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Infrared Singularities of multileg QCD amplitudes

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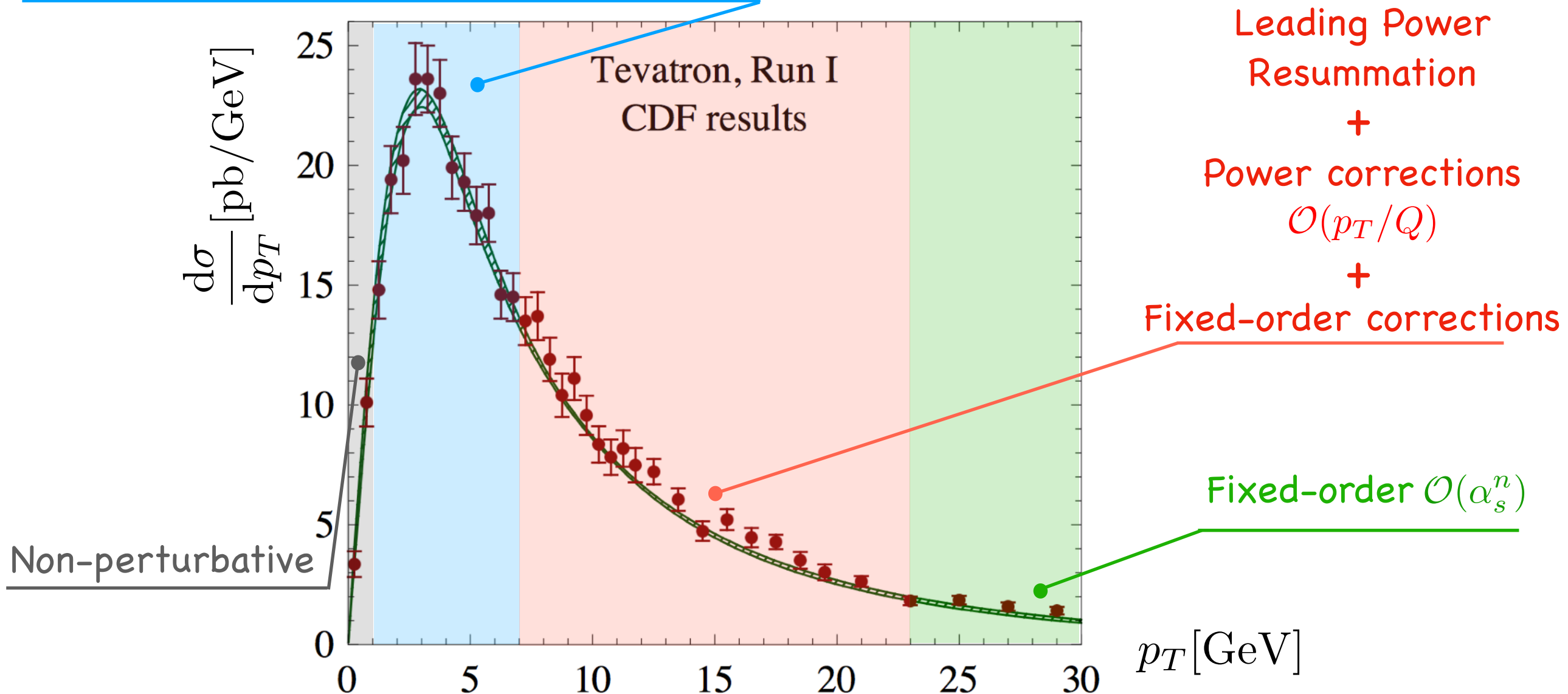
Outline

- Motivation
- Constraints to anomalous dimensions
 - Soft-collinear factorization – Kinematics
 - Non-Abelian Exponentiation Theorem – Color
 - Two-particle collinear limits
 - Small-mass limits
- Calculation of the Tripole Correlation
- Results

Precise Theoretical Prediction

Resummation of $\ln(p_T/Q)$ at leading power

Becher, Neubert, Wilhelm '11



$$\frac{d\sigma}{d\Omega} = \int dx_a \underbrace{f_{a/N_1}(x_a, \mu)}_{\text{PDF}} \int dx_b \underbrace{f_{b/N_2}(x_b, \mu)}_{\text{PDF}} \underbrace{\frac{d\hat{\sigma}_{ab \rightarrow Z}}{d\Omega}}_{\text{Cross section at parton level}} + \mathcal{O}(\Lambda_{\text{QCD}}/M_Z)$$

Resummation

- For $p_T \ll Q$, perturbative convergence is violated by $L = \ln(p_T/Q)$
- A good parameter for expansion is $1/L$, not α_s $\alpha_s L \sim 1$
- Large logs should be resummed to all orders in α_s

$$\sigma(p_T) \sim \left[1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi} \right)^n c_n \right] \exp \left[\underbrace{L g_1(\alpha_s L)}_{\text{LL}} + \underbrace{g_2(\alpha_s L)}_{\text{NLL}} + \underbrace{\alpha_s g_3(\alpha_s L)}_{\text{NNLL}} + \cdots \right] + \mathcal{O}(p_T/Q)$$

$$\begin{aligned} \sigma(p_T) \sim & \alpha_s (L^2 + L + c_1) \\ & + \alpha_s^2 (L^4 + L^3 + L^2 + L + c_2) \\ & + \alpha_s^3 (L^6 + L^5 + L^4 + L^3 + \cdots) \\ & + \alpha_s^4 (L^8 + L^7 + L^6 + L^5 + \cdots) \\ & + \vdots + \vdots + \vdots \end{aligned}$$

- Resummation can be achieved by renormalization-group (RG) evolution in **SCET**

$$\sigma \sim H \otimes S \otimes B \otimes B$$

$$\frac{d}{d \ln \mu} H = \left(\Gamma_{\text{cusp}} \ln \frac{Q}{\mu} + \gamma^H \right) H$$

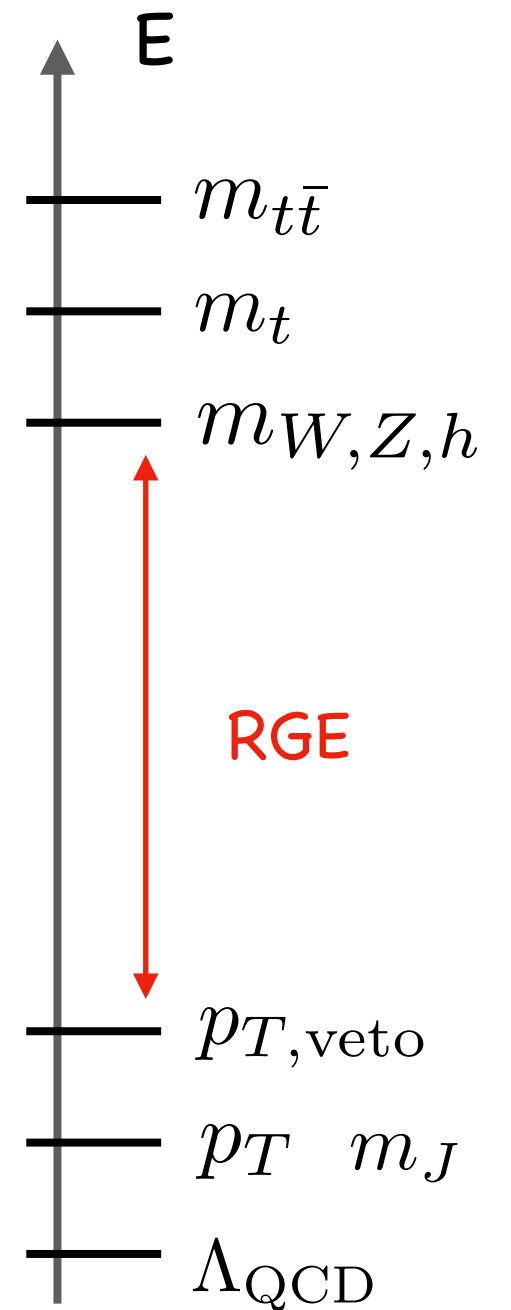
Precision Collider Observables

- Multi scales are involved in measurements
 - Fixed-order results are invalid due to large logarithms of scale ratios
 - Large logs need to be resummed to all orders in α_s
 - Renormalization-group evolutions are governed by anomalous dimensions

Effective field theory is powerful for scale separation and factorization

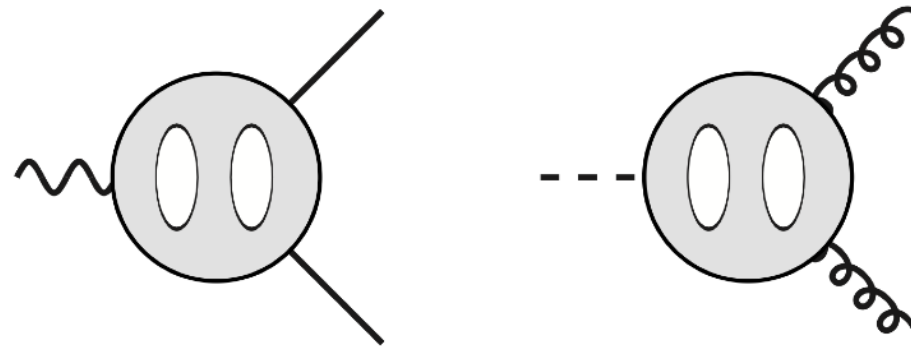
$$\sigma \sim \underset{\substack{\uparrow \\ \text{IR poles}}}{H} \otimes \underset{\substack{\nwarrow \nearrow \\ \text{UV poles in low energy matrix elements}}}{B_{a/N}} \otimes B_{b/N} \otimes S \otimes \prod_i J$$

In soft-collinear effective theory (SCET), IR poles of hard coefficients are in one-to-one correspondence to the UV poles of low-energy matrix elements.



