$$\int d\phi\, d\psi\, D^{9}\Phi \{Q, V\} = \int d\phi\, A'(\phi) = A\Big|_{-\infty}^{+\infty} = 0$$

$\Phi = (\phi, \psi)$

$Q\phi = \psi$

$Q\psi = f(\phi) \qquad \psi^2 = 0$

$V = A(\phi) + B(\phi)\psi$

$\{Q, V\} = A'(\phi)\psi + B'(\phi)\psi\psi + B(\phi) f(\phi)$

M n-dimensional C^∞ manifold

$\circlearrowleft_{U(1)}$ V^μ $\quad \forall g \in U(1)$ $x \in M$, $gx = x' \in M$

$g_{\mu\nu}$

Coordinates x^μ, $\mu = 1, \cdots, n$

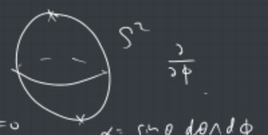
$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_k} \hookrightarrow \psi^{\mu_1} \psi^{\mu_2} \cdots \psi^{\mu_k}$

$\alpha = \sum f(x)\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \hookrightarrow f(x)\, \psi^{\mu_1} \cdots \psi^{\mu_k}$

Aim: prove Atiyah-Bott localization formula

$$\int_M \alpha = \sum_i \frac{\pi^n \alpha_0(x_i)}{\sqrt{\det \partial_\mu V^\nu(x_i)}}$$

α_0: $\alpha|_{\psi=0}$



$\alpha = \sin\theta \, d\theta \wedge d\phi$

$\alpha = \alpha_0 + \alpha_{1\mu}\psi^\mu + \alpha_{2\mu\wedge}\psi^\mu\psi^\nu + \cdots$

$\begin{cases} Q x^\mu = \psi^\mu \\ Q\psi^\mu = V^\mu(x) \end{cases}$ $\quad Q \hookleftarrow d + \iota_V$ $\quad Q^2 \hookleftarrow \mathcal{L}_V = 0$ $\quad \mathcal{L}_V g_{\mu\nu} = 0$

$$Z(t) = \int \alpha(x,\psi)\, d^n x\, d^n \psi \; e^{-t\, QW(x,\psi)}$$

T^*M

$Q\alpha = 0$ $\qquad \partial_\rho (V^\nu g_{\mu\nu})\, \psi^\rho$

$W = V^\mu(x) g_{\mu\nu} \psi^\nu$

$QW = V^\mu g_{\mu\nu} V^\nu + \partial_\rho (V^\mu g_{\mu\nu}) \psi^\rho \psi^\nu$

$$Z(0) = \lim_{t\to\infty} Z(t) = \sum_{\substack{x_i \\ V(x_i)=0}}$$

fixed point: $V^\mu(x_i) = 0$, $\psi_i = 0$

Assuming $x_i = 0$

$\sqrt{t}\, dx = d\tilde{x}$, $\quad t^{-n/2} d\psi = d\tilde{\psi}$

$\sqrt{t}\, x = \tilde{x}$, $\quad d^n x\, d^n \psi = d^n \tilde{x}\, d^n \tilde{\psi}$

$\sqrt{t}\, \psi = \tilde{\psi}$

$$tQW = H^{(V)}_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu + S^{(V)}_{\mu\nu} \tilde{\psi}^\mu \tilde{\psi}^\nu + O\left(\frac{1}{t^{n/2}}\right)$$

$\begin{cases} Q\tilde{x}^\mu = \tilde{\psi}^\mu \\ Q\tilde{\psi}^\mu = \partial_\nu V^\mu(0)\, \tilde{x}^\nu \end{cases}$

$\sqrt{t}\, Q\tilde{\psi}^\mu = \sqrt{t} V^\mu(x) = \sqrt{t} \underbrace{V^\mu(0)}_{=0} + \partial_\nu V^\mu(0) \tilde{x}^\nu + \frac{1}{\sqrt{t}} \partial_\rho \partial_\sigma V^\mu(0) \tilde{x}^\rho \tilde{x}^\sigma + \cdots$

$Q^2 W = 0 \Rightarrow H_{\mu\nu} = S_{\rho\mu} \partial_\nu V^\rho(0)$

$$Z(0) = \lim_{t\to\infty} Z(t) = \sum_{x_i} \frac{\alpha_0 \, Pf(S)}{\sqrt{\det H}} = \sum_{x_i} \frac{\alpha(x_i, 0)}{\sqrt{\det \partial_\mu V^\nu(x_i)}}$$

(S: $\tilde{\psi}$; H: $\tilde{x}$)

