SGEVP

A novel way to extend the power of GEVP

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Outset $C(t)$ Decomposition

$$
\begin{aligned}
C_{ij}(t) = \langle 0| \hat{O}_i(t)\hat{O}_j(0)^\dagger |0\rangle = \sum_{n=1}^{\infty} \langle 0| \hat{O}_i(t) |n\rangle \langle n| \hat{O}_j(0)^\dagger |0\rangle \\
\approx \sum_{n=1}^{\infty} \langle 0| \hat{O}_i |n\rangle \langle n| \hat{O}_j^\dagger |0\rangle e^{-E_n t} = \sum_{n=1}^{\infty} z_{in} z_{jn}^* e^{-E_n t},
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$$C(t) = \underbrace{\left[\begin{pmatrix} \\ \vec{z}_1 \\ \\ \end{pmatrix} \begin{pmatrix} & \vec{z}_1^\dagger & \end{pmatrix} \right]}_{A_1} e^{-E_1 t} + \underbrace{\left[\begin{pmatrix} \\ \vec{z}_2 \\ \\ \end{pmatrix} \begin{pmatrix} & \vec{z}_2^\dagger & \end{pmatrix} \right]}_{A_2} e^{-E_2 t} + \dots$$

$$= A_1 e^{-E_1 t} + A_2 e^{-E_2 t} + \dots$$

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$\boxed{\text{How to extract } E_n \text{ and } A_n\text{?}}$

Issues of Direct Fitting

$$C(t) = A_1 e^{-E_1 t} + A_2 e^{-E_2 t} + \dots$$
$$\tilde{C}(t) = C(t) + \delta(t), \quad \delta_{ij}(t) \sim \mathcal{N}\left(0, \mathbf{\Sigma}_{(N^2 \times N_T, N^2 \times N_T)}\right)$$

Since the expression is known, it is natural to do a direct χ^2 fit:

1. $\mathbf{\Sigma}$ can be extracted by resampling method, such as Bootstrap.
2. fix $[t_{\text{min}}, t_{\text{max}}]$ and state numbers N.
3. Define the χ^2, and let MINUIT or lsqfit go.

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However, this approach is cumbersome to answer:

- How to select $[t_{\mathrm{min}}, t_{\mathrm{max}}]$ and N?
- How to ensure it is numerically stable w.r.t initial values.
- $\hat{C}(t) \leftrightarrow \hat{C}(t) + A_\epsilon e^{-E_{\mathrm{max}} t}$

  (Can be partially solved by F-test, however not so decisive.)
- Less elegant compared to GEVP

Direct Fit V.S. Generalized Eigenvalue Problem (GEVP)

	Direct Fit	GEVP
Pros	General	LQCD Specific
Cons	Instability, Subjective	Expensive to add new $\langle \mathcal{O}(t)\mathcal{O}^\dagger(0)\rangle$
Style	Analytic	Algebraic

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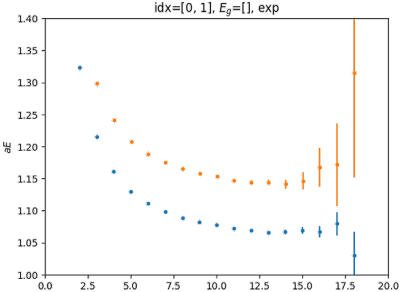
- Plateau is pronounced?
- Fit intervals?



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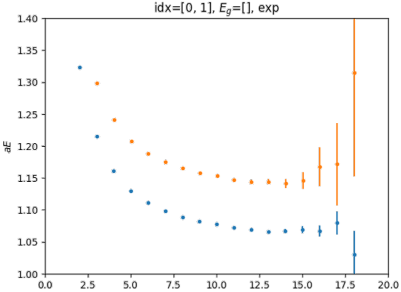

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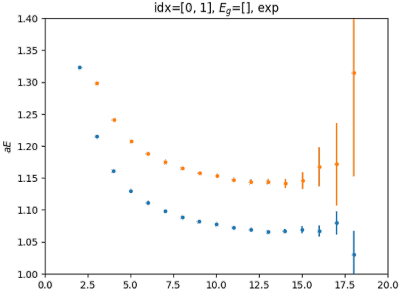
- $\Rightarrow$ Can we improve GEVP?
- Review GEVP



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$$\Rightarrow A_m v_n = 0 \leftrightarrow v_n \in \ker(A_m), \forall m \neq n$$

GEVP Short Summary

$$\begin{cases} C(t) = \sum_{m=1}^{\infty} A_m e^{-E_m t} \\ C(t)v_n = C(t_0)v_n e^{-E_n(t-t_0)} \end{cases} \Rightarrow \begin{cases} A_m v_n = 0, \forall m \neq n \\ v_n \in \cap_{m\neq n} \ker(A_m) \end{cases}$$

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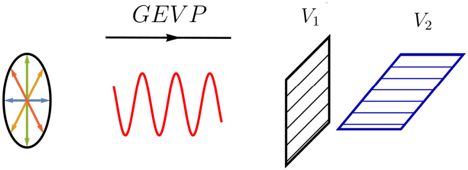

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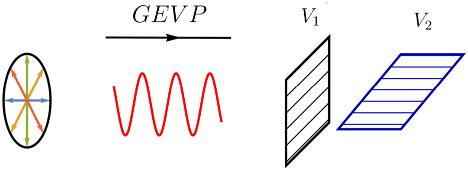

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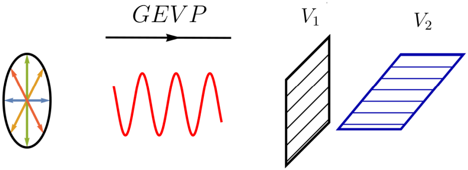

- N rank-1 $A_m \sim N$ filters $\Rightarrow N$ spectra.
- What if some A_ms have full rank? (e.g. due to noise)
  $\Rightarrow$ GEVP does not apply!

