Finite Volume Hamiltonian method for two-particle system

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conventional paradigm

Ensembles → correction function → lattice spectra $\xrightarrow{\text{Lüscher formula}}$ scattering amplitude

possible challenges for L\”uscher formula

1' Poisson formula : (for a regular function) summation – integration = $O(e^{-\alpha L})$, $\alpha \sim m_\pi$

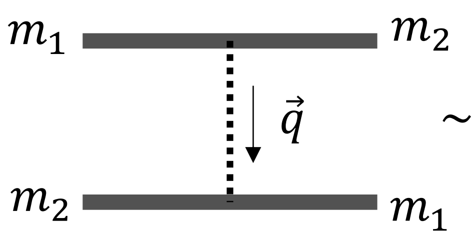
nonnegligible exponential ?

2' separation of on-shell contribution and analytical continuation below threshold

$$\frac{f(q)}{e-q^2/2\mu} = \underbrace{\frac{f(q)-f(\sqrt{2\mu e})}{e-q^2/2\mu}}_{\text{regular}} + \underbrace{\frac{f(\sqrt{2\mu e})}{e-q^2/2\mu}}_{\text{on-shell}}$$

additional analytical properties below threshold ?

both issues matter when One-Pion-Exchange is important!



$$\sim \frac{1}{\vec{q}^2 + m_{\rm eff}^2}, \quad m_{\rm eff}^2 = m_\pi^2 - (m_2 - m_1)^2$$

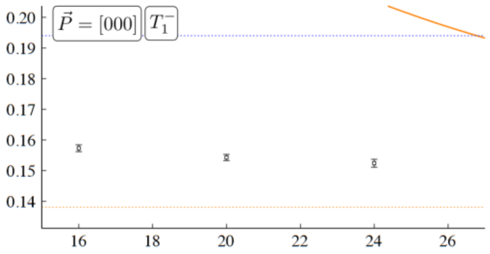
$m_{\rm eff} \ll m_\pi$ such that $m_{\rm eff} L \ll 1$

left-hand cut below threshold

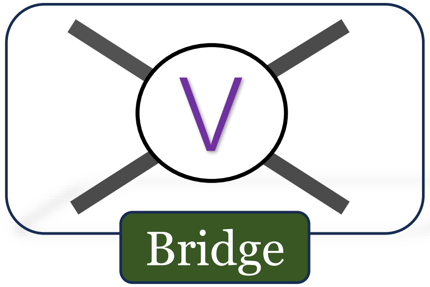
Finite Volume Hamiltonian method

a model-independent relationship if potential is SHORT-RANGED

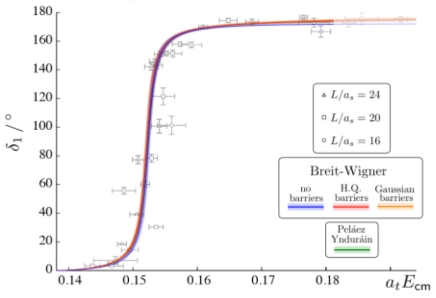
Standard Lüscher Formula



solve Eigenvalues



solve scattering equation



Hilbert space $\{|\vec{n}\rangle\}, (\vec{n} \in Z^3)$

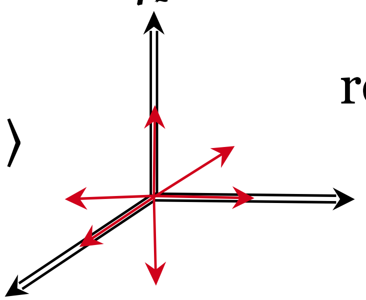
$$\begin{pmatrix} 1 & \cdots & 2 \\ \vdots & \ddots & \vdots \\ 3 & \cdots & 4 \end{pmatrix}$$

Large dimensional matrix

! irreducible representation decomposition **!**

Sketch : spinless system at rest

subspace: $|(0,0,\pm1)\rangle,\ |(0,\pm1,0)\rangle,\ |(\pm1,0,0)\rangle$



reference momentum ,e.g. $\vec{n}_{\text{ref}} = |(0,0,1)\rangle$

irrep basis $\quad |\vec{n}_{\text{ref}};\, T_1^-, \mu = 1, r = 1\rangle \propto \sum_{g\in O_h} \sum_{\nu=1}^{3} \left[c_{\vec{n}_{\text{ref}}}^{r=1}\right]_\nu D_{\mu=1,\nu}^{T_1^-}(g)\, |g\vec{n}_{\text{ref}}\rangle$

$$I_{\nu'\nu} = \sum_{g\in LG(\vec{n}_{\text{ref}})} D^{T_1^-}(g)$$ eigenvecs $[c^{r=1}], [c^{r=2}]$, ... associated to nonzero eigenvals

Hamiltonian element $\quad \left\langle \vec{n}'_{\text{ref}}, r' \left| V_L^{T_1^-} \right| \vec{n}_{\text{ref}}, r \right\rangle = \left(\frac{2\pi}{L}\right)^3 \sqrt{\frac{|LG(\vec{n}_o)|}{|LG(\vec{n}'_o)|}} \sum_{g\in LC(\vec{n}_o)} \underbrace{\left[c_{\vec{n}'_{\text{ref}}}^{r'}\right] \cdot D^{T_1^-}(g) \cdot \left[c_{\vec{n}_{\text{ref}}}^{r}\right]}_{\text{kinematical}} \underbrace{V(\vec{n}', g\vec{n})}_{\text{dynamical}}$

