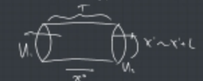
Exact solution of 2d Yang-Mills theory

$$Z(\epsilon) = \frac{1}{Vol(G)} \int DA \int D\phi \; e^{-S}$$

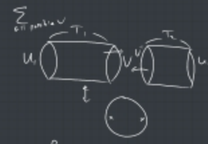
$$S = -\int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{g_{YM}^2}{2}\phi^2 \omega\right)$$

① Combinatorial method

$$H\,\chi_R(U) = \frac{1}{2} g_{YM}^2 \sum C_2(R)\,\chi_R(U), \quad \langle U|R\rangle = \chi_R(U)$$



$$Z_{0,2}(T, U_1, U_2) = \langle U_1 | e^{-TH} | U_2 \rangle$$
$$= \sum_{R,R_2} \langle U_1|R_1\rangle\langle R_1| e^{-TH} |R_2\rangle\langle R_2|U_2\rangle$$
$$= \sum_R \chi_R(U_1)\,\chi_R(U_1^{-1})\, e^{-\frac{1}{2}g_{YM}^2 L T C_2(R)}$$



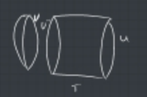
$$\int_G dV\, Z_{0,2}(T_1, U_1, V)\, Z_{0,2}(T_2, V^{-1}, U_2)$$
$$= Z_{0,2}(T_1 + T_2, U_1, U_2)$$

$$\int_G dV\, \chi_{R_1}(V)\,\chi_{R_2}(V^{-1}) = \delta_{R_1,R_2}$$

Disk partition function

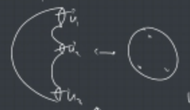
$$Z_{0,1}(\epsilon=0, U) = \delta(U, \mathbb{1}) = \sum_R d_R\,\chi_R(U)$$

$$d_R = \chi_R(\mathbb{1}) = \mathrm{Tr}_R\,\mathbb{1} = \dim R$$



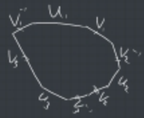
$$\int_G dV\, Z_{0,1}(\epsilon=0, V)\, Z_{0,2}(T, V^{-1}, U)$$
$$= \sum_R \dim R\;\chi_R(U)\, e^{-\frac{1}{2}g_{YM}^2 L T C_2(R)}$$
$$\epsilon = g_{YM}^2 \mathrm{Area}(\Sigma)$$

- A pair of pants



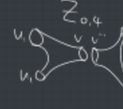
$$Z_{0,3}(\epsilon, U_1, U_2, U_3)$$
$$= \int_G dV_1\, dV_2\, dV_3\, Z_{0,1}(\epsilon, V_1^{-1}U_1V, V_2^{-1}U_2V_1, V_3^{-1}U_3V_2)$$

using identity $\int_G dV\, \chi_R(V^{-1}UVW) = \frac{\chi_R(U)\chi_R(W)}{\dim R}$



$$Z_{0,3} = \sum_R (\dim R)^{-2}\,\chi_R(U_1)\,\chi_R(U_2)\,\chi_R(U_3)\,\underbrace{\chi_R(\mathbb{1})}_{\dim R}\, e^{-\frac{1}{2}\epsilon C_2(R)}$$
$$= \sum_R (\dim R)^{-1}\,\chi_R(U_1)\,\chi_R(U_2)\,\chi_R(U_3)\, e^{-\frac{1}{2}\epsilon C_2(R)}$$

Example



$$Z_{0,4}(\epsilon, U_1, U_2, U_3, U_4)$$
$$= \int_G dV\, Z_{0,3}(\epsilon_1, U_1, U_2, V)\, Z_{0,3}(\epsilon-\epsilon_1, U_3, U_4, V^{-1})$$
$$= \sum_R (\dim R)^{-2} \prod_{i=1}^4 \chi_R(U_i)\; e^{-\frac{1}{2}\epsilon C_2(R)}$$

$$Z_{1,1}(\epsilon, U)$$
$$= \int_G dV\, Z_{0,3}(\epsilon, U, V, V^{-1}) = \sum_R (\dim R)^{-1}\,\chi_R(U)\, e^{-\frac{1}{2}\epsilon C_2(R)}$$

$$Z_{g,s}(\epsilon, U_1, \ldots, U_s) = \sum_R (\dim R)^{2-2g-s}\; e^{-\frac{1}{2}\epsilon C_2(R)} \prod_{i=1}^s \chi_R(U_i)$$

Cartan-Weyl basis $\{H^i, E^\alpha : \alpha \neq 0\}$ over $\mathbb{C}$ $[T^a, T^b] = f^{abc} T^c$

