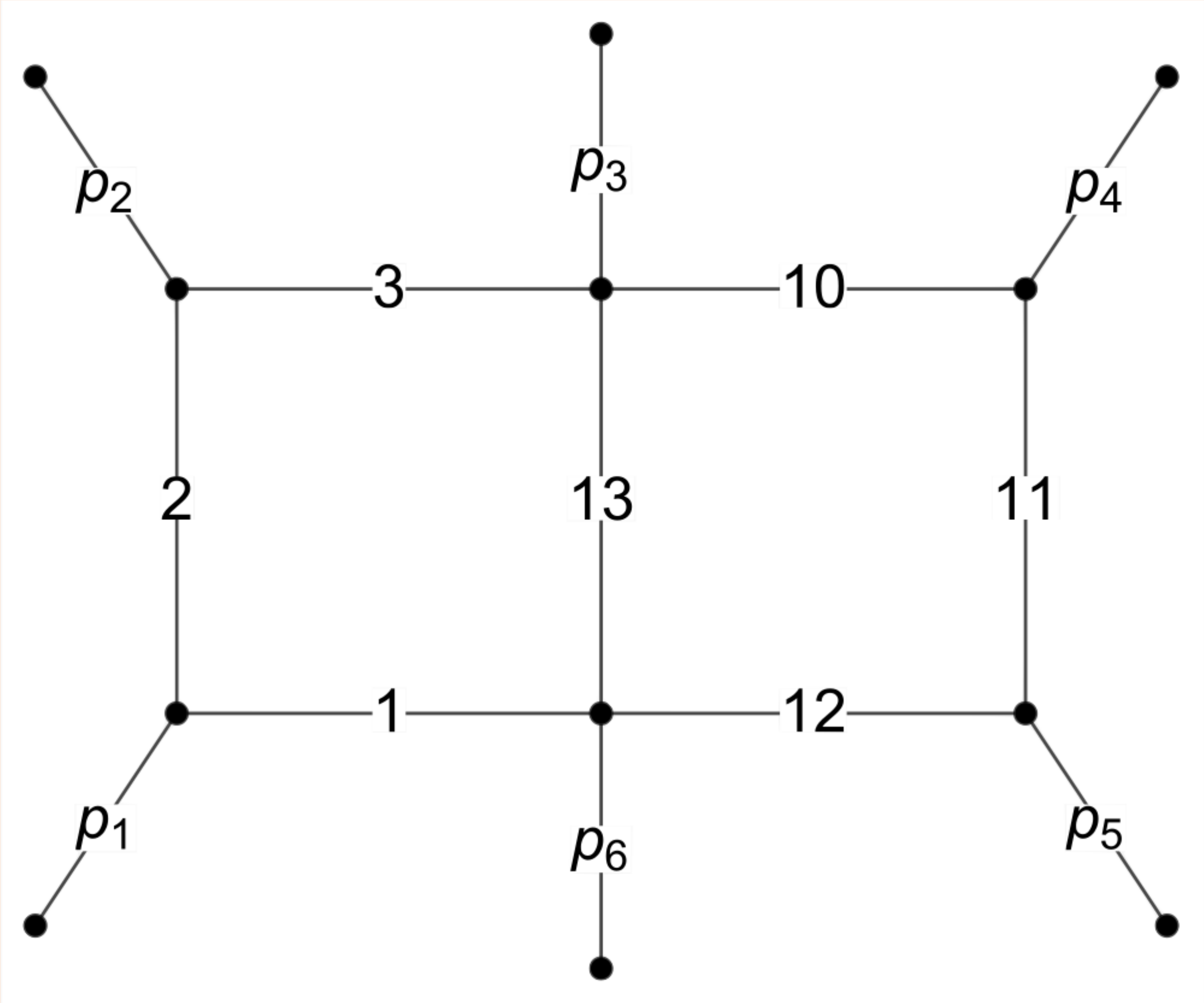
parity odd

$$\mathcal{N}_A - \mathcal{N}_B = \frac{-8 s_{45} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_5 & p_1 & p_2 & p_6 \end{pmatrix}}{\epsilon_{5126}},$$

Chiral numerator to UT integral numerators

quadratic combination

$$s_{24}\frac{\langle 15\rangle}{\langle 42\rangle}(l_1\cdot\lambda_2\tilde{\lambda}_1)(l_2'\cdot\lambda_4\tilde{\lambda}_5)$$



parity even

$$\frac{1}{8}G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2-p_6 & p_4 & p_5 \end{pmatrix}+\frac{D_2D_{11}(s_{123}+s_{126})}{8}$$

additional term added
from the canonical DE construction

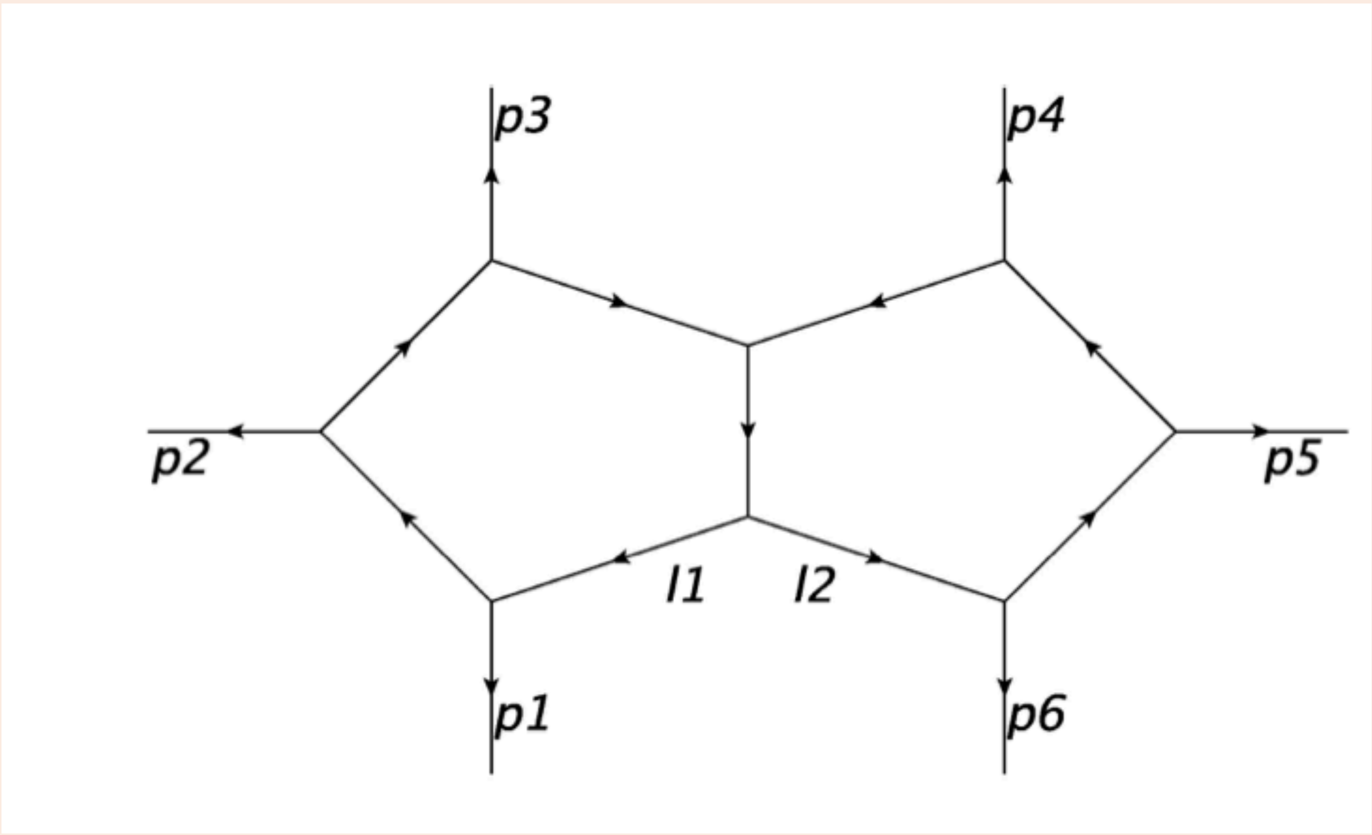
parity odd

One-loop
hexagon leading singularity

$$\Delta_6=\langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61]-\langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

$$-\frac{1}{2\epsilon}\frac{\Delta_6}{G(1,2,4,5)D_{13}}G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}$$

2loop 6point top sector, UT integrals



J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, *2501.01847*

UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9}$$ → evanescent

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9}$$ → evanescent

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9}$$ → evanescent, 6D weight-6 integral