Sketch : PV system at rest

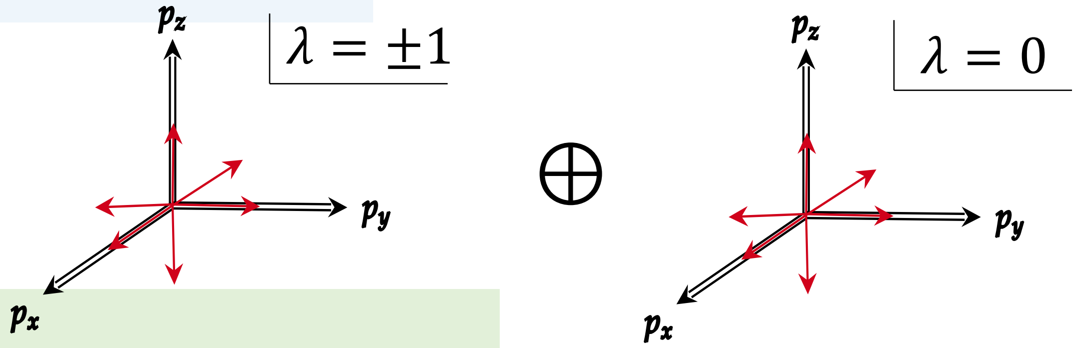
Hilbert space $\{|\vec{n},\sigma\rangle\}$, $(\vec{n} \in Z^3, \sigma = 0, \pm 1)$ polarization would mix.

convert to helicity representation $|\vec{n},\lambda\rangle = \sum_{\sigma} \left(e^{-iJ_z\phi_{\vec{n}}}\, e^{-iJ_y\theta_{\vec{n}}}\right)_{\sigma\lambda} |\vec{n},\sigma\rangle$

$$g|\vec{n},\lambda\rangle = e^{-i\lambda\phi_w}|g\vec{n},\lambda\rangle \ , \ g \in O_h^+$$

$$P|\vec{n},\lambda\rangle = |-\vec{n},-\lambda\rangle$$

different helicity $|\lambda|$ would not mix, as if spinless !



Sketch : others

- moving system : only substitution $O_h \to C_{4v}, C_{3v}, C_{2v}$ is needed
- fermionic system: take care of double cover group O_h^2
- isospin symmetry : incorporation of permutation group

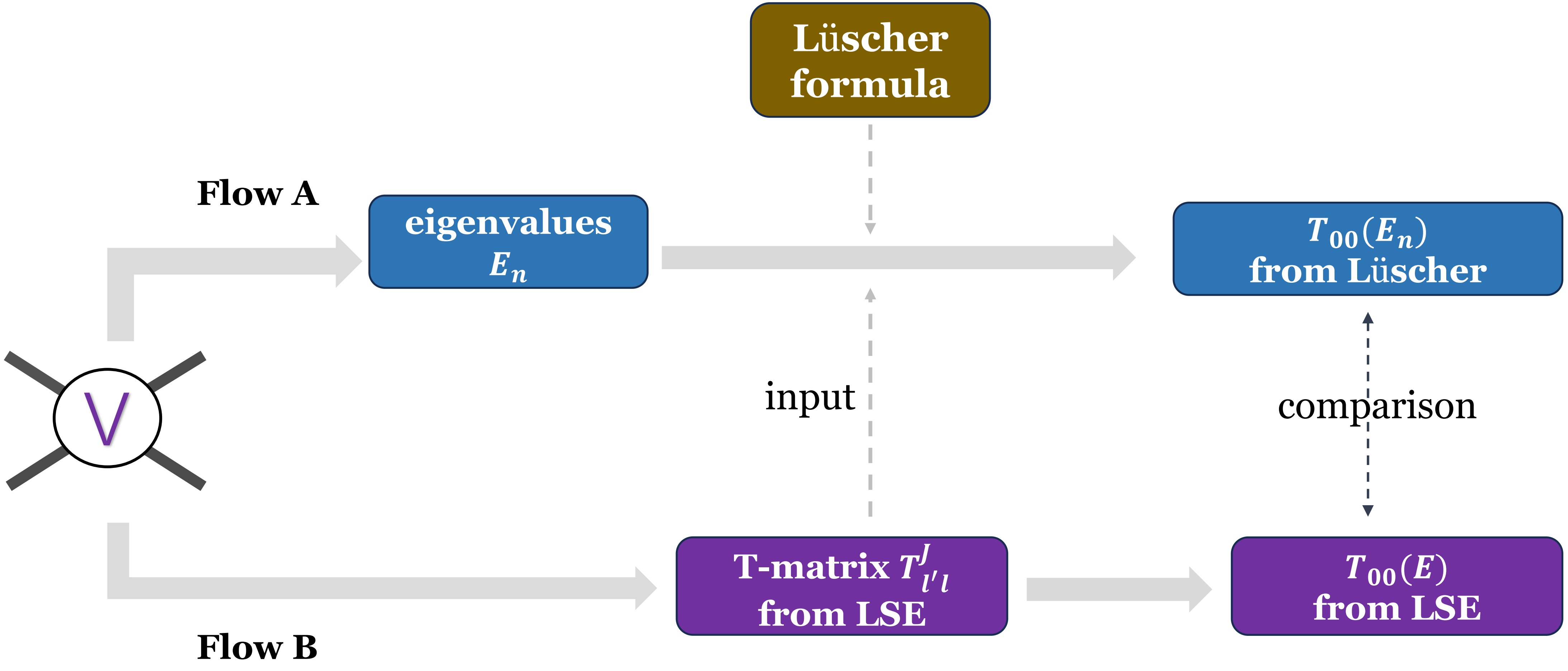
Application: Toy Model and issues from left-hand cut

