The strong CP problem revisited

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In collaboration with:

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14 August, ITP-CAS, Beijing

Outline

- A historic review of QCD instantons
- The strong CP problem
- Fermion correlation functions in the theta vacuum
- Some discussions
- Summary

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Chiral limit

If we consider the following triplet in flavor space

$$\psi^T = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

with

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \bar{\psi}_i \left(\mathrm{i} \not{D} - M_{ij} \right) \psi_j$$

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For $M_{ij} \to 0$ (chiral limit) , we have the symmetry

$$U(3)_L \times U(3)_R = SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$$

$$\begin{pmatrix} u_{L/R} \\ d_{L/R} \\ s_{L/R} \end{pmatrix} \to \exp\left(-\mathrm{i} \sum_{a=1}^{8} \theta^a_{L/R} \frac{\lambda^a}{2} \right) e^{-\mathrm{i}\theta_{L/R}} \begin{pmatrix} u_{L/R} \\ d_{L/R} \\ s_{L/R} \end{pmatrix}$$

Vector: $\theta_L = \theta_R$

Axial: $\theta_L = -\theta_R$

The U(1) problem

Quark condensate

$$\langle \bar{\psi}\psi \rangle \neq 0$$

Chiral symmetry breaking

$$SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A \rightarrow SU(3)_V \times U(1)_V$$

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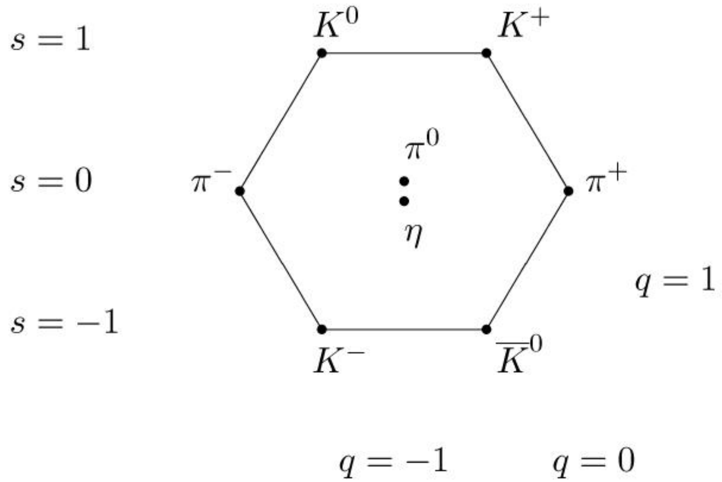
$$SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A \to SU(3)_V \times U(1)_V$$

This shall lead to 9 (pseudo)-Goldstone bosons



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S. Weinberg, PRD 11 (1975) 3583

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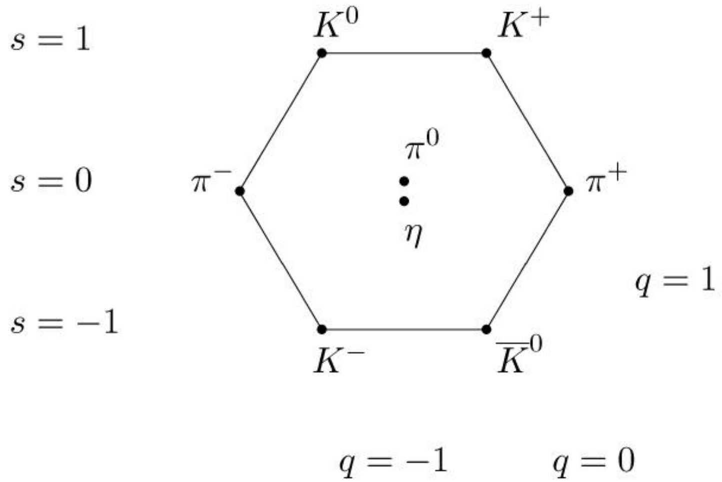

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Resolution: additional explicit symmetry breaking of $U(1)_A$ by instantons

't Hooft, PRL 37 (1976) 8

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When instanton effects are "integrated out", we have the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar{\psi}_i\left(\mathrm{i}\not{D} - M_{ij}\right)\psi_j + \underbrace{\left(\Gamma e^{-\mathrm{i}\theta}\det(\bar{\psi}_L\psi_R) + \Gamma e^{\mathrm{i}\theta}\det(\bar{\psi}_R\psi_L)\right)}_{\text{'t Hooft vertex}}$$

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Dirac mass term

In general, the Dirac mass matrix can have complex phases

$$M_{ij} = \mathrm{diag}(e^{\mathrm{i}\alpha_1\gamma^5} m_u, e^{\mathrm{i}\alpha_2\gamma^5} m_d, e^{\mathrm{i}\alpha_3\gamma^5} m_s)$$

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Write the Dirac mass term as

$$\bar{\psi}_i \left(m_i e^{\mathrm{i}\alpha_i\gamma^5} \right) \psi_i = \bar{\psi}_{iR} \left(m_i e^{-\mathrm{i}\alpha_i} \right) \psi_{iL} + \bar{\psi}_{iL} \left(m_i e^{\mathrm{i}\alpha_i} \right) \psi_{iR}$$

