Towards First-Principles Determination of TMDs

Recent Lattice QCD Progress on the Collins-Soper Kernel, Intrinsic Soft Function, and TMDPDFs

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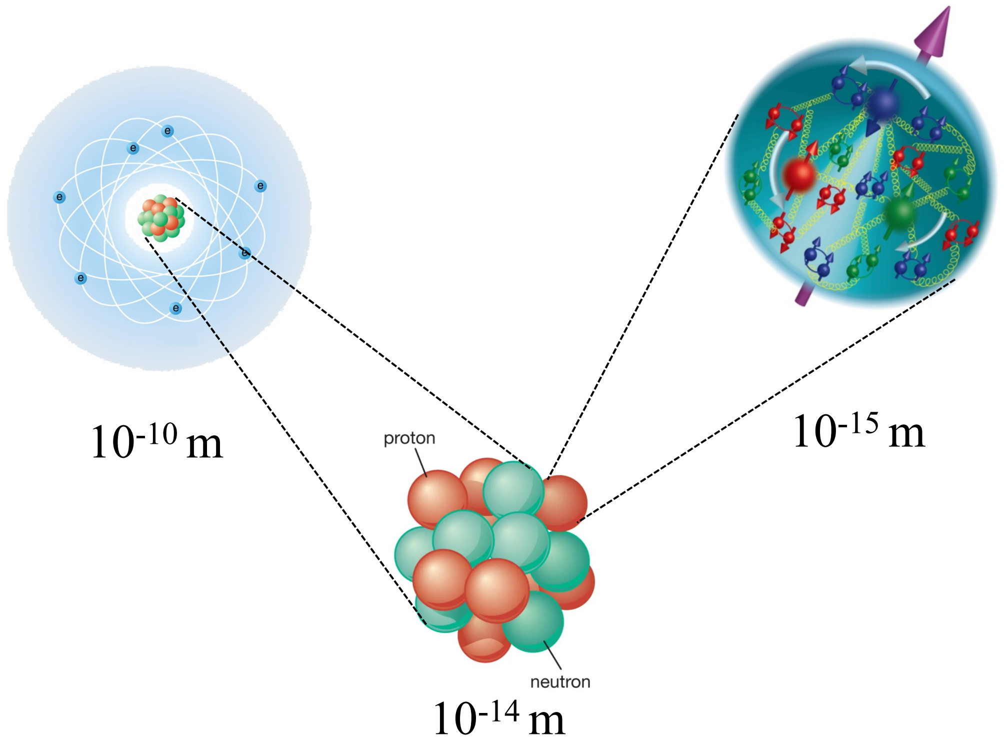
What do we know about the nucleon?

Despite more than half a century of exploration, the nucleon is still a very mysterious object, and the most abundant piece of matter in the visible Universe



Parton Dynamics

- Intrinsic motion of quarks and gluons?
- Transverse motion and spin-$k_\perp$ correlations?

Spatial Imaging

- 3-D tomography of nucleon?

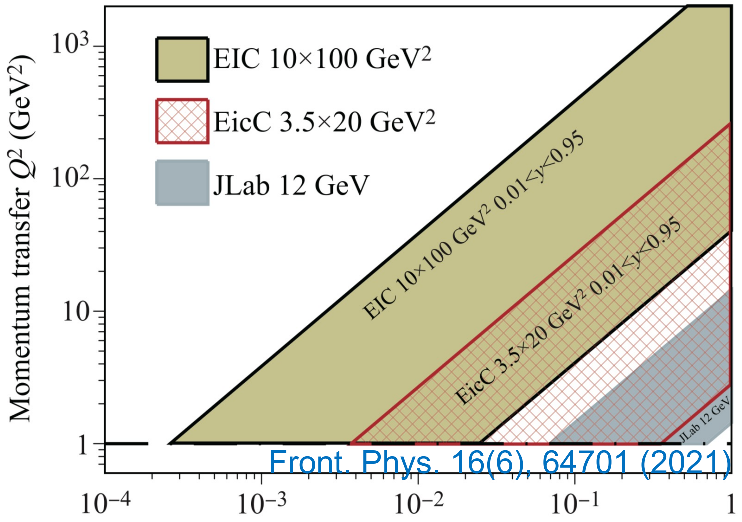
Spin and Orbital Angular Momentum

- How nucleon spin emerge from quark and gluon d.o.f?
- What role do orbiting quark and gluon polarization play?