PV → PV $$V_{\lambda'\lambda}(\vec{p},\vec{k}) = \frac{1}{(2\pi)^3}\delta_{\lambda'\lambda}V(\vec{p},\vec{k})$$

- Model 1: $V(\vec{p},\vec{k}) = c_S\left(\frac{\Lambda^2}{\vec{k}^2+\Lambda^2}\right)^2\left(\frac{\Lambda^2}{\vec{p}^2+\Lambda^2}\right)^2$ S-wave short
- Model 2: $V(\vec{p},\vec{k}) = c_D\left(\frac{1}{3}\left(\vec{p}\cdot\vec{k}\right)^2 - \vec{p}^2\,\vec{k}^2\right)\frac{1}{\Lambda^4}\left(\frac{\Lambda^2}{\vec{k}^2+\Lambda^2}\right)^4\left(\frac{\Lambda^2}{\vec{p}^2+\Lambda^2}\right)^4$ D-wave short
- Model 3: $V(\vec{p},\vec{k}) = c_S\left(\frac{\Lambda^2}{\vec{k}^2+\Lambda^2}\right)^2\left(\frac{\Lambda^2}{\vec{p}^2+\Lambda^2}\right)^2 + \frac{1}{2}\int d\cos\theta\,\frac{-g\,m_{\rm eff}^2}{(\vec{p}+\vec{k})^2+m_{\rm eff}^2}$ S-wave short + S-wave long
- Model 4: $V(\vec{p},\vec{k}) = c_S\left(\frac{\Lambda^2}{\vec{k}^2+\Lambda^2}\right)^2\left(\frac{\Lambda^2}{\vec{p}^2+\Lambda^2}\right)^2 + \frac{-g\,m_{\rm eff}^2}{(\vec{p}+\vec{k})^2+m_{\rm eff}^2}$ S-wave short + full long

