4d $\mathcal{N}=2$ Supersymmetric Gauge Theory in the Ω-background

$SO(4)$ rotation symmetry of $\mathbb{R}^4$

$$\mathcal{Z}(a, m, q; \underbrace{\epsilon_1, \epsilon_2}_{\text{Coord for Cartan subgroup of } SO(4)})$$

a: Coulomb parameter

m: mass

q: instanton counting para.

$$q_\Gamma = \begin{cases} \Lambda_{UV}^{\#} & \text{A.F} \\ e^{2\pi i \tau_{UV}} & \text{A. CFT} \end{cases}$$

$$\tau_{UV} = \frac{\theta_{UV}}{2\pi} + i\frac{4\pi}{g^2_{UV}}$$

$$\mathcal{Z} = \mathcal{Z}^{cl}\,\mathcal{Z}^{1\text{-loop}}\,\underbrace{\mathcal{Z}^{\text{instanton}}}$$

$$-\lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log \mathcal{Z}(a, m, q; \epsilon_1, \epsilon_2) = \mathcal{F}(a, m, q_\Gamma)$$

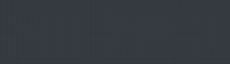
$\mathcal{F}$: LEET prepotential

Seiberg-Witten solution

ADHM Construction of instanton moduli space on $\mathbb{R}^4$

$$\mathcal{M}_{k,SU(n)} = \left[A \,\middle|\, \begin{matrix} F_{\mu\nu}+[\bar\eta_{ab}[x_a,x_b]]=0 \\ \sum_a [x_a,x_a^\dagger]=0 \end{matrix} \right] / G_{\mathbb{C}} \leftarrow \{\beta(x) \mid \text{gauge trivial}\}$$

$$= \{(B, B_2, I, J) \mid \vec{\mu}=0\}/U(k)$$



$B_1, B_2 \quad k\times k$
$I \quad k\times n$
$J \quad n\times k$

$\vec{\mu} = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}, \mu_{\mathbb{C}}^\dagger) \qquad \mu_{\mathbb{R}} = [B_1,B_1^\dagger]+[B_2,B_2^\dagger]+II^\dagger-J^\dagger J$

$\mu_{\mathbb{C}} = [B_1,B_2]+IJ$

$$Z_{inst}^{SU(n)} = \sum_k \mathfrak{f}^k \int_{\mathcal{M}_{k,n}} 1$$

← equivariant integral w.r.t. $U(1)^2 \subset SU(4)$

UV div small instanton $\longrightarrow$ $M_k = \sum_a k_a$ $k_a>0 \Rightarrow U(n)$ instanton in $\mathbb{R}^4$

IR div zero-energy instanton $\longrightarrow$ Ω background

$\mathcal{M}_{1,n} \supset \mathbb{R}^{2n}$

$$\int_{\mathbb{C}} \text{density}\; e^{-\frac{\pi}{\mathfrak{f}}|z|^2}$$

$\mathfrak{f}$ ↔ Instanton $\in \mathbb{C}$

Symmetries

$B_a \to B_a \quad I \to I e^{i\epsilon} \quad J \to e^{i\epsilon} J \qquad t \in U(n)$
$t = e^{\phi}, \; \phi = \text{diag}(\phi_1 \cdots \phi_n)$

$U(1)^2 \subset SU(4)$

$B_a \to q_1 B_a \quad I \to I \quad J \to q_2 q_1 J$

$q_1 \in U(1) \quad q_1 = e^{i\epsilon_1}$

$U(k)$

$B_a \to g B_a g^{-1} \quad I \to gI, \; J \to Jg^{-1} \quad g = e^{\phi}$

$$Z_k = \int DB\, DI\, DJ\, D\bar\psi\, D\chi\, D\mu\, D\eta\, D\bar\eta\, D\phi\, e^{-S}$$

$$S = \left\{ Q, \mathrm{Tr}\left(-i\bar\chi \vec{\mu} - \cdots \right) \right\}$$

$\vec{\chi}\,\vec{\mu} = \chi_{\mathbb{R}}\mu_{\mathbb{R}} + \chi_{\mathbb{C}}\mu_{\mathbb{C}} + \chi_{\mathbb{C}}^\dagger \mu_{\mathbb{C}}^\dagger$