IR Singularities of Form Factors

The IR singularities of quark/gluon form factor



Renormalization $|\mathcal{M}(\{\underline{s}\}, \mu)\rangle = \lim_{\epsilon \rightarrow 0} \mathbf{Z}^{-1}(\epsilon, \{\underline{s}\}, \mu) |\mathcal{M}(\epsilon, \{\underline{s}\})\rangle$

Renormalization-group equation (RGE) gives

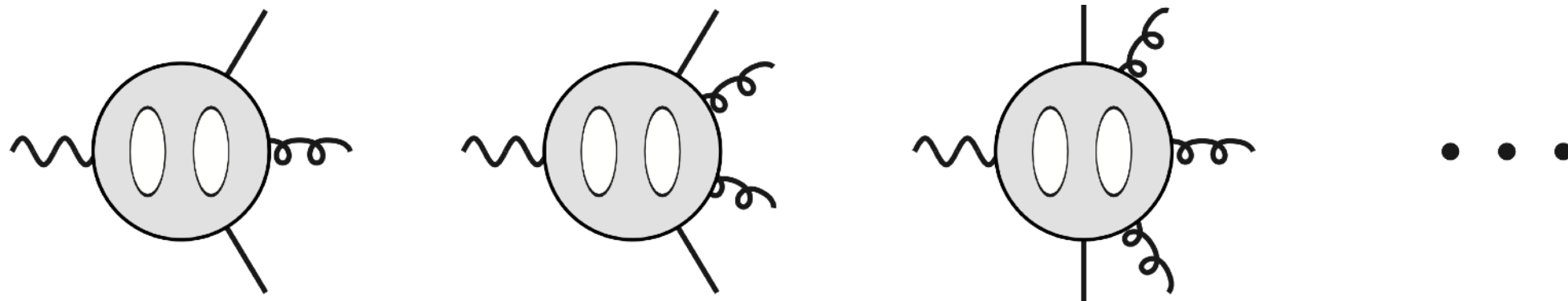
$$\mathbf{Z}(\epsilon, \{\underline{p}\}, \mu) = \mathbf{P} \exp \left[\int_{\mu}^{\infty} \frac{d\mu'}{\mu'} \Gamma(\{\underline{p}\}, \mu') \right] \leftarrow \text{anomalous dimension}$$

With $\Gamma(\mu) = -C_{R_i} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{12}} + 2\gamma^i(\alpha_s)$ $C_{R_i} \equiv \mathbf{T}_{q/g}^2 = C_{F/A}$

$\Gamma_{\text{cusp}}^i(\alpha_s)$ has Casimir scaling: $\frac{\Gamma_{\text{cusp}}^q(\alpha_s)}{C_F} = \frac{\Gamma_{\text{cusp}}^g(\alpha_s)}{C_A}$ violated at 4 loops

Dipole Structure of IR Singularities

Can we predict IR singularities of amplitudes with 3, 4, 5, ... partons ?



Dipole formula of anomalous dimensions can describe the IR singularities of multi-leg scattering “almost” correctly

$$\Gamma(\{\underline{p}\}, \mu) = \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s) \quad s_{ij} = \pm 2p_i \cdot p_j$$

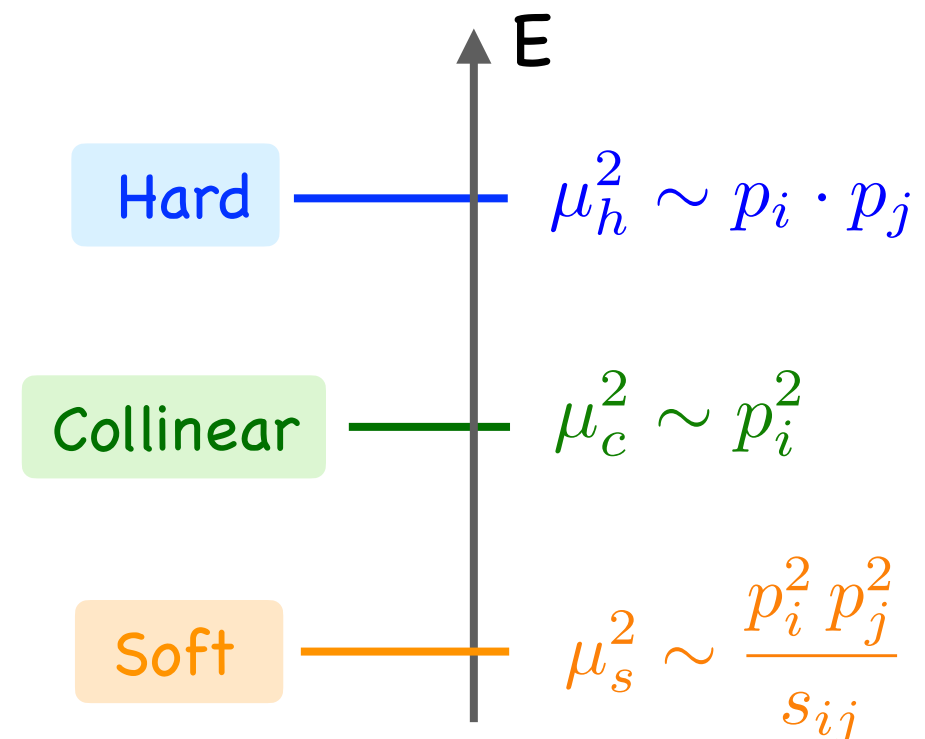
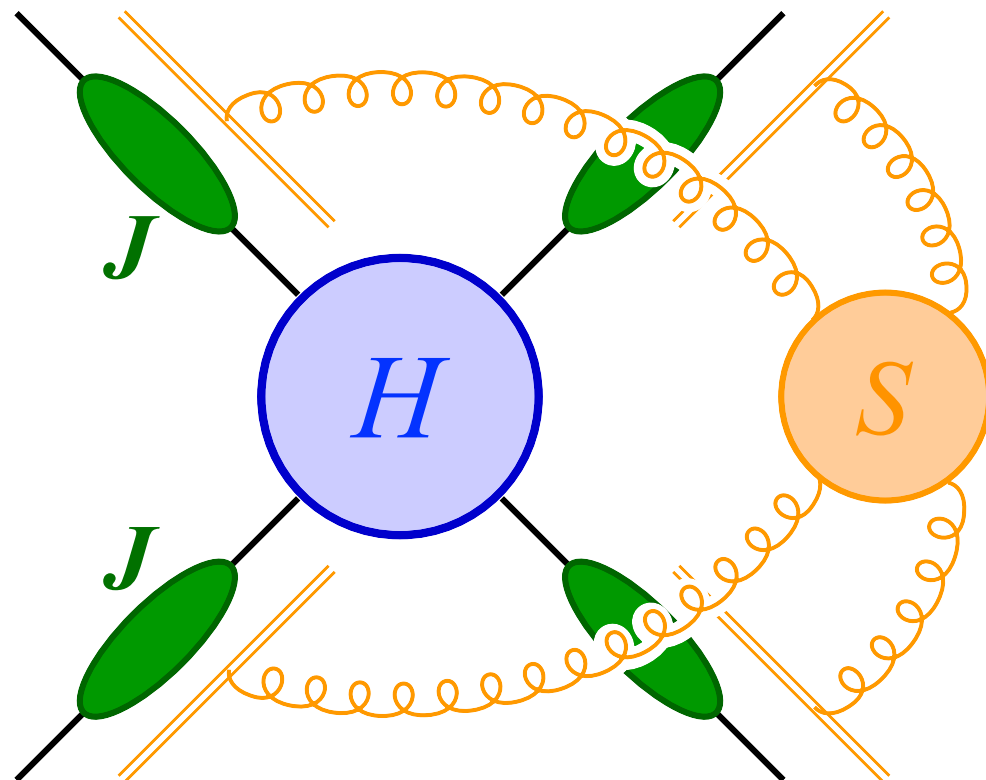
universal

But new structures arise at three loops and beyond !

Soft-Collinear Factorization

For n-jet amplitudes:

$$\frac{\text{Off-Shell Green's function}}{\text{UV renormalized free of IR poles}} = \underbrace{S(\{\underline{\beta}\}, \epsilon)}_{\substack{\text{Matrix in color space} \\ \beta_{ij} = \ln \frac{(-s_{ij}) \mu^2}{(-p_i^2)(-p_j^2)} \\ \text{cusp angles}}} \prod_i \underbrace{J(L_i^2, \epsilon)}_{\ln \frac{\mu^2}{-p_i^2}} \underbrace{|\mathcal{M}(\{\underline{s}\}, \epsilon)\rangle}_{s_{ij} = \pm 2p_i \cdot p_j}$$



Soft-Collinear Factorization

RG invariance implies:

$$\Gamma(\{\underline{s}\}, \mu) = \Gamma_s(\{\underline{\beta}\}, \mu) + \sum_{i=1}^n \Gamma_c^i(L_i, \mu) \mathbf{1}$$

Hard:

$$\Gamma(\{\underline{s}\}) = \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma_{\text{cusp}} \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i \quad \text{No } p_i^2$$

Collinear:

$$\Gamma_c^i = -\Gamma_{\text{cusp}}^i \ln \frac{\mu^2}{-p_i^2} + \gamma_c^i$$

Becher, Hill, Lange, Neubert, '03

The cancellation of collinear scales in $\Gamma_s + \sum_i \Gamma_c^i \mathbf{1}$ gives

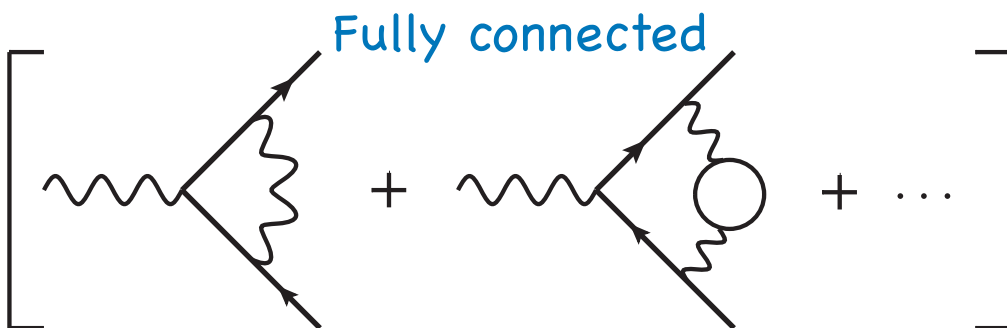
$$\Gamma_s(\{\underline{\beta}\}) \sim \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma_{\text{cusp}} \ln \frac{\mu^2(-s_{ij})}{(-p_i^2)(-p_j^2)} + \dots$$

Cusp angles β_{ij}

Non-Abelian Exponentiation Theorem

- UV poles of soft matrix elements can be written as exponentials of simpler quantities, which only receive contribution from color connected **webs**

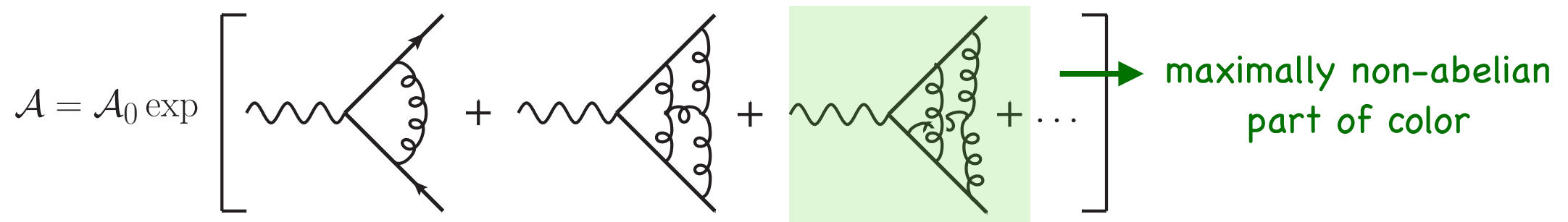
Abelian:

$$\mathcal{A} = \mathcal{A}_0 \exp \left[\text{Fully connected} + \dots \right]$$