GEVP Exploration

$$C(t) = \begin{pmatrix} 2 & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & 3 & 3 \\ \sqrt{6} & 3 & 3 \end{pmatrix} e^{-0.1t} + \begin{pmatrix} 3 & \sqrt{6} & 3 \\ \sqrt{6} & 2 & \sqrt{6} \\ 3 & \sqrt{6} & 3 \end{pmatrix} e^{-0.2t} + \begin{pmatrix} 3 & {\color{red}4} & \sqrt{6} \\ {\color{red}4} & 3 & {\color{red}5} \\ \sqrt{6} & {\color{red}5} & 2 \end{pmatrix} e^{-0.3t}$$

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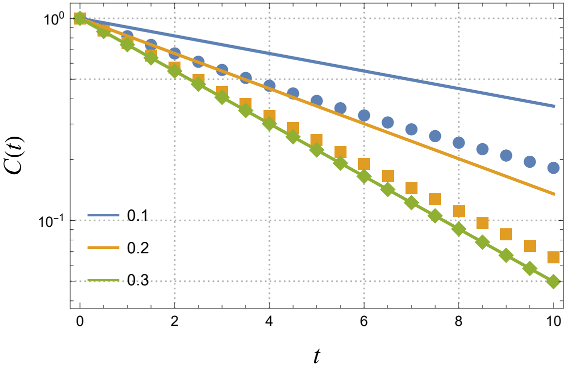

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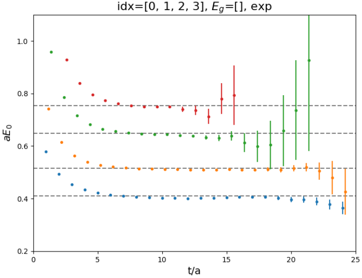

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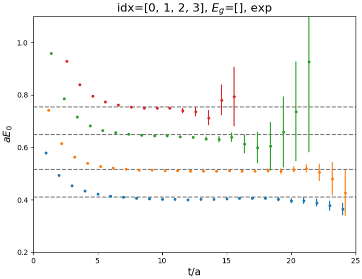
1. How to suppress the excited states contamination?
2. If the SNR is bad or $E_{\text{effective}}$ plateau is short/questionable, is it possible to enhance the signal or recover the missing states?



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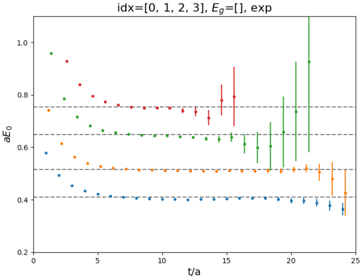

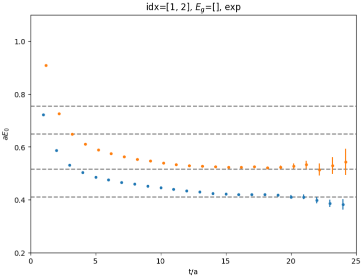

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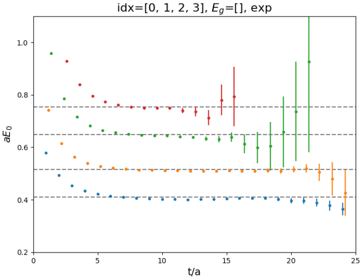
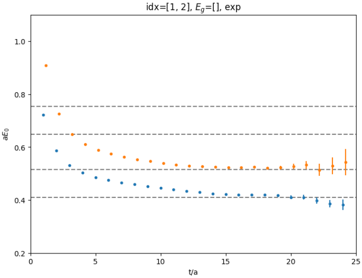
3. How to push GEVP to its limit?
(e.g. Extract $N+1$ or more spectra from $N \times NC(t)$?)





Subtraction + GEVP = SGEVP

$$C(t) = \sum_{n=1}^{\infty} A_n e^{-E_n t}$$

$$\boxed{\text{Tension: Infinite } A_n \text{ V.S. Finite } N \times N \; C(t) \text{ matrix.}}$$

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$$C'(t) := C(t) - e^{E_g} C(t+1) = \sum_n (1 - e^{E_g - E_n}) A_n e^{-E_n t}$$

$$= \sum_n \epsilon(E_g, E_n) A_n e^{-E_n t}$$

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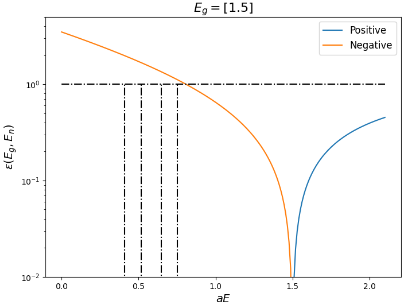
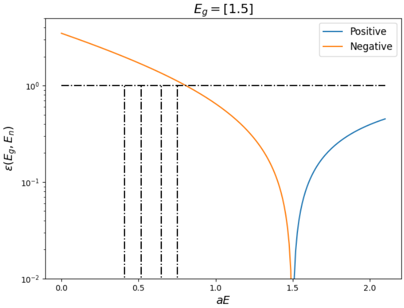
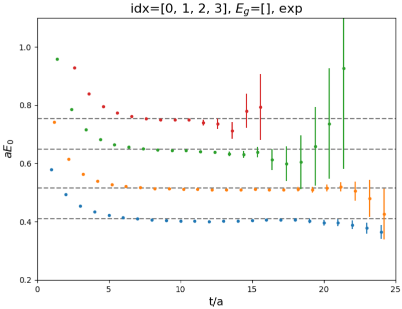
Tune $\epsilon(E_g, E_n)$
make it larger/smaller if we want to enhance/suppress some channels' contribution

Subtraction Factor ϵ

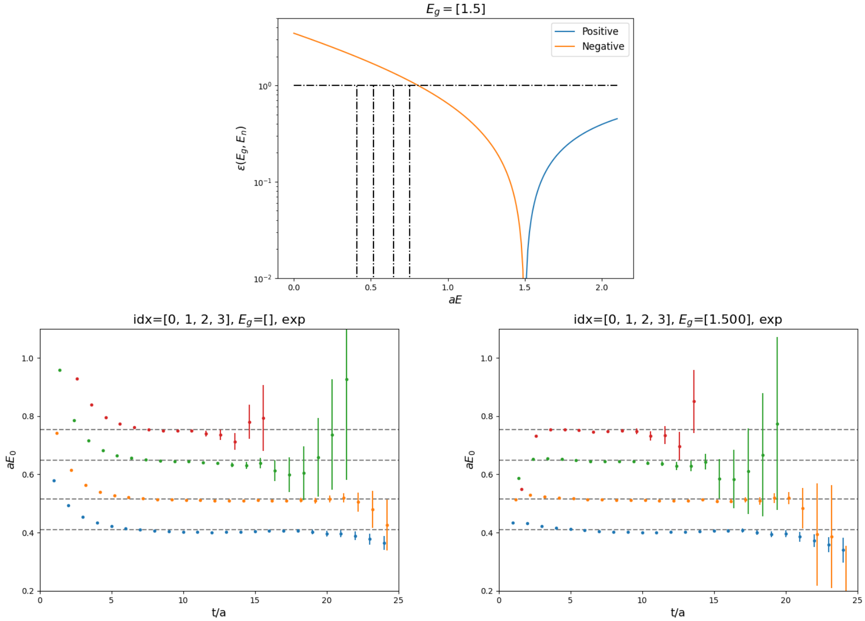




$E_g = [1.5]$
Positive
Negative
$\epsilon(E_g, E_n)$
10^0
10^{-1}
10^{-2}
0.0
0.5
1.0
1.5
2.0
aE



