Automatic Calculations of Multi-loop Feynman Amplitudes

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Outline

Motivation

Representative state-of-the-art multi-loop calculations:

- Four-loop $2 \to 1$
  H
- Three-loop $2 \to 2$
  Drell-Yan, $\gamma\gamma$, $q\bar{q}$, gg, γj, Hj, HH, ...
- Two-loop $2 \to 2$ with finite top mass
  $\gamma\gamma$, Hj, ZZ, WW, HH, ...
- Two-loop $2 \to 3$
  $\gamma\gamma\gamma$, $\gamma\gamma j$, γjj, three-jet, Wbb^{-}, $W\gamma\gamma$, $W\gamma j$, $b\bar{b}H$, Vjj, $t\bar{t}j$, ttW, ttH, ...

General-purpose algorithms and automatic calculations are indispensable for multi-loop calculations.

Feynman amplitudes

General structure of Feynman amplitudes

dimensional regularization

$$T_{\alpha\beta\gamma\cdots} \times \int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \frac{l_1^\alpha l_1^\beta \cdots l_2^\gamma \cdots}{(q_1^2 - m_1^2)^{i_1} (q_2^2 - m_2^2)^{i_2} \cdots}. \quad (1)$$

Workflow of loop calculations

- Amplitude generation and tensor algebras
  Qgraf, FeynArts, FORM, FeynCalc, ...
- Integral reduction (to master integrals)
- Calculation of master integrals

One-loop calculation

Passarino-Veltman reduction Passarino & Veltman (1979) NPB

Tensor reduction

$$\int \mathrm{d}^d l \frac{l^\mu l^\nu}{l^2 - m^2} = g^{\mu\nu} I \,,$$

$$I = \frac{1}{d} \int \mathrm{d}^d l \frac{l^2}{l^2 - m^2} \,.$$

Integral reduction

$$\frac{l \cdot p}{l^2 (l-p)^2} = \frac{1}{2} \left[\frac{p^2}{l^2 (l-p)^2} + \frac{1}{(l-p)^2} - \frac{1}{l^2} \right] \,.$$

Propagators of one-loop integrals are complete.

One-loop calculation

Other method of one-loop calculation

- Generalized unitarity Bern et al. (1995) NPB, Britto et al. (2005) NPB, Giele et al. (2008) JHEP, Ellis et al. (2008) JHEP
- Integrand reduction del Aguila & Pittau (2004) JHEP
- OPP method Ossola et al. (2007) NPB
- Generating function Feng et al. (2022) JHEP, Feng (2023) CTP, Hu et al. (2025) PRD, ...

The one-loop calculation is a solved problem.

Multi-loop integral reduction

Integration by parts (IBP) Tkachov (1981) PLB, Chetyrkin & Tkachov (1981) NPB

IBP relations

$$\int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \frac{\partial}{\partial l_i^\mu} \frac{l_j^\mu}{(q_1^2 - m_1^2)^{i_1} (q_2^2 - m_2^2)^{i_2} \cdots} = 0. \tag{2}$$

The number of linearly independent integrals (master integrals) is finite. *Smirnov & Petukhov (2011) LMP*

Multi-loop integral reduction

Laporta algorithm Laporta (2000) IJMP

The number of equations increases faster than the number of integrals when increasing the indices.

Packages: FIRE, REDUZE, KIRA, ...

Problem: intermediate expression swell

Solution: finite field Kauers (2008) NPB, Kant (2014) CPC, von Manteuffel & Schabinger (2014) PLB

Other solutions:

Syzygy equations Gluza et al. (2011) PRD, Schabinger (2012) JHEP, Böhm et al. (2018) PRD

NeatIBP

Block-triangular form Liu & Ma (2019) PRD

Blade

Calculation of master integrals

Methods of master integral calculation

- Sector decomposition Hepp (1966) CMP, Binoth & Heinrich (2000) NPB, Bogner & Weinzierl (2008) CPC
- Mellin-Barnes method Boos & Davydychev (1991) TMP, Usyukina & Davydychev (1993) PLB
- Difference-equation method Tarasov(1996)PRD, Laporta(2000)IJMPA
- Differential-equation method Kotikov(1991)PLB, Remiddi(1997)NCA
- Direct integration Panzer (2015) CPC, von Manteuffel et al. (2015) JHEP

Mellin-Barnes

$$\int \mathrm{d}^d l \frac{1}{l^2[(l+p)^2+m^2]} = \int \frac{\mathrm{d}z}{2\pi i} \frac{1}{\Gamma(-z)\Gamma(z+1)} \int \mathrm{d}^d l \frac{(-m^2)^z}{l^2[(l+p)^2]^{z+1}} \, .$$

Packages: MB Tools, AMBRE, ...

The integration converges slowly for multi-fold Mellin-Barnes integrals due to the highly oscillating integrands.