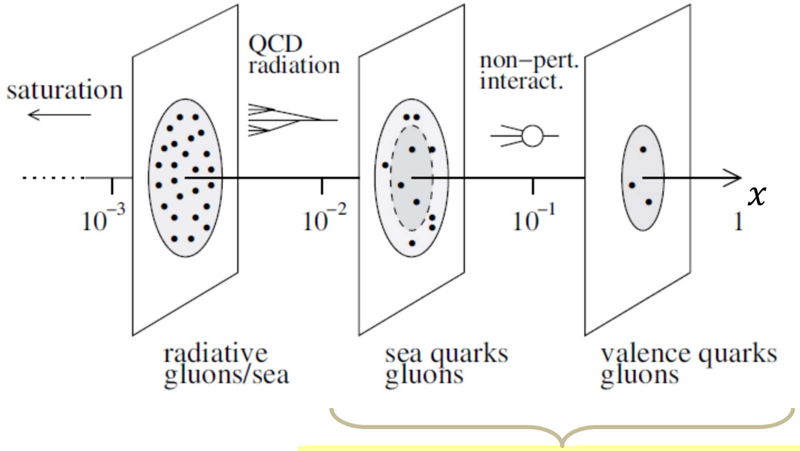
Mass Origin

- How does mass arise from QCD dynamics and confinement?
- Connection between mass generation and parton structure?

The Nucleon Landscape



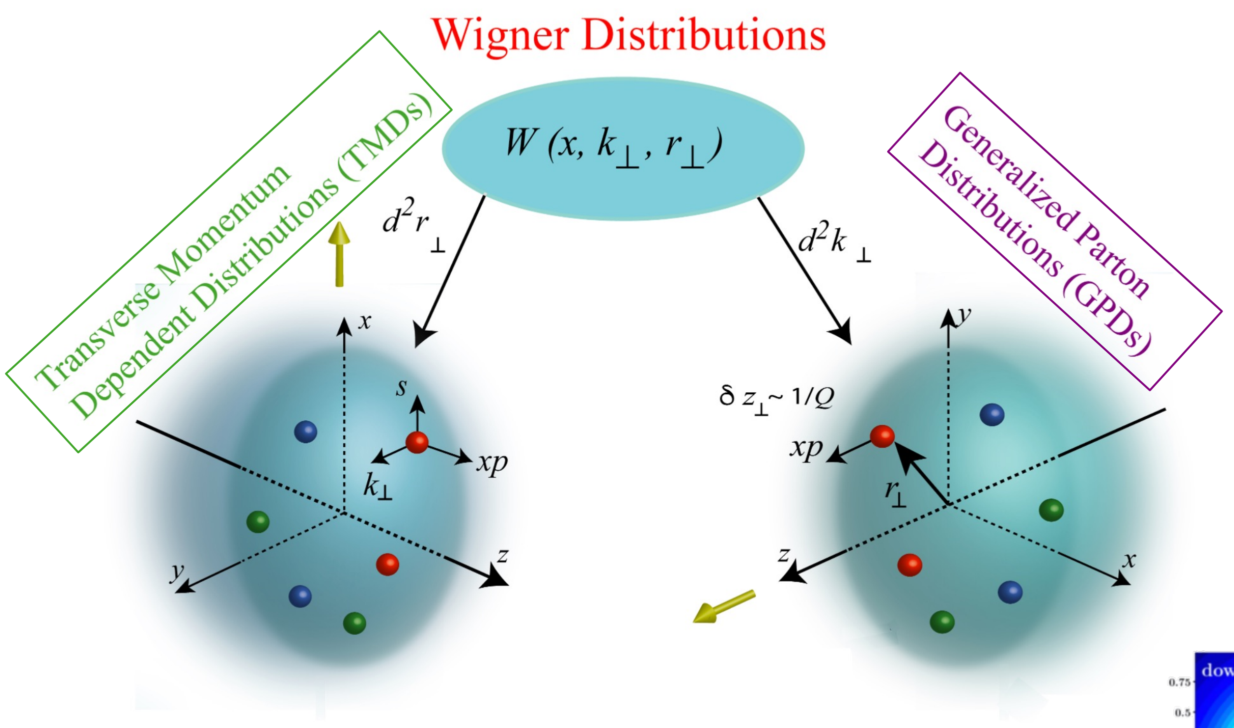
Front. Phys. 16(6), 64701 (2021)