idx=[0, 1, 2, 3], E_g=[], exp
aE_0
1.0
0.8
0.6
0.4
0.2
0
5
10
15
20
25
t/a



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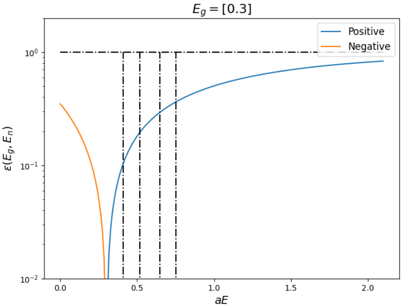
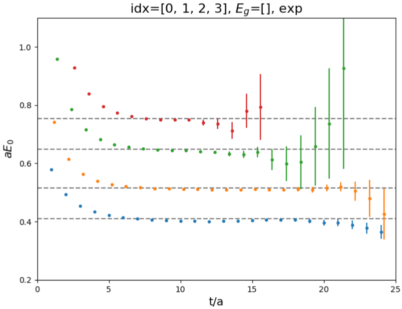
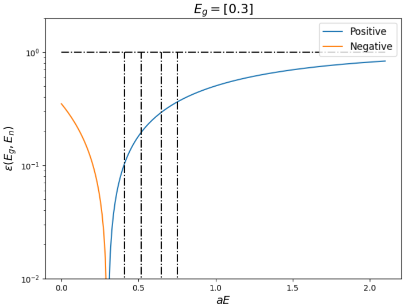
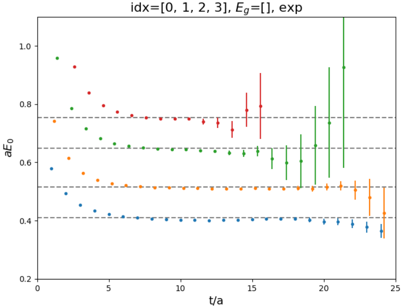
Positive
Negative
$\varepsilon(E_g, E_n)$
10^0
10^{-1}
10^{-2}
0.0
0.5
1.0
1.5
2.0
aE
idx=[0, 1, 2, 3], E_g=[], exp
idx=[0, 1, 2, 3], E_g=[1.500], exp
aE_0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
t/a

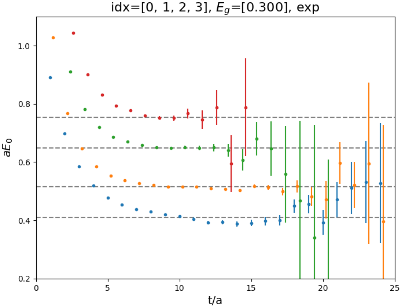
Subtraction Factor ϵ






$E_g = [0.3]$
Positive
Negative
$\epsilon(E_g, E_n)$
aE


idx=[0, 1, 2, 3], E_g=[], exp
aE_0
t/a


idx=[0, 1, 2, 3], E_g=[0.300], exp
aE_0
t/a

Single Point 1st-Order Subtraction:

$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2{-}1) \end{pmatrix}}_{\vec{C}'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{C}}$$

Single Point 1st-Order Subtraction:

$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-1) \end{pmatrix}}_{\vec{C}'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{C}}$$

Multi Points n-th Order Subtraction: $(M_g)^k, (M_{g_1})^{k_1} \cdot (M_{g_2})^{k_2} \dots$

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$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-1) \end{pmatrix}}_{\vec{C}'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{C}}$$

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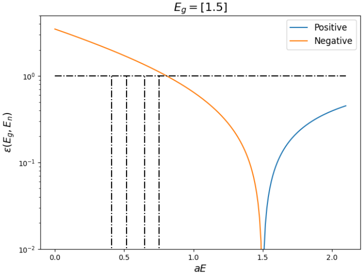
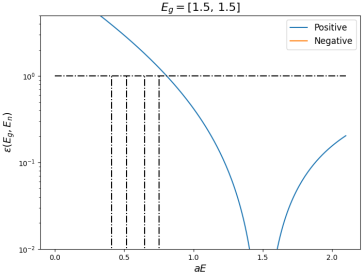


Matrix Form & Spectral Density

Single Point 1st-Order Subtraction:

$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-1) \end{pmatrix}}_{\vec{C}'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{C}}$$

Multi Points n-th Order Subtraction: $(M_g)^k, (M_{g_1})^{k_1} \cdot (M_{g_2})^{k_2} \dots$



Single Point 1st-Order Subtraction:

$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-1) \end{pmatrix}}_{\vec{C}'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{C}}$$

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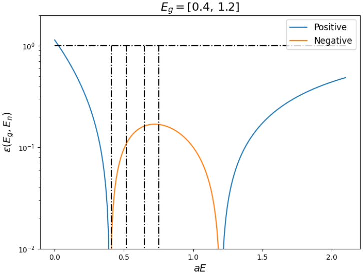

Matrix Form & Spectral Density

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$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-1) \end{pmatrix}}_{\vec{C}'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{C}}$$

Multi Points *n*-th Order Subtraction: $(M_g)^k, (M_{g_1})^{k_1} \cdot (M_{g_2})^{k_2} \dots$

$$\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & c_1 & c_2 & c_3 & \dots & \dots \\ 0 & 1 & c_1 & c_2 & c_3 & \dots \\ 0 & 0 & 1 & c_1 & c_2 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}}_{p(x)=1+c_1x+c_2x^2+c_3x^3+\dots} \begin{pmatrix} C(0) \\ C(1) \\ \vdots \end{pmatrix}$$

Single Point 1st-Order Subtraction:

$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-1) \end{pmatrix}}_{\vec{C}'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{C}}$$

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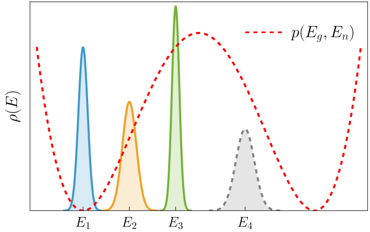
Make roots $\approx e^{-E_n}$

Single Point 1st-Order Subtraction:

$$\underbrace{\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-1) \end{pmatrix}}_{\vec{c}\,'} = \underbrace{\begin{pmatrix} 1 & -e^{E_g} & 0 & \dots & 0 \\ 0 & 1 & -e^{E_g} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{M_g} \underbrace{\begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}}_{\vec{c}}$$