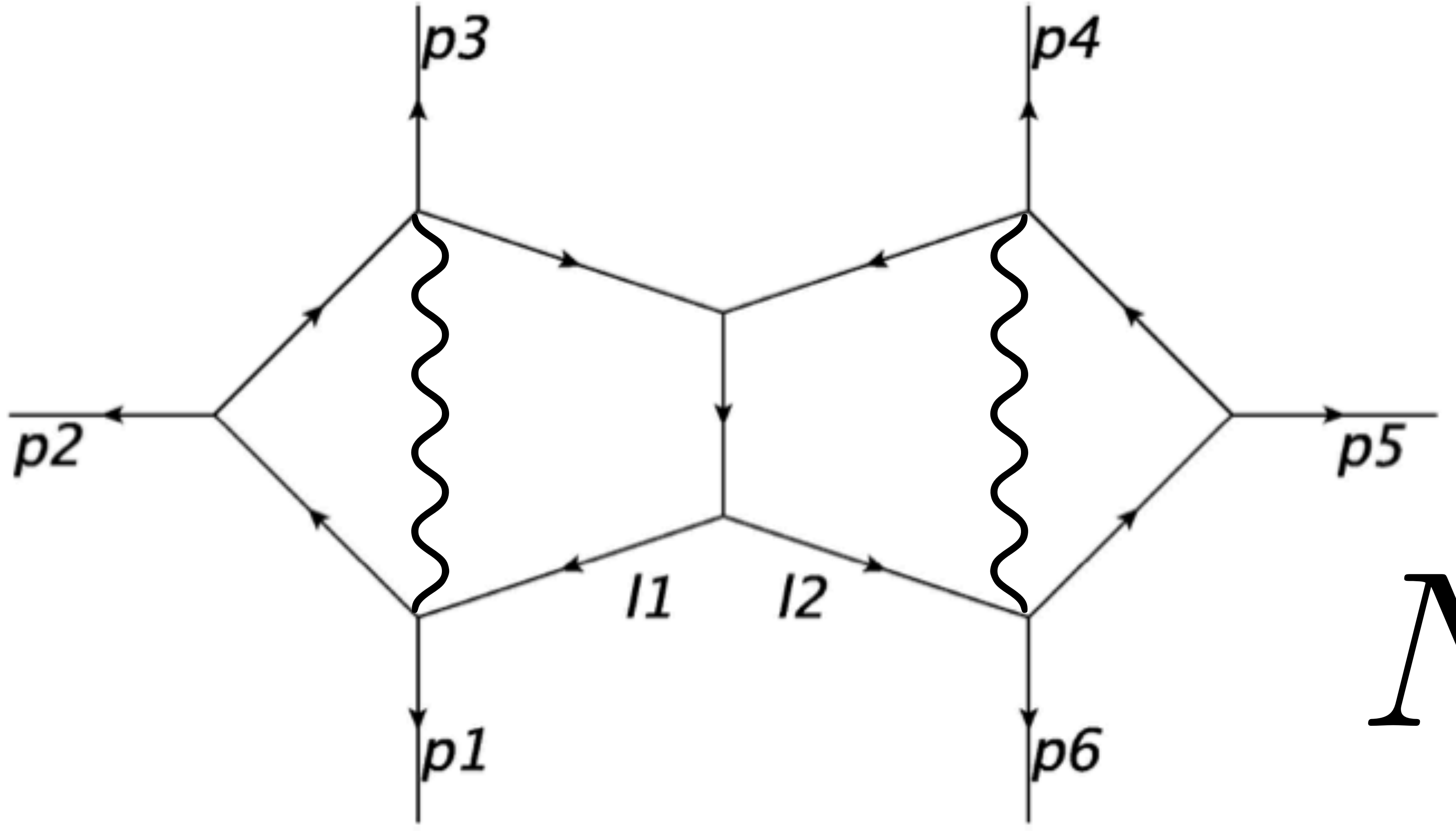
$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5\mu_{12}}{D_1 \dots D_9}$$

5 MIs (this sector)
267 MIs (whole family)

245 letters in total
except the 6D ones

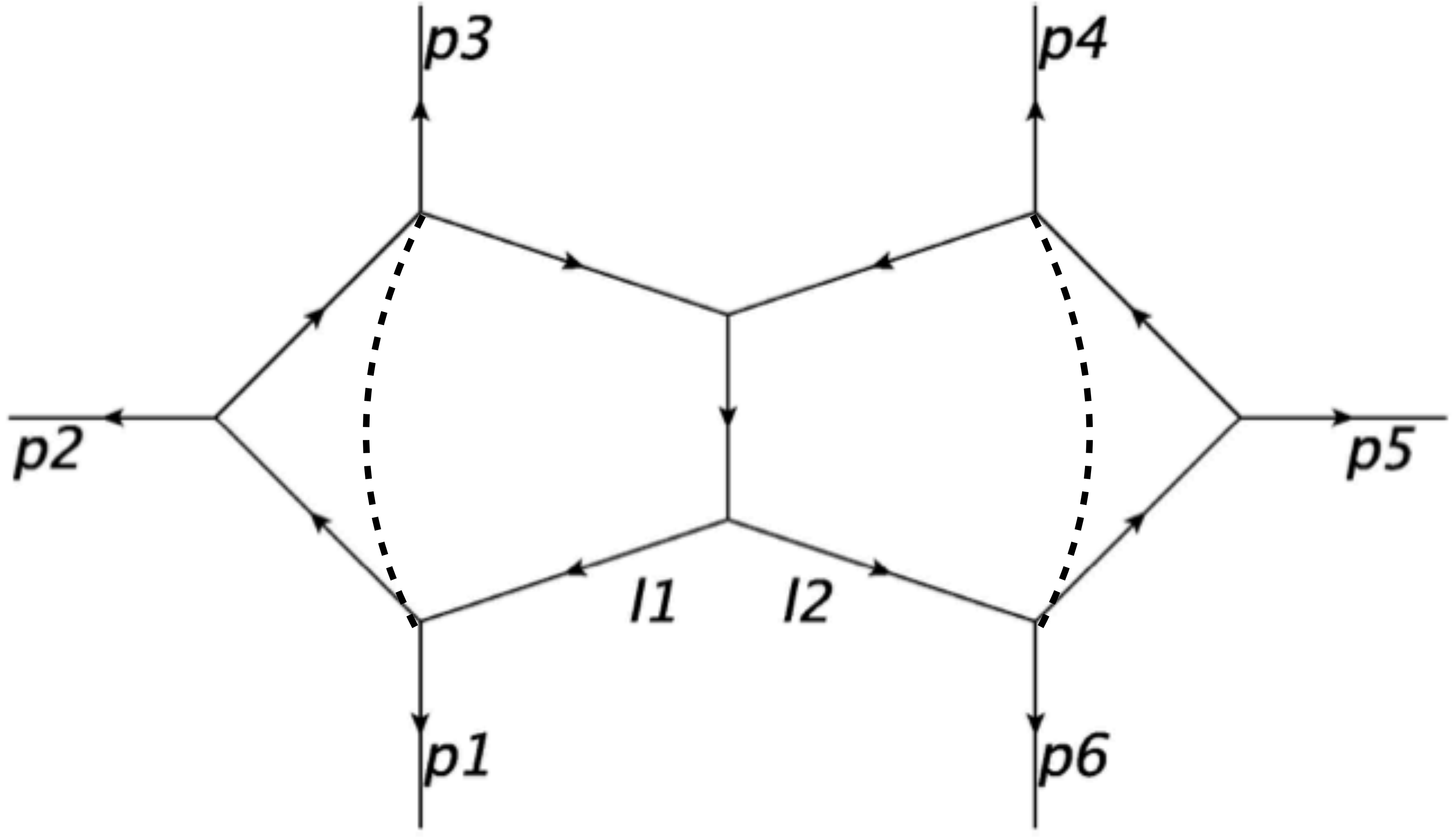
"evanescent": vanishing up to ϵ^0

N_1, N_2, N_3 and N_4 are chiral numerators — Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010