Hadron Structure Dominated

- Nucleon is a many body dynamical system of quarks and gluons
- Changing x we probe different aspects of nucleon wave function
- How partons move and how they are distributed in space?
- Theoretically such information is encoded into the 3D distributions of the nucleon: Transverse Momentum Dependent distributions (TMDs) and Generalized Parton Distributions (GPDs)

The Nucleon Landscape



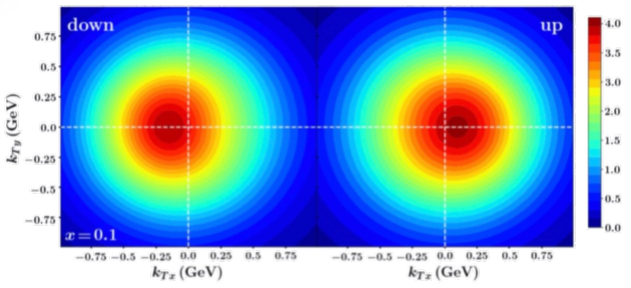
$$\mathbf{TMDs}(x, k_\perp) = \int d^2 r_\perp\, W(x, k_\perp, r_\perp)$$

$$\mathbf{GPDs}(x, r_\perp) = \int d^2 k_\perp\, W(x, k_\perp, r_\perp)$$

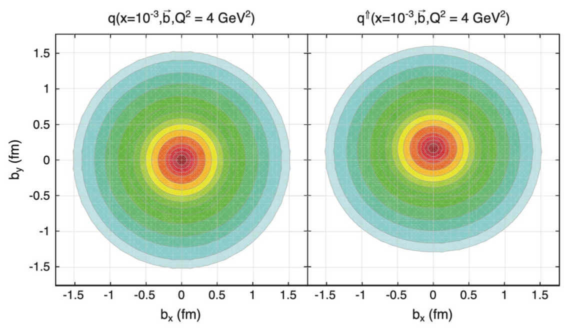
3-D Structures of Nucleon:

TMDs: Spin-dependent 3D momentum space images from semi-inclusive scattering

GPDs: Spin-dependent 2D coordinate space (transverse) + 1D (longitudinal momentum) images from exclusive scattering