Multi Points n-th Order Subtraction: $(M_g)^k, (M_{g_1})^{k_1} \cdot (M_{g_2})^{k_2} \dots$

$$\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & c_1 & c_2 & c_3 & \dots & \dots \\ 0 & 1 & c_1 & c_2 & c_3 & \dots \\ 0 & 0 & 1 & c_1 & c_2 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}}_{p(x)=1+c_1x+c_2x^2+c_3x^3+\dots} \begin{pmatrix} C(0) \\ C(1) \\ \vdots \end{pmatrix}$$



Make roots $\approx e^{-E_n}$

1. Conventional GEVP, get **good initial guess** E_g
   (from $E_{\mathrm{effective}}$ plots or key thresholds like free energies)

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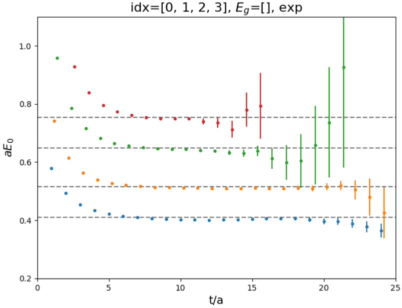
SGEVP Workflow

1. Conventional GEVP, get **good initial guess** E_g
   (from $E_{\text{effective}}$ plots or key thresholds like free energies)
2. Tune to suppress the contamination
   (**both** excited/low-lying states).
3. Fine-tune the $\{E_g\}$ to make both E_{eff} and v_n, z time-invariant.
4. Possible error analysis (not easy, call for experts >_<)

- C32P29, $L^3 \times T = 32^3 \times 64, a = 0.10530(18)$
- $m_\pi = 292.4(1.1)\mathrm{MeV}, m_\pi L = 5.01$
- four operators: $\{\mathcal{O}_\rho, \mathcal{O}_{\pi\pi(p=1,2,3)}\}$
- $C(t)$ data from Zhengli Wang et.al. [JHEP 08 (2025) 064]

Realistic Examples

- C32P29, $L^3 \times T = 32^3 \times 64, a = 0.10530(18)$
- $m_\pi = 292.4(1.1)\mathrm{MeV}, m_\pi L = 5.01$
- four operators: $\{\mathcal{O}_\rho, \mathcal{O}_{\pi\pi(p=1,2,3)}\}$
- $C(t)$ data from Zhengli Wang et.al. [JHEP 08 (2025) 064]



- C32P29, $L^3 \times T = 32^3 \times 64$, $a = 0.10530(18)$
- $m_\pi = 292.4(1.1)\mathrm{MeV}$, $m_\pi L = 5.01$
- four operators: $\{\mathcal{O}_\rho, \mathcal{O}_{\pi\pi(p=1,2,3)}\}$
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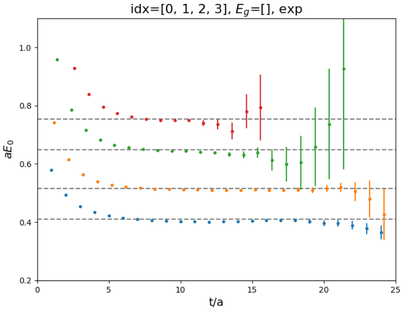
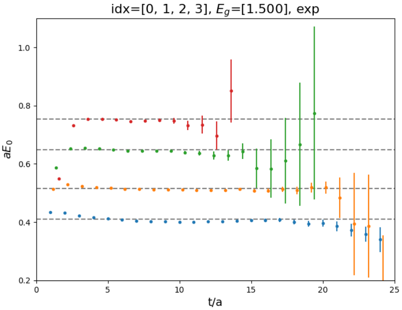
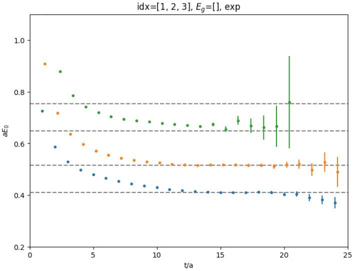




What if lowest $\mathcal{O}_\rho$ is dropped? GEVP still works?

Check $z(t)$'s time-dependence

What if lowest $\mathcal{O}_\rho$ is dropped? GEVP still works?



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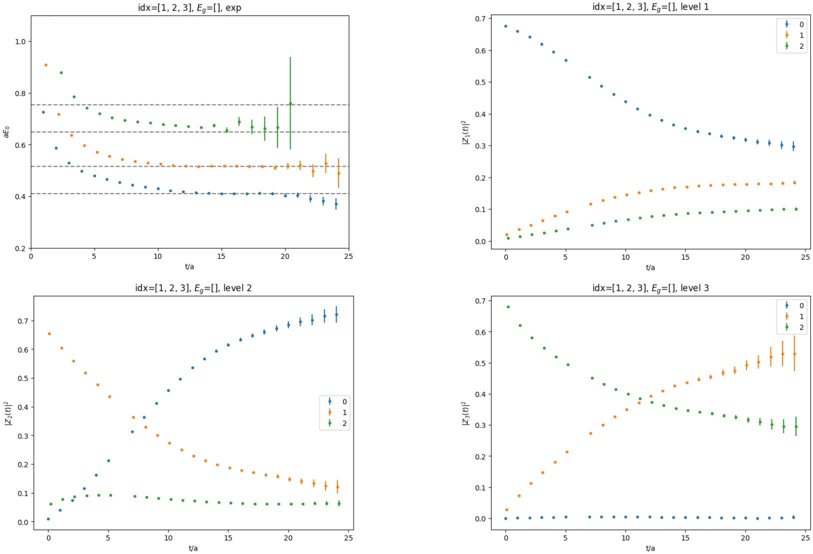
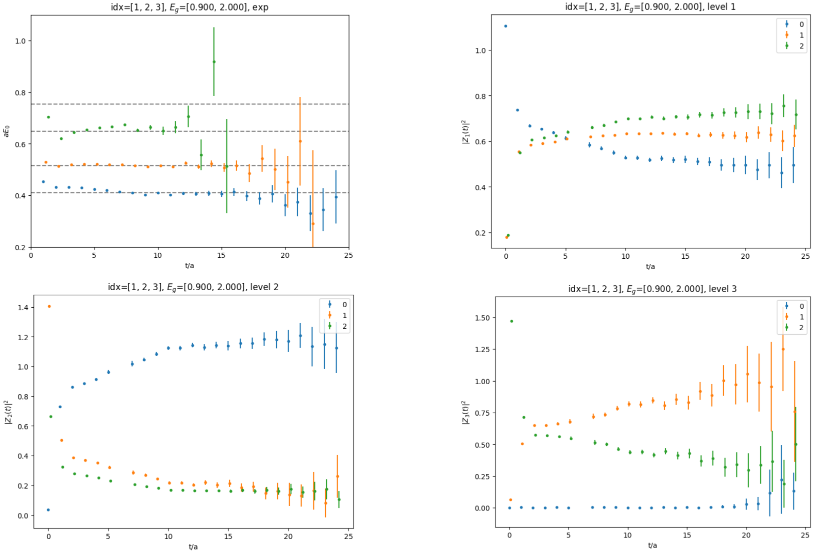
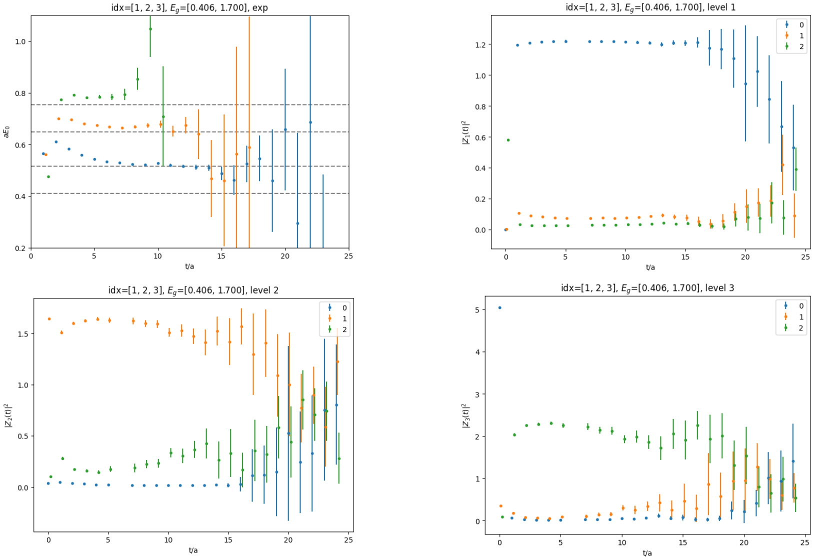