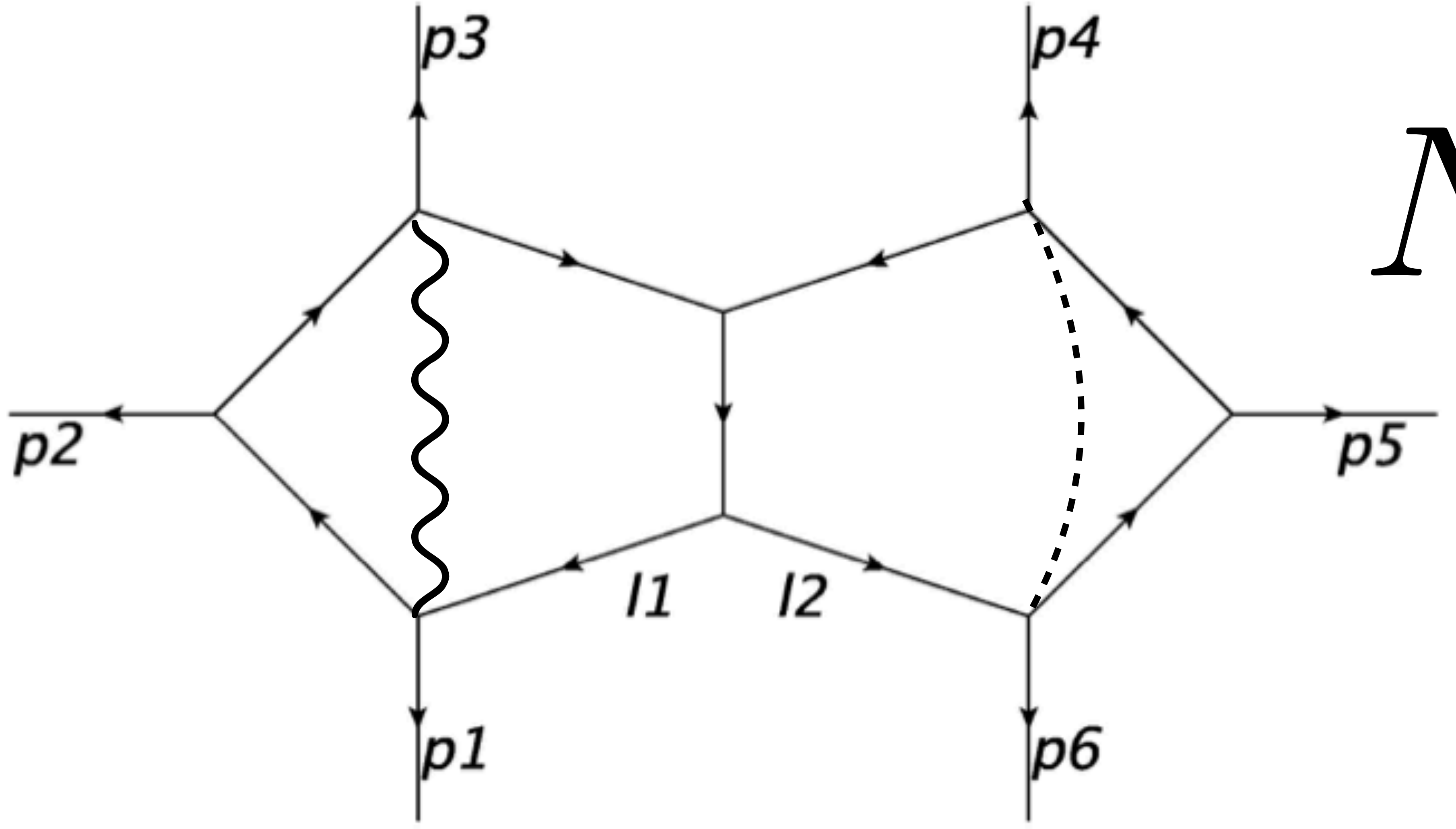
N_1

N_4



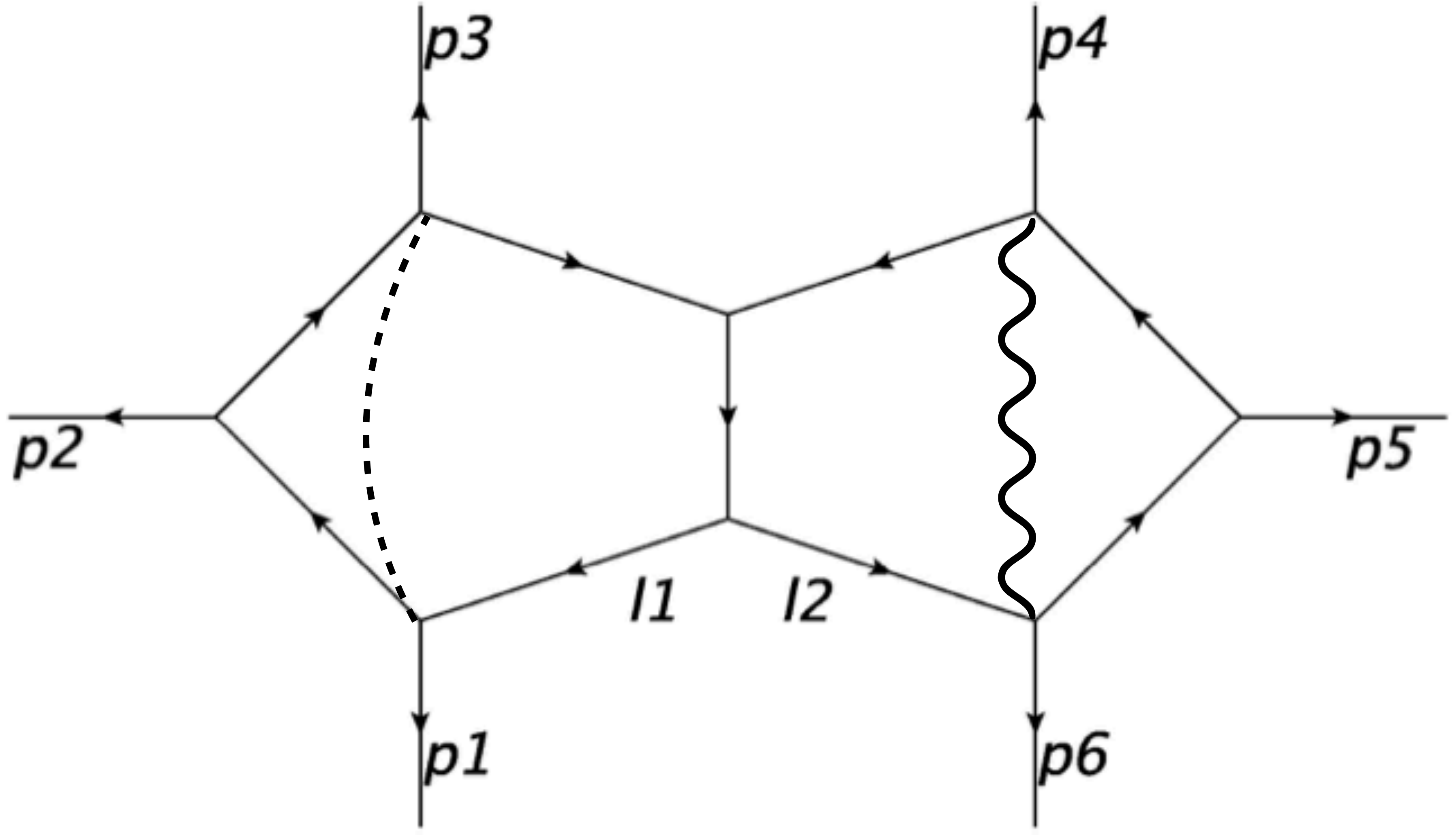
$$ut_2 = I[N_2 - N_3] = -2\tilde{\Omega}_{\text{odd}} + O(\epsilon)\,,$$
$$ut_5 = I[N_1 + N_4] = 2\Omega_{\text{even}} + O(\epsilon)$$

chiral-numerator integrals are finite and calculated to weight-4, Dixon, Drummond, Henn 2011



N_2

N_3



Complete canonical differential equation for 2l6p planar integrals

Use momentum twistor Variables

267 × 267 for double pentagon
202 × 202 for hexagon box

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

$$A_i = \frac{\partial}{\partial x_i}\tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

use alphabet to fit the canonical differential equation

Even letter, Odd letter and the more complicated …

Even letter $F(s)$ a polynomial in Mandelstam variables or homogeneously linear in square roots

Conjecture: a Feynman integrals' even letters are all from Landau singularity?

Odd letter $\frac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

"square roots": ϵ_{ijkl}, Δ_6, , $\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo scalar

leading singularity hexagon

Källin function from massive triangle diagrams

More complicated letter $\frac{P(s)-\sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s)+\sqrt{Q_1(s)}\sqrt{Q_2(s)}}$

A new algorithm to search for odd letters

Odd letter $\frac{P(s) - \sqrt{Q(s)}}{P(s) + \sqrt{Q(s)}}$

$$P^2 - Q = c \prod_i W_i^{e_i}, \quad c \in \mathbb{Q}, \quad e_i \in \mathbb{N}$$

Even letter

An observation (and conjecture) from Heller, von Manteuffel, Schabinger 2020

Algorithm to solve for e_i

Matijasic, J. Miczajka, to appear

Effortless

https://github.com/antonela-matijasic/Effortless

Even letter, Odd letter and the more complicated ...

245 letters

156 Even letters

$$s_{12}, \quad s_{123}$$
$$s_{12} - s_{123}$$
$$\cdots$$
$$-s_{12}s_{45} + s_{123}s_{345}$$
$$\cdots$$
$$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}$$

79 Odd letters

$$\frac{s_{12} + s_{34} - s_{56} - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{s_{12} + s_{34} - s_{56} + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}$$
$$\cdots$$
$$\frac{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) - \epsilon_{1234}}{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) + \epsilon_{1234}},$$
$$\cdots \quad \ldots$$
$$\frac{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) - \Delta_6}{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) + \Delta_6}$$

10 More complicated letters

$$\frac{P - \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \quad \ldots$$

some letters not found by PLD.jl
Fevola, Mizera, Telen
or
BaikovLetter
Jiang, Liu, Xu, Yang,

but after our paper,
the recent package
SOFIA claims to find all letters
Correia, Giroux, Mizera

Then the canonical DE is derived analytically after ~200 times of numeric IBP running

Boundary Values

Numeric boundary values

It is fine to use the package AMFlow to get ~100 digits as the boundary value for double-box, pentagon-triangle, hexagon-bubble diagrams

Liu, Wang, Ma, 2018
Liu, Ma 2022

Analytic boundary values

It is still possible to get *fully analytic* boundary values

$$X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1,-1,-1,-1,-1,-1,-1,-1,-1\}$$

Solve the canonical DE on a curve starting with X_0 and require the finite solution
Some known integrals' boundary values

} analytic boundary value

boundary value for a point in the physical region also obtained

Boundary Values

Analytic boundary values

Boundary values at the initial point, are combination of poly-logarithm of roots of unity

$$
\begin{aligned}
\epsilon^4 I_{\mathrm{db},1}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4, \\
\epsilon^4 I_{\mathrm{db},2}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4, \\
I_{\mathrm{db},3}(X_0) &= I_{\mathrm{db},4}(X_0) = I_{\mathrm{db},5}(X_0) = 0, \\
\epsilon^4 I_{\mathrm{db},6}(X_0) &= -\left(\frac{\pi^4}{540} + \frac{4}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4, \\
\epsilon^4 I_{\mathrm{db},7}(X_0) &= 0.
\end{aligned}
$$