$QB_a = \psi_a$
$Q\psi_a = [\phi, B_a] + \epsilon_1 B_a$
$QI = \psi_3$
$Q\psi_3 = \phi I - Ia$
$QJ = \psi_4$
$Q\psi_4 = -J(a+\epsilon_1+\epsilon_2) + \epsilon_2 J$
$Q\chi_{\mathbb{R}} = H_{\mathbb{R}}$
$QH_{\mathbb{R}} = [\phi, \chi_{\mathbb{R}}]$
$Q\chi_{\mathbb{C}} = H_{\mathbb{C}}$
$QH_{\mathbb{C}} = [\phi, \chi_{\mathbb{C}}] + \epsilon\chi_{\mathbb{C}}$
$Q\phi = \eta$
$Q\eta = [\phi, \eta]$
$Q\bar\eta = H$

$\epsilon = \epsilon_1 + \epsilon_2$

$\begin{cases} QX = Y \\ QY = V(X) \end{cases}$

$$S' = \left\{ Q, \chi_{\mathbb{R}}^\dagger H_{\mathbb{R}} - \bar\eta\chi_{\mathbb{C}} + B_a^\dagger \psi_a + I^\dagger \psi_3 - \psi_4 J^\dagger \right\}$$

$$= H_{\mathbb{R}}^\dagger H_{\mathbb{R}} + \bar\eta H_{\mathbb{C}} + B_a^\dagger ([\phi, B_a] + \epsilon B_a) + I^\dagger(\phi I - Ia) + (-J(a+\epsilon_1+\epsilon_2) + \epsilon_2 J)J^\dagger$$
$$+ \chi_{\mathbb{C}}^\dagger ([\phi, \chi_{\mathbb{C}}] + \epsilon\chi_{\mathbb{C}}) + \eta\chi_{\mathbb{R}} + \psi_a^\dagger \psi_a + \psi_3^\dagger \psi_3 - \psi_4 \psi_4^\dagger$$

$\phi = \text{diag}(\phi_1 \cdots \phi_k)$

$$Z_k = \frac{1}{k!} \oint \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} \prod_{i\neq j} (\phi_i - \phi_j)^2 \frac{1}{\prod_{i=1}^{k}\prod_{a=1}^{n} (\phi_i - a_a)(-\phi_i + a_a + \epsilon)}$$

$$\prod_{i,j} \frac{(\phi_{ij} + \epsilon)}{(\phi_{ij} + \epsilon)(\phi_{ij} - \epsilon)}$$

$\phi_{ij} = \phi_i - \phi_j$

$$= \frac{1}{k!} \left(\frac{\epsilon}{\epsilon_1 \epsilon_2}\right)^k \int \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} \frac{1}{\prod_{i=1}^{k}\prod_{a=1}^{n} (\phi_i - a_a)(\phi_i - a_a - \epsilon)}$$

(residue)

$$\prod_{i,j} S(\phi_{ij})$$

$$S(x) = \frac{x(x+\epsilon)}{(x-\epsilon)(x+\epsilon)}$$

Jeffrey-Kirwan residue formula $\left(\begin{matrix} q_3 > 0 \text{ or} \\ \text{Re}\, a_3 < 0 \end{matrix}\right)$

$$\int \frac{A(u)}{B(u)} du_1 \cdots du_n$$

u: poles of $B(u)$

$$= \sum_{u_0} \text{JkRes}_{u=u_0} (Q(u), \eta)\, \omega$$

$B(u) = \prod (Q \cdot u + b)$

$Q \cdot u = \sum_{j=1}^{n} Q_j u_j$

Singularity when k hyperplanes $H_i = \{u \mid Q \cdot u + b = 0\}$ meet at a point $u = u_0$

$\{H_i\}$ linearly independent

$$\text{JkRes}_{u=u_0}(Q(u), \eta) \frac{du_1 \cdots du_n}{\prod_{i=1}^{n} (Q_i \cdot u + b_i)} = \begin{cases} \dfrac{1}{|\det(Q_{\cdots})|} & \eta \in \text{Cone}(Q_{\cdots}) \\ 0 & \text{otherwise} \end{cases}$$