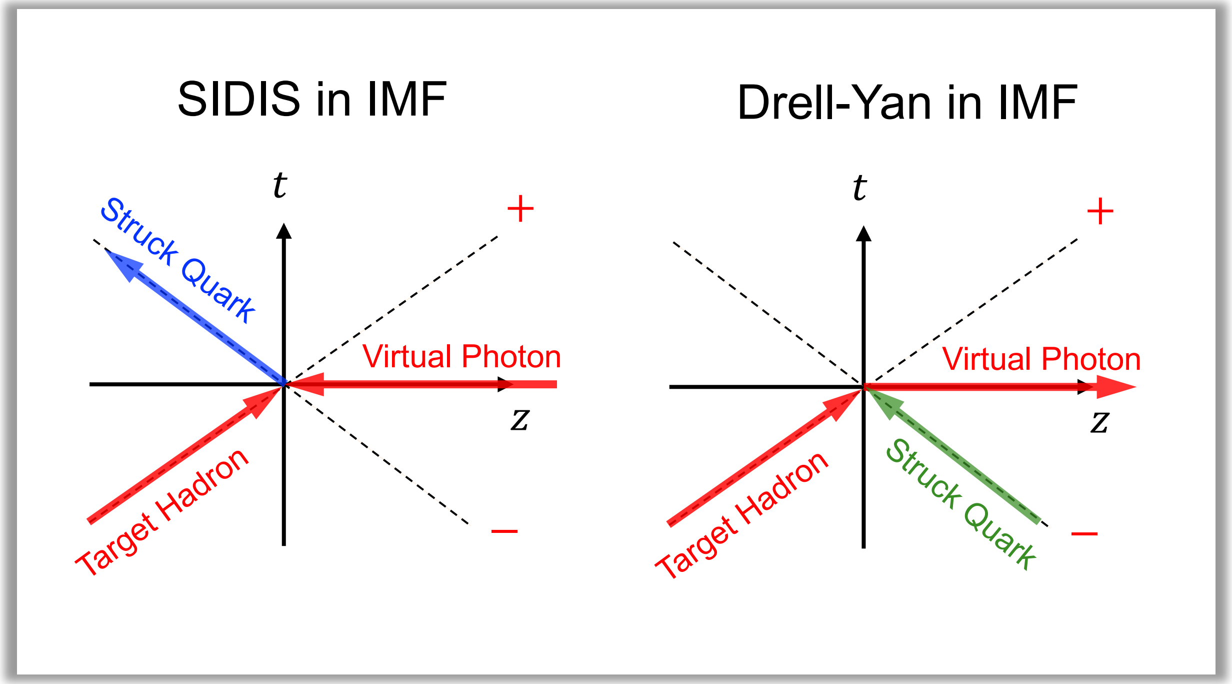
Sivers, PRD102, 054002 (2020)



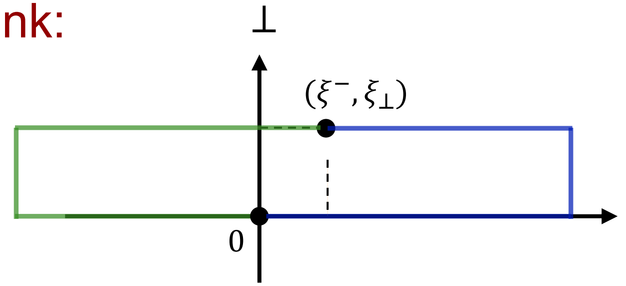
GPD, PRD70, 117504 (2004)

Transverse Momentum Dependent distributions

$$\Phi_{ij}(x, k_\perp) = \int \frac{d\xi^-}{2\pi}\frac{d^2\xi_\perp}{2\pi} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp} \langle P, S | \bar{\psi}_j(0) \mathcal{U}(0,\xi) \psi_i(\xi) | P, S \rangle |_{\xi^+=0}$$



Gauge link:

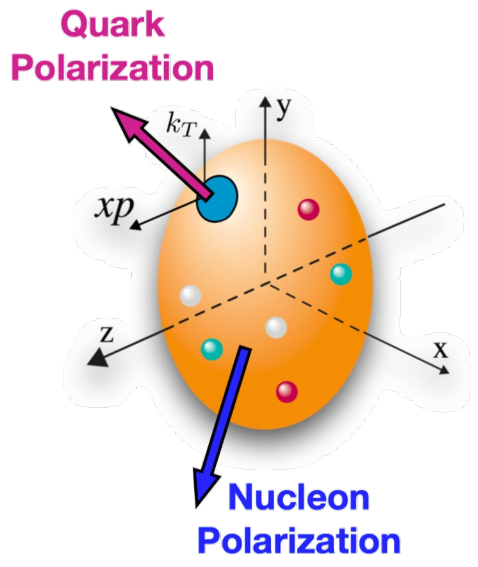


$$\mathcal{U}_n(a,b) = \exp\left[-ig\int_a^b d\lambda\, n\cdot A^\alpha(\lambda n) t^\alpha\right]$$

- Ensures gauge invariance of the distribution.
- Gauge links generate initial and final state interactions.