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Consider the chiral rotation

$$\psi_i \to e^{\mathrm{i}\beta_i\gamma^5} \psi_i, \quad \bar{\psi}_i \to \bar{\psi}_i \, e^{\mathrm{i}\beta_i\gamma^5}$$

one has

$$\bar{\psi}_i m_i e^{\mathrm{i}\alpha_i\gamma^5} \psi_i \to \bar{\psi}_{iR} \left(m_i e^{-\mathrm{i}(\alpha_i + 2\beta_i)} \right) \psi_{iL} + \bar{\psi}_{iL} \left(m_i e^{\mathrm{i}(\alpha_i + 2\beta_i)} \right) \psi_{iR}$$

Remaining chiral phase

Comparing the 't Hooft vertex with the Dirac mass term

$$\Gamma e^{-\mathrm{i}\theta} \det(\bar{\psi}_L \psi_R) + \Gamma e^{\mathrm{i}\theta} \det(\bar{\psi}_R \psi_L) = \Gamma \det\left(\bar{\psi} e^{-\mathrm{i}\theta\gamma^5/3} \psi\right)$$

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We can choose $2\beta_i = -\alpha_i$ to make the Dirac mass term real. Then

$$\theta \to \theta + \sum_i \alpha_i = \theta + \mathrm{Arg}\det M \equiv \bar{\theta}$$

Nonvanishing $\bar{\theta}$ causes CP violation

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$$\bar{\theta} < 10^{-10}!$$

Why so small?

Strong CP problem!

neutron electric dipole moment (nEDM):

$$|d_n| \approx (0.0 \pm 1.1) \times 10^{-26} e \cdot cm$$

Possible solutions

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R. D. Peccei, H. R. Quinn, PRL 38 (1977) 1440

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- Alignment $\Gamma \det\left(\bar{\psi} e^{-i\theta\gamma^5/3}\psi\right) \to \Gamma \det\left(\bar{\psi} e^{+i(\sum_i \alpha_i)\gamma^5/3}\psi\right)$ If yes, why and how?

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Our suggestion

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$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \bar{\psi}_i \left(\mathrm{i} \not{D} - M_{ij} \right) \psi_j + \Gamma \det \left(\bar{\psi} e^{+\mathrm{i}(\sum_\mathrm{i} \alpha_\mathrm{i}) \gamma^5 / 3} \psi \right)$$

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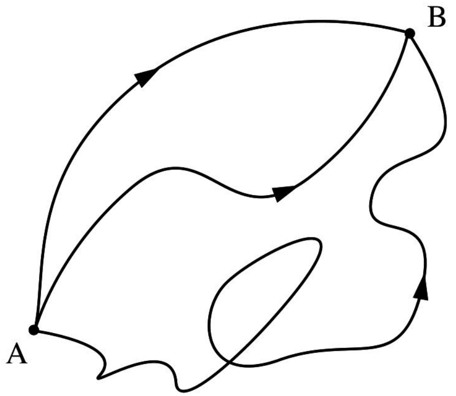
We only need to derive the form of the effective operator; the specific coefficients are irrelevant

Why a different result?

We will calculate the correlation function using path integral



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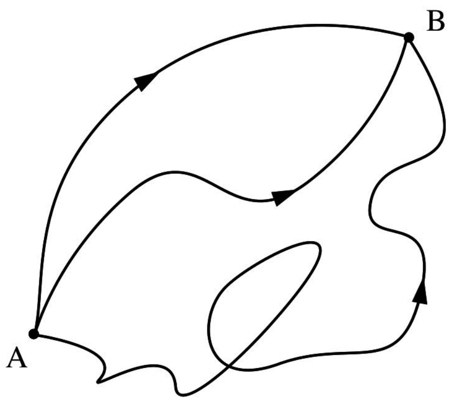

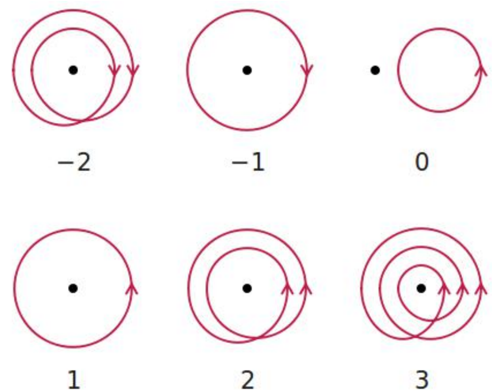

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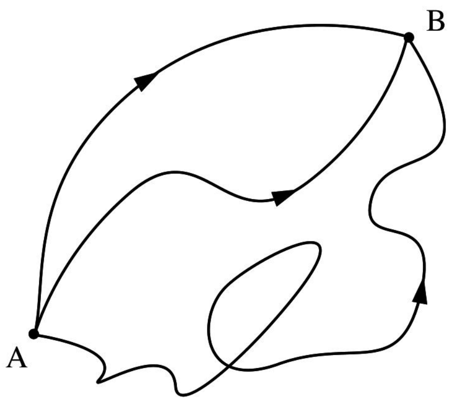

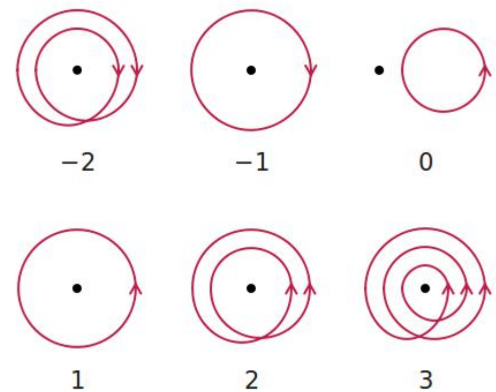

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In each topological sector, there is alignment between the chiral phases in the Dirac mass term and that in the 't Hooft vertex.