Sector decomposition

$$
\begin{aligned}
&\int_0^1 \mathrm{d}x\mathrm{d}y\ (x+y)^{-2-\epsilon} \\
=&\int_0^1 \mathrm{d}x\mathrm{d}y\ (x+y)^{-2-\epsilon}\left(\Theta(x-y)+\Theta(y-x)\right) \\
=&\int_0^1 \mathrm{d}x\ x^{-1-\epsilon}\int_0^1 \mathrm{d}t\ (1+t)^{-2-\epsilon} + \int_0^1 \mathrm{d}y\ y^{-1-\epsilon}\int_0^1 \mathrm{d}t\ (t+1)^{-2-\epsilon} \\
=&2\int_0^1 \mathrm{d}x\ x^{-1-\epsilon}\int_0^1 \mathrm{d}t\ (1+t)^{-2-\epsilon}\ .
\end{aligned}
$$

Packages: pySecDec, FIESTA, ...

Differential-equation method

$$\frac{\partial}{\partial m_1^2}\int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \frac{1}{\left(q_1^2-m_1^2\right)^{i_1}\left(q_2^2-m_2^2\right)^{i_2}\cdots}$$
$$= -\,i_1\int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \frac{1}{\left(q_1^2-m_1^2\right)^{i_1+1}\left(q_2^2-m_2^2\right)^{i_2}\cdots}. \quad (3)$$

$$\frac{\partial}{\partial x} I_i = \epsilon \sum_j M_{ij} I_j \ .$$

The boundary conditions can be determined by matching the asymptotic solutions of the differential equations to the asymptotic expansions of the master integrals.

Differential-equation method

Canonical differential equation Henn (2013) PRL

$$M = \epsilon \sum_i \frac{M_i}{x - x_i} \ .$$

$$\mathrm{d}I = \epsilon \sum_i M_i \mathrm{d} \log(\mathrm{x} - \mathrm{x_i}) I \equiv \epsilon \mathrm{d}A \ I \ .$$

Iterated integrals

$$I = \mathrm{P} \exp \left(\epsilon \int \mathrm{d}A \right) I \ .$$

Packages: Canonica, Fuchsia, epsilon, Libra, INITIAL, ...

Numerical differential equations and automatic computation

AMFlow, AmpRed

Method of region

Beneke & Smirnov (1998) NPB, Pak & Smirnov (2011) EPJC, Jantzen (2011) JHEP

$$I = \int \mathrm{d}^d l \frac{1}{(l-p)^2(l^2-m^2)} \ .$$

$$m^2 \ll p^2 \ .$$

- hard region: $m^2 \ll l^2 \sim p^2$

$$I_1 = \int \mathrm{d}^d l \frac{1}{(l-p)^2 l^2} + m^2 \int \mathrm{d}^d l \frac{1}{(l-p)^2 l^4} + \cdots$$

- soft region: $m^2 \sim l^2 \ll p^2$

$$I_1 = \int \mathrm{d}^d l \frac{1}{l^2(l^2-m^2)} + \int \mathrm{d}^d l \frac{2l \cdot p}{l^4(l^2-m^2)} + \cdots$$

$$I = I_1 + I_2 \ .$$

Parametric representation

Advantages of parametric representation:

- Integrands are functions of Lorentz invariants.
- No need to introduce auxiliary propagators.
- More symmetric under permutations of indices.
- Tensor reduction is trivial.
- can be applied for direct integration.
- A systematic algorithm for asymptotic expansion is available
- can be used to reduce integrals with phase-space cuts (delta functions and theta functions)
- ...

Parametrization

Schwinger parametrization

$$\frac{1}{D_i^{\lambda_i+1}} = \frac{e^{-\frac{\lambda_i+1}{2}i\pi}}{\Gamma(\lambda_i+1)} \int_0^\infty dx_i\ e^{ix_i D_i} x_i^{\lambda_i}, \qquad \mathrm{Im}\{D_i\} > 0.$$

Parametrization of scalar integrals

$$\begin{aligned} J(-\frac{d}{2},\lambda_1,\lambda_2,\cdots,\lambda_n) &\equiv \pi^{-Ld/2} \int \prod_{i=1}^{L} \mathrm{d}^d l_i \frac{1}{D_1^{\lambda_1+1} D_2^{\lambda_2+1} \dots D_n^{\lambda_n+1}} \\ &\to \frac{\Gamma(\frac{d}{2})}{\prod_{i=1}^{n+1}\Gamma(\lambda_i+1)} \int \mathrm{d}\Pi^{(n+1)} \mathcal{F}^{-\frac{d}{2}} \prod_{i=1}^{n+1} x_i^{\lambda_i} \\ &\equiv \int \mathrm{d}\Pi^{(n+1)} \mathcal{I}^{(-n-1)} \equiv I(-\frac{d}{2},\lambda_1,\lambda_2,\cdots,\lambda_n) \end{aligned} \tag{4}$$