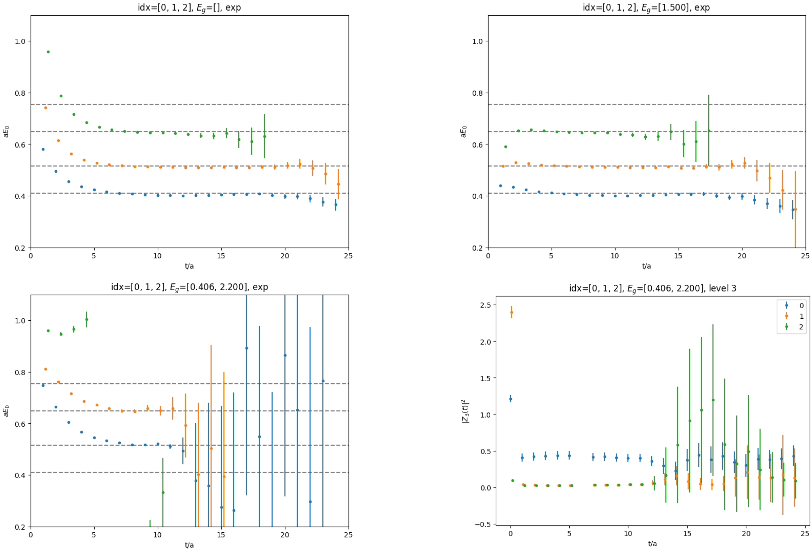
SGEVP. Extend the Plateau of E_{eff}, Z



In $\mathcal{O}_{\pi\pi(p=1,2,3)}$ Suppress $\rho(1S)$?



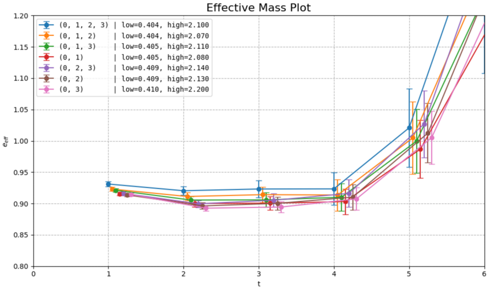
With $\mathcal{O}_\rho$, suppress $\rho(1S)$. Glimpse of $\rho(2S)$?



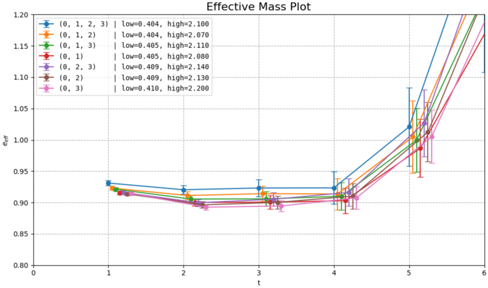
$\rho(2S)$ More Evidence?

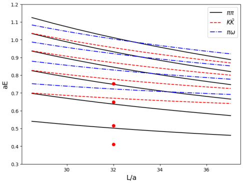
- Include $\mathcal{O}_\rho$, explore all combinations with $\{\mathcal{O}_{\pi\pi(\boldsymbol{p}=1,2,3)}\}$

- Include $\mathcal{O}_\rho$, explore all combinations with $\{\mathcal{O}_{\pi\pi(\boldsymbol{p}=1,2,3)}\}$
- Results somewhat consistent $aE \sim 0.925$
  $\rightarrow \rho(2S) \sim 1733$ MeV V.S. PDG *educated guess* 1465 MeV



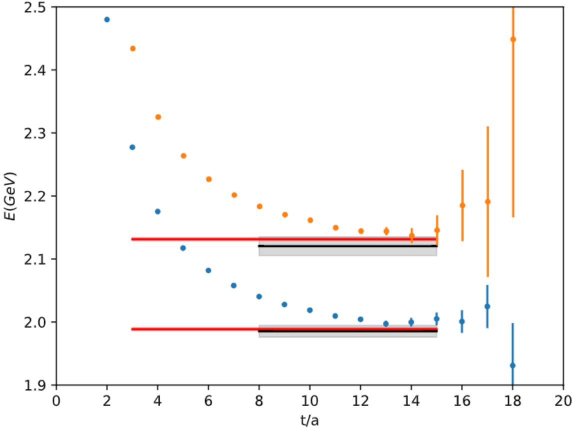
- Include $\mathcal{O}_\rho$, explore all combinations with $\{\mathcal{O}_{\pi\pi(\boldsymbol{p}=1,2,3)}\}$
- Results somewhat consistent $aE \sim 0.925$
  $\rightarrow \rho(2S) \sim 1733$ MeV V.S. PDG *educated guess* 1465 MeV
- No $K\bar{K}, \pi\omega$ operators, so take the result with caveat.