Spin-Dependent Quark TMDPDFs

Individual TMDs can be projected out of the correlator:



- Unpolarized Quarks:

$$\frac{1}{2}\mathrm{Tr}[\gamma^+\Phi(x,k_\perp)] \;=\; f_1(x,k_\perp^2) - \frac{\varepsilon_\perp^{ij}k_{\perp i}S_{\perp j}}{M} f_{1T}^\perp(x,k_\perp^2)$$

- Longitudinally Polarized Quarks:

$$\frac{1}{2}\mathrm{Tr}[\gamma^+\gamma_5\,\Phi(x,k_\perp)] \;=\; \lambda_N g_{1L}(x,k_\perp^2) - \frac{k_\perp\cdot S_\perp}{M} g_{1T}^\perp(x,k_\perp^2)$$

- Transversely Polarized Quarks:

$$\frac{1}{2}\mathrm{Tr}[i\sigma^{i+}\gamma_5\,\Phi(x,k_\perp)] \;=\; S_\perp^i h_1(x,k_\perp^2) + \frac{k_\perp^2}{M^2}\Big(\frac{1}{2}g_\perp^{ij} - k_\perp^i k_\perp^j\Big)S_{\perp j}h_{1T}^\perp(x,k_\perp^2)$$
$$+\frac{\lambda_N}{M}k_\perp^i h_{1L}^\perp(x,k_\perp^2) - \frac{\varepsilon_\perp^{ij}k_\perp^j}{M} h_1^\perp(x,k_\perp^2)$$

Spin-Dependent Quark TMDPDFs

f_1: unpolarized TMDPDF

g_{1L}: helicity TMDPDF

h_1: transversity TMDPDF

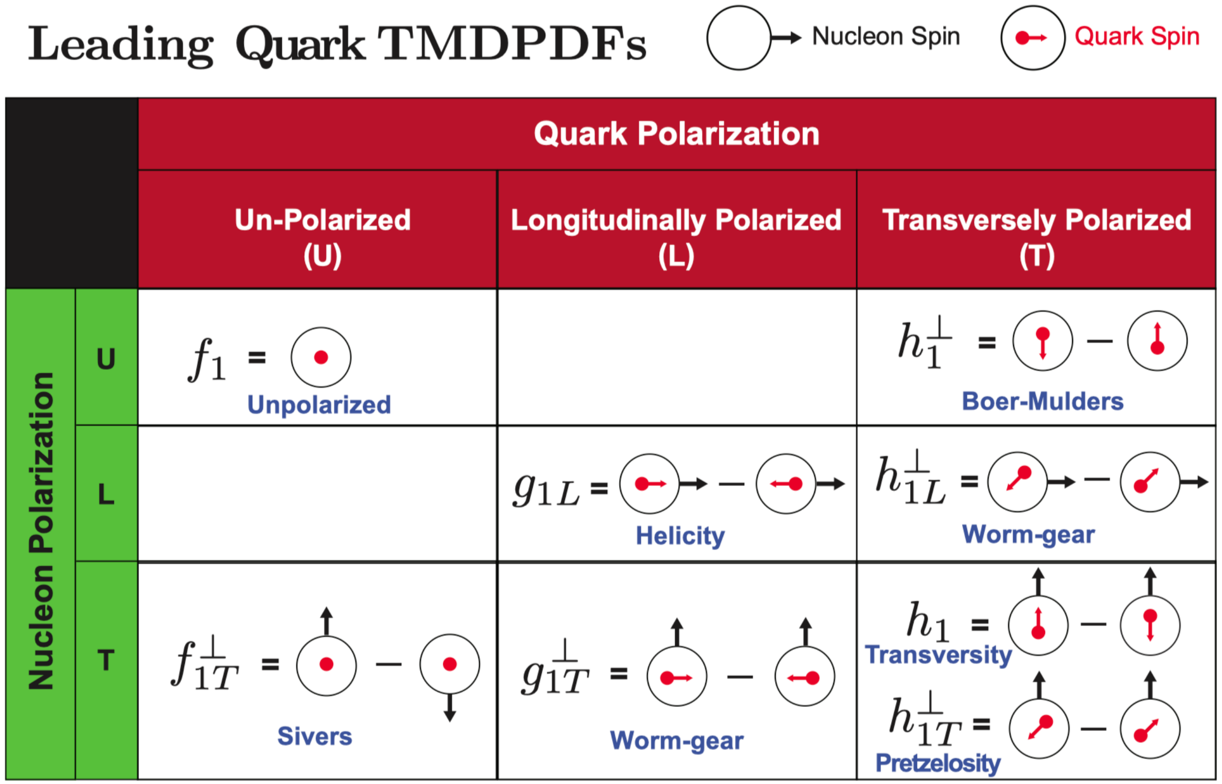
$f_{1T}^{\perp}$: Sivers function (T-odd)