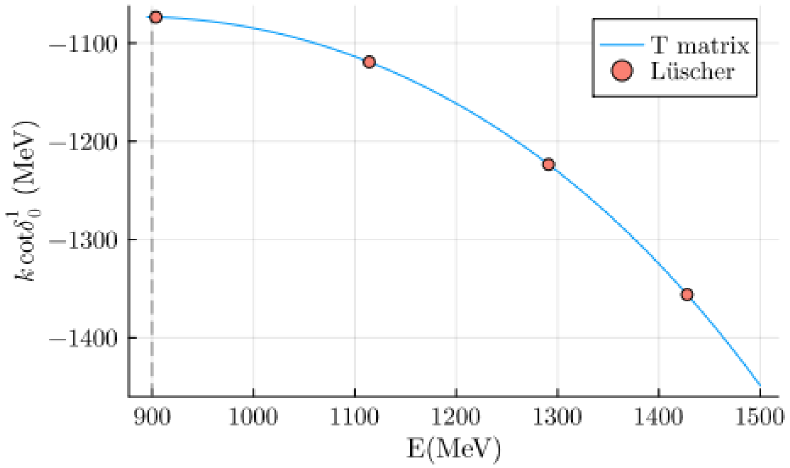
$m_{\rm eff}L = 2$

Application: Toy Model and issues from left-hand cut

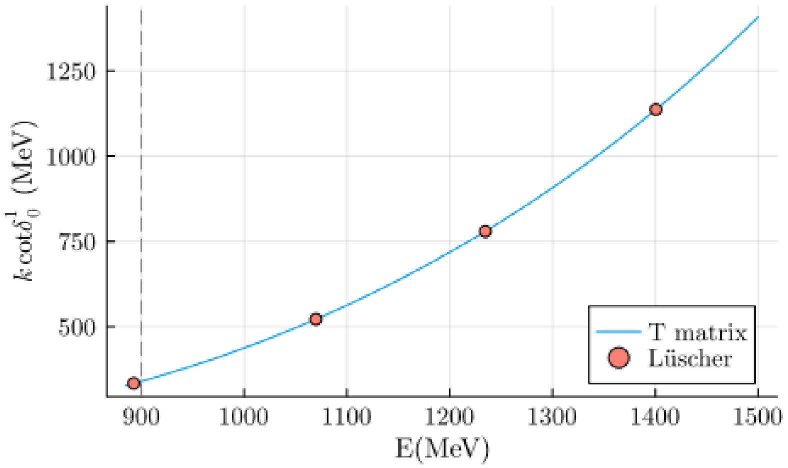


Application: Toy Model 1 and Model 2

pure S-wave short-range

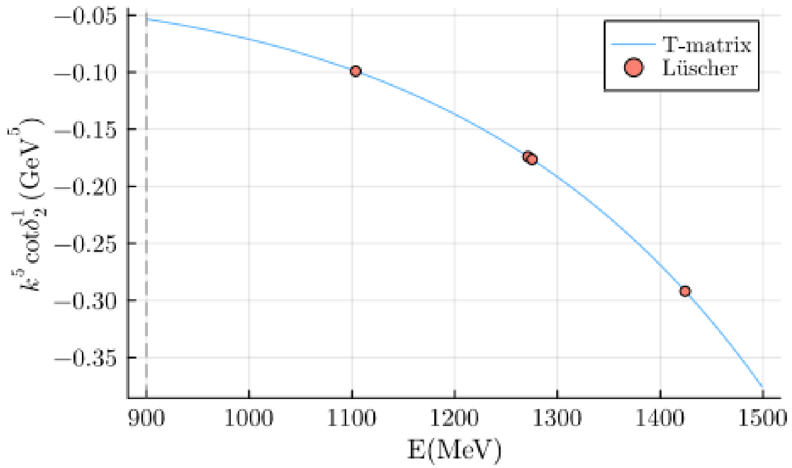


(a) Model I: repulsive potential: $c_S > 0$

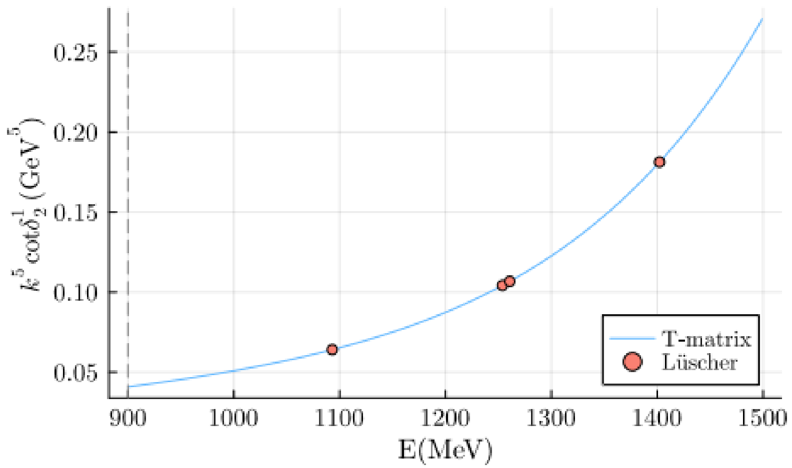


(b) Model I: attractive potential: $c_S < 0$

pure D-wave short-range



(c) Model II: repulsive potential: $c_D > 0$



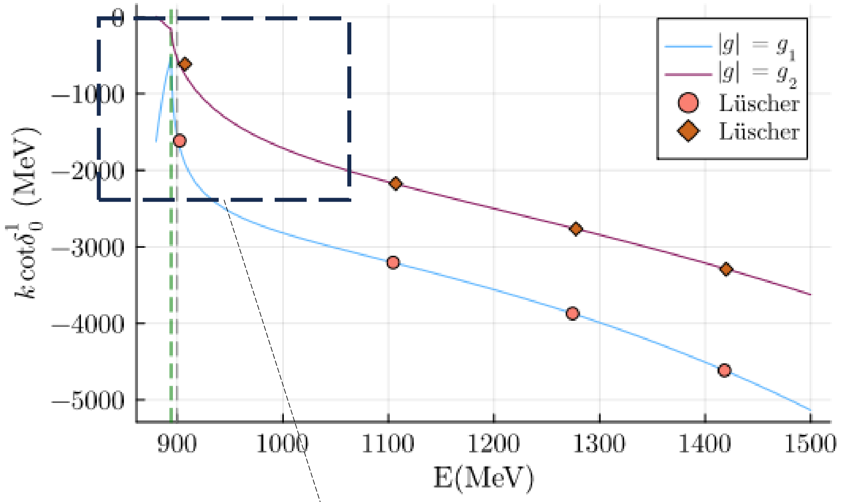
(d) Model II: attractive potential: $c_D < 0$