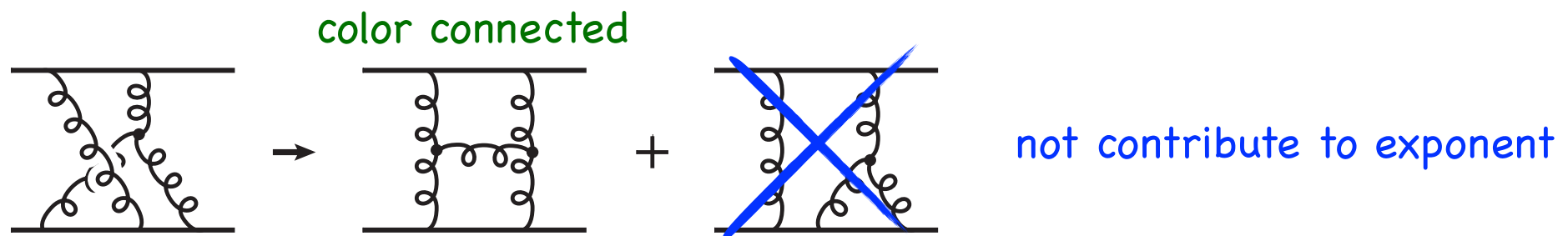
non-Abelian:

Gatheral, '83

Frenkel & Taylor, '84

$$\mathcal{A} = \mathcal{A}_0 \exp \left[\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \right] \rightarrow \text{maximally non-abelian part of color}$$


color connected



not contribute to exponent

Non-Abelian Exponentiation Theorem

- Generalize to multi-leg scattering amplitudes

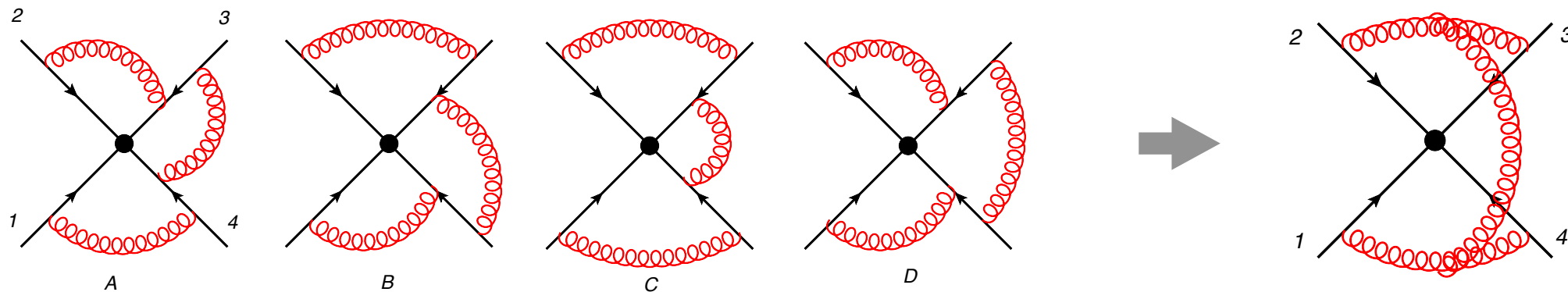
Gardi et al. '09-'14

$$S = \exp \left[\sum_W \sum_{D,D'} \overset{\text{kinematic}}{\downarrow} \boxed{F(D)} \overset{\text{conventional color}}{\downarrow} R_{DD'}^{(W)} \boxed{C(D')} \right]$$

Mixing matrix as a projection operator

- An example of **web** : $W_{(1,1,2,2)}$

Figure from Gardi, Smillie, White, 1304.7040



$$R^{(1,1,2,2)} = \frac{1}{6} \begin{bmatrix} 2 & 2 & -2 & -2 \\ 2 & 2 & -2 & -2 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

obtained by replica trick

Gardi, Laenen, Stavenga, White '10

$$\sum_{D,D'} F(D) R_{DD'}^{(W)} C(D') = \frac{1}{6} (-2F_A - 2F_B + F_C + F_D) \times \underline{f^{abd} f^{bce} T_1^c T_2^a T_3^d T_4^e}$$

Color connected

Construct Soft Anomalous Dimensions

- Kinematic dependences

- ▶ Cusp angles – **depend on collinear scales**

$$\beta_{ij} = L_i + L_j - \ln \frac{\mu^2}{-s_{ij}}$$

$$\beta_{Ij} = L_j - \ln \frac{m_I \mu}{-s_{Ij}}$$

$$\beta_{IJ} = \cosh^{-1} \left(\frac{-s_{IJ}}{2m_I m_J} \right)$$

$$s_{ij} = \pm 2p_i \cdot p_j \quad L_i = \frac{\mu^2}{-p_i^2}$$

- ▶ Conformal cross ratios – **independent on collinear scales**

Purely massless:

$$\beta_{ijkl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

Becher, Neubert '09

Gardi, Magnea '09

One massive:

$$r_{ijI} \equiv \frac{v_I^2 (n_i \cdot n_j)}{2 (v_I \cdot n_i)(v_I \cdot n_j)}$$

ZLL, Schalch '22

$$Y_{v_i}(x) = \mathbf{P} \exp \left(-ig_s \int_0^\infty dt \, v_i \cdot A_s^a(x + tv_i) \mathbf{T}^a \right) \text{ is invariant} \quad v_i \rightarrow \alpha v_i$$

Construct Soft Anomalous Dimensions

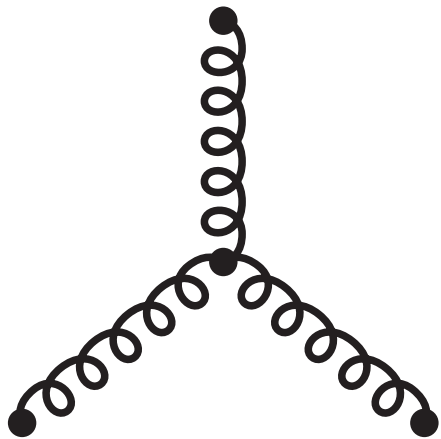
- Color

- Only color connected

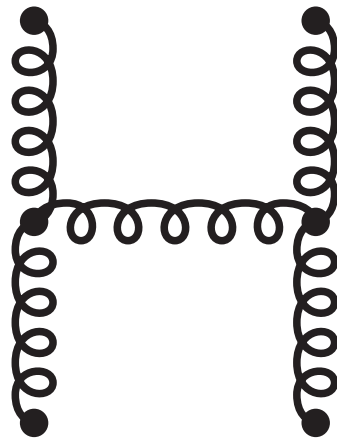
$$T_i^a T_j^a$$



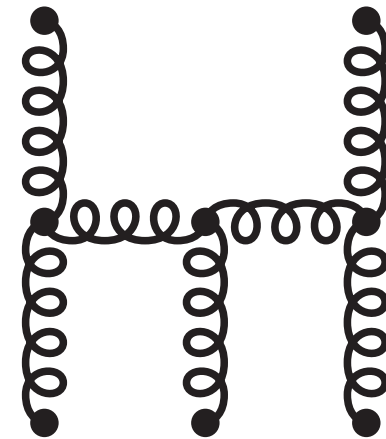
$$i f^{abc} T_i^a T_j^b T_k^c$$



$$f^{abe} f^{cde} T_i^a T_j^b T_k^c T_l^d$$



$$i f^{adf} f^{bcg} f^{efg} T_i^a T_j^b T_k^c T_l^d T_m^e$$



apply $[T_i^a, T_j^b] = i f^{abc} T_i^c$

- Symmetrization with external legs

$$T_i^a T_j^a$$

color dipole

$$\mathcal{T}_{ijk} \equiv i f^{abc} (T_i^a T_j^b T_k^c)_{+}$$

$$\mathcal{T}_{ijkl} \equiv f^{ade} f^{bce} (T_i^a T_j^b T_k^c T_l^d)_{+}$$

Sum over all the permutations of i,j,k,...

Becher, Neubert '09

Construct Soft Anomalous Dimensions

- Only linearly depends on cusp angles due to RG invariance

$$\frac{\partial \Gamma_s}{\partial L_i} = -\frac{\partial \Gamma_c}{\partial L_i} \mathbf{1} = \Gamma_{\text{cusp}}^i \mathbf{1}$$

No terms like $\beta_{ij}^2, \beta_{ij}^3, \dots$

Except for conformal cross ratios

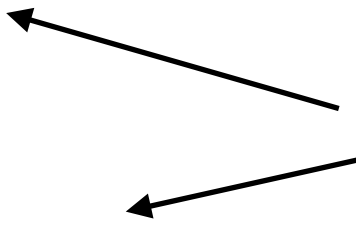
- Symmetry properties

$$\mathcal{T}_{ijkl} = \mathcal{T}_{jilk} = -\mathcal{T}_{ikjl} = -\mathcal{T}_{ljki} = \mathcal{T}_{klij}$$

- Jacobi identity

$$\mathcal{T}_{iklj} + \mathcal{T}_{iljk} + \mathcal{T}_{ijkl} = 0$$

also for β_{ijkl}



only two of twenty-four are linearly independent

Construct Soft Anomalous Dimensions

- When kinematic functions hit color structures Becher, Neubert, '09

- ▶ Tripole correlations $i f^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c$ and $i f^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c$ vanish

impossible to construct an anti-symmetric kinematic functions

Explain only dipole structures for two-loop massless amplitudes !!!