from the ordinary differential equation

spurious pole asymptotic analysis

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Solution of canonical DE

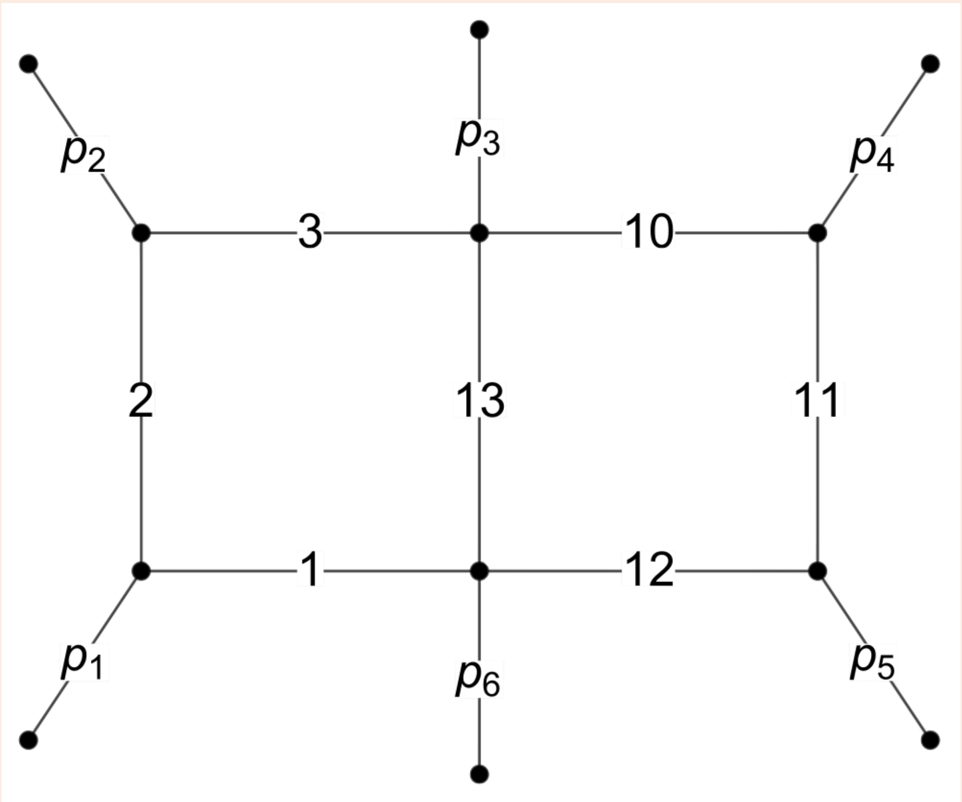
$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-1, weight-2

All in logarithm and classical poly-logarithm



$$\begin{aligned}
I^{(2)}_{\text{db},1} = & -\log(-v_1)\log(-v_2) - \log(-v_1)\log(-v_3) + \log(-v_1)\log(-v_4) - \log(-v_1)\log(-v_5) - \\
& \log(-v_1)\log(-v_6) + 4\log(-v_1)\log(-v_8) + \frac{1}{2}\log^2(-v_1) + \log(-v_2)\log(-v_3) - \\
& \log(-v_2)\log(-v_4) - \mathrm{Li}_2\left(1 - \frac{v_2 v_5}{v_7 v_8}\right) + \log(-v_2)\log(-v_6) + \log(-v_2)\log(-v_7) - \\
& 2\mathrm{Li}_2\left(1 - \frac{v_2}{v_8}\right) - \log(-v_2)\log(-v_8) - \log^2(-v_2) - \log(-v_3)\log(-v_4) + \log(-v_3)\log(-v_5) - \\
& \mathrm{Li}_2\left(1 - \frac{v_3 v_6}{v_8 v_9}\right) - 2\mathrm{Li}_2\left(1 - \frac{v_3}{v_8}\right) - \log(-v_3)\log(-v_8) + \log(-v_3)\log(-v_9) - \\
& \log^2(-v_3) - \log(-v_4)\log(-v_5) - \log(-v_4)\log(-v_6) + 4\log(-v_4)\log(-v_8) + \\
& \frac{1}{2}\log^2(-v_4) + \log(-v_5)\log(-v_6) + \log(-v_5)\log(-v_7) - 2\mathrm{Li}_2\left(1 - \frac{v_5}{v_8}\right) - \log(-v_5)\log(-v_8) - \\
& \log^2(-v_5) - 2\mathrm{Li}_2\left(1 - \frac{v_6}{v_8}\right) - \log(-v_6)\log(-v_8) + \log(-v_6)\log(-v_9) - \log^2(-v_6) - \\
& \log(-v_7)\log(-v_8) - \frac{1}{2}\log^2(-v_7) - \log(-v_8)\log(-v_9) + 3\log^2(-v_8) - \frac{1}{2}\log^2(-v_9) + \\
& \frac{\pi^2}{6}
\end{aligned}$$

Solution of canonical DE

$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-3, weight-4

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1} \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2} \vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{f}^{(2)}(t)\right).\end{aligned}$$

one-fold integration

It takes minutes on a laptop to get 14 digits from our analytic solution

in both Euclidean and Physical regions

from the request of John Ellis …

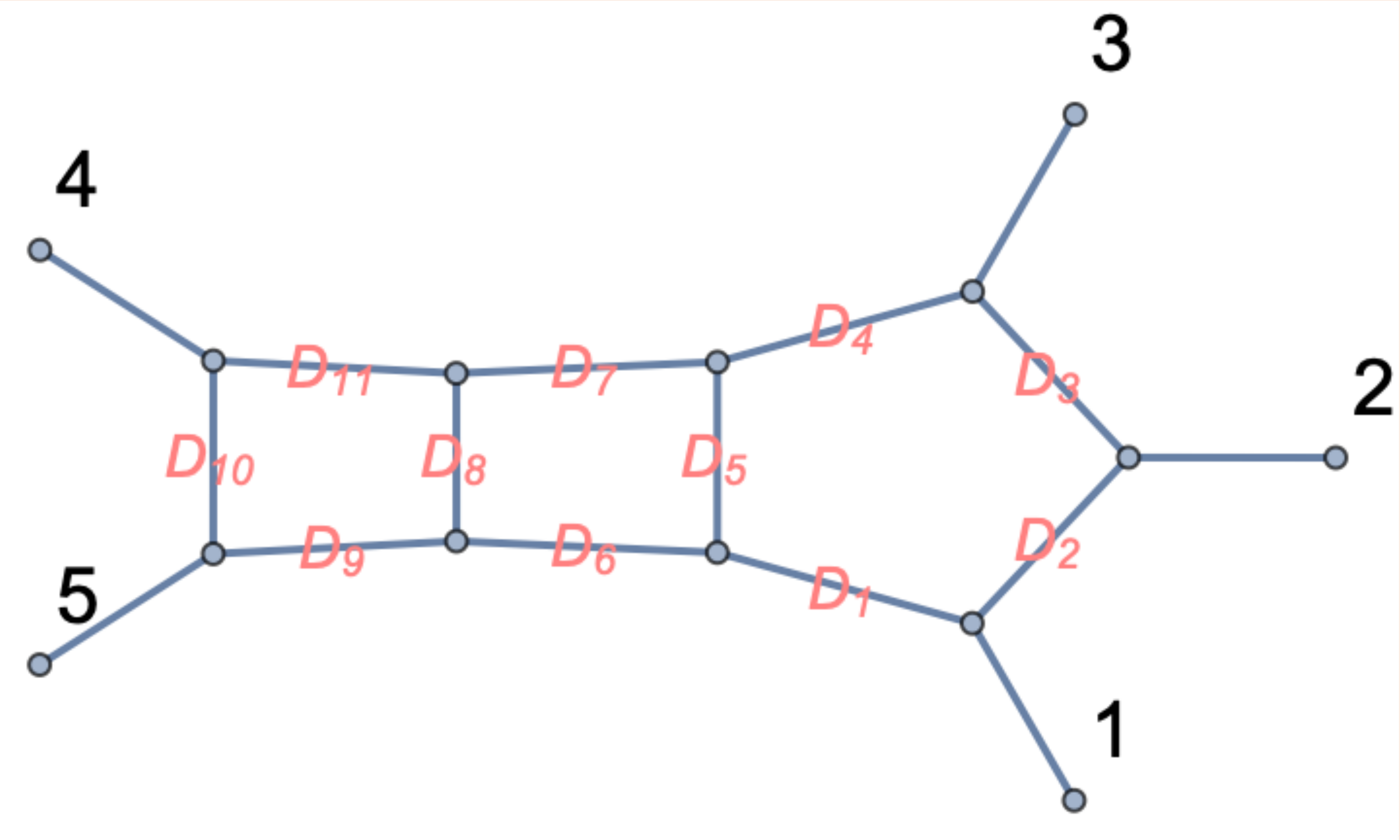
3loop 5point Feynman integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, arXiv: 2411.18697 PRD editor's suggestion

↑

"State-of-the-art calculations exploiting this method cover classes of FIs with up to three-loop and five-scales", Prisco, Ronca, Tramontano

3loop 5point planar family



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, arXiv: 2411.18697

UT basis found!

Baikov analysis
Gram determinant

Canonical differential equation complicated ?

We use NeatIBP to find the differential equation

hard to integrate to weight-6?

A novel one-fold representation

Using NeatIBP

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

Using algebraic geometry (module intersection) to find short IBP system,
2 or 3 orders of magnitudes shorter than that from Laporta algorithm.