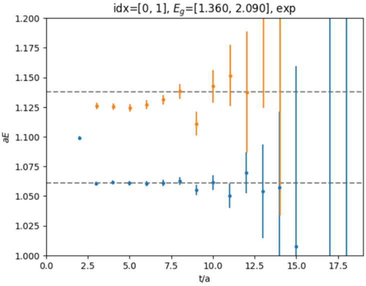


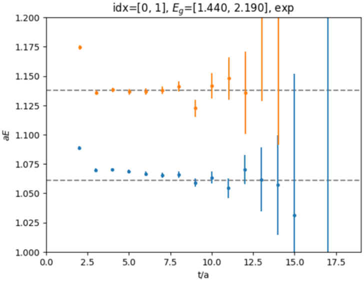


Preliminary results for *PN* (Data from Zhang Kuan)

- GEVP two state fit $E_1 = 1985.2(9.2)$, $E_2 = 2120(15)$
- SGEVP one-state-fit $E_1 = 1988.6(2.1)$, $E_2 = 2131.1(2.0)$
- Strict error analysis is on going (call for experts >_<)



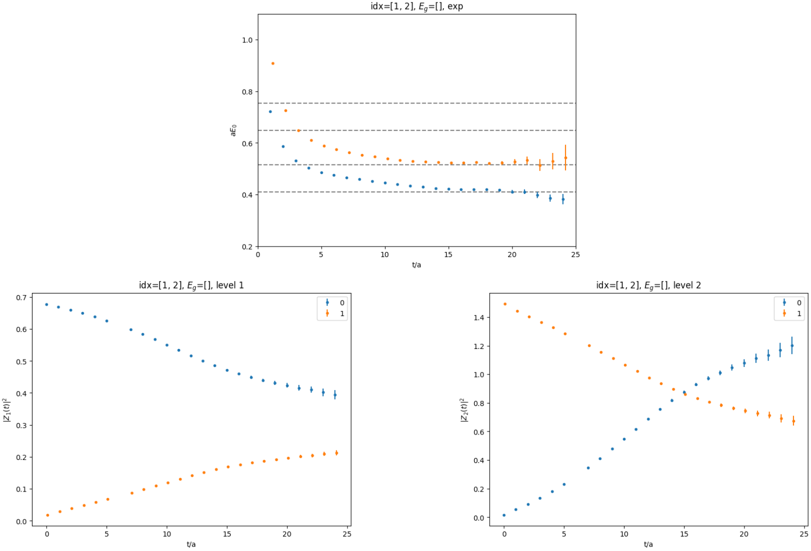

idx=[0, 1], E_g=[1.360, 2.090], exp
aE
t/a
1.200
1.175
1.150
1.125
1.100
1.075
1.050
1.025
1.000
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5


idx=[0, 1], E_g=[1.440, 2.190], exp
aE
t/a
1.200
1.175
1.150
1.125
1.100
1.075
1.050
1.025
1.000
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5

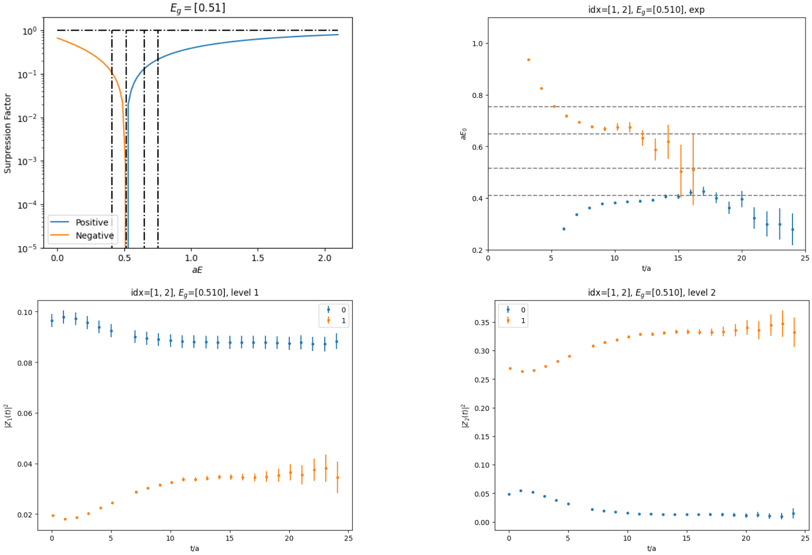
- Realness of λ, V is NOT ensured by Hermitian $C(t)$, but by (semi-)positive-definite $C(t)$

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} v = \begin{pmatrix} 1 & 3 \\ 3 & 8 \end{pmatrix} v\lambda \Rightarrow \begin{cases} \lambda = (1 \pm i\sqrt{3})/2 \\ \\ \langle v_n | \, C(t) \, | v_n \rangle = 0, v_n \text{ is complex.} \end{cases}$$

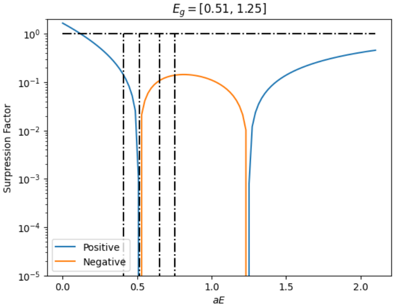
- In some subtractions, matrix may be indefinite /(ㄒ o ㄒ)/

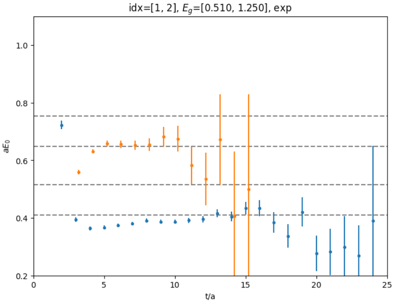

idx=[1, 2], E_g=[], exp
aE_0
t/a
idx=[1, 2], E_g=[], level 1
$|Z_1(t)|^2$
t/a
0
1
idx=[1, 2], E_g=[], level 2
$|Z_2(t)|^2$
t/a
0
1

$E_g = 0.51$ is just fine?