- ▶ $i f^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c F \left(\beta_{IJ}, \ln \frac{v_I \cdot p_k}{v_J \cdot p_k} \right)$ and $i f^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c F(\beta_{IJ}, \beta_{IK}, \beta_{JK})$

are allowed starting at two-loop order Ferroglia, Neubert, Pecjak, Yang, '09

- ▶ Kinematic functions correspond to $\mathcal{T}_{ijkl}$ and $\mathcal{T}_{ijkI}$ are odd functions

Construct Soft Anomalous Dimensions

- Non-dipole building blocks up to three loops (only one massive parton)

$$\mathcal{T}_{iijj}\beta_{ij}, \quad \mathcal{T}_{iijj}, \quad \mathcal{T}_{jjII}\beta_{Ij}, \quad \mathcal{T}_{jjII}, \quad \mathcal{T}_{iijk}\beta_{ij}, \quad \mathcal{T}_{iijk}\beta_{jk}, \quad \mathcal{T}_{iijk}, \quad \mathcal{T}_{ijII}\beta_{ij},$$

$$\mathcal{T}_{ijII}\beta_{Ii}, \quad \mathcal{T}_{iijI}\beta_{ij}, \quad \mathcal{T}_{iijI}\beta_{Ii}, \quad \mathcal{T}_{iijI}\beta_{Ij}, \quad \mathcal{T}_{ijkl}\beta_{ij}, \quad \mathcal{T}_{ijkI}\beta_{ij}, \quad \mathcal{T}_{ijkI}\beta_{Ij},$$

$$\mathcal{T}_{ijII}\bar{F}_{h2}^{[A]}(r_{ijI}), \quad \mathcal{T}_{iijI}\bar{F}_{h2}^{[B]}(r_{ijI}), \quad \mathcal{T}_{ijkI}\bar{F}_{h3}(r_{ijI}, r_{ikI}, r_{jkI}), \quad \mathcal{T}_{ijkl}\bar{F}_4(\beta_{ijkl}, \beta_{ijkl} - 2\beta_{ilkj})$$

not linear independent, so vanish

Symmetry property of $\mathcal{T}_{ijkl}$ has been taken into consideration

- Color conservation: $\sum_i \mathbf{T}_i + \sum_I \mathbf{T}_I = 0$ in the color-space formalism
massless massive Catani, Seymour, '96

leads to

$$\sum_{i \neq j} \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{I,j} \mathbf{T}_I \cdot \mathbf{T}_j = - \sum_j \mathbf{T}_j^2 = - \sum_j C_{R_j} \quad \text{Casimir invariants}$$

$$\mathcal{T}_{ijII} = \frac{1}{2} (\mathcal{T}_{jjii} + \mathcal{T}_{iijI}) - \frac{1}{2} \sum_{k \neq i,j} (\mathcal{T}_{ijkI} + \mathcal{T}_{jikI}) - \frac{1}{2} \sum_{J \neq I} (\mathcal{T}_{ijIJ} + \mathcal{T}_{jiIJ})$$

and a few more similar relations, which help to reduce the building blocks.

Construct Soft Anomalous Dimensions

- The detailed derivation is given by

$$\begin{aligned}
 \bar{\Gamma}_s^{(3)} = & \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} (\beta_{ij} \bar{f}_1 + \bar{h}_1) + \sum_{I,j} \mathbf{T}_I \cdot \mathbf{T}_j \left[\left(\bar{f}_1 + \frac{C_A^2}{8} \bar{f}_2 \right) \beta_{Ij} + \bar{h}_2 \right] + \sum_i \bar{c}_i + \sum_I \bar{c}_I \\
 & + \sum_{(i,j)} \mathcal{T}_{iijj} (\beta_{ij} \bar{f}_3 + \bar{h}_3) + \sum_{I,j} \mathcal{T}_{IIjj} (\beta_{Ij} \bar{f}_4 + \bar{h}_4) + \sum_{(i,j,k)} \mathcal{T}_{iijk} (\beta_{jk} \bar{f}_5 + \bar{h}_5) + \sum_I \sum_{(i,j)} \mathcal{T}_{iijI} \beta_{Ij} \bar{f}_6 \\
 & + \sum_I \sum_{(i,j)} \mathcal{T}_{ijII} [\beta_{ij} \bar{f}_7 + \bar{F}_{h2}(r_{ijI})] + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl} [\beta_{ijkl} \bar{f}_8 + \bar{F}_4(\beta_{ijkI}, \beta_{ijkl} - 2\beta_{ilkj})] \\
 & + \sum_I \sum_{(i,j,k)} \mathcal{T}_{ijkI} \bar{F}_{h3}(r_{ijI}, r_{ikI}, r_{jkI}) + \dots
 \end{aligned}$$

Using constraint from soft-collinear factorization

$$\begin{aligned}
 \frac{\partial \bar{\Gamma}_s^{(3)}}{\partial L_i} = & C_{Ri} \left(\frac{C_A^2}{4} \bar{f}_5 - \bar{f}_1 \right) + \frac{C_A^2}{8} \sum_I \mathbf{T}_I \cdot \mathbf{T}_i (\bar{f}_2 + 2\bar{f}_5 - 2\bar{f}_7) + 2 \sum_{j \neq i} \mathcal{T}_{iijj} (\bar{f}_3 - \bar{f}_5) + \sum_I \mathcal{T}_{iiII} (\bar{f}_4 - 2\bar{f}_7) \\
 & + \sum_I \sum_{j \neq i} \mathcal{T}_{jjii} (\bar{f}_6 - 2\bar{f}_5) + \dots
 \end{aligned}$$

We have

$$\bar{f}_4 = 2\bar{f}_7 = \bar{f}_2 + 2\bar{f}_3, \quad \bar{f}_5 = \bar{f}_3, \quad \bar{f}_6 = 2\bar{f}_3$$

General Structure of Anomalous Dimension

General formula of anomalous dimensions up to three loops

$$\Gamma(\{\underline{p}\}, \{\underline{m}\}, \mu) = \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s) \mathbf{1}$$

Becher, Neubert, 0901.0722, 0903.1126
Gardi, Magnea, 0901.1091

Becher, Neubert
0904.1021

$$+ \sum_{I,j} \mathbf{T}_I \cdot \mathbf{T}_j \gamma_{\text{cusp}}(\alpha_s) \ln \frac{m_I \mu}{-s_{Ij}} - \sum_{(I,J)} \frac{\mathbf{T}_I \cdot \mathbf{T}_J}{2} \gamma_{\text{cusp}}(\beta_{IJ}, \alpha_s) + \sum_I \gamma^I(\alpha_s) \mathbf{1}$$

Almelid, Duhr, Gardi
1507.00047

$$+ f(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{ijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl} F_4(\beta_{ijkl}, \beta_{ijkl} - 2\beta_{ilkj}, \alpha_s)$$

← Starting at 3L

ZLL, Schalch
2207.02864

$$+ \sum_I \sum_{(i,j)} \mathcal{T}_{ijII} F_{h2}(r_{ijI}, \alpha_s) + \sum_I \sum_{(i,j,k)} \mathcal{T}_{ijkI} F_{h3}(r_{ijI}, r_{ikI}, r_{jkI}, \alpha_s)$$

+ [non-dipole contributions involving two or more massive partons]

+ $\mathcal{O}(\alpha_s^4)$

$$if^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_k^c F \left(\beta_{IJ}, \ln \frac{v_I \cdot p_k}{v_J \cdot p_k} \right)$$

$$if^{abc} \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c F(\beta_{IJ}, \beta_{IK}, \beta_{JK})$$

up to two loops

Ferrogia, Neubert, Pecjak, Yang, '09

General Structure of Anomalous Dimension

Massless anomalous dimensions up to four loops

Becher & Neubert, '19

$$\Gamma^{(4)}(\{p\}, \mu) = \sum_R g^R(\alpha_s) \left[\sum_{(i,j)} (\mathcal{D}_{iijj}^R + 2\mathcal{D}_{iiij}^R) \ln \frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}_{ijkk}^R \ln \frac{\mu^2}{-s_{ij}} \right]$$

Violate Casimir scaling

$$+ \sum_R \sum_{(i,j,k,l)} \mathcal{D}_{ijkl}^R G^R(\beta_{ijlk}, \beta_{iklj}; \alpha_s) + \sum_{(i,j,k,l)} \mathcal{T}_{ijkli} H_1(\beta_{ijlk}, \beta_{iklj}; \alpha_s)$$

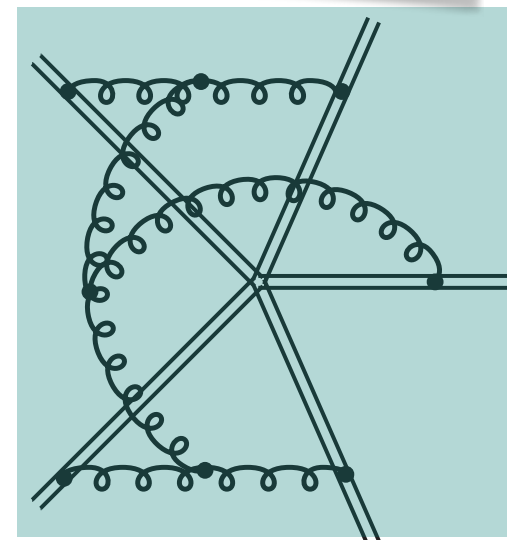
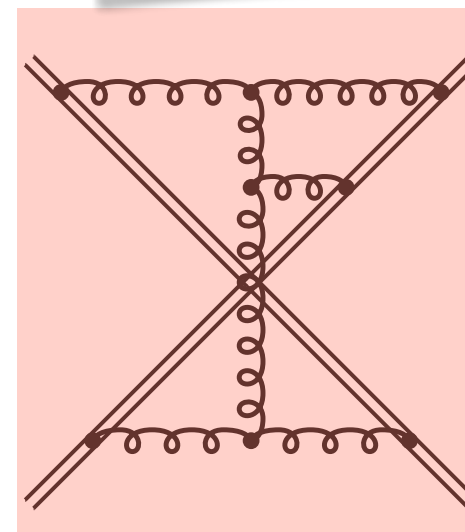
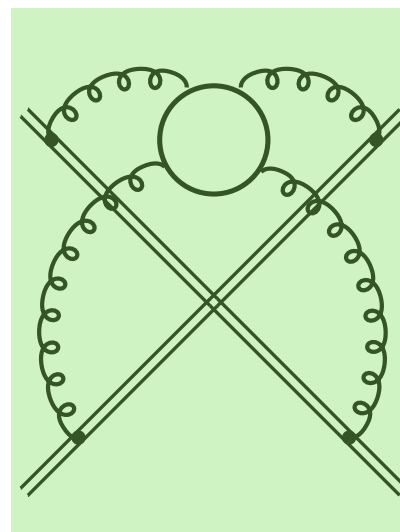
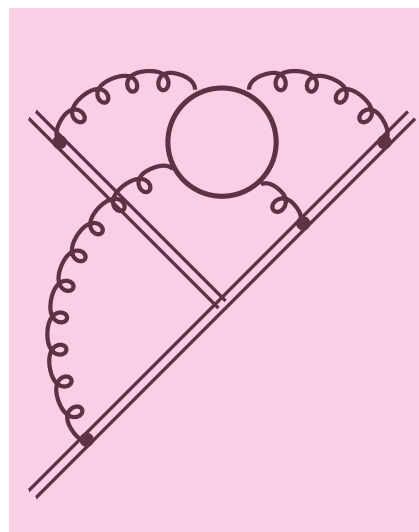
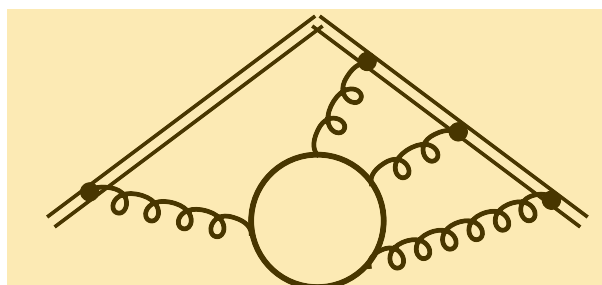
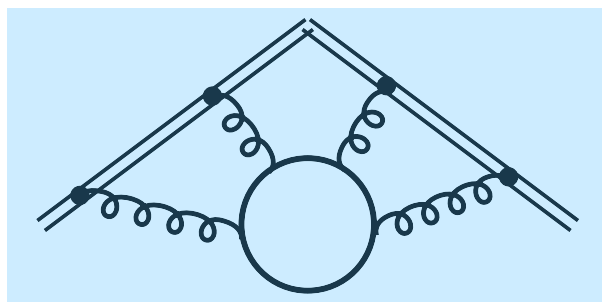
$$+ \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm} H_2(\beta_{ijkl}, \beta_{ijmk}, \beta_{ikmj}, \beta_{jiml}, \beta_{jlmj}; \alpha_s) + \mathcal{O}(\alpha_s^5).$$