Application: Toy Model 3

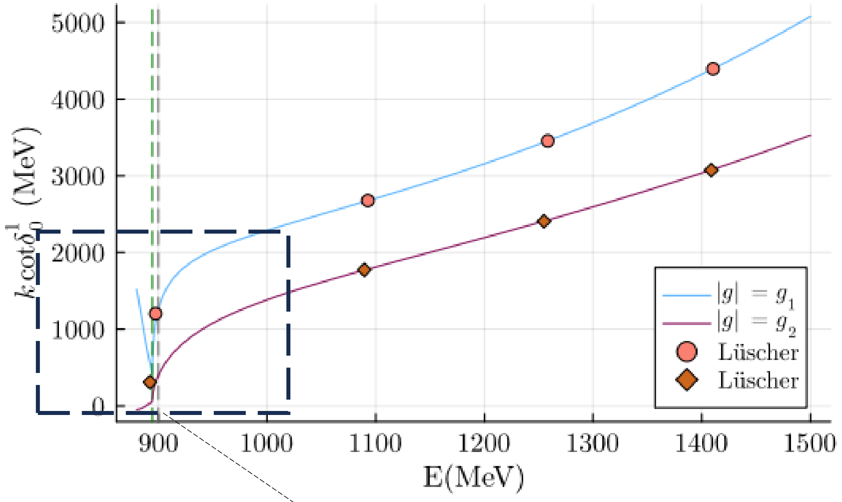
pure S-wave short + pure S-wave long

$m_{\text{eff}}L = 2$

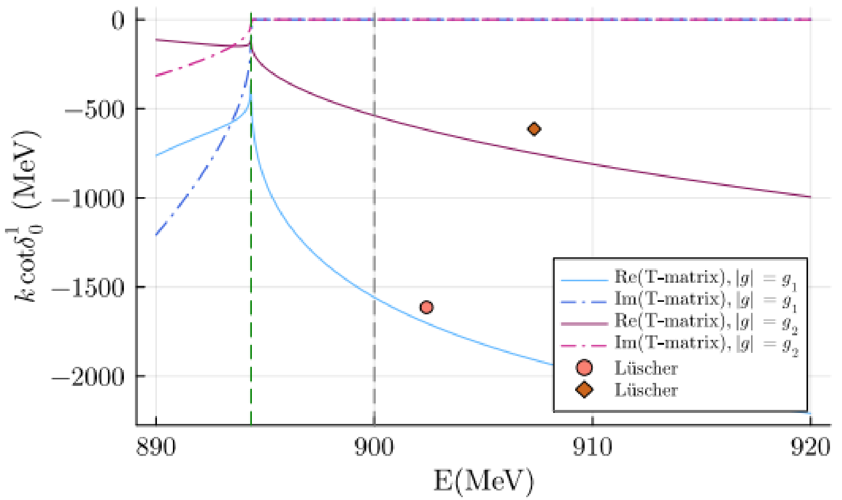
Blue: weaker g
Red: stronger g



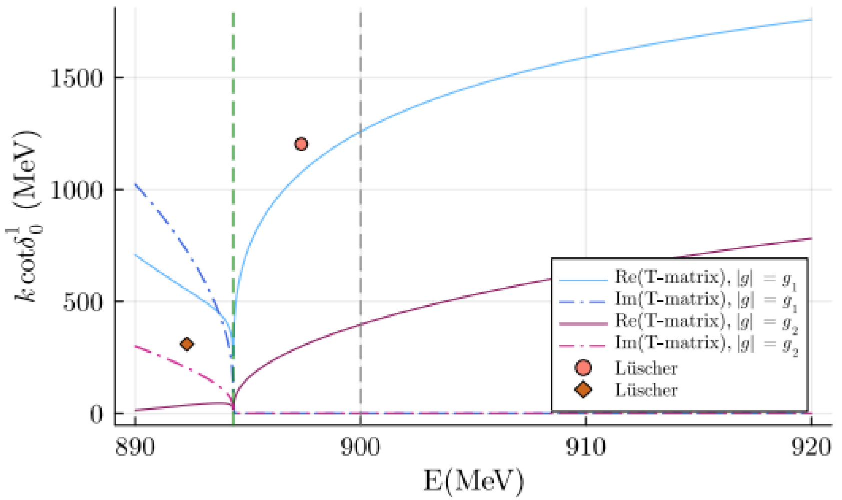
(a) repulsive potential: $c_S > 0$, $g < 0$