Studies that used NeatIBP	**Reference**
Differential Equations for Energy Correlators in Any Angle	arXiv:2506.02061
One-loop amplitudes for $t\bar{t}j$ and $t\bar{t}\gamma$ productions at the LHC through $O(\epsilon^2)$	arXiv:2505.10406
Two-loop Feynman integrals for leading color $Wt\bar{t}$ production	JHEP 07 (2025) 001
Two-loop QCD helicity amplitudes for $gg \to gt\bar{t}$ at leading color	JHEP 03 (2025) 070
Full-color double-virtual amplitudes for $q\bar{q} \to b\bar{b}H$	JHEP 03 (2025) 066
Three-loop five-point pentagon-box-box Feynman diagram	arXiv:2411.18697
Two-loop QCD corrections for $pp \to t\bar{t}j$	arXiv:2411.10856
Two-loop amplitudes for $W\gamma\gamma$ production at LHC	JHEP 12 (2025) 221
NLO corrections to $J/\Psi c\bar{c}$ photoproduction	Phys.Rev.D 110 (2024) 9, 094047
Two-loop five-point two-mass planar integrals	JHEP 10 (2024) 167
Two-loop integrals for $t\bar{t}j$ production at hadron colliders in the leading color approximation	JHEP 07 (2024) 073

phenomenology application of NeatIBP

A novel representation of iterative integrals

$$\mathbf{I}^{(n+2)}(x) = \mathbf{I}^{(n+2)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}(t)}{\mathrm{d}t}\mathbf{I}^{(n+1)}(x_0)\mathrm{d}t + \int_0^1 (\tilde{A}(1) - \tilde{A}(t))\frac{\mathrm{d}\tilde{A}(t)}{\mathrm{d}t}\mathbf{I}^{(n)}(t)\mathrm{d}t.$$

Weight $+2$

A novel formula

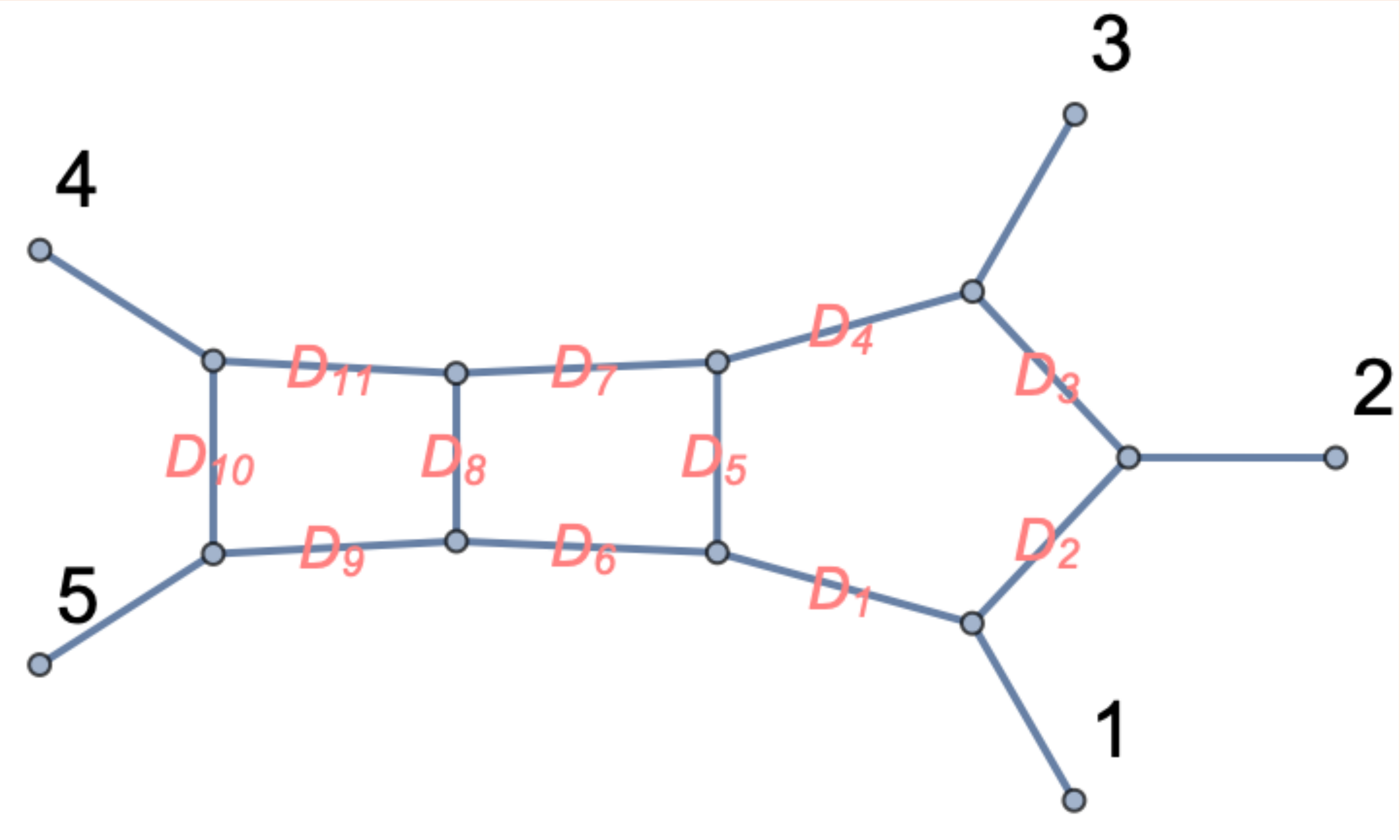
$$\mathrm{d}\tilde{B} = (\mathrm{d}\tilde{A})\tilde{A},$$

$\tilde{B}$ exists due to Poincare lemma

$$\begin{aligned}\mathbf{I}^{(n+3)}(x) =& \mathbf{I}^{(n+3)}(x_0) + \int_0^1 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n+2)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(1) - \tilde{A}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n+1)}(x_0)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{A}(t) - \tilde{A}(1)\right)\tilde{A}(t)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n)}(t)\mathrm{d}t \\ &+ \int_0^1 \left(\tilde{B}(1) - \tilde{B}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\mathbf{I}^{(n)}(t)\mathrm{d}t.\end{aligned}$$

Weight $+3$

What we achieved



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

✓ UT basis found!

✓ Canonical differential equation found with NeatIBP

31 letters …

All boundary values up to weight-6 are obtained by spurious pole analysis as GPL values

✓ First 3loop 5point integral family evaluated
weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes 2 minutes on a laptop to get 10 digits from our analytic solution

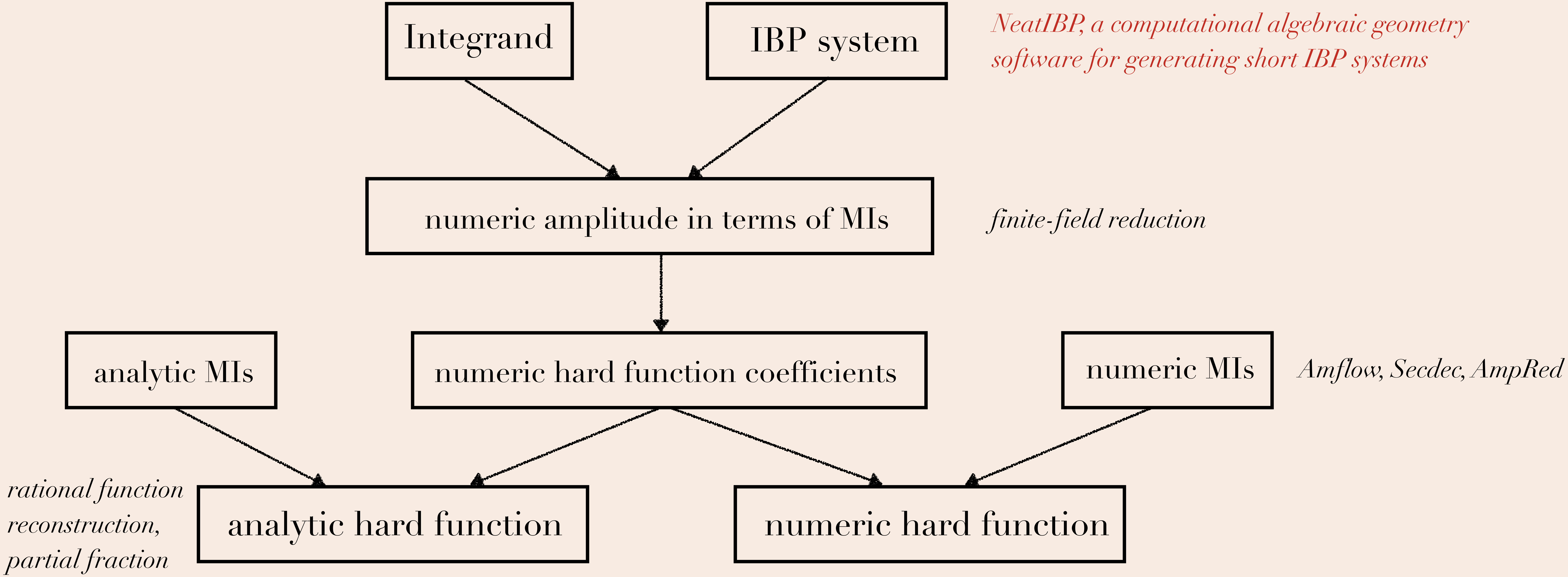
Liu, Matijasic, Miczajka, Xu, Xu, YZ, *arXiv: 2411.18697*