$[H^i, H^j] = 0$ $\quad$ $\mathrm{Tr}(H^i H^j) = \delta^{ij}$

$[H^i, E^\alpha] = \alpha^i E^\alpha$ $\quad$ $\mathrm{Tr}(E^\alpha E^\beta) = \delta^{\alpha+\beta,0}$

$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta} & \alpha+\beta \text{ root} \\ \alpha \cdot H & \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases}$ $\quad$ $\mathrm{Tr}(H^i E^\alpha) = 0$

$A = \sum_{i=1}^r A_i H^i + \sum_{\alpha \in \Delta} A_\alpha E^\alpha$

$\phi = \sum_{i=1}^r \phi_i H^i + \sum_{\alpha \in \Delta} \phi_\alpha E^\alpha$

① $\mathrm{su}(1,1) \subset L_+(G)$ $\quad$ $\phi \to u \phi u^\dagger + u \partial u^\dagger$ $\quad$ $A \to u A u^\dagger + u d u^\dagger$

$\phi(z) \in L_+(T)$ $\quad$ T Cartan subgroup of G generated by ϕ

BRST operator Q $\quad$ $Q^2 = 0$

$$\begin{cases} Q\phi_i = 0 \\ Q\phi_\alpha = \sum_{i=1}^r (\alpha \cdot \phi) C_i \\ QC_i = 0 \\ QC_\alpha = b_\alpha \\ Qb_\alpha = 0 \end{cases}$$

Add $\{Q \int_\Sigma \sum_{\alpha \in \Delta_+} \bar{C}_{-\alpha} \phi_\alpha \omega\} = \int_\Sigma \sum_{\alpha \in \Delta_+} \left(b_{-\alpha} \phi_\alpha - \sum_{i=1}^r \bar{C}_{-\alpha} (\alpha \cdot \phi) C_i \right) \omega$

Integrating over $b_{-\alpha} \Rightarrow$ set $\phi_\alpha = 0$

$S_\phi = \int_\Sigma \left(\frac{i}{2\pi} \sum_{i=1}^r \phi_i dA_i - \frac{1}{4\pi} \sum_{\alpha \in \Delta_+} (\alpha \cdot \phi) A_\alpha \wedge A_{-\alpha} \right)$

$\quad + \int_\Sigma \left(\frac{1}{2} \sum_{i,j=1}^r \phi_i \phi_j - \sum_{\alpha \in \Delta_+} \sum_{i=1}^r \bar{C}_{-\alpha} (\alpha \cdot \phi) C_i \right) \omega$

$\frac{1}{2\pi} \int_\Sigma dA_i = n_i \in \mathbb{Z}$

$A_i = \bar{A}_i + a_i$, $\quad d\bar{A}_i = 2\pi n_i \omega$ $\quad$ $\int_\Sigma \omega = 1$

$\frac{i}{2\pi} \sum_{i=1}^r \phi_i dA_i = i \sum_{i=1}^r \phi_i (2\pi n_i \omega + da_i)$ $\quad$ $\Omega^1(\Sigma, L_+(T))$

Integrate over $a_i \Rightarrow d\phi_i = 0 \Rightarrow \phi_i(x) = \text{const}$

Integrate over $\bar{C}, C$: $\sum_{\alpha \in \Delta_+} \sum_{i=1}^r \bar{C}_{-\alpha} (\alpha \cdot \phi) C_i \Rightarrow \det(\alpha \cdot \phi) \Big|_{\phi \neq 0}$

Integrate over $A_\alpha, A_{-\alpha}$: $\frac{i}{4\pi} \sum_{\alpha \in \Delta_+} (\alpha \cdot \phi) A_\alpha \wedge A_{-\alpha} \Rightarrow \frac{1}{|\det(\alpha \cdot \phi)|_{\phi \neq 0}}$

Using Riemann-Roch theorem

$\dim_{\mathbb{C}} \Omega^0(\Sigma, \mathcal{L}_\alpha) - \dim_{\mathbb{C}} \Omega^1(\Sigma, \mathcal{L}_\alpha) = 1 - 3 + C(\mathcal{L}_\alpha)$

For $G = U(N)$ $\quad$ $\alpha = e_i - e_j$ $\quad$ $(i \neq j)$ $\quad$ $i, j = 1, \dots, N$

$C(\mathcal{L}_\alpha) = n_i - n_j$

$(\alpha_1, \dots, \alpha_N)$ $\quad$ constant generators of $C(\mathcal{L}_\alpha)$

Then determinant $\Rightarrow \prod_{\alpha \in \Delta_+} (\alpha \cdot \phi)^{2-2g}$

$$Z(\epsilon) = \frac{1}{|W|} \prod_{i=1}^r \int d\phi_i \sum_{\vec{n}} \prod_{\alpha \in \Delta_+} (\alpha \cdot \phi)^{1-2g} \exp\left(-\frac{\epsilon}{2} \sum_{i=1}^r m_i \phi_i - \frac{\epsilon^2}{2} \sum_{i=1}^r \phi_i^2 \right)$$

