$$\langle \mathcal{O}_{\Delta_1}(x_1)\, \mathcal{O}_{\Delta_2}(x_2)\, \mathcal{O}_{\Delta_3}(x_3)\, \mathcal{O}_{\Delta_4}(x_4)\rangle = \frac{g(u,v)}{(x_{12})^{\Delta_1+\Delta_2}\,(x_{34})^{\Delta_3+\Delta_4}} \left(\frac{x_{24}^2}{x_{12}^2}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}}$$

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① CFTs $\longrightarrow$ $\underbrace{\{\Delta_i, f_{123}\}}$ $\longrightarrow$ Reconstruct 2 and 3 pt fncs

② Conformal symm. $\longrightarrow$ 2 and 3 pt function space
time dep.

+

4 pt $\longrightarrow$ $g(u,v)$

N - higher point function $\longrightarrow$ $\frac{N(N-3)}{2}$ cross-ratios

OPERATOR PRODUCT EXPANSION (OPE)

$$\boxed{\begin{matrix} O_i(x_1) \\ \\ O_j(x_2) \end{matrix}} \longrightarrow \sum_k O_k(y)$$

$$\langle O_A(x_3)\, \underbrace{O_i(x_1)\, O_j(x_2)}_{\text{PRIMARY OPERATORS}} \rangle = \langle \sum_k \boxed{\lambda_{ijk}(x_1, x_2, y, \partial_y)}\, O_k(y)\, O_A(x_3) \rangle$$

$$y = \frac{1}{2}(x_1 + x_2)$$

– PRIMARY OPERATORS

- Compatible with conformal invariance

$$\lambda_{ijk}(x_1, x_2, y, \partial_y) = \underset{\substack{\uparrow \\ \text{3 pt function} \\ \text{coefficient}}}{f_{ijk}} \underbrace{\hat{\lambda}_{ijk}(x_1, x_2, y, \partial_y)}_{\substack{\text{FUNCTION} \\ \text{FULLY FIXED} \\ \text{BY CONF. SYMMETRY}}}$$

- OPE in CFTs has a finite radius of convergence which is determined by the distance to the next operator insertion

$$\langle O_1(x_1) O_2(x_2) \ldots O_n(x_n) \rangle$$

OPE will converge if
$$|x_1 - y|, |x_2 - y| \ldots < \min_{i=1,\ldots,n} |x_i - y|$$

Any correlation function $\xrightarrow[\text{OPE}]{}$ 3pt functions
n-point $\qquad\qquad\qquad$ 2pt

$\mathcal{O}_i = \mathcal{O} \qquad \Delta_i = \Delta_{\mathcal{O}}$

$$\langle \underbrace{\mathcal{O}(x_1)\,\mathcal{O}(x_2)}\,\underbrace{\mathcal{O}(x_3)\,\mathcal{O}(x_4)} \rangle = \boxed{\frac{g(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}}\,x_{34}^{2\Delta_{\mathcal{O}}}}} =$$

$$\mathcal{O}(x_1)\,\mathcal{O}(x_2) = \sum_k f_{\mathcal{O}\mathcal{O}\mathcal{O}_k}\,\hat{\lambda}_{\mathcal{O}\mathcal{O}\mathcal{O}_k}(x_{12}, y, \partial_y)\,\mathcal{O}_k(y) \qquad y = \frac{x_1+x_2}{2}$$

symm traceless rep

$$\mathcal{O}(x_3)\,\mathcal{O}(x_4) = \sum_{k'} f_{\mathcal{O}\mathcal{O}\mathcal{O}_{k'}}\,\hat{\lambda}_{\mathcal{O}\mathcal{O}\mathcal{O}_{k'}}(x_{34}, y', \partial_{y'})\,\mathcal{O}_{k'}(y') \qquad y' = \frac{x_3+x_4}{2}$$

$$= \sum_{k,k'} f_{\mathcal{O}\mathcal{O}\mathcal{O}_k}\, f_{\mathcal{O}\mathcal{O}\mathcal{O}_{k'}}\,\hat{\lambda}_{\mathcal{O}\mathcal{O}\mathcal{O}_k}(x_{12}, y, \partial_y)\,\hat{\lambda}_{\mathcal{O}\mathcal{O}\mathcal{O}_{k'}}(x_{34}, y', \partial_{y'})\,\langle\mathcal{O}_k(y)\,\mathcal{O}_{k'}(y')\rangle$$

SPACE TIME DER. $\quad$ FULLY FIXED BY CONF.