$E_g = 0.51, 1.25$ is problematic



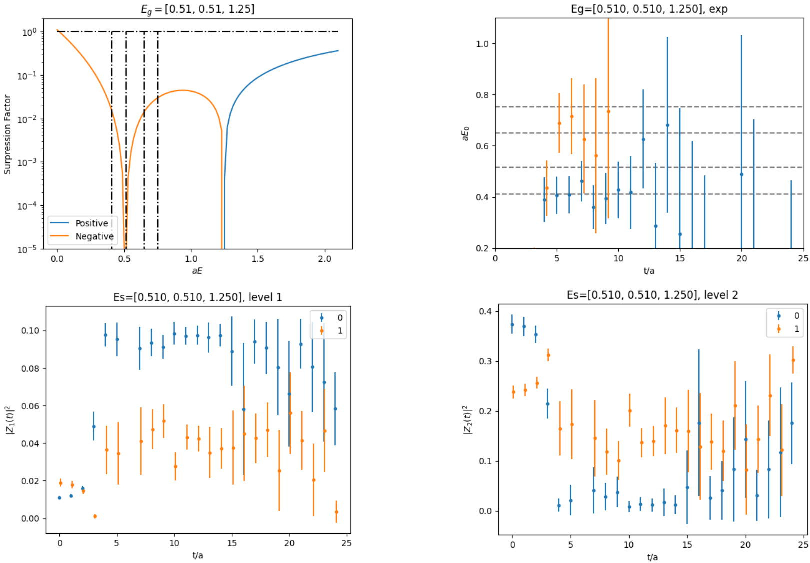



```
num=0, t=2,
 C[2]=
 [[7035.93673059 2446.00313703]
 [2446.00313703 -975.54334924]],
 C[6]=
 [[1165.52387762  527.89528302]
 [ 527.89528302  -19.85263655]],
 Eigenvalues=
 [[-5382.43372607+0.j           -37.08836965+0.j
     6.39332773+1.30034047j     0.          +0.j
     0.          +0.j           0.          +0.j
```

Discard the imaginary part to get $E_{\rm eff}$ plot

Z is not well defined.

[0.51, 0.51, 1.25] is too noisy



Summary

- Conventional GEVP should check both E_{eff} and V_n.
- SGEVP can suppress the contamination from excited states & restore the missing states.
- SGEVP can extract $N+1$ from N, “nearly” for free.
  (such as $\rho(2S)$)
- Complex eigenvalues (due to indefinite) hinder free subtraction
- DOF from E_g challenges strict error analysis
  (call for experts in the audience $>\_<$)

Thank You

$$\begin{cases} C_x := \cosh x,\, S_x := \sinh x \\ \cosh(x \pm \Delta) = C_x C_\Delta \pm S_x S_\Delta \\ \sinh(x \pm \Delta) = S_x C_\Delta \pm C_x S_\Delta \end{cases} \Rightarrow \boxed{C_{x+\Delta} - 2C_{E_g} C_x + C_{x-\Delta} = 2(C_\Delta - C_{E_g}) C_x}$$

For cosh-type correlation function

$$C_{ij}(t) = \sum_n a_{in} a_{jn}^* 2e^{-E_n T/2} \cosh(E_n(t - T/2))$$

$$\Downarrow$$

$$C'_{ij}(t) = \sum_n a_{in} a_{jn}^* 2e^{-E_n T/2} 2(\cosh E_n - \cosh E_g) \cosh(E_n(t - T/2))$$

We make the following subtraction (for all C_{ij} along t-axis)

$$\begin{pmatrix} C'(0) \\ C'(1) \\ \vdots \\ C'(T/2-2) \end{pmatrix} = \begin{pmatrix} 1 & -2C_{E_g} & 1 & 0 & \dots \\ 0 & 1 & -2C_{E_g} & 1 & \dots \\ 0 & 0 & 1 & \ddots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} C(0) \\ C(1) \\ \vdots \\ C(T/2) \end{pmatrix}$$

This discards contribution from E_g, and suppress $E_g \pm \delta$.

Z and Normalization Issue

$$Z_i^n := \langle n|i\rangle := \langle n|\, O_i^\dagger \,|0\rangle$$

$$1 = v_n^\dagger C(t_0) v_n = \sum_m (v_n^\dagger A_m v_n) e^{-E_m t_0} = (v_n^\dagger A_n v_n) e^{-E_n t_0}$$

Z and Normalization Issue

$$Z_i^n := \langle n|i\rangle := \langle n|\, O_i^\dagger\, |0\rangle$$

$$\begin{aligned} 1 = v_n^\dagger C(t_0) v_n &= \sum_m (v_n^\dagger A_m v_n) e^{-E_m t_0} = (v_n^\dagger A_n v_n) e^{-E_n t_0} \\ &= \sum_{ij} v_n^{i*} \langle i|n\rangle \langle n|i\rangle\, v_n^i e^{-E_n t_0} = |\sum_i v_n^{i*} \langle i|n\rangle\, e^{-E_n t_0/2}|^2 \end{aligned}$$

Z and Normalization Issue

$$Z_i^n := \langle n|i\rangle := \langle n|\, O_i^\dagger \,|0\rangle$$

$$\begin{aligned}
1 = v_n^\dagger C(t_0) v_n &= \sum_m (v_n^\dagger A_m v_n) e^{-E_m t_0} = (v_n^\dagger A_n v_n) e^{-E_n t_0} \\
&= \sum_{ij} v_n^{i*} \langle i|n\rangle \langle n|i\rangle\, v_n^i e^{-E_n t_0} = |\sum_i v_n^{i*} \langle i|n\rangle\, e^{-E_n t_0/2}|^2
\end{aligned}$$

$$\begin{aligned}
e^{E_n t_0/2} \sum_j v_n^{j*} C_{ji}(t_0) &= e^{E_n t_0/2} \sum_j v_n^{j*} \langle j|n\rangle\, e^{-E_n t_0} \langle n|i\rangle = \left(\sum_j v_n^{j*} \langle j|n\rangle\, e^{-E_n t_0/2} \right) \langle n|i\rangle \\
&= 1 * \langle n|i\rangle = Z_i^n
\end{aligned}$$

$$Z_i^n := \langle n|i\rangle := \langle n|\, O_i^\dagger\, |0\rangle$$

$$1 = v_n^\dagger C(t_0) v_n = \sum_m (v_n^\dagger A_m v_n) e^{-E_m t_0} = (v_n^\dagger A_n v_n) e^{-E_n t_0}$$

$$= \sum_{ij} v_n^{i*} \langle i|n\rangle \langle n|i\rangle\, v_n^i e^{-E_n t_0} = |\sum_i v_n^{i*} \langle i|n\rangle\, e^{-E_n t_0/2}|^2$$

$$e^{E_n t_0/2} \sum_j v_n^{j*} C_{ji}(t_0) = e^{E_n t_0/2} \sum_j v_n^{j*} \langle j|n\rangle\, e^{-E_n t_0} \langle n|i\rangle = \left(\sum_j v_n^{j*} \langle j|n\rangle\, e^{-E_n t_0/2}\right) \langle n|i\rangle$$

$$= 1 * \langle n|i\rangle = Z_i^n$$

$$\langle i|j\rangle = \sum_{n=1}^{\infty} \langle i|n\rangle \langle n|j\rangle = C_{ij}(0)$$

$$\overset{?}{\approx} \sum_{n=1}^{N} \langle i|n\rangle \langle n|j\rangle$$

Z and Normalization Issue

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$$e^{E_n t_0/2} \sum_j v_n^{j*} C_{ji}(t_0) = e^{E_n t_0/2} \sum_j v_n^{j*} \langle j|n\rangle e^{-E_n t_0} \langle n|i\rangle = \left(\sum_j v_n^{j*} \langle j|n\rangle e^{-E_n t_0/2}\right) \langle n|i\rangle$$

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- Is $\infty \to N$ good enough?