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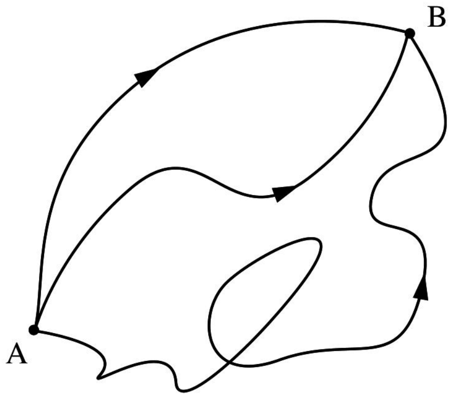

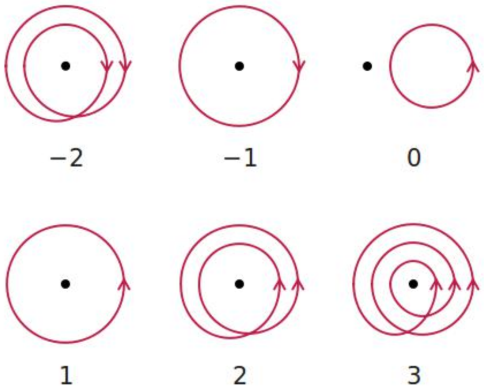

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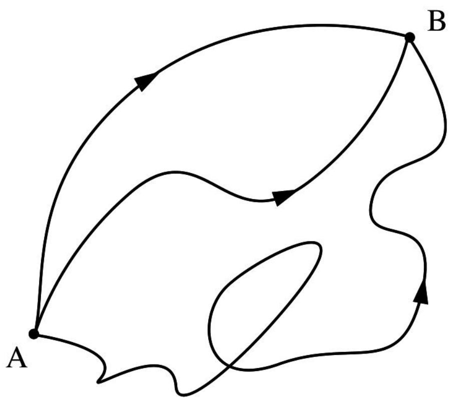

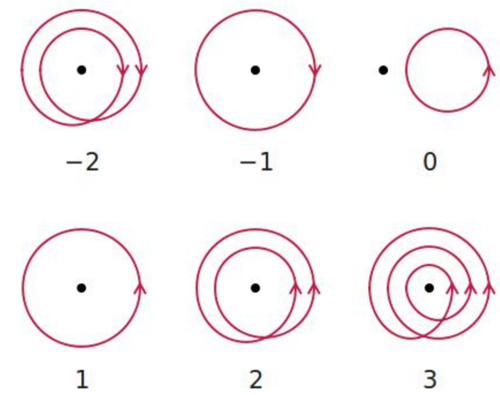

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Interferences between different topological sectors ⟶ "Standard" result!

However, we found that the interferences disappear if the we take the infinity spacetime volume limit! ⟶ New result!

The procedure

Step 0: Start with the Euclidean action that contain the instanton effects

↓

Step 1: Calculate the fermion correlation function using path integral

↓

Step 2: Compute the path integral using the method of steepest descent

↓

Step 3: After obtaining the correlation function, one can infer the effective 't Hooft vertex

Euclidean action and the path integral

Starting point: Euclidean action

$$S_E = \int \mathrm{d}^4x \left[\frac{1}{4}\hat{G}^a_{\mu\nu}\hat{G}^a_{\mu\nu} + \hat{\bar{\psi}}_i \left(\hat{\gamma}_\mu \hat{D}_\mu + M_{ij}\right)\hat{\psi}_j - \mathrm{i}\theta \frac{g^2}{32\pi^2}\hat{G}^a_{\mu\nu}\tilde{\hat{G}}^a_{\mu\nu}\right]$$

Encode the theta vacuum

where

$$\tilde{\hat{G}}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\hat{G}^a_{\alpha\beta}$$

The term $\hat{G}^a_{\mu\nu}\tilde{\hat{G}}^a_{\mu\nu}$ is topological and the integral over it gives the winding number (integers)

$$\nu = \frac{g^2}{32\pi^2}\int \mathrm{d}^4x\, \hat{G}^a_{\mu\nu}\tilde{\hat{G}}^a_{\mu\nu}, \quad \nu = n(t=+\infty) - n(t=-\infty)$$

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For simplicity we will consider $N_f = 1$. In terms of path integral

$$\langle\hat{\psi}(x)\hat{\bar{\psi}}(x')\rangle = \frac{1}{Z}\int \mathcal{D}\hat{A}_\mu\mathcal{D}\hat{\psi}\mathcal{D}\hat{\bar{\psi}}\,\hat{\psi}(x)\hat{\bar{\psi}}(x')e^{-S_E}$$

where

$$Z = \int \mathcal{D}\hat{A}_\mu\mathcal{D}\hat{\psi}\mathcal{D}\hat{\bar{\psi}}\,e^{-S_E}$$