(b) attractive potential: $c_S < 0$, $g > 0$

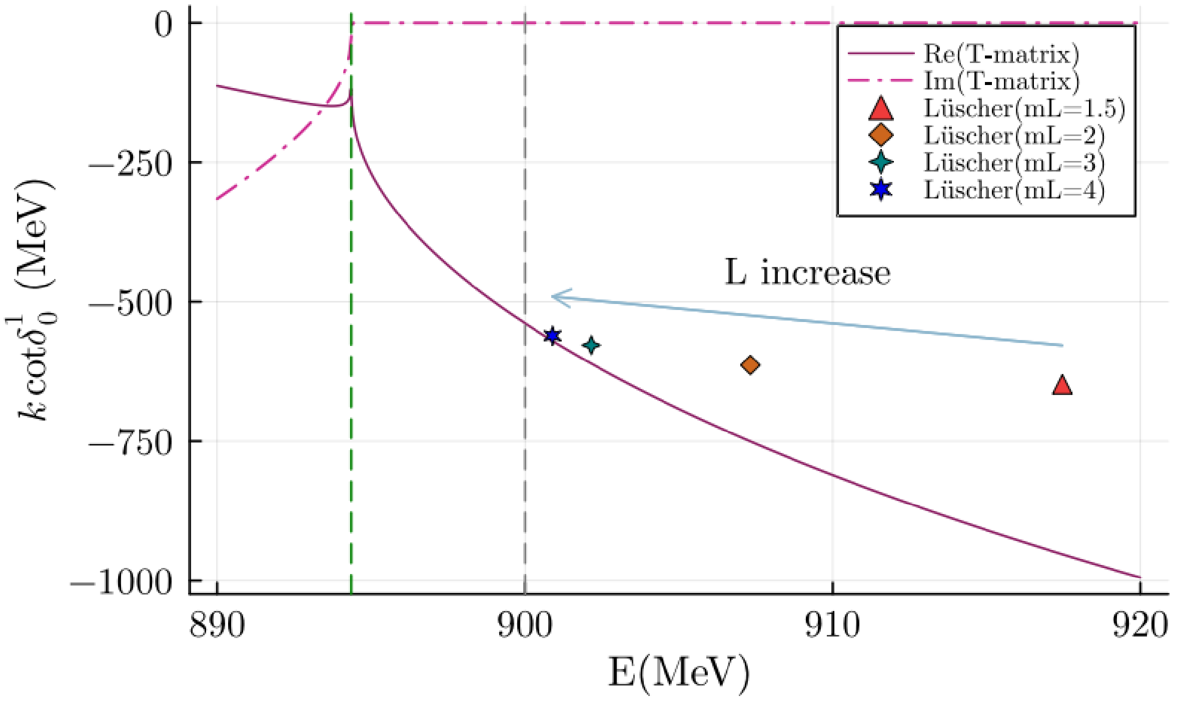


(c) detail near the threshold for (a)

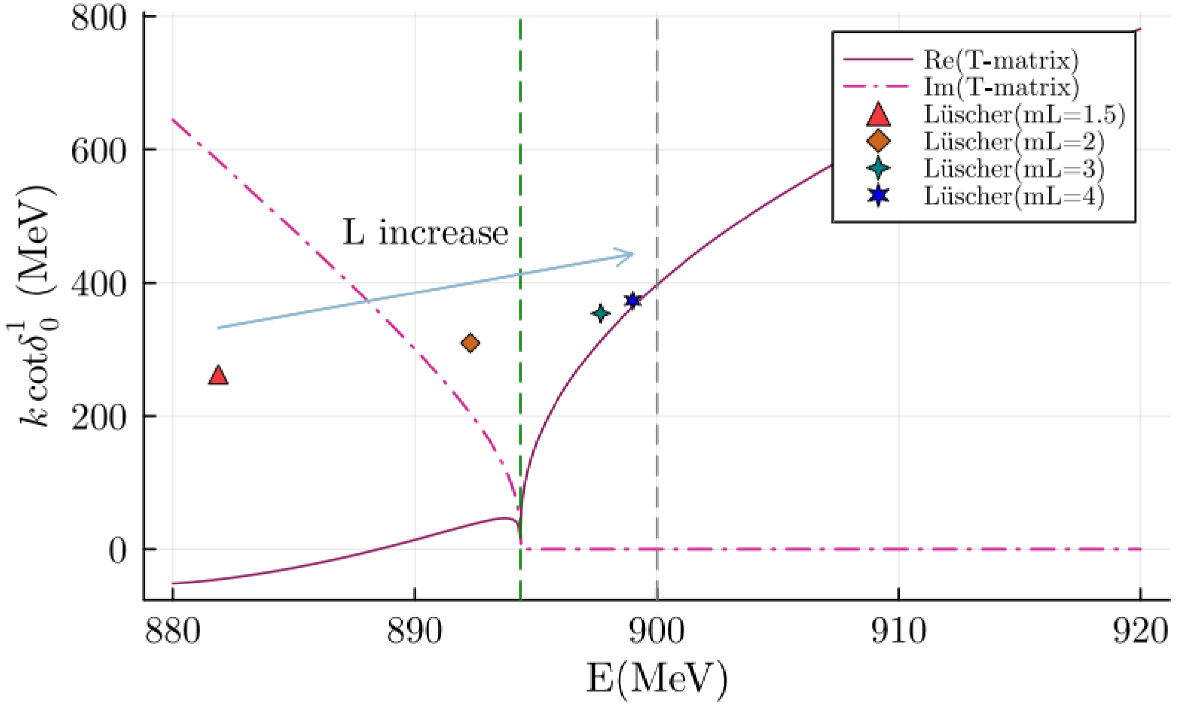


(d) detail near the threshold for (b)