Boels, Huber, Yang, '17

Symmetric Tensors:

$$\mathcal{D}_{ijkl}^R = d_R^{abcd} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d$$

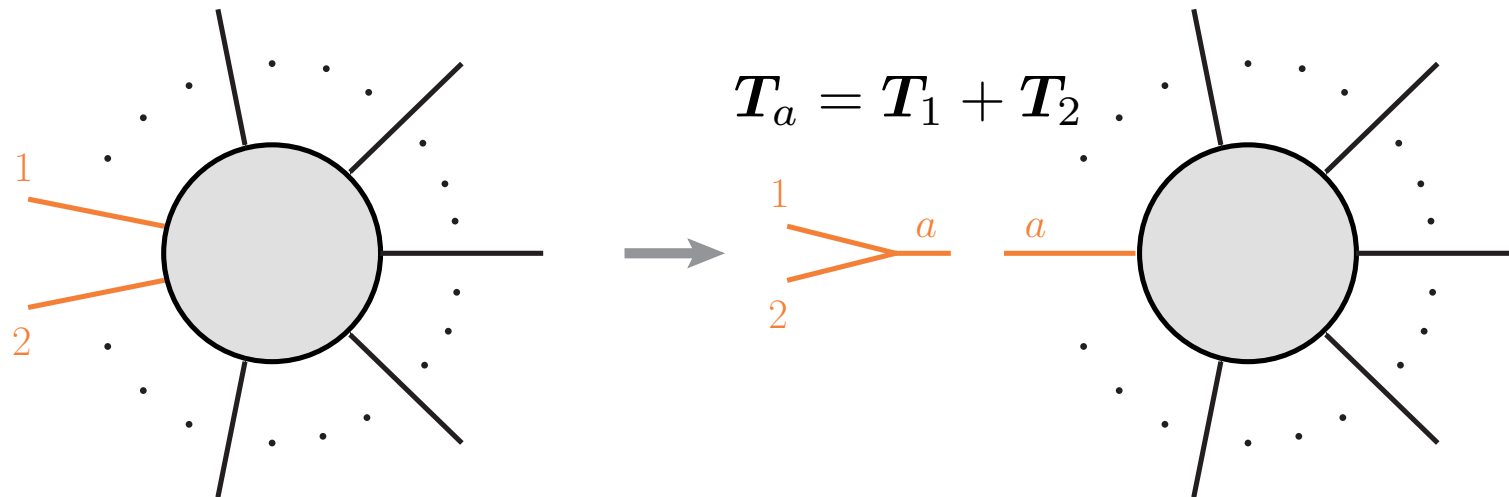
$$d_R^{a_1 \dots a_n} = \text{Tr}_R (\mathbf{T}^{a_1} \dots \mathbf{T}^{a_n})_+$$



Two-Particle Collinear Limit

- When particle 1 and 2 become collinear

Berends, Giele, '89, Bern, Chalmers, '95
Kosower, '99, Mangano, Parke, '05



n-particle scattering

Splitting function (n-1)-particle scattering

$$|\mathcal{M}(\{p_1, p_2, p_3, \dots, p_n\}, \epsilon)\rangle \simeq \mathbf{Sp}(\{p_1, p_2\}) |\mathcal{M}(\{p_a, p_3, \dots, p_n\}, \epsilon)\rangle$$

leads to $\Gamma_{\text{Sp}}(\{p_1, p_2\}, \mu) = \Gamma(\{p_1, p_2, \dots, p_n\}, \{\underline{m}\}, \mu) - \Gamma(\{p_a, \dots, p_n\}, \{\underline{m}\}, \mu)$

must be independent of color generators for particles other than 1 and 2, so

$$\lim_{\omega \rightarrow -\infty} F_4(\omega, \omega, \alpha_s) = \frac{f(\alpha_s)}{2},$$

Almelid, Duhr, Gardi, '15

$$F_{h2}(0, \alpha_s) = 3f(\alpha_s),$$

$$F_{h3}(0, r, r, \alpha_s) = 2f(\alpha_s)$$

ZLL, Schalch, '22

Two-Particle Collinear Limit

- The detailed derivation is given by

$$\mathbf{T}_a = \mathbf{T}_1 + \mathbf{T}_2$$

$$\Gamma_{\text{Sp}}(\{p_1, p_2\}, \mu) = \Gamma(\{p_1, p_2, \dots, p_n\}, \{\underline{m}\}, \mu) - \Gamma(\{p_a, \dots, p_n\}, \{\underline{m}\}, \mu)$$

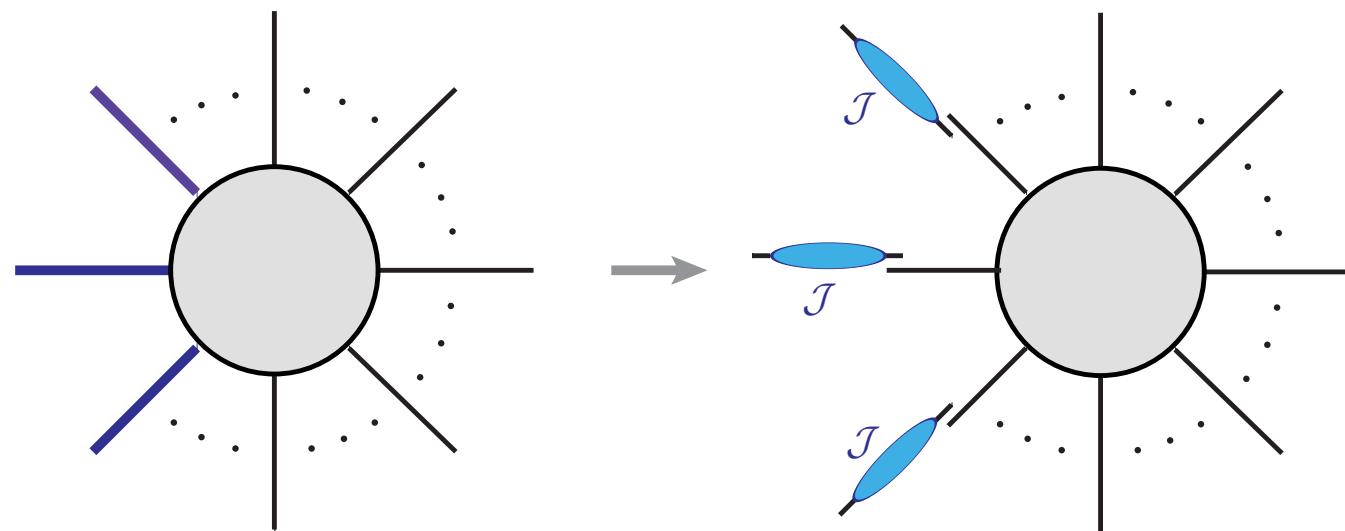
$$\begin{aligned} &= \gamma_{\text{cusp}}(\alpha_s) \left[\mathbf{T}_1 \cdot \mathbf{T}_2 \left(\ln \frac{\mu^2}{-s_{12}} + \ln[z(1-z)] \right) + C_{R_1} \ln z + C_{R_2} \ln(1-z) \right] \\ &\quad + [\gamma^1(\alpha_s) + \gamma^2(\alpha_s) - \gamma^a(\alpha_s)] \mathbf{1} \\ &\quad + \left[f(\alpha_s) + 4F_4(\omega_{ij}, \omega_{ij}, \alpha_s) \right] \left(-\frac{C_A^2}{4} \mathbf{T}_1 \cdot \mathbf{T}_2 - 2\mathcal{T}_{1122} \right) \\ &\quad + 4 \sum_{i \neq 1,2} \mathcal{T}_{12ii} \left[f(\alpha_s) - 2F_4(\omega_{ij}, \omega_{ij}, \alpha_s) \right] \quad \text{Almelid, Duhr, Gardi, '15} \\ &\quad + 2 \sum_I \mathcal{T}_{12II} \left[F_{h2}(0, \alpha_s) - f(\alpha_s) - 4F_4(\omega_{ij}, \omega_{ij}, \alpha_s) \right] \quad \text{ZLL, Schalch, '22} \\ &\quad + 2 \sum_I \sum_{i \neq 1,2} (\mathcal{T}_{12iI} + \mathcal{T}_{21iI}) \left[F_{h3}(0, r_{1iI}, r_{1iI}, \alpha_s) - 4F_4(\omega_{ij}, \omega_{ij}, \alpha_s) \right] \\ &\quad + \dots \end{aligned}$$

where we have used the relation:

$$\sum_{(i,j)}^{i,j \neq 1,2} \mathcal{T}_{12ij} = -\frac{C_A^2}{8} \mathbf{T}_1 \cdot \mathbf{T}_2 - \mathcal{T}_{1122} - \sum_{i \neq 1,2} \mathcal{T}_{12ii} - \sum_I \mathcal{T}_{12II} - \sum_{I,i}^{i \neq 1,2} (\mathcal{T}_{12iI} + \mathcal{T}_{12Ii}) - \sum_{(I,J)} \mathcal{T}_{12IJ}$$

Small-Mass Limit

- When masses of external legs are much smaller than the hard scales



Mitov, Moch, '06
Becher, Melnikov, '07

collinear singularities
regularized by masses

purely massless amplitude

$$\lim_{m \rightarrow 0} |\mathcal{M}(\{\underline{p}\}, \{\underline{m}\}, \epsilon)\rangle \simeq \prod_I \mathcal{J}_I(\{\underline{m}\}, \epsilon) S(\{\underline{m}, \epsilon\}) |\mathcal{M}(\{\underline{p}\}, \{\underline{0}\}, \epsilon)\rangle$$

Heavy quark loops

$$\Gamma(\{\underline{p}\}, \{\underline{m} \rightarrow 0\}, \mu) - \Gamma(\{\underline{p}\}, \{\underline{0}\}, \mu) = \sum_I \left[C_{R_I} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu}{m_I} + \gamma^Q - \gamma^q \right]$$

There is no color exchange between different external legs, so

$$F_{h2}(0, \alpha_s) = 3f(\alpha_s), \quad \lim_{v_I^2 \rightarrow 0} F_{h3}(r_{ijI}, r_{ikI}, r_{jkI}, \alpha_s) = 2f(\alpha_s) + 4F_4(\beta_{ijkI}, \beta_{ijkI} - 2\beta_{kjiI}, \alpha_s)$$