Integral reduction and differential equations

Parametric IBP WC (2020)JHEP, (2020,2021)EPJC

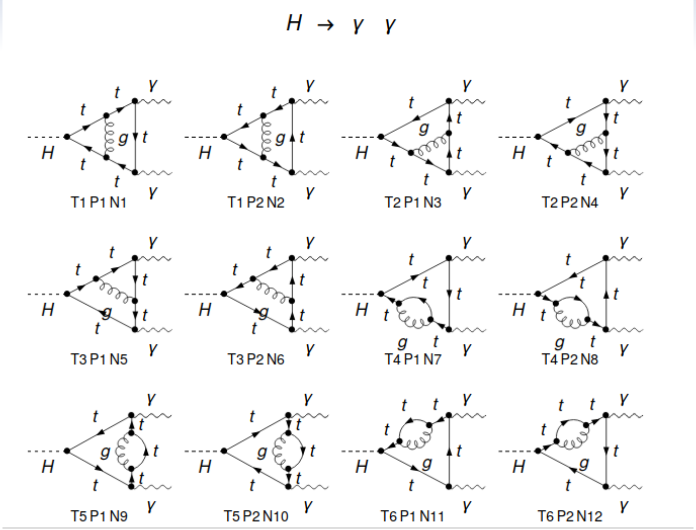
$$0 = \int \mathrm{d}\Pi^{(n+1)} \frac{\partial}{\partial x_i} \mathcal{I}^{(-n)} + \delta_{\lambda_i 0} \int \mathrm{d}\Pi^{(n)} \, \mathcal{I}^{(-n)} \Big|_{x_i=0}, \qquad i = 1, 2, \ldots . \tag{5}$$

Equation (5) is a consequence of the homogeneity of the integrand.

Differential equations

$$\frac{\partial}{\partial y} I = \int \mathrm{d}\Pi^{(n+1)} \frac{\partial}{\partial y} \mathcal{I}^{(-n-1)} \tag{6}$$

An example: Higgs decay
Generate amplitudes with FeynArts



Define the kinematics

```
SetKinematics[{P} → {p[1], p[2]}, {FA["MH"]} → {0, 0}];
FA["MH"] = FA["EL"] = FA["GS"] = 1;
FA["MT"] = 2; FA["MW"] = 1 / 2;
FA["SW"] = 1 / 2;
```

Convert the amplitudes

```
amp = FA2AR[Get["HiggsDecay_TwoLoop_Amplitude.m"], ToFeynmanInt → True] /. {_Incoming → P, Outgoing → p} // Total;
```

Tensor algebras

```
amp1 = amp // SpinorChainSimplify // ColourSimplify;
```

Integral reduction

```
amp2 = AlphaReduce[amp];
```

Numerical evaluation of master integrals

```
amp3 = AlphaIntEvaluateN[amp1, 8, {}, PrecisionGoal → 40, Print → Print];
```

Result

```
Chop[Collect[amp3[[1]] // epsSeries, {_Pair, eps}], 10^-20]
```

$$\left(\frac{0.6555969719636989339927156929097 7\, i\, C_F}{\epsilon} + 63.108561600245894557465525711895 2\, i\, C_F\right) p_1 \cdot \epsilon^*_{p_2}\; p_2 \cdot \epsilon^*_{p_1} +$$
$$\left(\frac{0.0956155694439145384129445922411 2\, i\, C_F}{\epsilon} - 0.181268820276402967819607796164 8\, i\, C_F\right) p_1 \cdot \epsilon^*_{p_1}\; p_2 \cdot \epsilon^*_{p_2} +$$
$$\left(-\frac{0.3277984859818494669963578464549\, i\, C_F}{\epsilon} - 31.554280800122947278732762855947 6\, i\, C_F\right) \epsilon^*_{p_1} \cdot \epsilon^*_{p_2}$$

Benchmarks

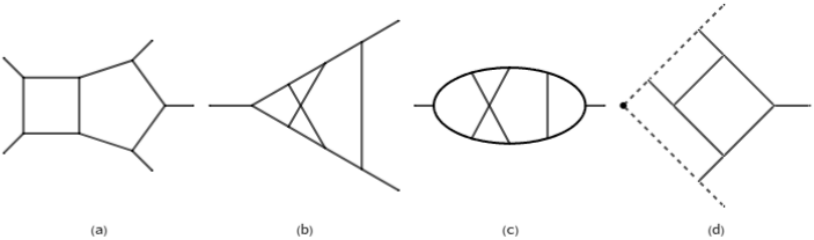


FIG. 1: Diagrams corresponding to the test integrals. All the internal lines are massless, and dashed lines represent Wilson lines.

CPU:	AMD EPYC 7R32 / 2.8Ghz-3.3Ghz
RAM:	520 GB
Threads:	24
IBP solver:	`KIRA2.3`
Precision goal:	20

diagram	a	b	c	d
AmpRed	5.27×10^3	2.86×10^3	3.39×10^4	1.54×10^4
`AMFlow`	2.70×10^3	8.64×10^3	1.31×10^5	?

TABLE I: Computation times (in seconds) for the test integrals in Fig. 1.

Outlook

- Intermediate expression swell for multi-scale amplitudes (multi-legs, internal masses)
- Direct integration of amplitudes?
- Infrared structure of Feynman amplitudes
- Parallel computing and usage of GPU
- Application of AI 2502.09544

The End