W: Weyl group $\quad$ $\vec{n} = (n_1, \dots, n_r)$

① Strong coupling expansion

Poisson summation formula $\quad \sum_{n \in \mathbb{Z}} e^{2\pi i n x} = \sum_{m \in \mathbb{Z}} \delta(x - m)$

$$Z_{SU}(\epsilon) = \frac{1}{|W|} \sum_{\vec{m} \in \mathbb{Z}^r} \prod_{\alpha \in \Delta_+} (\alpha \cdot m)^{2-2g} \exp\left(-\frac{\epsilon}{2} \sum_{i=1}^r m_i^2 \right)$$

Define $\lambda = \sum_{i=1}^r m_i e_i$ $\quad$ $\lambda \in \Lambda_W$ (weight lattice) $\quad$ $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ (Weyl vector)

$$Z(\epsilon) = \frac{1}{|W|} \sum_{\lambda \in \Lambda_W} \prod_{\alpha \in \Delta_+} \langle \alpha, \lambda + \rho \rangle \exp\left(-\frac{\epsilon}{2} \langle \lambda + \rho, \lambda + \rho \rangle \right)$$

$$= \sum_{\lambda \in \Lambda_W} \left(\prod_{\alpha \in \Delta_+} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle} \right)^{2-2g} \exp\left(-\frac{\epsilon}{2} \langle \lambda, \lambda + \rho \rangle \right)$$

$\lambda \in \Lambda_W$: $\sum_{\vec{n}}$ $\quad$ $\alpha \in \Delta_+$: $\prod_{i<j}$ $\quad$ $\exp(\dots)$: $G(\epsilon)$

$$\prod_{\alpha \in \Delta_+} \langle \alpha, \rho \rangle^{2-2g} \exp\left(-\frac{\epsilon}{2} \langle \rho, \rho \rangle \right)$$

② Weak coupling expansion

$G = SU(1)$ $\quad$ $g = 0$

$$Z = \frac{1}{2} \sum_{m \in \mathbb{Z}} \int d\phi \; e^{-\frac{\epsilon}{2} m \phi} \cdot e^{-\frac{\epsilon}{2} \phi^2}$$

$$= \sqrt{\frac{\pi}{2}} \sum_{m \in \mathbb{Z}} \underbrace{e^{\frac{(\epsilon m)^2}{8\epsilon}}}_{e^{-\frac{\epsilon}{2} m^2}} \left(\underbrace{\epsilon^{\frac{1}{2}}}_{\text{path integration around weak coupling point}} \cdot \underbrace{e^{\frac{1}{2}\epsilon^2}}_{\text{classical}} \right)$$

$S_{BF} = \int F \wedge F$ $\quad$ $e^{-\frac{\epsilon}{2}}$

$\delta S = 0 \Rightarrow D \cdot F = 0$

$F = 2\pi n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad$ Poisson summation

* Equivalence between YMs and $\frac{\text{N=(2,2) SCFT}}{\text{Topological twist}}$

$Q^2 = 0$

$$\begin{cases} QA = \psi \\ Q\psi = D_A \sigma \\ Q\sigma = 0 \\ Q\chi = H \\ QH = [\sigma, \chi] \\ Q\bar{\psi} = \eta \\ Q\eta = [\sigma, \eta] \end{cases}$$

$S_{YM} = \frac{1}{2g_{YM}^2} \int \{ Q (\bar{\psi}(F - \frac{1}{2}\eta)) + \frac{\theta}{8\pi} \cdot \frac{1}{2} \cdot (*1) \}$

Add $\quad i t Q \int \chi \cdot \sigma *$

give mass to $\chi, H, \bar{\psi}, \eta$

$t \to \infty$

$S = \int \frac{1}{2g^2} (*F - \frac{\theta}{2\pi} F + \frac{t}{2} *H) \wedge H$

$$Z = \int DA D\psi D\sigma \; H(\sigma)^2 \exp\left(-\int \frac{8\pi^2}{g^2} F \wedge *F + \frac{\partial W}{\partial \sigma} \psi \wedge \psi + \frac{\partial^2 W}{\partial \sigma^2} \bar{\psi} \psi \right)$$

$\int_\Sigma \frac{F}{2\pi} \in \mathbb{Z}$

$\frac{\partial W}{\partial \sigma} = n_\sigma \in \mathbb{Z}$ $\quad \Leftrightarrow$ Quantum integrable systems

Bethe ansatz

$\Rightarrow \exp(2\pi i \frac{\partial W}{\partial \sigma}) = 1$ $\quad$ Bethe/gauge correspondence