Analytic Scattering Amplitudes

Badger, Hartanto, Wu, YZ, Zoia, JHEP12(2024) 221

Badger, Hartanto, Poncelet, Wu, YZ, Zoia, JHEP03(2025) 066

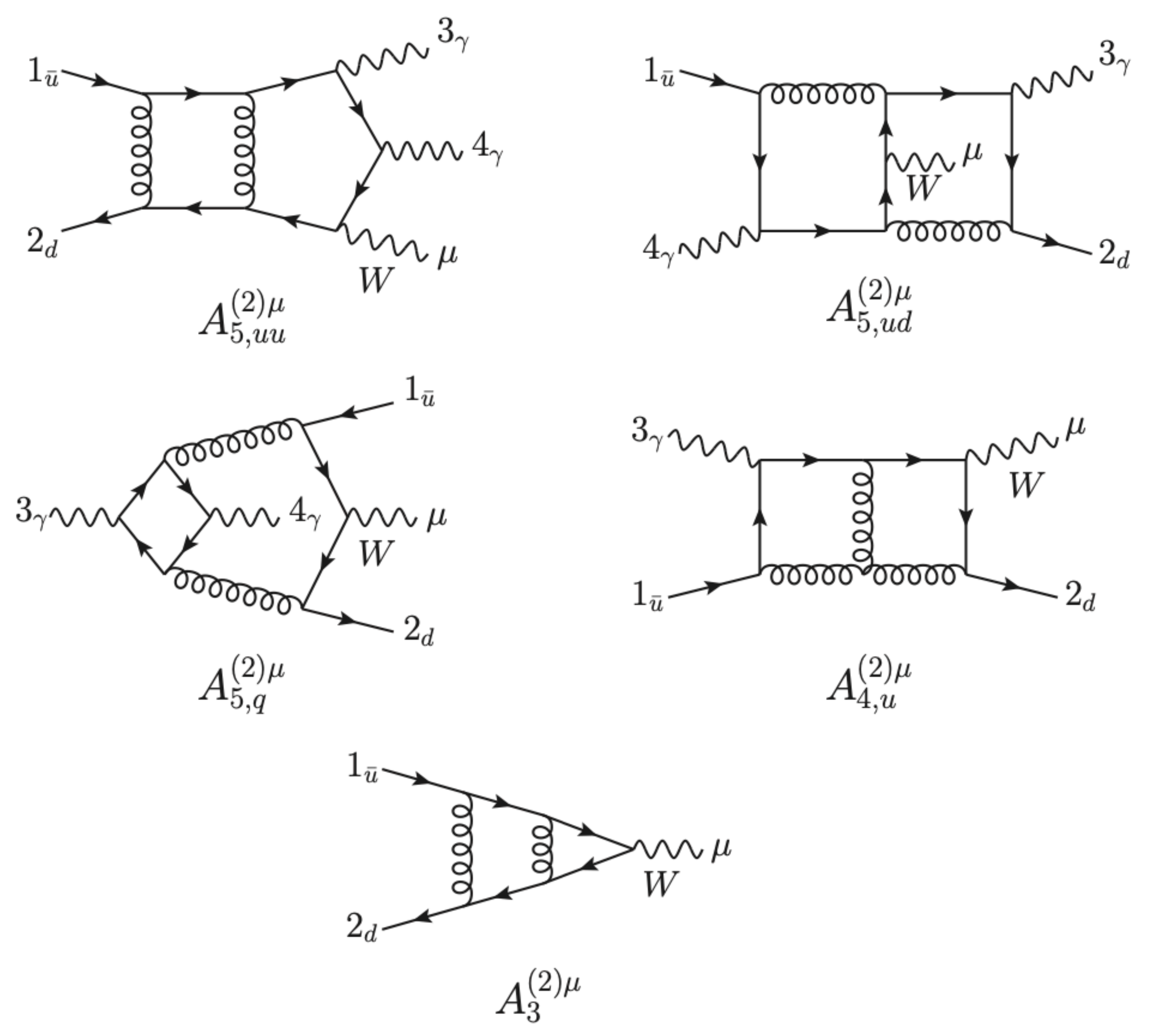
Scattering Amplitudes, a modern workflow



also see Bayu Hartanto's Amplitudes 2025 talk

NNLO QCD correction to W + 2 photon production

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221



Most complicated diagram would be the two-loop five-point with one massive external leg (W boson)

6 scales: 5 Mandelstam + 1 mass

Laporta algorithm needs >10 million IBP reductions
Traditional IBP reduction is difficult …

NNLO QCD correction to W + 2 photon production

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221

Using NeatIBP 1.0 to generate short IBP systems

family	deg.	# IBPs	# integrals	IBP disk size	running time
DPmz	5	26673	27432	71.4 MB	7h5m
DPzz	5	61777	63880	375.8 MB	21h54m*
HBmzz	5	15428	15916	17.0 MB	5h38m
HBzmz	5	10953	11289	13.4 MB	5h45m
HBzzz	5	21126	21766	38.0 MB	9h32m
PBmzz	5	10224	10329	7.8 MB	2h23m
PBzmz	5	11610	11791	6.5 MB	2h50m
PBzzz	5	8592	8752	5.8 MB	2h50m
HTmzzz	4	3120	3176	1.5 MB	1h12m
HTzmzz	4	6594	6650	2.5 MB	1h31m
HTzzzz	4	4680	4631	4.0 MB	2h31m

Laporta algorithm needs >10 million IBP reductions

$\sim$10 million $\rightarrow$ $\sim$10000

NeatIBP 1,0's IBP system has the size $\sim$10000

Planar:
NeatIBP+ Finiteflow 8 times faster,
3 times lower RAM usage than Finiteflow

non-Planar:
NeatIBP+ Finiteflow works
Finiteflow itself does not provide the result

NNLO QCD correction to W + 2 photon production

Coefficients in hard functions

$$F^{(2)} = \lim_{\epsilon \to 0} \left[A^{(2)} - 4 I_2(\epsilon) A^{(0)} - 2 \Big(I_1(\epsilon) + \frac{\beta_0}{\epsilon} \Big) A^{(1)} \right]$$

From each probe, hard function is a linear combination of MIs with finite-field numeric coefficients

univariate reconstruction

↓

univariate partial fraction to get many terms

↓

for each term, multivariate reconstruction

↓

combine terms, multivariate partial fraction

Only reconstruct linearly independent coefficients …
denominators are known …

$\sim$ 30000 points needed for planar diagrams
$\sim$ 130000 points needed for non-planar diagrams

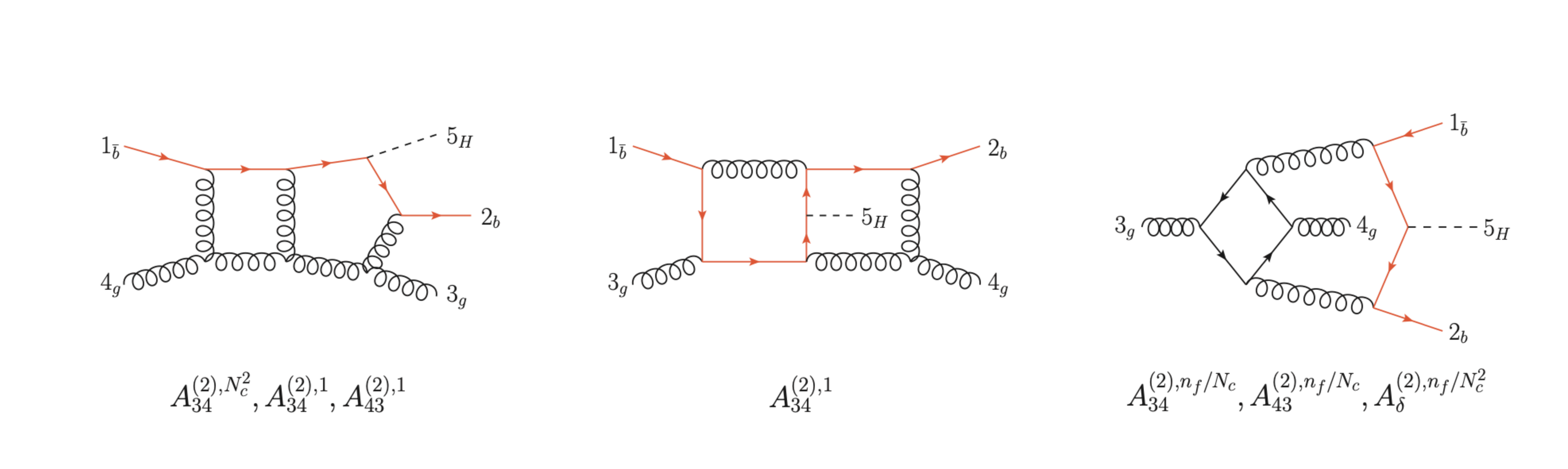
so far, we fully reconstructed the planar amplitude to the analytic form

pfd-parallel package https://github.com/singular-gpispace/pfd-parallel

Bendle, Boehm, Heymann, Ma, Rahn, Wu, Xu, YZ 2021

Full color double virtual H + bbar production

Badger, Hartanto, Poncelet, Wu, *YZ*, Zoia,   JHEP 03 (2025) 066



Reuse the same IBP system of NeatIBP for W + 2 photon production

Wilson Loop Bootstrap

Carrôlo, Chicherin, Henn, Yang, YZ

arXiv: 2505.01245, accepted in JHEP

Wilson loop

Colleagues, including fellow Nobel laureate Hans Bethe,
toast Kenneth Wilson's Nobel Prize in win, October 1982 (Cornell).