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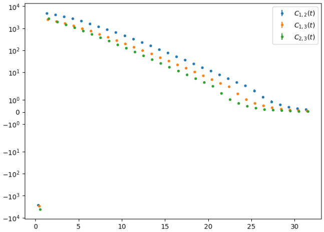
$$= 1 * \langle n|i\rangle = Z_i^n$$



- Is $\infty \to N$ good enough?
  - $E_m \gg 1, e^{-E_m t}$ decays quickly may cause sign-flip in $C(t)$ at short time.

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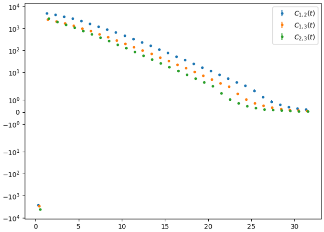
$$e^{E_n t_0/2} \sum_j v_n^{j*} C_{ji}(t_0) = e^{E_n t_0/2} \sum_j v_n^{j*} \langle j|n\rangle e^{-E_n t_0} \langle n|i\rangle = \left(\sum_j v_n^{j*} \langle j|n\rangle e^{-E_n t_0/2} \right) \langle n|i\rangle$$

$$= 1 * \langle n|i\rangle = Z_i^n$$



- Is $\infty \to N$ good enough?
  - $E_m \gg 1, e^{-E_m t}$ decays quickly may cause sign-flip in $C(t)$ at short time.
  - Explain why GEVP requires $t_0, t \to \infty$

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$$= \sum_{ij} v_n^{i*} \langle i|n\rangle \langle n|i\rangle\, v_n^i e^{-E_n t_0} = |\sum_i v_n^{i*} \langle i|n\rangle\, e^{-E_n t_0/2}|^2$$

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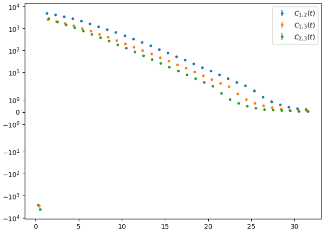
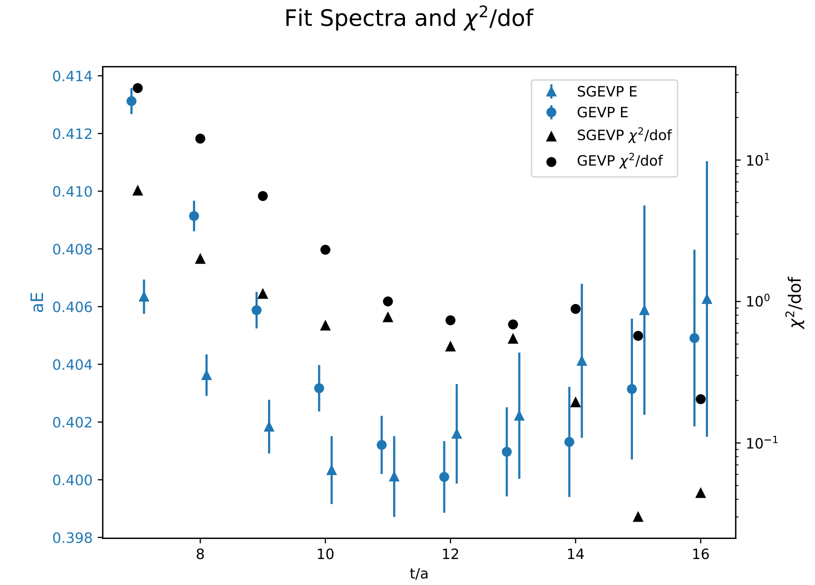
$$= 1 * \langle n|i\rangle = Z_i^n$$

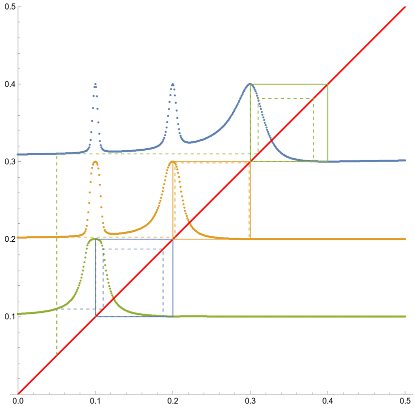


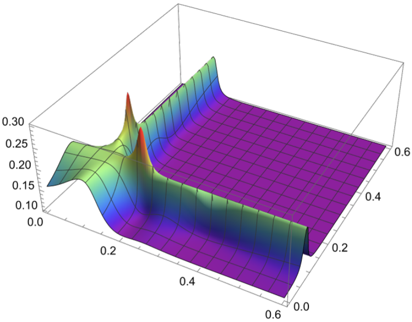
- Is $\infty \to N$ good enough?
  - $E_m \gg 1, e^{-E_m t}$ decays quickly may cause sign-flip in $C(t)$ at short time.
  - Explain why GEVP requires $t_0, t \to \infty$
  - Select $C(t=1)$ is less contaminated, but
  - Leads to a systematic error in Z. Impact the BS Wave function

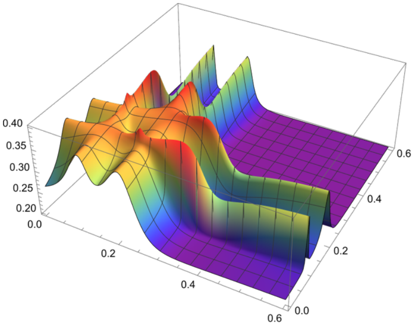


Fit Spectra and χ^2/dof
SGEVP E
GEVP E
SGEVP χ^2/dof
GEVP χ^2/dof
aE
0.414
0.412
0.410
0.408
0.406
0.404
0.402
0.400
0.398
χ^2/dof
10^1
10^0
10^{-1}
t/a
8
10
12
14
16

$3 \times 3 \to 4 \times 4$ Noise Free




0.30
0.25
0.20
0.15
0.10
0.0
0.2
0.4
0.6
0.0
0.2
0.4
0.6


0.40
0.35
0.30
0.25
0.20
0.0
0.2
0.4
0.6
0.0
0.2
0.4
0.6