ZLL, Schalch, '22

Small-Mass Limit

- In small-mass limit

$$\lim_{m \rightarrow 0} \beta_{IJ} = \lim_{m \rightarrow 0} \cosh^{-1} \left(\frac{-s_{IJ}}{2m_I m_J} \right) \simeq \ln \frac{\mu}{m_I} + \ln \frac{\mu}{m_J} - \ln \frac{\mu^2}{-s_{IJ}}$$

we have

$$\begin{aligned} \Gamma(\{\underline{p}\}, \{\underline{m} \rightarrow 0\}, \mu) - \Gamma(\{\underline{p}\}, \{\underline{0}\}, \mu) = & \sum_{I,i} \mathbf{T}_I \cdot \mathbf{T}_i \ln \frac{m_I}{\mu} + \sum_{(I,J)} \mathbf{T}_I \cdot \mathbf{T}_J \ln \frac{m_I}{\mu} + \sum_I (\gamma^Q - \gamma^q) \\ & + \sum_I \sum_{(i,j)} \left[\mathcal{T}_{ijII} F_{h2}(0, \alpha_s) - (\mathcal{T}_{ijII} + \mathcal{T}_{iiIj} + \mathcal{T}_{jjiiI}) f(\alpha_s) \right] \\ & + \sum_I \sum_{(i,j,k)} \mathcal{T}_{ijkI} \left[\lim_{v_I^2 \rightarrow 0} F_{h3}(r_{ijI}, r_{ikI}, r_{jkI}, \alpha_s) - 4F_4(\beta_{ijkI}, \beta_{ijkI} - 2\beta_{kjiI}, \alpha_s) \right] \end{aligned}$$

using

$$\mathcal{T}_{ijII} = \frac{1}{2} (\mathcal{T}_{jjiiI} + \mathcal{T}_{iiijI}) - \frac{1}{2} \sum_{k \neq i,j} (\mathcal{T}_{ijkI} + \mathcal{T}_{jikI}) - \frac{1}{2} \sum_{J \neq I} (\mathcal{T}_{ijIJ} + \mathcal{T}_{jiIJ})$$

$$F_{h2}(0, \alpha_s) = 3f(\alpha_s), \quad \lim_{v_I^2 \rightarrow 0} F_{h3}(r_{ijI}, r_{ikI}, r_{jkI}, \alpha_s) = 2f(\alpha_s) + 4F_4(\beta_{ijkI}, \beta_{ijkI} - 2\beta_{kjiI}, \alpha_s)$$

ZLL, Schalch, '22

State of Art of Anomalous Dimensions

- **Four-loop γ_{cusp} and $\gamma^{q/g}$**

Henn, Smirnov, Smirnov, Steinhauser, '16; + Lee, '16;

massless form factors

Lee, Smirnov, Smirnov, Steinhauser, '17,'19;

γ_{cusp} and $\gamma^{q/g}$ are complete

von Manteuffel, Schabinger, '19, + Panzer '20, + Agarwal '21;

Davies, Vogt, Ruijl, Ueda, Vermaseren, '16;

splitting functions for γ_{cusp}

Moch, Ruijl, Ueda, Vermaseren, Vogt, '17, '18

Grozin, '18; Henn, Peraro, Stahlhofen, Wasser, '19;

Wilson loop for γ_{cusp}

Henn, Korchemsky, Mistlberger, '19;

Brüser, Grozin, Henn, Stahlhofen, '19;

angle-dependent $\gamma_{\text{cusp}}(\beta)$

Brüser, Dlapa, Henn, Yan, '20

still incomplete at 4 loops

- **Three-loop $\gamma_{\text{cusp}}(\beta)$ and γ^Q**

Grozin, Henn, Korchemsky, Marquard, '14,'15

$\gamma_{\text{cusp}}(\beta)$ and γ^Q are complete at 3 loops

Brüser, ZLL, Stahlhofen, '19 only for γ^Q

- **Three-loop non-dipole terms**

Almelid, Duhr, Gardi, '15; + McLeod, White, '17

f and F_4

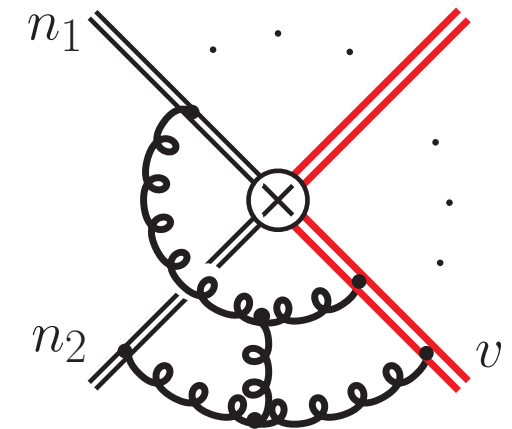
ZLL, Stahlhofen, '20 only for f

Calculation of Tripole Correlation

- We calculate three-loop tripole correlation

$$F_{h2}(r, \alpha_s) = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathcal{F}_{h2}(r) + \mathcal{O}(\alpha_s^4)$$

$$\text{with } r = \frac{v^2(n_1 \cdot n_2)}{2(v \cdot n_1)(v \cdot n_2)}$$



- Regularization of IR poles

- Configuration space with exponential regulators

Gardi, '13;
Almelid, Duhr, Gardi, '15

$$ig_s \int_0^\infty dt n_i \cdot A(tn_i) \rightarrow ig_s \int_0^\infty dt e^{-i\omega t \sqrt{n_i^2 - i0}} n_i \cdot A(tn_i)$$

Calculation in Feynman gauge and Mellin-Barnes representation

Perform asymptotic expansion $(n_i \cdot n_j) / \sqrt{n_i^2 n_j^2} \rightarrow \infty$

- Momentum space with off-shellness

Ferrogia, Neubert, Pecjak, Yang, '09

$$\frac{1}{n \cdot k} \rightarrow \frac{1}{n \cdot k + \delta} \quad \frac{1}{v \cdot k} \rightarrow \frac{1}{v \cdot k + \delta'}$$

Gauge invariance is not guaranteed !

IR Regulator

- In SCET, low-energy matrix elements are free of IR poles

Construct a soft function at cross section level

$$S(\omega) = \langle 0 | \mathbf{Y}_{n_1}^\dagger \mathbf{Y}_{n_2}^\dagger \mathbf{Y}_v^\dagger \delta(\omega - v \cdot \hat{p}) \mathbf{Y}_v \mathbf{Y}_{n_1} \mathbf{Y}_{n_2} | 0 \rangle$$

Pick up momenta of all soft emissions

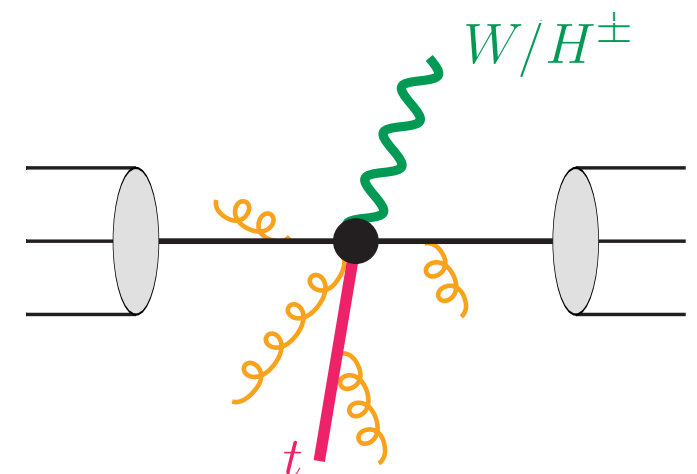
IR poles are regularized by the low-energy measurement

Soft function has applications in phenomenology, e.g.

$$\sigma(pp \rightarrow tW) \sim f_{a/N} \otimes f_{b/N} \otimes H_{ab \rightarrow tW} \otimes S$$

describe the soft-gluon effect near threshold in tW associated production

One-particle inclusive



- Several advantages

- ▶ ω is the only dimensionful variable, does NOT increase complexity of integrals
- ▶ Gauge invariance is preserved, calculation can be in general covariant gauge
- ▶ Calculation in momentum space, IBP and differential equation (DE) can be used

Phase Space to Loop Integration

- Soft function is defined at cross section level

$$S(\omega) = \langle 0 | \mathbf{Y}_{n_1}^\dagger \mathbf{Y}_{n_2}^\dagger \mathbf{Y}_v^\dagger \delta(\omega - v \cdot \hat{p}) \mathbf{Y}_v \mathbf{Y}_{n_1} \mathbf{Y}_{n_2} | 0 \rangle$$

Standard procedure is to compute VVR + VRR + RRR V: virtual, R: real

Can we find a compact way?

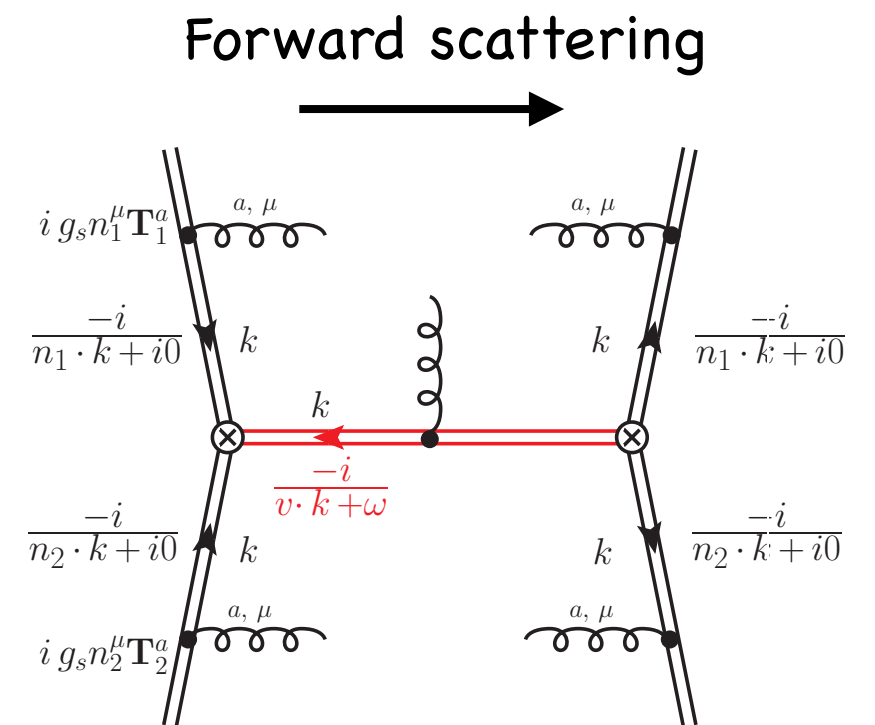
- Transform phase-space integrations to loop integrals ZLL, Stahlhofen, '20

$$\mathbf{Y}_{n_i} = \text{P exp} \left(i g_s \int_{-\infty}^0 dt n_i \cdot A^a(x + t n_i) \mathbf{T}_i^a \right)$$

$$\mathbf{Y}_v^\dagger \delta(\omega + i v \cdot \partial) \mathbf{Y}_v = \delta(\omega + i v \cdot D)$$

$$\delta(x) = \frac{1}{2\pi i} \left(\frac{1}{x - i0} - \frac{1}{x + i0} \right) \quad \text{reverse unitarity}$$

$$S(\omega) = \text{Re} \left[\text{Disc}_\omega \langle 0 | \mathbf{Y}_{n_1}^\dagger \mathbf{Y}_{n_2}^\dagger \frac{1}{\omega + i v \cdot D} \mathbf{Y}_{n_1} \mathbf{Y}_{n_2} | 0 \rangle \right]$$