$$\mathrm{tr} P \exp\left\{ ig \oint \mathcal{A} \right\}$$

A crucial tool in gauge theory studies

Wilson loop also has deep connections with amplitudes

Wilson loop and scattering amplitudes

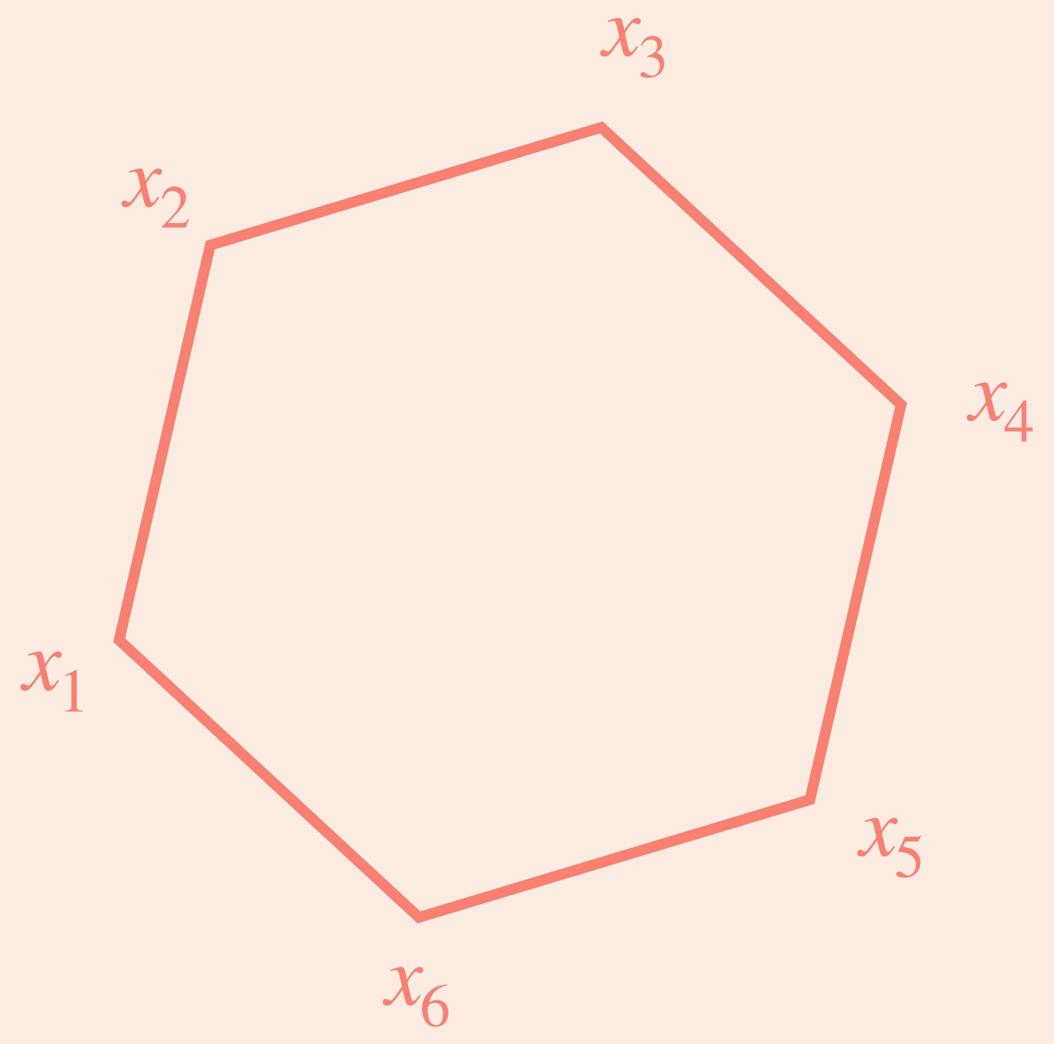
Alday and Maldacena, 2007

N=4 planar sYM,
Lightlike boundary Wilson loop expectation value is dual to MHV scattering amplitude

$$\log \frac{A_n^{\text{MHV}}(p_1,\ldots,p_n)}{A_n^{\text{MHV, tree}}(p_1,\ldots,p_n)} = \log\langle W(x_1,\ldots,x_n)\rangle \qquad p_i = x_{i+1} - x_i$$

infrared divergence

UV divergence from the gluon exchange near the corner



In the x-space, it has anomalous (dual) conformal invariance DCI

$$K^\mu \ \log\langle W(x_1,\ldots,x_n)\rangle_{\text{finite}} = \frac{1}{2}\Gamma_{\text{cusp}} \sum_{i=1}^{n} x^\mu_{i,i+1} \log\left(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\right)$$

Bern-Dixon-Smirnov (BDS) Ansatz is a solution for this equation,
However the physical solution is BDS Ansatz plus a DCI remainder function

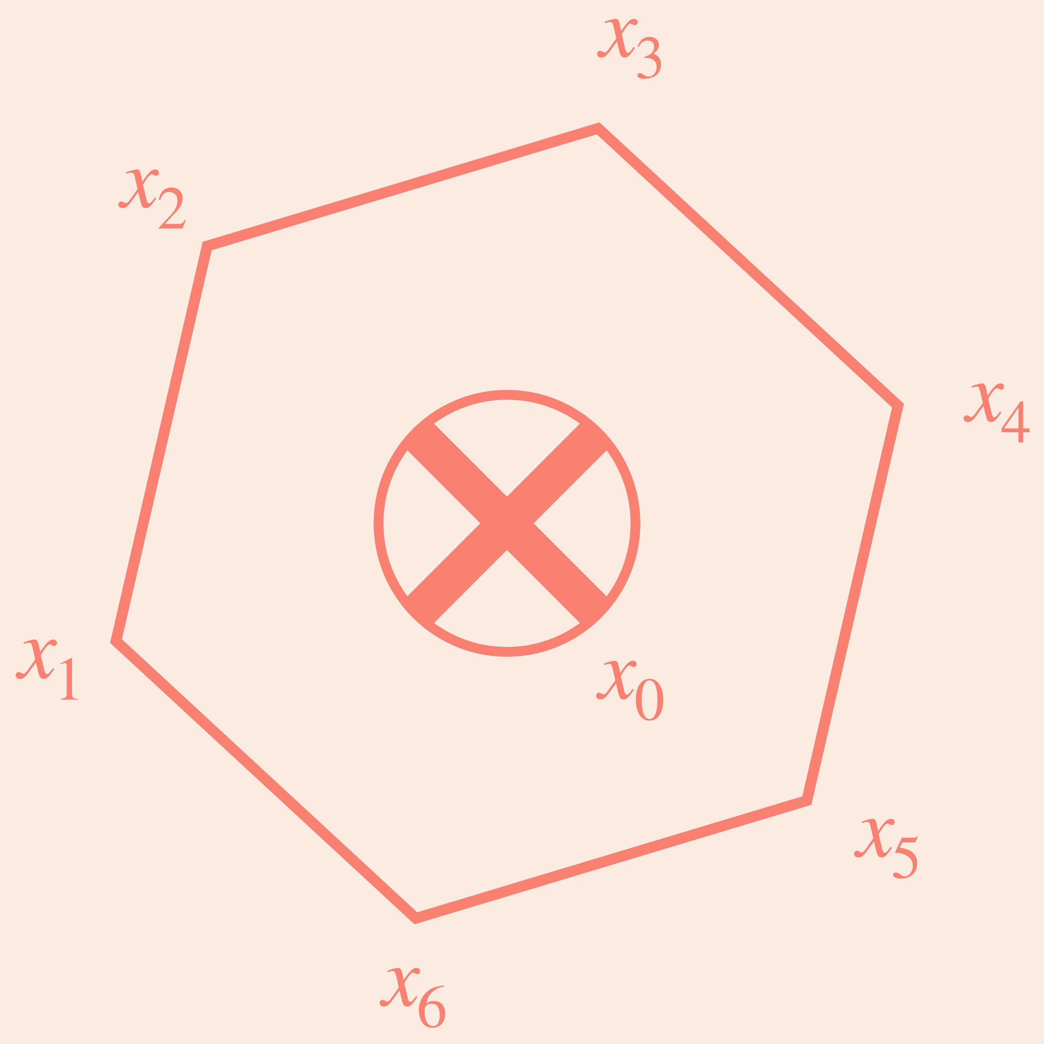
Wilson loop + Lagrangian insertion and scattering amplitudes

$$F_n(x_1,\ldots,x_n;x_0) \equiv \pi^2 \frac{\langle W(x_1,\ldots,x_n)\mathcal{L}(x_0)\rangle}{\langle W(x_1,\ldots,x_n)\rangle}$$

Alday, Buchbinder and Tseytlin 2011
Alday, Heslop and Sikorowski 2013

N=4 sYM on-shell Lagrangian

This quantity is finite



Surprisingly, it is dual to the maximally transcendental part of all-plus helicity amplitude in pure Yang-Mills theory!

Chicherin and Henn, 2022

L-loop F_n is dual to (L+1)-loop all-plus helicity amplitude in pure Yang-Mills theory

Bootstrap

Set up an ansatz and fit the coefficients

ansatz from dual conformal invariants

for N=4 sYM planar amplitudes bootstrap

Caron-Huot, Dixon, McLeod, von Hippel, 2016 (5loop)

Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou, 2020 (6loop & 7loop)

ansatz from all functions (all integrable symbols)

usually too many functions …

ansatz from functions in Feynman integrals!

leading singularity (rational function)

$$F_n = \sum_{i=1}^{22} \sum_{j} c_{ij} R_i f_j$$

transcendental function from Feynman integrals

9 letters only for six points

Wilson loop + Lagrangian leading singularity

Chicherin and Henn, 2022
Brown, Henn, Mazzucchelli, Trnka 2025

$$B_{ijklm} := \frac{\langle AB(mij) \cap (jkl)\rangle^2}{\langle ABjm\rangle\langle ABij\rangle\langle ABjk\rangle\langle ABlj\rangle\langle ABmi\rangle\langle ABkl\rangle},$$

$(abc) \cap (def) := (ab)\langle cdef\rangle - (ac)\langle bdef\rangle + (bc)\langle adef\rangle$

$$B_{ijkl} := \frac{\langle ijkl\rangle^2}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABli\rangle}.$$

Kermit functions

Take the gauge, $\langle ABij\rangle \to \langle ij\rangle$ rational function in momentum twistor variables

There are only 20 (22) linearly independent leading singularities

Function space: planar 2loop 6point massless integrals

A counting of functions, with the dihedral symmetry

Transcendental weight	1	2	3	4
# All symbols	9	62	319	945
# Two-loop six-point symbols	9	62	266	639
# Two-loop five-point one-mass symbols	9	59	263	594
# One-loop squared symbols	9	59	221	428
# Genuine two-loop six-point symbols	0	0	3	45

Surprisingly,
the number of genuinely two-loop six-point functions are very small …
It is a good news for bootstrap

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2501.01847

First Application:
Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 sYM

Carrôlo, Checherin, Henn, Yang, YZ, 2505.01245

$$F_n(x_1, \ldots, x_n; x_0) := \pi^2 \frac{\langle 0|W_n[x_1, \ldots, x_n]L(x_0)|0\rangle}{\langle 0|W_n[x_1, \ldots, x_n]|0\rangle}$$

Using group representation to construct an Ansatz with dihedral symmetry →

weight	0	1	2	3	4
unknowns in dihedral ansatz	5	22	139	644	1892
genuine **unknowns**	4	20	125	585	1718
constraints:					
soft	3	20	116	515	1439
collinear	3	20	121	551	1539
spurious $s_{24} = 0$	1	12	76	360	1044
spurious $s_{25} = 0$	1	6	36	165	483
scaling dimension	0	4	20	125	585
triple collinear	1	5	31	134	353
total **constraints**	4	20	125	585	1718
unfixed **unknowns**	0	0	0	0	0

Bootstrap

$F_2(x_1 \ldots x_6; x_0)$ is fixed in the symbol level!