$\alpha(x,\psi) = \alpha(x_0, 0) + O\left(\frac{1}{\sqrt{t}}\right)$

2d Yang-Mills theory

G: compact, connected (not necessarily simply connected) Lie group

P: a principal G-bundle over an oriented closed Riemann surface Σ w/ genus g

$F_{\mu\nu}F^{\mu\nu}$ $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$

$$S = -\frac{1}{2g_{YM}^2}\int_\Sigma \mathrm{Tr}(F\wedge * F) \qquad F = dA + A\wedge A$$

A, F: anti-Hermitian

For $SU(N)$: $\mathrm{Tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}$

① no local degrees of freedom

$\mathcal{L}_\xi (\mathrm{Tr} f^2 \omega) = 0$

$(\mathcal{L}_\xi \mathrm{Tr} f^2)\omega + \mathcal{L}_\xi \omega$

$F = * f \qquad \mathrm{Tr}(F\wedge * F) = \omega\, \mathrm{Tr} f^2 \qquad d^2x\sqrt{\det G_{ij}}$

$\omega = * 1$

Any diffeomorphism transformation that keeps the area form invariant is a symmetry of the action.

$\mathcal{L}_\xi (f\omega) =$

$\phi(x) = \phi'(x)$

$\phi(x) = \phi'(x')$

~~$\phi(x) = \phi$~~

$\phi(x) = \varphi$

$S = \int f\omega$

$S' = \int \varphi^* (f\omega)$

$\delta S = S' - S = 0$

$\phi(x) \to \phi(x')$

$\int dx\, \phi(x) = \int dx'\, \phi(x')$

S $\qquad$ S'

② First-order: introduce a scalar field ϕ

$$S = -\int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{g_{YM}^2}{2}\phi^2 \omega\right)$$

$\int_\Sigma \omega = \mathrm{Area}(\Sigma) \qquad g_{YM}^2\, \mathrm{Area}(\Sigma) = \epsilon$

$\omega \to t\omega$

$\epsilon \to 0 \qquad g_{YM}^2 \to t^{-1} g_{YM}^2$

$$Z = \frac{1}{\mathrm{Vol}(G)}\int DA\, \delta(F) = \mathrm{Vol}(\mathcal{M}_{\text{flat }G}(\Sigma))$$

③ 2d YM is super-renormalizable: no UV divergence associated w/ diagrams involving only A, ϕ legs.

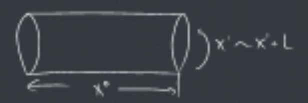
But when we work on compact manifolds, there can UV divergence w/ diagrams involving $g_{\mu\nu}$ legs.

$$\Delta S = \alpha_1 \underbrace{g_{YM}^2\int_\Sigma \omega}_{\epsilon} + \alpha_2 \underbrace{\int_\Sigma \frac{R}{4\pi}}_{2-2g = \chi(\Sigma)}$$

$$Z' = Z e^{\alpha_1 \epsilon + \alpha_2(2-2g)}$$

Solve 2d YM

① Cutting and gluing



- Use x^0-dependent gauge transf $\Rightarrow$ gauge $A_0 = 0$

$\Rightarrow \Psi[A_1(x)]$ obeys $\frac{\delta S}{\delta A_0} = 0 \Rightarrow D_1 F_{10} = 0$

- Use $\{x^0$-independent, x^1-dependent$\}$ gauge transf $\Rightarrow A_1 = \mathrm{const}$

gauge invariant observables: holonomy $U = P\exp\left(\int_0^L A_1(x)\,dx^1\right) \in G$

- Use x^1-independent sym. $\Rightarrow \Psi[U] = \Psi[hUh^{-1}]$, $h \in G$

- a linear combination of the characters in irrep R

$$\langle U | R\rangle = \chi_R(U) = \mathrm{Tr}_R U$$

$$\int dU\, \chi_{R_1}(U)\,\chi_{R_2}(U^{-1}) = \delta_{R_1, R_2} \qquad A = \sum_a A^a T^a$$

Haar measure

- conjugate moment: $\Pi_{A_1^a} = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_1^a)} = \phi^a = -i\frac{\delta}{\delta A_1^a}$

$$H = \frac{1}{2} g_{YM}^2 \int_0^L dx^1\, \frac{\delta}{\delta A^a}\frac{\delta}{\delta A^a}$$

$$H\chi_R(U) = \frac{1}{2} g_{YM}^2 \int_0^L dx^1\, \underbrace{T^a T^a}_{C_2(R)} \exp\int_0^L A\,dx^1 = \frac{1}{2} g_{YM}^2 L\, C_2(R)\,\chi_R(U)$$