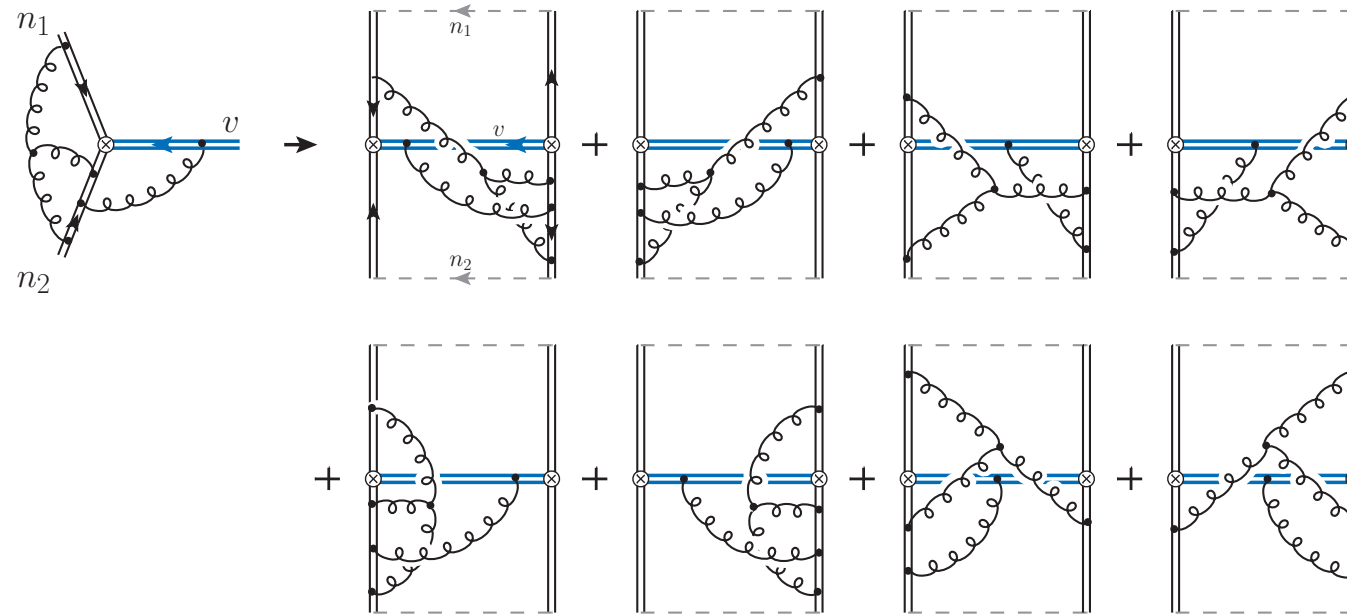


Imaginary part from branch cut in $(-n_1 \cdot n_2 - i0)^{-n_\epsilon}$ cancels between left-right mirror diagrams

Calculation

- Only consider the diagrams contribute to $\mathcal{F}_{h2}(r)$

Replica trick for evaluating diagrammatic contribution to the exponent is compatible



Nogueira, '93

- Using **QGraf** to generate diagrams, **976** in total
- Perform partial fraction, the integrals are mapped onto **30** topologies
- IBP reduction by **FIRE6** and **Kira**, there are **173** master integrals (MIs)
Smirnov, Chuharev, '19 Usovitsch et al., '20
- Using **CANONICA** and **DlogBasis**, convert DE to a **canonical form**
Meyer, '17 Henn, Mistlberger, Smirnov, Wasser, '20 **Henn, '13**
- Determine boundary conditions at $r = 1$ ($v^\mu = n_1^\mu + n_2^\mu$)

Solve the DEs

- Symbol alphabet is $\{r, r-1, r-2, (r-1)\sqrt{r}, \sqrt{r(r-1)}\}$
rationalized by $u = \sqrt{r} - 1$

- Solve the DEs order-by-order in ϵ in terms of **GPL** and **GHPL**

GPL

$$G(a_1, \dots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

GHPL

$$G(-\rho, \vec{a}; x) = \int_0^x \frac{dt}{\sqrt{t(t+4)}} G(\vec{a}; t)$$

Aglietti, Bonciani, '04

- Validity of the calculation
 - ✓ Gauge parameter ξ cancels out
 - ✓ **GHPLs** drop out, because there is no heavy quark pair threshold
 - ✓ All the poles from $1/\epsilon^5$ to $1/\epsilon$ cancels out

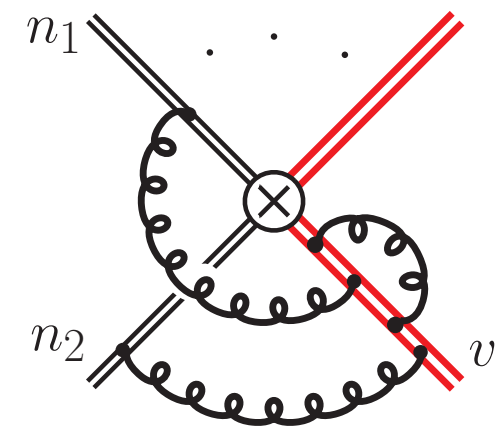
Result

- Three-loop tripole correlation can be simplified as $\hat{r} = \sqrt{r}$

$$\begin{aligned} \mathcal{F}_{h2}(r) = & 128 \left[H_{-1,0,0,0}(\hat{r}) + H_{-1,1,0,0}(\hat{r}) + H_{1,-1,0,0}(\hat{r}) - H_{1,0,0,0}(\hat{r}) \right] \text{ weight 4} \\ & + 128 (\zeta_2 + \zeta_3) \left[H_{1,0}(\hat{r}) - H_{-1,0}(\hat{r}) \right] + 96 (\zeta_3 + \zeta_4) \left[H_{-1}(\hat{r}) - H_1(\hat{r}) \right] \\ & + 128 \zeta_2 \left[H_{-2,0}(\hat{r}) - H_{2,0}(\hat{r}) + H_{-1,0,0}(\hat{r}) - H_{1,0,0}(\hat{r}) \right] \\ & + 256 \left[H_{1,2,0,0}(\hat{r}) + H_{2,0,0,0}(\hat{r}) - H_{-2,0,0,0}(\hat{r}) + H_{-1,-2,0,0}(\hat{r}) \right. \\ & \quad \left. - H_{-1,2,0,0}(\hat{r}) - H_{1,-2,0,0}(\hat{r}) - H_{-1,0,0,0,0}(\hat{r}) + H_{1,0,0,0,0}(\hat{r}) \right] \\ & + 48 (2\zeta_2\zeta_3 + \zeta_5) \end{aligned}$$

- ▶ **HPLs** are sufficient to describe the result
- ▶ Returns to f in massless case when $r \rightarrow 0$
- ▶ Does **NOT** have a uniform transcendental weight $2L - 1$, unlike f and F_4

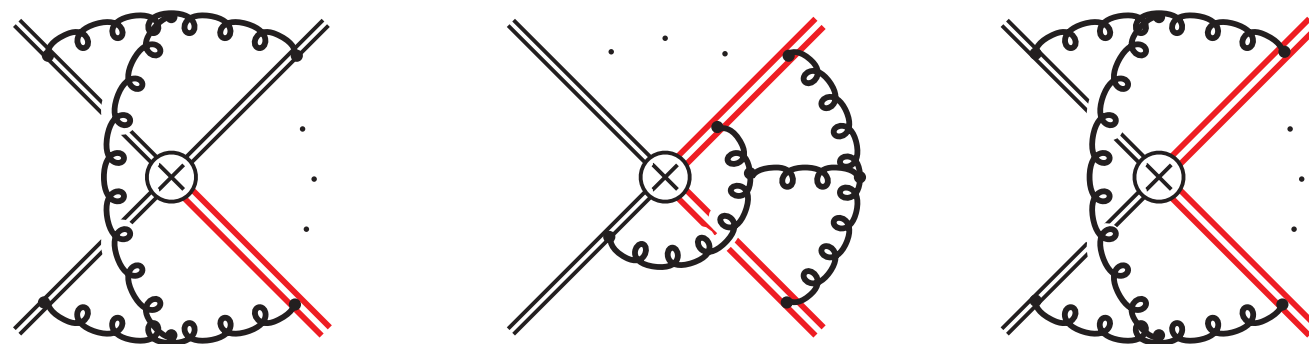
Boomerang webs



Gardi, Harley, Lodin, Palusa,
Smillie, White, Yeomans, '21

Summary & Outlook

- Complete the general structure of three-loop anomalous dimensions for QCD amplitudes with a massive and an arbitrary number of external legs
- Obtain the relations in two-particle collinear and small-mass limits
- Calculate the contribution from the tripole correlation between one massive and two massless legs
- Wish list



Henn, Milloy, Yan, '23

Thanks for your attention!