The method of steepest descent

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- Find stationary points, i.e. classical solutions to field EoM
- Expand about the stationary points

There are indeed nontrivial classical solutions in Euclidean space: BPST instantons

A. A. Belavin etl. Phys.Lett.B 59 (1975) 85

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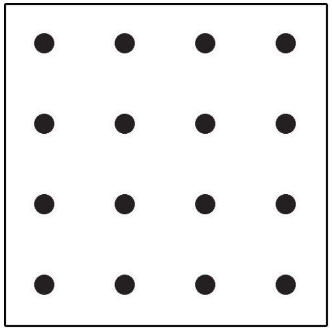
BPST instanton and anti-instanton are classical solutions to Yang-Mills equation with $\nu = \pm 1$, respectively.

$$S_E^{(\mathrm{YM})} = |\nu| \frac{8\pi^2}{g^2} \equiv S_1$$

Path integral (II)

For general winding number ν , we use the dilute-gas approximation:

n instantons $+$ $\bar{n}$ anti-instantons



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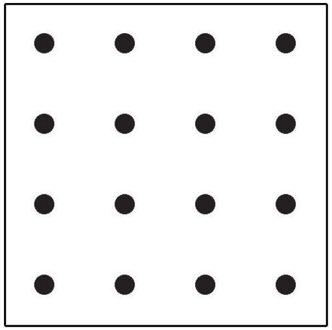

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The path integral can be written as

$$\langle \hat{\psi}(x)\hat{\bar{\psi}}(x')\rangle = \frac{\sum_{\nu}\left(\sum_{\substack{n,\bar{n}\geq 0\\ n-\bar{n}=\nu}}\int \mathcal{D}\hat{A}_{\mu}^{(n,\bar{n})}\mathcal{D}\hat{\psi}\mathcal{D}\hat{\bar{\psi}}\,\hat{\psi}(x)\hat{\bar{\psi}}(x')e^{-S_E}\right)}{\sum_{\nu}\left(\sum_{\substack{n,\bar{n}\geq 0\\ n-\bar{n}=\nu}}\int \mathcal{D}\hat{A}_{\mu}^{(n,\bar{n})}\mathcal{D}\hat{\psi}\mathcal{D}\hat{\bar{\psi}}\,e^{-S_E}\right)} \equiv \frac{\sum_{\nu}\langle \hat{\psi}(x)\hat{\bar{\psi}}(x')\rangle_{\nu}}{\sum_{\nu} Z_{\nu}}$$

The path integral is a sum over different topological sectors with different winding number ν .

Path integral in $(n,\bar{n})$ topological sector

The semiclassical part

$$e^{-(n+\bar{n})S_1}\, e^{\mathrm{i}\nu\theta}$$

Integrating the fluctuations about the background, we have

$$\int \mathcal{D}\hat{A}_\mu^{(n,\bar{n})} \mathcal{D}\hat{\psi} \mathcal{D}\hat{\bar{\psi}}\, \hat{\psi}(x)\hat{\bar{\psi}}(x') e^{-S_E} = \det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)_{n,\bar{n}} \left(\det{}_{\hat{\bar{A}}_{n,\bar{n}}}\right)^{-1/2} S_{n,\bar{n}}(x,x')$$

Three pieces:

Fermionic fluctuations

Gauge and ghost fluctuations

Fermionic Green's function

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Three pieces: Fermionic fluctuations; Gauge and ghost fluctuations; Fermionic Green's function

The gauge fluctuations contain zero modes corresponding to the collective coordinates of the BPST instantons

$$\left(\det{}_{\bar{\hat{A}}_{n,\bar{n}}}\right)^{-1/2} = \left(\det{}'_{\bar{\hat{A}}_{n,\bar{n}}}\right)^{-1/2}\frac{1}{n!\bar{n}!}\left(\prod_{\bar{m}=1}^{\bar{n}}\int_{V\mathcal{T}}\mathrm{d}^4x_{0,\bar{m}}\mathrm{d}\Omega_{\bar{m}}J_{\bar{m}}\right)\left(\prod_{m=1}^{n}\int_{V\mathcal{T}}\mathrm{d}^4x_{0,m}\mathrm{d}\Omega_mJ_m\right)$$

To operate on the Green's function

Remaining steps

- Construct the fermionic Green's function in the $(n, \bar{n})$ topological sector. Using the dilute-gas approximation, this can be obtained from the Green's functions in the BPST instanton backgrounds

- Carefully track the chiral phases in the fermonic functional determinants

$$\det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)_{n,\bar{n}} \sim \left(\det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)\Big|_{\nu=-1}\right)^{\bar{n}} \left(\det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)\Big|_{\nu=1}\right)^{n}$$

- Carry out the integral over the collective coordinates of the instantons

We will show that the formulae given later can give us the “standard” result and how one can get a different result with a new observation