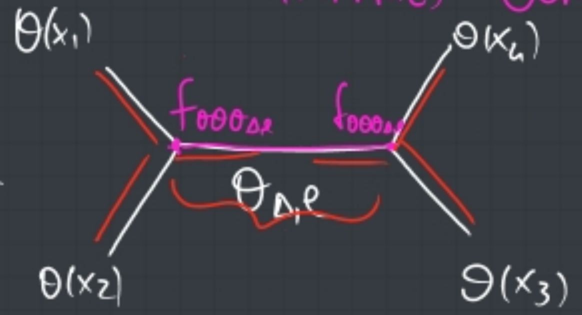
$\delta_{kk'}$ $\qquad y = \frac{x_1+x_2}{2}$, $\quad y' = \frac{x_3+x_4}{2}$

$k = \Delta, \ell$ — ℓ: spin, Δ: conf. dim

$$= \sum_k f^2_{\mathcal{O}\mathcal{O}\mathcal{O}_k} \qquad \hat{\lambda}\,\hat{\lambda}\,\langle\mathcal{O}_k(y)\,\mathcal{O}_k(y')\rangle = \sum_{\Delta,\ell} \frac{f^2_{\mathcal{O}\mathcal{O}\mathcal{O}_{\Delta,\ell}}\, G_{\Delta,\ell}(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}}\,x_{34}^{2\Delta_{\mathcal{O}}}}$$

PRIMARIES $\qquad$ CONFORMAL BLOCKS

$$= \sum_{\Delta,\ell}$$



$d = 4 \qquad u = z\bar{z} \qquad v = (1-z)(1-\bar{z})$

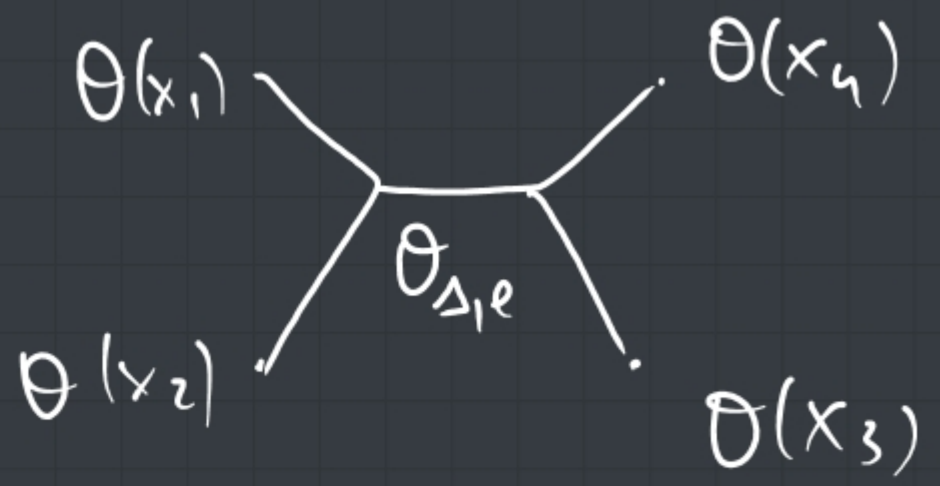
$$G^{\Delta_{12},\Delta_{34}}_{\Delta,\ell}(z,\bar{z}) = \frac{1}{(-2)^\ell}\,\frac{z\bar{z}}{z-\bar{z}}\Big(k_{\Delta+\ell}(z)\,k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z})\Big)$$

$$k_\beta(x) = x^{\beta/2}\,{}_2F_1\left(\frac{\beta-\Delta_{12}}{2}, \frac{\beta+\Delta_{34}}{2}, \beta, x\right)$$