Application: Toy Model 3



(a) repulsive potential: $c_S > 0\,, g < 0$

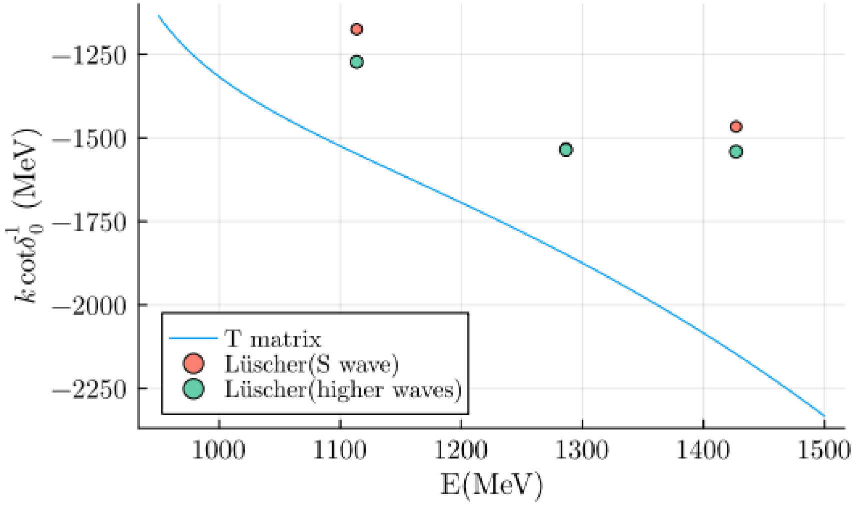


(b) attractive potential: $c_S < 0\,, g > 0$

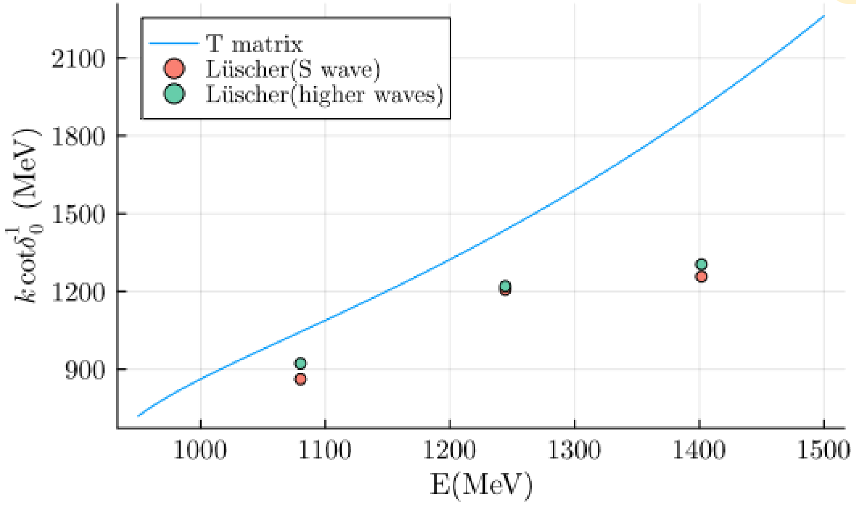
Application: Toy Model 4

pure S-wave short + full long

$m_{\text{eff}}L = 2$



(c) repulsive potential: $c_S > 0$, $g < 0$



(d) attractive potential: $c_S < 0$, $g > 0$

- Luscher formula (only S-wave)
- Luscher formula (input some higher $T_{l'l}^{J\leq 5}$)

nucl-th/0104027 W. Glockle et al.

wave sums for increasing j_{max}. It should be noted that the convergence of the partial wave sums is especially slow for the backward angle differential cross section. A sum up to j_{max}=16 is needed to obtain convergence within 1 %. In Fig. 4 A_y is displayed, which

Application: Toy Model

Universal issues

- **For short-ranged system, FVH method is equivalent to L\”uscher formula**

- **When long-range potential is important, then**

  1’ Lüscher formula cannot be applied below left-hand cut

  2’ Finite volume correction may be under-estimated

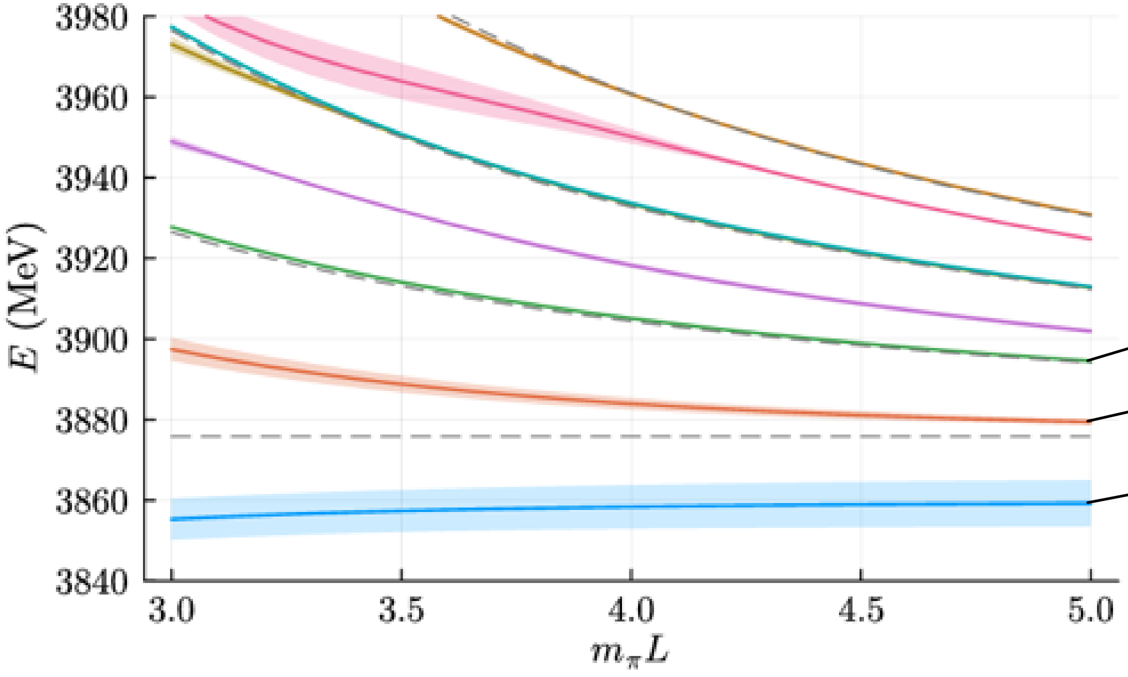
  3’ contribution from high partial-wave may be significant

  FVH method can resolve them simultaneously.