Results in each topological sector

For the partition function within a fixed topological sector

$$Z_\nu = \sum_{\substack{n,\bar{n}\geq 0 \\ n-\bar{n}=\nu}} \frac{1}{n!\bar{n}!}(V\mathcal{T})^{n+\bar{n}}\kappa^{n+\bar{n}}(-1)^{n+\bar{n}}e^{\mathrm{i}\nu(\alpha+\theta)}$$

where

$$\kappa = \int \mathrm{d}\Omega\, J\Theta\varpi e^{-S_1}$$

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For the correlation function within a fixed topological sector

$$\langle\hat{\psi}(x)\hat{\bar{\psi}}(x')\rangle_\nu$$
$$= \sum_{\substack{n,\bar{n}\geq 0 \\ n-\bar{n}=\nu}} \frac{1}{n!\bar{n}!}\left[h(x,x')\left(\frac{\bar{n}}{me^{-\mathrm{i}\alpha}}P_L + \frac{n}{me^{\mathrm{i}\alpha}}P_R\right)(V\mathcal{T})^{n+\bar{n}-1} + S_{0\mathrm{inst}}(x,x')(V\mathcal{T})^{n+\bar{n}}\right]\kappa^{n+\bar{n}}(-1)^{n+\bar{n}}e^{\mathrm{i}\nu(\alpha+\theta)}$$

where

$$S_{0\mathrm{inst}}(x,x') = \int \frac{\mathrm{d}^4p}{(2\pi)^4}e^{-\mathrm{i}p(x-x')}\frac{\mathrm{i}\gamma_\mu k_\mu + me^{-\mathrm{i}\alpha\gamma^5}}{k^2+m^2}(P_L+P_R)$$

Standard result

If we keep $V\mathcal{T}$ to be finite in each topological sector, then

$$\sum_{\nu} \sum_{\substack{n,\bar{n}\geq 0 \\ n-\bar{n}=\nu}} \to \sum_{n,\bar{n}\geq 0}$$

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For the chiral pieces induced by instanton effects, look at e.g.

$$\sum_{n,\bar{n}\geq 0} \frac{1}{n!\bar{n}!} h(x,x')\left(\frac{\bar{n}}{me^{-\mathrm{i}\alpha}}P_L\right)(V\mathcal{T})^{n+\bar{n}-1}\kappa^{n+\bar{n}}(-1)^{n+\bar{n}}e^{\mathrm{i}(n-\bar{n})(\alpha+\theta)}$$
$$= \left(-e^{-\mathrm{i}\theta}\frac{\kappa}{m}P_L h(x,x')\right) \sum_{n\geq 0,\bar{n}\geq 1}\left(\frac{1}{n!}(V\mathcal{T})^n(-1)^n\left(e^{\mathrm{i}(\alpha+\theta)}\right)^n\right)\left(\frac{1}{(\bar{n}-1)!}(V\mathcal{T})^{\bar{n}-1}(-1)^{\bar{n}-1}\left(e^{-\mathrm{i}(\alpha+\theta)}\right)^{\bar{n}-1}\right)$$
$$= -\left(e^{-\mathrm{i}\theta}P_L\right)\frac{\kappa}{m}h(x,x')e^{-2\kappa V\mathcal{T}\cos(\alpha+\theta)}$$

Standard result

Finally, we obtain the “standard” result

$$\langle\hat{\psi}(x)\hat{\bar{\psi}}(x')\rangle = -\left(e^{-\mathrm{i}\theta}P_L + e^{\mathrm{i}\theta}P_R\right)\frac{\kappa}{m}h(x,x') + S_{0\mathrm{inst}}(x,x') = -\frac{\kappa}{m}e^{\mathrm{i}\theta\gamma^5}h(x,x') + S_{0\mathrm{inst}}(x,x')$$

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Misalignment of the chiral phases unless $\theta = -\alpha$

Strong CP problem!

New result

We first do the sum in each topological sector

$$\sum_{\substack{n,\bar{n}\geq 0 \\ n-\bar{n}=\nu}} \frac{1}{n!\bar{n}!} \left[h(x,x') \left(\frac{\bar{n}}{me^{-\mathrm{i}\alpha}} P_L + \frac{n}{me^{\mathrm{i}\alpha}} P_R \right) (V\mathcal{T})^{n+\bar{n}-1} + S_{0\mathrm{inst}}(x,x')(V\mathcal{T})^{n+\bar{n}} \right] \kappa^{n+\bar{n}} (-1)^{n+\bar{n}} e^{\mathrm{i}\nu(\alpha+\theta)}$$

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Note that $\nu = n - \bar{n}$ and

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$$I_\alpha(x) = \sum_{\bar{n}=0}^{\infty} \frac{1}{\bar{n}!(\bar{n}+\nu)!}\left(\frac{x}{2}\right)^{2\bar{n}+\nu}$$

The sum gives

$$\left[h(x,x')\frac{\kappa}{m}\left(e^{\mathrm{i}\alpha}P_L I_{\nu+1}(2\kappa V\mathcal{T}) + e^{-\mathrm{i}\alpha}P_R I_{\nu-1}(2\kappa V\mathcal{T})\right) + S_{0\mathrm{inst}}(x,x')I_\nu(2\kappa V\mathcal{T})\right](-1)^{\nu}e^{\mathrm{i}\nu(\alpha+\theta)}$$