Summary and Outlook

今兵威已振，譬如破竹，数节之后，皆迎刃而解
《晋书 · 杜预传》

Analytic computation of all 2loop 6point planar massless integrals is done
The first computation on 3loop 5point family is done;

NeatIBP, a powerful package for cutting-edge IBP reduction

Analytic computation of amplitudes: two-loop $W + \gamma\gamma$, $H + b\bar{b}$ production

The dawn of bootstrap

amplitude bootstrap on the way

two-loop hexagonal Wilson loop + Lagrangian insertion
with the help of the function space of Feynman integrals


Dawn of analytic multi-loop computations
A lot of new result will come soon
Thank you
Qingdao, Shandong

Side plot: 2loop N=4 spacelike splitting amplitude

Space-like collinear: generalized factorization (factorization violation)

Collinear particles one incoming one outgoing $p_a \cdot p_b < 0$

Tree

$$|\mathcal{M}^{(0)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle$$

One loop

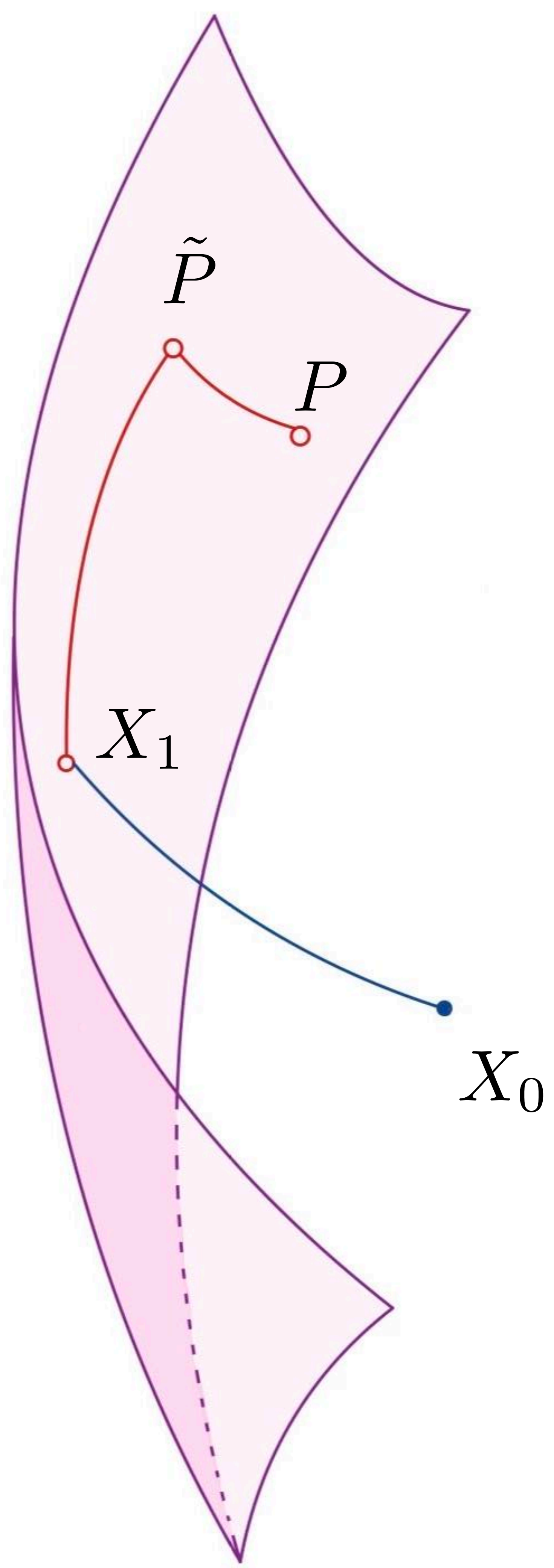
$$|\mathcal{M}^{(1)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(1)}(p_a, p_b; P; p_1, \ldots, \widehat{p_a}, \ldots \widehat{p_b}, \ldots p_n) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(1)}(P, \ldots, p_n)\rangle \; .$$

$$\mathbf{SP}^{(1)}(p_a, p_b; P; p_1, \ldots, \widehat{p_a}, \ldots \widehat{p_b}, \ldots p_n) \supset \frac{2}{\epsilon} \sum_{j \neq a,b} \mathbf{T}_b \cdot \mathbf{T}_j f(\epsilon, z_2 - i s_{j,b} 0^+)$$

Two-loop $\mathbf{Sp}^{(2)}$ was not completely known at that time ...

Catani, de Florian, Rodrigo, 2012

Side plot: 2loop N=4 spacelike splitting amplitude



Sotnikov, Chicherin 2020

Solve canonical DE for 2loop 5point master integrals

$$X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{3, -1, 1, 1, -1\}$$

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{\frac{4}{\lambda^2+1}, \frac{-4\lambda^2}{\lambda^2+1}, 1, \frac{2-2\lambda^2}{\lambda^2+1}, -1\}$$

$$X_1: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{4, -4\delta^2, 1, 2, -1\}$$

$$P: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{sz, -4\delta^2, (1-z)xs, s, xs + c\delta\}$$

29040 master integrals solved in terms of s, z, x, δ, y in terms of GPL functions up to weight 4.

With the integrand given in *Carrasco and Johansson 2012*, the 2loop N=4 (planar + nonplanar) 5-point SYM amplitude is obtained in spacelike collinear region as GPL→ HPL → Li functions.

Henn, Ma, Xu, Yan, YZ, Zhu, arXiv 2406.14604

Side plot: 2loop N=4 spacelike splitting amplitude

Henn, Ma, Xu, Yan, YZ, Zhu, arXiv 2406.14604

in memorial of Stefano Catani (1958-2024)

$$\mathbf{Sp}^{(2)} = \left[\frac{\mu^2\, z}{s_{ab}\,(1-z)}\right]^{2\epsilon} \Big\{ 4N_c^2\, \bar{r}_S^{(2)}(z+i0)$$

dipole $$+ N_c\, \mathbf{T}_a \cdot \mathbf{T}_{\text{in}}\ (2\pi i) \left[c_2(\epsilon)\, \frac{1}{\epsilon^3} + c_1^2(\epsilon) \left(-\frac{2}{\epsilon^2} \ln z + \frac{2}{\epsilon} \ln z \ln\left(\frac{z}{z-1}\right) \underline{-2\, \mathrm{Li}_3\left(1-\frac{1}{z}\right) - \ln(z) \ln^2\left(\frac{z}{z-1}\right)} \right) \right]$$

tripole $$+ \sum_{I\in\text{outgoing}} [\mathbf{T}_a \cdot \mathbf{T}_{\text{in}}, \mathbf{T}_a \cdot \mathbf{T}_I]\, (2\pi i) \left[\left(\frac{1}{2\epsilon^2} - \underline{\frac{1}{2}\zeta_2} \right) (\ln|z_I|^2 + i\pi) + \underline{\frac{1}{6}\left(\ln^2 \frac{z_I}{\bar{z}_I} + 4\pi^2 \right) \ln \frac{z_I}{\bar{z}_I} + 2\zeta_3} \right]$$

$$+ \sum_{I\in\text{outgoing}} \{\mathbf{T}_a \cdot \mathbf{T}_{\text{in}}, \mathbf{T}_a \cdot \mathbf{T}_I\}\, (2\pi^2) \left[\frac{1}{2\epsilon^2} - \underline{\frac{1}{2}\zeta_2} \right] \Big\}\, \mathbf{Sp}^{(0)}\,.$$

The ε-pole terms were given in *Catani, de Florian, Rodrigo 2012.*

We computed the finite part, from the fully analytic computation of 2loop 5point Feynman integrals.

Side plot: 2loop N=4 spacelike splitting amplitude

Time-like collinear: strict factorization

Collinear particles are both outgoing $p_a \cdot p_b > 0$, with $+ - - -$ signature

Tree $$|\mathcal{M}^{(0)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle$$

One loop $$|\mathcal{M}^{(1)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(1)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(1)}(P, \ldots, p_n)\rangle \; .$$

Two loop $$|\mathcal{M}^{(2)}(p_a, p_b, \ldots, p_n)\rangle \sim \mathbf{SP}^{(2)}(p_a, p_b; P) \; |\mathcal{M}^{(0)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(1)}(p_a, p_b; P) \; |\mathcal{M}^{(1)}(P, \ldots, p_n)\rangle + \mathbf{SP}^{(0)}(p_a, p_b; P) \; |\mathcal{M}^{(2)}(P, \ldots, p_n)\rangle \; .$$

Catani, Grazzini, 1999
Catani, de Florian, Rodrigo, 2012