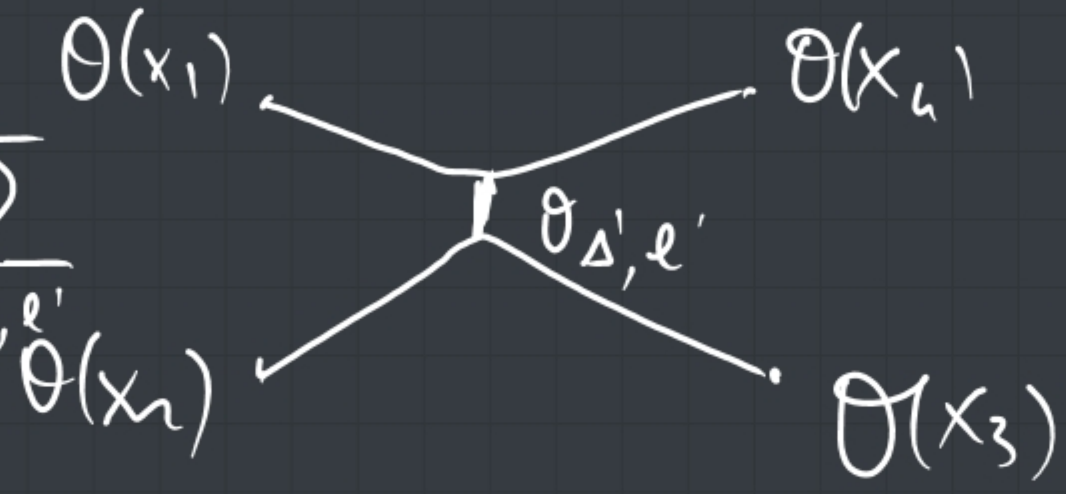
↳ hypergeom. function

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\, \mathcal{O}(x_3)\, \mathcal{O}(x_4) \rangle = \langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\, \mathcal{O}(x_3)\, \mathcal{O}(x_4) \rangle$$

$\sum_{\Delta,\ell}$



$= \sum_{\Delta',\ell'}$



2+3 pt funct

$$\frac{g(u,v)}{x_{12}^{2\Delta_O}\, x_{34}^{2\Delta_O}} = \frac{g(u',v')}{x_{23}^{2\Delta_O}\, x_{14}^{2\Delta_O}} = \frac{g(v,u)}{x_{23}^{2\Delta_O}\, x_{14}^{2\Delta_O}}$$

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} \qquad v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}$$

$$u' \xrightarrow[x_2 \to x_4]{x_1 \to x_3} \frac{x_{23}^2\, x_{14}^2}{x_{13}^2\, x_{24}^2} = v$$

$$v' \longrightarrow \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} = u$$

$$\boxed{g(u,v)} = \left(\frac{x_{12}^2\, x_{34}^2}{x_{23}^2\, x_{14}^2}\right)^{\Delta_O} g(v,u)$$

$$= \boxed{\left(\frac{u}{v}\right)^{\Delta_O} g(v,u)}$$

CROSSING RELATION

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta_0} g(v,u)$$

BOOTSTRAP EQUATION

$$\boxed{\sum_{\Delta,\ell} f^2_{OOO_{\Delta,\ell}} \, G_{\Delta,\ell}(u,v) = \left(\frac{u}{v}\right)^{\Delta_0} \sum_{\Delta',\ell'} f^2_{OOO_{\Delta',\ell'}} \, G_{\Delta',\ell'}(v,u)}$$

FIXED

≥ 0

≥ 0

$$f_{ijk} \in \mathbb{R}$$