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We finally arrive at the correlation function

$$\langle\hat{\psi}(x)\hat{\bar{\psi}}(x')\rangle$$

$$= \frac{\sum_\nu\left\{\left[h(x,x')\frac{\kappa}{m}\left(e^{\mathrm{i}\alpha}P_L I_{\nu+1}(2\kappa V\mathcal{T}) + e^{-\mathrm{i}\alpha}P_R I_{\nu-1}(2\kappa V\mathcal{T})\right) + S_{0\mathrm{inst}}(x,x')I_\nu(2\kappa V\mathcal{T})\right](-1)^{\nu}e^{\mathrm{i}\nu(\alpha+\theta)}\right\}}{\sum_\nu\left\{I_\nu(2\kappa V\mathcal{T})(-1)^{\nu}e^{\mathrm{i}\nu(\alpha+\theta)}\right\}}$$

The infinity limit for spacetime volume

Taking the limit $V\mathcal{T} \to \infty$, we have

$$\lim_{x\to\infty} \frac{I_\nu(x)}{I_0(x)} = 1$$

and obtain (new result)

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Physical reason for this different result:

Disappearance of interferences between the different topological sectors when spacetime volume is taken to be infinity!

Outline

- A historic review of QCD instantons
- The strong CP problem
- Fermion correlation functions in the theta vacuum
- Some discussions
- Summary

Some discussions

First, the derivation was carried out with general chiral phase of the Dirac mass term and the angle theta. Under chiral rotations,

$$\psi \to e^{i\beta\gamma^5}\psi, \quad \bar{\psi} \to \bar{\psi}\, e^{i\beta\gamma^5}$$

we have

K. Fujikawa, PRL 42 (1979) 1195; PRD 21 (1980) 2848

$$\theta \frac{g^2}{32\pi^2} \int d^4x\, G^a_{\mu\nu} \widetilde{G}^{a\mu\nu} \to (\theta - 2\beta) \frac{g^2}{32\pi^2} \int d^4x\, G^a_{\mu\nu} \widetilde{G}^{a\mu\nu}$$

so that we can rotate the topological term away at the beginning.
But the conclusion will not be affected.

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so that we can rotate the topological term away at the beginning. But the conclusion will not be affected.

Second, our new form of the effective operator does not contradict the selection rule from the chiral Ward identity:

$$\psi \to e^{i\beta\gamma^5}\psi, \quad \bar{\psi} \to \bar{\psi}\, e^{i\beta\gamma^5};$$

$$\alpha \to \alpha - 2\beta, \quad \theta \to \theta + 2\beta.$$

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Third, even the topological term has been rotated away, the instanton effects are still there and need to be integrated out before constructing further effective theories.

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Contradiction with the literature? An example: Baluni's operator. Start with

$$\mathcal{L}_\theta = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \bar{\psi}_i \left(\mathrm{i} \not{D} - M_{ij} \right) \psi_j + \theta \frac{g^2}{32\pi^2} \int \mathrm{d}^4 x \, G^a_{\mu\nu} \widetilde{G}^{a\mu\nu}$$

Performing chiral rotations, we have

$$\mathcal{L}_\theta \to \widetilde{\mathcal{L}}_\theta = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \bar{\psi}_i \left(\mathrm{i} \not{D} - \widetilde{M}_{ij} \right) \psi_j$$

where $\mathrm{Arg}\,\det \widetilde{M} = \bar{\theta}$. With this Lagrangian, one constructs the chiral Lagrangian. For example, one has (assuming $\bar{\theta} \ll 1$)

$$\delta \mathcal{L}^M_{\not{C}\not{P}} = -\mathrm{i}\bar{\theta} \frac{m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s} \left(\bar{u}\gamma^5 u + \bar{d}\gamma^5 d + \bar{s}\gamma^5 s \right)$$

V. Baluni, PRD 19 (1979) 2227

Some comments

This process implicitly assumes the starting point:

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar{\psi}_i\left(\mathrm{i}\not{D} - M_{ij}\right)\psi_j + \left(\Gamma e^{-\mathrm{i}\theta}\det(\bar{\psi}_L\psi_R) + \Gamma e^{\mathrm{i}\theta}\det(\bar{\psi}_R\psi_L)\right)$$

while we suggest to start with

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar{\psi}_i\left(\mathrm{i}\not{D} - M_{ij}\right)\psi_j + \Gamma\det\left(\bar{\psi}e^{+\mathrm{i}(\sum_{\mathbf{i}}\alpha_{\mathbf{i}})\gamma^5/3}\psi\right)$$

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- The multiple flavor case has also been carried out
- The calculations were also carried out in Minkowski space

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- Constructed fermionic Green's functions in BPST instantons with general complex mass
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axion +  or no strong CP problem at all?

Functional determinants

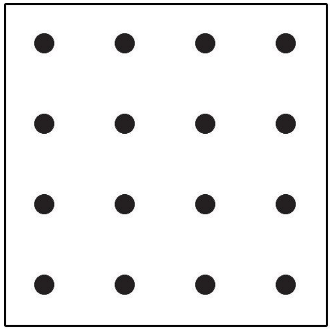
One might think

$$\det{}'_{\bar{\hat{A}}_{n,\bar{n}}} = \left(\det{}'_{\bar{\hat{A}}}\right)^{n+\bar{n}}$$

Over counting!