Replica Trick

- Soft exponent can be extracted from Gardi, Laenen, Stavenga, White '10

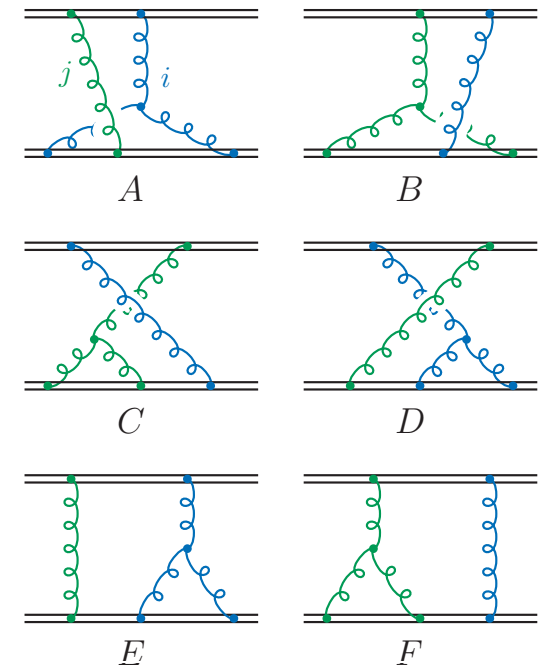
$$\ln S = \lim_{N \rightarrow 0} \frac{S^N - 1}{N}$$

Then consider a theory with N non-interacting copies of gauge fields

$\mathcal{R} \mathbf{P} \exp \left[i \sum_{j=1}^N \int ds_k n_k \cdot A^{k(j)}(s_k n_k) \right]$ with $\mathcal{R} \left[\mathbf{T}_k^{(i)} \mathbf{T}_k^{(j)} \right] = \begin{cases} \mathbf{T}_k^{(i)} \mathbf{T}_k^{(j)} & i \leq j \\ \mathbf{T}_k^{(j)} \mathbf{T}_k^{(i)} & i > j \end{cases}$

$\uparrow$
replica ordering operator

Diagram	replica	color	multiplicity	Diagram	replica	color	multiplicity
A	$i = j$	$C(A)$	N	D	$i = j$	$C(D)$	N
	$i > j$	$C(E)$	$N(N-1)/2$		$i > j$	$C(E)$	$N(N-1)/2$
	$i < j$	$C(F)$	$N(N-1)/2$		$i < j$	$C(F)$	$N(N-1)/2$
B	$i = j$	$C(B)$	N	E	$i = j$	$C(E)$	N
	$i > j$	$C(F)$	$N(N-1)/2$		$i > j$	$C(E)$	$N(N-1)/2$
	$i < j$	$C(E)$	$N(N-1)/2$		$i < j$	$C(F)$	$N(N-1)/2$
C	$i = j$	$C(C)$	N	F	$i = j$	$C(F)$	N
	$i > j$	$C(F)$	$N(N-1)/2$		$i > j$	$C(F)$	$N(N-1)/2$
	$i < j$	$C(E)$	$N(N-1)/2$		$i < j$	$C(E)$	$N(N-1)/2$



$$\sum_{D'} R_{DD'} C(D') = \left(f^{bde} f^{cae} \mathbf{T}_1^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_2^d, f^{bde} f^{cae} \mathbf{T}_1^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_2^d, \frac{C_A^2}{4} \mathbf{T}_1^a \mathbf{T}_2^a, \frac{C_A^2}{4} \mathbf{T}_1^a \mathbf{T}_2^a, 0, 0 \right)^T$$

Boundary Conditions of the MIs

- The scale dependence of integrals is trivial at boundary $r = 1$

- ▶ Mellin-Barnes representation + PSLQ

Difficult to compute 3-loop integrals analytically

- ▶ Sector decomposition

- ▶ Dimensional Recurrence Relations (DRR) is adopted

von Manteuffel, Panzer, Schabinger, '14 '15

$$\begin{array}{ccccc}
 G_i(d+2) & \xrightarrow[\text{LiteRed}]{\text{DRR}} & \sum_j c_j(d) F_j(d) & \xrightarrow{\text{IBP}} & \sum_j \tilde{c}_j(d) G_j(d) \quad \text{MIs} \\
 \downarrow & & & & \\
 & & & & \text{evaluated by HyperInt} \quad \text{Panzer, '15}
 \end{array}$$

Lee, '12

Related to a set of finite integrals $F_i(d = n - 2\epsilon)$ with $n = 4, 6, 8, \dots$

increase the dimension to $D=6$ $\rightarrow$ decrease IR poles

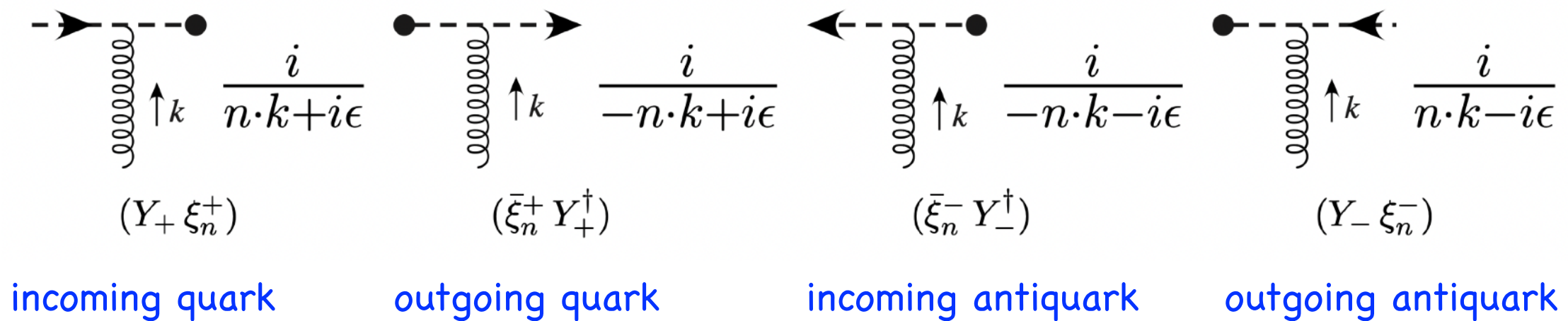
increase the power of propagators $\rightarrow$ decrease UV poles

Each MI can be expressed by a linear combination of finite integrals

Backup

- Eikonal $i\epsilon$ prescriptions for incoming/outgoing quarks and anti-quarks

Figure from Arnesen, Kundu, Stewart, 0508214



$$Y_+(x) = \bar{P} \exp \left[-ig_s \int_{-\infty}^0 dt n \cdot A^a(x + tn) \mathbf{T}^a \right]$$

$$Y_-(x) = P \exp \left[+ig_s \int_0^{\infty} dt n \cdot A^a(x + tn) \mathbf{T}^a \right]$$

$$Y_{\pm}^\dagger = (Y_{\mp})^\dagger$$

Phase Space to Loop Integration

- Soft function is defined at cross section level

$$S(\omega) = \langle 0 | \bar{T} [\mathbf{Y}_{n_1}^\dagger \mathbf{Y}_{n_2}^\dagger \mathbf{Y}_v^\dagger] \delta(\omega - v \cdot \hat{p}) T [\mathbf{Y}_{n_1} \mathbf{Y}_{n_2} \mathbf{Y}_v] | 0 \rangle$$

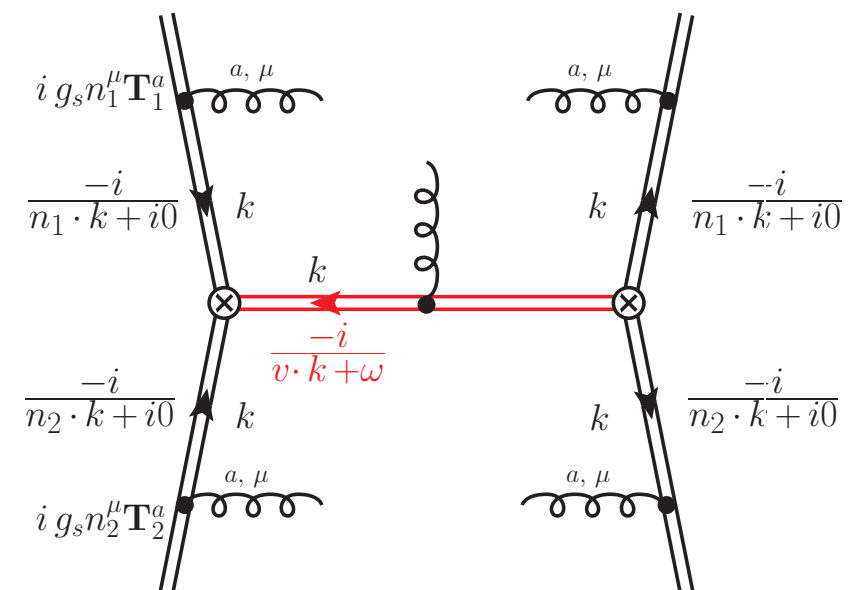
- Transform phase-space integrations to loop integrals

ZLL, Stahlhofen, '20

$$\mathbf{Y}_{i,+}(x) = \bar{P} \exp \left[-ig_s \int_{-\infty}^0 dt n_i \cdot A^a(x + tn_i) \mathbf{T}^a \right]$$

$$\mathbf{Y}_{i,-}(x) = P \exp \left[+ig_s \int_0^{\infty} dt n_i \cdot A^a(x + tn_i) \mathbf{T}^a \right]$$

$$S(\omega) = \frac{1}{2\pi} \text{Re} [\mathcal{S}(\omega + i0) - \mathcal{S}(\omega - i0)]$$



$$\mathcal{S}(\omega) = \int_0^\infty dt e^{i\omega t} \langle 0 | \bar{T} [\mathbf{Y}_{n_1}^\dagger(tv) \mathbf{Y}_{n_2}^\dagger(tv)] P \exp \left[ig \int_0^t ds v \cdot A^c(sv) \mathbf{T}_v^c \right] \mathbf{Y}_{n_1}(0) \mathbf{Y}_{n_2}(0) | 0 \rangle$$

outgoing

incoming

Result

- Three-loop tripole correlation can be simplified as $\hat{r} = \sqrt{r}$

$$\begin{aligned} \mathcal{F}_{h2}(r) = & 128 \left[H_{-1,0,0,0}(\hat{r}) + H_{-1,1,0,0}(\hat{r}) + H_{1,-1,0,0}(\hat{r}) - H_{1,0,0,0}(\hat{r}) \right] \\ & + 128 (\zeta_2 + \zeta_3) \left[H_{1,0}(\hat{r}) - H_{-1,0}(\hat{r}) \right] + 96 (\zeta_3 + \zeta_4) \left[H_{-1}(\hat{r}) - H_1(\hat{r}) \right] \\ & + 128 \zeta_2 \left[H_{-2,0}(\hat{r}) - H_{2,0}(\hat{r}) + H_{-1,0,0}(\hat{r}) - H_{1,0,0}(\hat{r}) \right] \\ & + 256 \left[H_{1,2,0,0}(\hat{r}) + H_{2,0,0,0}(\hat{r}) - H_{-2,0,0,0}(\hat{r}) + H_{-1,-2,0,0}(\hat{r}) \right. \\ & \quad \left. - H_{-1,2,0,0}(\hat{r}) - H_{1,-2,0,0}(\hat{r}) - H_{-1,0,0,0,0}(\hat{r}) + H_{1,0,0,0,0}(\hat{r}) \right] \\ & + 48 (2\zeta_2\zeta_3 + \zeta_5) \end{aligned}$$

- Does $\mathcal{F}_{h2}(r_{ijI})$ have an imaginary part?

$$r_{ijI} \equiv \frac{v_I^2 (n_i \cdot n_j)}{2 (v_I \cdot n_i)(v_I \cdot n_j)} = -e^{\beta_{ij} - \beta_{Ii} - \beta_{Ij}} \quad 0 < r_{ijI} < 1 \text{ in full kinematic region}$$

$$F_{h2}(0, \alpha_s) = 3f(\alpha_s) = \left(\frac{\alpha_s}{4\pi} \right)^3 48(2\zeta_2\zeta_3 + \zeta_5) + \mathcal{O}(\alpha_s^4)$$