Application: X(3872)

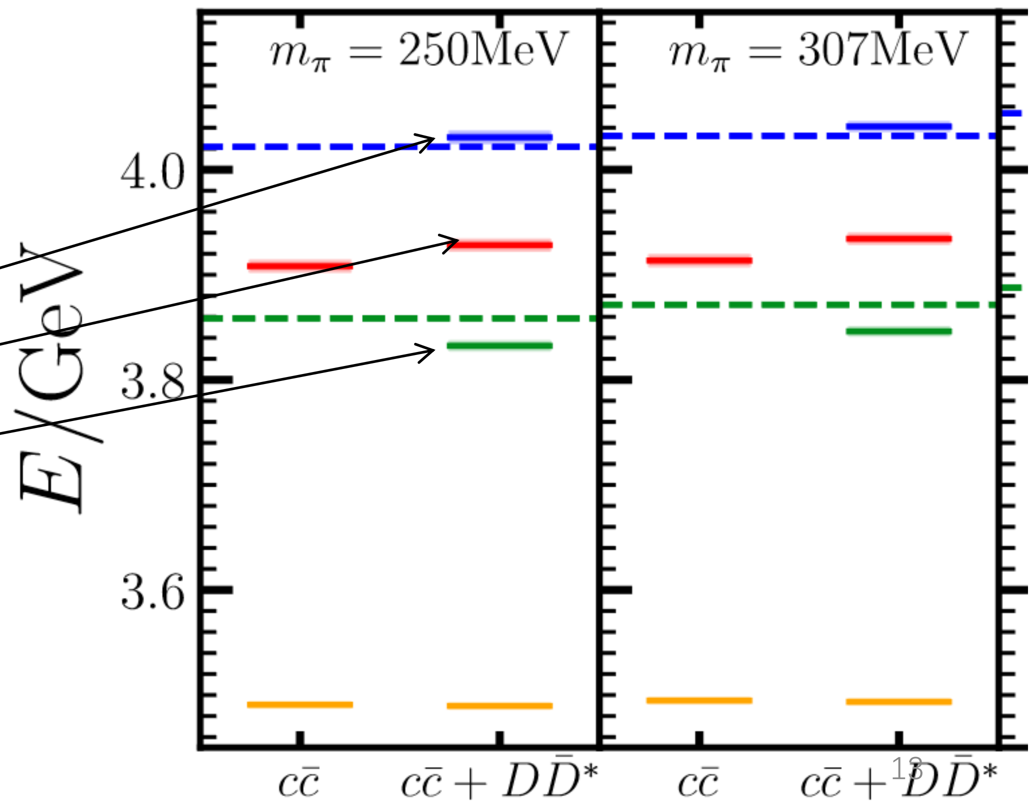
$$V = V^{D\bar{D}^*} + V^{c\bar{c}}$$

short-range part $V^{c\bar{c}}$: $3P0$ model
long-range part $V^{D\bar{D}^*}$: OPE at hadronic level



(a) full model

Chen et al. 2402.14541

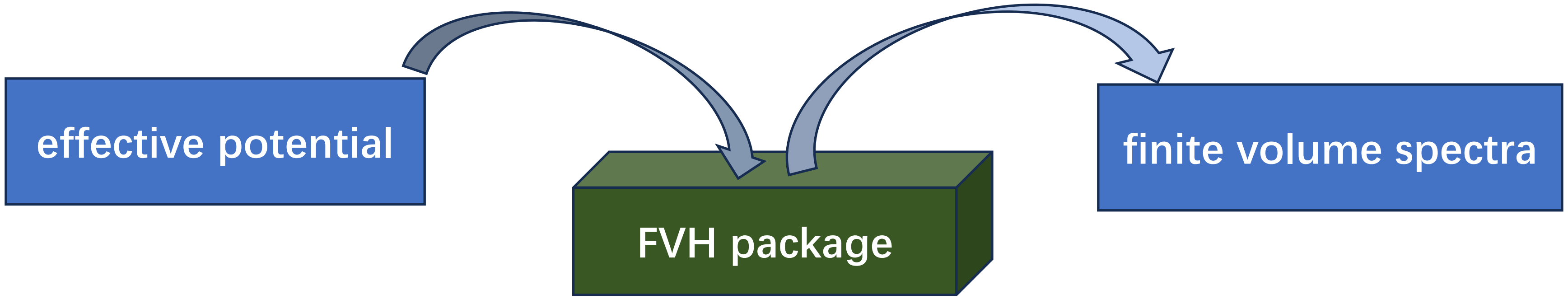


Prospect

1' a practical method to analyze lattice spectra of system whose long-range dynamic may be significant

- DD^* scattering (T_{cc}) see Xie Jia-Jun's report at 14:50 tomorrow
- NN scattering (Almost done...)

2' package



3' Extension to three-particle system

- $\omega \to 3\pi$ system (Almost done...)
- Roper ($N^*(1440)$) resonance (in preparation...)