$2 \to 2$ $\quad k = 2n^2 + 1$

$A(s,t,u)$

$S, t : \quad -s, S, \cos\theta$

$A \to 2, \ \cos\theta : \ z = \frac{s}{4}$

$A(z, \cos\theta) = 32\pi \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right) \underbrace{f_l(s)}_{\text{unitarity}} P_l(\cos\theta)$ — expand Legendre's

$\int_{-1}^{1} d(\cos\theta)\, P_l(\cos\theta)\, P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}$

$|f_l(s)| < 1$ — unitarity

$\mathrm{Im} f_l(s) > 0$

$S^\dagger S = 1 \implies \underbrace{2\,\mathrm{Im} f_l(s) \geq |f_l(s)|^2}$

$\sigma_{tot} = \frac{1}{32\pi s} \int_{-1}^{1} d(\cos\theta)\, |A(s,t)|^2 = \frac{1}{32\pi s} \sum_{l,l'} \int_{4m^2}^{\infty} dt\, (\dots) \int_{-1}^{1} d(\cos\theta)\, P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{32\pi}{s} \sum \left(l + \frac{1}{2}\right) |f_l(s)|^2$

$\mathrm{Im}\, A(s, 0, 0) = \sigma_{tot}(s) \sqrt{s(s-4m^2)}$ $\qquad \sigma_{tot} = \sigma_{el} + \sigma_{inel}$

$32\pi \sum_l \left(l + \frac{1}{2}\right) \mathrm{Im} f_l(s) = \left(\frac{32\pi}{s} \sum_l \left(l + \frac{1}{2}\right) |f_l(s)|^2 + \sigma_{inel}\right) \frac{1}{2}\sqrt{s(s-4m^2)}$

$\mathrm{Im} f_l(s) = \frac{1}{2}\left(\frac{1}{s} |f_l(s)|^2\right) \sqrt{s(s-4m^2)} + \frac{1}{2} \underbrace{\sigma_{inel}}_{>0}$

$2\,\mathrm{Im} f_l(s) \geqslant \sqrt{\frac{s-4m^2}{s}}\, |f_l(s)|^2$ $\quad (l=0)$

$A(s,t) = \frac{i}{2} \sum_l (2l+1) f_l(s) P_l^{(0)}\left(1 + \frac{2t}{s-4m^2}\right)$ $\quad h = \frac{(n-1)^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}(2\pi)^{\frac{n}{2}}}{\pi\, \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}$ $\quad z \neq 1$

$P_\ell^{(0)} = \frac{\Gamma(\ell+\alpha)\Gamma(\ell+1)}{\Gamma(\ell+2\alpha)\Gamma(\alpha)} C_\ell^{\alpha}(z) = {}_2F_1\left(-\ell, \ell+2\alpha; \frac{2\alpha}{2}; \frac{1-z}{2}\right)$

$2\,\mathrm{Im} f_l(s) \geqslant \frac{(s-4m^2)^{\frac{1}{2}}}{s} |f_l(s)|^2, \quad |f_l(s)| < \frac{\sqrt{s}}{(s-4m^2)^{\frac{1}{2}}}$

$A_{\lambda_3\lambda_4\lambda_1\lambda_2} = 32\pi \sum_l (2l+1)\, d^l_{\lambda\mu}(\cos\theta) f^l_{\lambda_3\lambda_4;\lambda_1\lambda_2}$

Wigner-d functions $\quad \lambda = \lambda_3 - \lambda_4, \ \ \mu = \lambda_1 - \lambda_2$

$\mathrm{Im} f_l(s) > 0$

1) Unitarity

2) Causality (asymptotic causality) $\implies A(s,t)$ — analytic in cut s-plane, real s, t

cf. Bros et al. for short — JS Toll Phys Rev 104 1760 (1956)

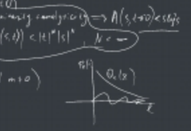
$t > 0$ $\qquad \theta(t)$

$A \in S^{1+\epsilon}_{\mathrm{Lang}}(t \geqslant 0, \mathrm{Sym})$

$\frac{1}{s-m^2} = \frac{1}{s} + \frac{m^2}{s^2} + \dots$

3) Locality $\quad A(s, \mathrm{cos}\,\theta) < S^1$

Mandelstam-Froissart bounds



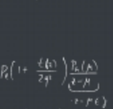
$A(s,t) = 16\pi \sum_l (2l+1) f_l(s) P_l\left(1 + \frac{2t}{s-4m^2}\right)$

$\frac{1}{\rho} = \frac{1}{2}\sqrt{1 - \frac{4m^2}{s}}$

$l \to \infty \quad f_l \to \exp(-\ell\phi) \quad (\text{in all } m > 0)$

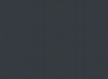
$\sum_{l=0}^{\infty} \quad f_l = ?$

$Q_l^{(s)} = \frac{1}{2}\int_{-1}^{1} dp\, \frac{P_l(p)}{z-p}$



$\oint_{PW} |\cdot Q_l(s)|\, dz$

$\oint A(s,t(s))\, dz\, Q_l(z) = \frac{1}{2} \sum (2l+1) f_l(s) \int_{-1}^{1} dp \oint dz\, P_l\left(1 + \frac{t(s)}{2q^2}\right) \frac{P_l(p)}{z-p}$



$= (2\pi i) \frac{1}{2} \sum (2l+1) f_l(s) \int_{-1}^{1} dp\, P_l(p) P_l\left(1 + \frac{t(s)}{2q^2}\right) =$

$= (2\pi i)\, f_l(s)$

Neumann expansion $\quad f_l(s) = \frac{1}{2\pi i} \left(\oint dz\, A(s, z^2(s,z)) Q_l(z)\right)$

$Q_\ell(z) \underset{z \to \infty}{\sim} K_\nu\left(\ell\sqrt{z(z-1)}\right) = e^{\ell\sqrt{z(z-1)}} \sqrt{\frac{\pi}{\ell\sqrt{z(z-1)}}}$

$|f_l(s)| < \frac{\pi}{2\ell\hbar} e^{-\frac{\ell}{2}(\ell\hbar)^2} \left\| \frac{\pi\hbar}{2\ell\hbar} \sim S^{-\ell} \right\| \sum e^{-\ell\hbar} \sim e^{\ell}$

$|f_l(s)| < 1$ (unitarity)

$\ell_0 = \frac{1}{\ell\hbar} N \log S \quad S \to \infty$

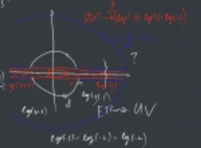
$A(s,t) = \sum_{l=0}^{\ell_0} P_l(s)(\dots) \sim \ell_0^2 < \frac{1}{\ell^2} N \log^2 S \sim \frac{1}{\ell^2} S \log^2 S \quad \text{as } \ell \to 0$

$A \sim S^{2-\epsilon\ell}$

$A_\ell \sim S^{\epsilon\ell}$

$(S \to \infty, \ t \to 0)$

$A < S^1$



$\mathrm{Im} A(s,t) \sim \frac{1}{16\pi^2} \oint \frac{A(s,t)}{(s')^3} = \frac{1}{\pi i} \left(\int_{\gamma} + \int_{-\infty}^{0}\right) \frac{A(s',t)}{s'^3}\, ds'$

$= \int_0^\infty \frac{ds'}{s'} \left(\frac{3\,\mathrm{Im} A}{s'^3}\right) > 0$

$A_{ER} \sim g_2 S^2 + g_1 S + g_0 + \dots \qquad g_2 > 0$

$\hbar \sim 4\hbar / 0.607\pi$

$t \to 0$