We should have

$$\det{}'_{\bar{\hat{A}}_{n,\bar{n}}} = \det{}_{\bar{\hat{A}}=0} \left(\frac{\det{}'_{\bar{\hat{A}}}}{\det{}_{\bar{\hat{A}}=0}} \right)^{n+\bar{n}}$$



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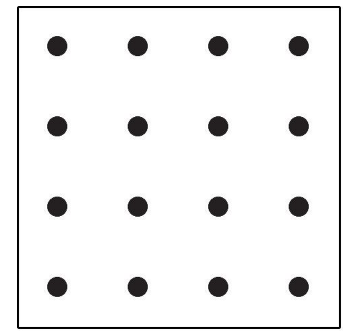

Similarly,

$$\det\left(-\hat{\gamma}_\mu\hat{D}_\mu - me^{\mathrm{i}\alpha\gamma^5}\right)_{n,\bar{n}}$$

$$= \det\left(-\hat{\gamma}_\mu\hat{\partial}_\mu - me^{\mathrm{i}\alpha\gamma^5}\right)\left(\left.\frac{\det\left(-\hat{\gamma}_\mu\hat{D}_\mu - me^{\mathrm{i}\alpha\gamma^5}\right)}{\det\left(-\hat{\gamma}_\mu\hat{\partial}_\mu - me^{\mathrm{i}\alpha\gamma^5}\right)}\right|_{\nu=-1}\right)^{\bar{n}}\left(\left.\frac{\det\left(-\hat{\gamma}_\mu\hat{D}_\mu - me^{\mathrm{i}\alpha\gamma^5}\right)}{\det\left(-\hat{\gamma}_\mu\hat{\partial}_\mu - me^{\mathrm{i}\alpha\gamma^5}\right)}\right|_{\nu=1}\right)^{n}$$

contain chiral phases

Functional determinants

Separating the chiral phase, we have

$$\frac{\det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)}{\det\left(-\hat{\gamma}_\mu \hat{\partial}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)} = -e^{\mathrm{i}\nu\alpha} \left| \frac{\det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)}{\det\left(-\hat{\gamma}_\mu \hat{\partial}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)} \right| \equiv e^{i\nu\alpha}(-\Theta)$$

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In total we get the path integral

$$\int \mathcal{D}\hat{A}_\mu^{(n,\bar{n})}\mathcal{D}\hat{\psi}\mathcal{D}\hat{\bar{\psi}}\,\hat{\psi}(x)\hat{\bar{\psi}}(x')e^{-S_E} = \frac{1}{n!\bar{n}!}\left(\prod_{\bar{m}=1}^{\bar{n}}\int_{V\mathcal{T}} \mathrm{d}^4\, x_{0,\bar{m}}\mathrm{d}\Omega_{\bar{m}} J_{\bar{m}}\right)\left(\prod_{m=1}^{n}\int_{V\mathcal{T}} \mathrm{d}^4\, x_{0,m}\mathrm{d}\Omega_m J_m\right) S_{n,\bar{n}}(x,x')$$
$$\times \det\left(-\hat{\gamma}_\mu \hat{\partial}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)\left(\det{}_{\hat{\bar{A}}=0}\right)^{-1/2} e^{-(n+\bar{n})S_1} e^{\mathrm{i}\nu(\theta+\alpha)}\varpi^{n+\bar{n}}(-\Theta)^{n+\bar{n}}$$

Common factors

Functional determinants

Separating the chiral phase, we have

$$\frac{\det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)}{\det\left(-\hat{\gamma}_\mu \hat{\partial}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)} = -e^{\mathrm{i}\nu\alpha}\left|\frac{\det\left(-\hat{\gamma}_\mu \hat{D}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)}{\det\left(-\hat{\gamma}_\mu \hat{\partial}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)}\right| \equiv e^{i\nu\alpha}(-\Theta)$$

In total we get the path integral

$$\int \mathcal{D}\hat{A}_\mu^{(n,\bar{n})}\mathcal{D}\hat{\psi}\mathcal{D}\hat{\bar{\psi}}\,\hat{\psi}(x)\hat{\bar{\psi}}(x')e^{-S_E} = \frac{1}{n!\bar{n}!}\left(\prod_{\bar{m}=1}^{\bar{n}}\int_{V\mathcal{T}}\mathrm{d}^4\,x_{0,\bar{m}}\mathrm{d}\Omega_{\bar{m}}J_{\bar{m}}\right)\left(\prod_{m=1}^{n}\int_{V\mathcal{T}}\mathrm{d}^4\,x_{0,m}\mathrm{d}\Omega_m J_m\right)S_{n,\bar{n}}(x,x')$$
$$\times \det\left(-\hat{\gamma}_\mu \hat{\partial}_\mu - m e^{\mathrm{i}\alpha\gamma^5}\right)\left(\det{}_{\bar{\hat{A}}=0}\right)^{-1/2} e^{-(n+\bar{n})S_1} e^{\mathrm{i}\nu(\theta+\alpha)}\varpi^{n+\bar{n}}(-\Theta)^{n+\bar{n}}$$

Common factors

The last piece: the fermionic Green's functions in a fixed background

$$S_{n,\bar{n}}(x,x')$$

Fermionic Green's functions

In the single BPST instanton background, the Green's function satisfies

$$\left(\hat{\gamma}_\mu \hat{D}_\mu + m e^{\mathrm{i}\alpha\gamma^5}\right) S(x,x') = \delta^{(4)}(x,x')$$