$\sum_{i=1}^{n} \lambda_i Q_i = \eta \quad \lambda_i > 0$

Classify all poles contributing to JkRes

H^{B}_{ij} $\quad \phi_{ij} + \varepsilon_a = 0$ $\qquad h^{B}_{ij} = e_i - e_j$

H^{I}_{i} $\quad \phi_i - a_\infty = 0$ $\qquad h^{I}_{i} = e_i$

H^{J}_{i} $\quad -\phi_i + a_\infty + \varepsilon = 0$ $\qquad h^{J}_{i} = -e_i$

$\{e_i\}$, $\quad e_i = (0, 0, \dots, \overset{i}{1}, 0, 0, \dots, \overset{k}{0})$

$$Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_k \end{pmatrix} \qquad Q_i \in \{h^{B}, h^{I}, h^{J}\}$$

$$\eta = (1, 1, \dots, 1) = \sum_{i=1}^{k} e_i$$

JkRes $\neq 0$ if $\sum_{i=1}^{k} \lambda_i Q_i = \sum_{i=1}^{k} e_i$, $\lambda_i > 0$

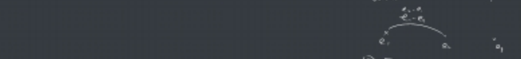
① h^{J} is not allowed

$-e_i$, $e_i - e_j$, $e_j - e_k$, ...

② There are $M > 1$ h^{I}

$e_1, e_2, \dots, e_M$

divide into M subsets, each set contains 1 e_i



③ Focus on e_i no $e_i - e_j$

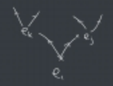
$$\lambda_i e_i + \lambda_j (e_i - e_j) + \sum \lambda_i Q_i = \eta = \sum_{i=1}^{k} e_i$$

$$(\lambda_i + \lambda_j) e_i + \sum \lambda_i Q_i = \underbrace{\lambda_j^{\infty} e_j}_{} + \sum e_i$$

$Q_i \ni e_j - e_k$

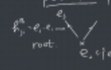
e_i, $e_i - e_j$, $e_j - e_k$, ..., $e_k - e_f$

④ allow $e_j - e_i$



Tree

<u>potential</u> poles $\quad$ <u>M trees</u>



Claim all poles are completely classified by

$$\vec{Y} = (Y^{(1)}, \dots, Y^{(n)})$$

poles $\quad \Phi_{m, \underset{\in Y^{(m)}}{\square = (x,y)}} = a_m + (1-x)\varepsilon_1 + (1-y)\varepsilon_2$

prove by induction

① $k = 1$ $\quad \vec{Y} = (\phi, \phi, \dots, \phi, \overset{(1,1)}{\square}, \phi, \dots, \phi)$

$Z_1 \neq 0$

② $k > 1$ $\quad \vec{Y} = (\phi, \square, \square, \phi, \square, \dots, \phi) \leftarrow h^{\pm}$

nonzero ✓

③ h^{I}, h^{B}

$(\phi_1, \dots, \phi_{k-1})$ contributes to Z_{k-1}

$$\text{Integrand} = (\ \ )^{\text{regular}} \frac{f(0) f(\varepsilon)}{f(\varepsilon_1) f(\varepsilon_2)}$$

$$f(x) = \prod_{\square \in Y_k} (\phi_k - c_\square - x)(c_\square - \phi_k + x)$$

$$c_\square = a_k + (1-x)\varepsilon_1 + (1-y)\varepsilon_2$$

ϕ_k connected by $e_k - e_j$

If $\phi_j = a_k$ then $\phi_k = a_k - \varepsilon_k$ ✓

If $\phi_j \to \square \in Y_k \setminus \{(1,1)\}$

then potential poles at $\phi_k = a_k + (1-x')\varepsilon_1 + (1-y')\varepsilon_2$

$(x, y') = \{(x+1, y), (x, y+1)\}$

If $y = 1$ $\quad$ If $y \neq 1$ $\quad$ poles $\begin{cases} \phi_k = c_{(x,y)} + \varepsilon_1 \\ \phi_k = c_{(x+1,y+1)} + \varepsilon_2 \end{cases}$ if $(x+1, y+1) \in Y_k$

zero $\quad \phi_k = c_{(x,y+1)} + \varepsilon_1 + \varepsilon_2$

✓