Spectral representation (when the mass is small)

$$S(x,x') = \frac{\hat{\psi}_{0,L/R}(x)\hat{\psi}^\dagger_{0,L/R}(x')}{m e^{\mathrm{i}\nu\alpha}} + \sum\!\!\!\!\!\!\int \frac{\hat{\psi}_\lambda(x)\hat{\psi}^\dagger_\lambda(x')}{\lambda}$$

M. A, Shifman, A. I. Vainshtein, and
V. I. Zakharov, NPB 163 (1980) 46

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For general complex mass, we obtained similar structure

arXiv: 2001.07152, WA, J. S. Cruz, B. Garbrecht, C. Tamarit

For the general background, we have

$$S_{n,\bar{n}}(x,x') \approx S_{0\mathrm{inst}}(x,x') + \sum_{\bar{m}=1}^{\bar{n}} \frac{\hat{\psi}_{0,L}(x-x_{0,\bar{m}})\hat{\psi}^\dagger_{0,L}(x'-x_{0,\bar{m}})}{m e^{-\mathrm{i}\alpha}} + \sum_{m=1}^{n} \frac{\hat{\psi}_{0,R}(x-x_{0,m})\hat{\psi}^\dagger_{0,R}(x'-x_{0,m})}{m e^{\mathrm{i}\alpha}}$$

Integral over the positions of instantons

Now we have the following integral

$$\left(\prod_{\bar{m}=1}^{\bar{n}}\int_{V\mathcal{T}}\mathrm{d}^4x_{0,\bar{m}}\right)\left(\prod_{m=1}^{n}\int_{V\mathcal{T}}\mathrm{d}^4x_{0,m}\right)$$
$$\times\left(S_{0\mathrm{inst}}(x,x')+\sum_{\bar{m}=1}^{\bar{n}}\frac{\hat{\psi}_{0,L}(x-x_{0,\bar{m}})\hat{\psi}^\dagger_{0,L}(x'-x_{0,\bar{m}})}{me^{-\mathrm{i}\alpha}}+\sum_{m=1}^{n}\frac{\hat{\psi}_{0,R}(x-x_{0,m})\hat{\psi}^\dagger_{0,R}(x'-x_{0,m})}{me^{\mathrm{i}\alpha}}\right)$$

For every instanton, we need to do the integral once. For example

$$\int_{V\mathcal{T}}\mathrm{d}^4x_{0,\bar{m}}\left(S_{0\mathrm{inst}}(x,x')+\sum_{\bar{m}=1}^{\bar{n}}\frac{\hat{\psi}_{0,L}(x-x_{0,\bar{m}})\hat{\psi}^\dagger_{0,L}(x'-x_{0,\bar{m}})}{me^{-\mathrm{i}\alpha}}+\cdots\right)$$
$$=V\mathcal{T}\left(S_{0\mathrm{inst}}(x,x')+\cdots\right)+m^{-1}e^{\mathrm{i}\alpha}h(x,x')P_L$$

Integral over the positions of instantons

Now we have the following integral

$$\left(\prod_{\bar{m}=1}^{\bar{n}}\int_{V\mathcal{T}}\mathrm{d}^4 x_{0,\bar{m}}\right)\left(\prod_{m=1}^{n}\int_{V\mathcal{T}}\mathrm{d}^4 x_{0,m}\right)$$
$$\times\left(S_{0\mathrm{inst}}(x,x') + \sum_{\bar{m}=1}^{\bar{n}}\frac{\hat{\psi}_{0,L}(x-x_{0,\bar{m}})\hat{\psi}^{\dagger}_{0,L}(x'-x_{0,\bar{m}})}{me^{-\mathrm{i}\alpha}} + \sum_{m=1}^{n}\frac{\hat{\psi}_{0,R}(x-x_{0,m})\hat{\psi}^{\dagger}_{0,R}(x'-x_{0,m})}{me^{\mathrm{i}\alpha}}\right)$$

For every instanton, we need to do the integral once. For example

$$\int_{V\mathcal{T}}\mathrm{d}^4 x_{0,\bar{m}}\left(S_{0\mathrm{inst}}(x,x') + \sum_{\bar{m}=1}^{\bar{n}}\frac{\hat{\psi}_{0,L}(x-x_{0,\bar{m}})\hat{\psi}^{\dagger}_{0,L}(x'-x_{0,\bar{m}})}{me^{-\mathrm{i}\alpha}} + \cdots\right)$$
$$= V\mathcal{T}\left(S_{0\mathrm{inst}}(x,x') + \cdots\right) + m^{-1}e^{\mathrm{i}\alpha}h(x,x')P_L$$

In total, for fixed winding number we have

$$\sum_{\substack{n,\bar{n}\geq 0\\ n-\bar{n}=\nu}}\frac{1}{n!\bar{n}!}\left[h(x,x')\left(\frac{\bar{n}}{me^{-\mathrm{i}\alpha}}P_L + \frac{n}{me^{\mathrm{i}\alpha}}P_R\right)(V\mathcal{T})^{n+\bar{n}-1} + S_{0\mathrm{inst}}(x,x')(V\mathcal{T})^{n+\bar{n}}\right]\kappa^{n+\bar{n}}(-1)^{n+\bar{n}}e^{\mathrm{i}\nu(\alpha+\theta)}$$

where

$$\kappa = \int \mathrm{d}\Omega\, J\Theta e^{-S_1}$$