$h_1^{\perp}$: Boer-Mulders function (T-odd)

$g_{1T}^{\perp}$: worm-gear T

$h_{1L}^{\perp}$: worm-gear L

$h_{1T}^{\perp}$: pretzelosity TMDPDF



arXiv:2304.03302 [hep-ph]

Progresses in the Study of Quark TMDPDFs

- **Theoretical analysis**
  - **TMD factorization, evolution and resummation:**

    Boussarie et al., TMD handbook, 2304.03302;

    Collins, Foundations of perturbative QCD; ……

- **Phenomenological parametrizations and extractions**
  - **Unpolarized:**

    Moos, JHEP05 (2024); Bacchetta, JHEP10 (2022); Bury, JHEP10 (2022);

    Scimem, JHEP06 (2020); Bacchetta, JHEP06 (2017); ……

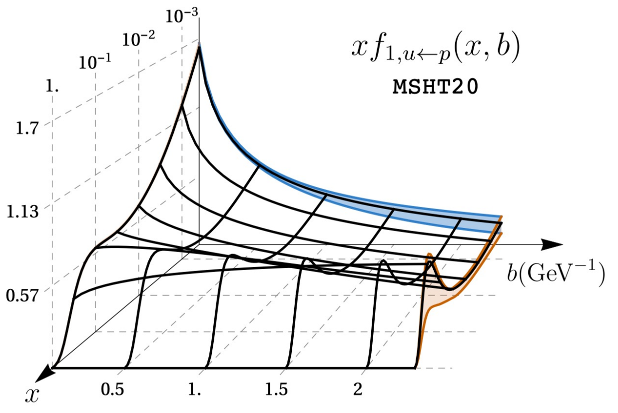
  - **Sivers, Boer-Mulders：**

    Bury, PRL126 (2021), JHEP05 (2021) ; Cammarota, PRD102(2020);

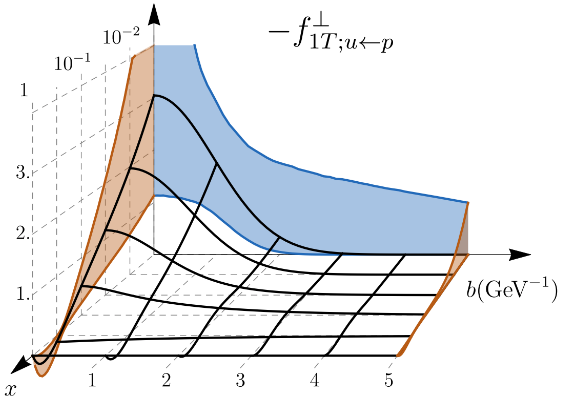
    Zhang, PRD77 (2008), Lu, PRD81 (2010) ; ……

  - **Others: helicity, worm-gear, gluon TMDs, ……**

    Yang, PRL134 (2025); ……



u-quark unpolarized TMDPDF, JHEP10 (2022)



u-quark Sivers function, PRL126 (2021)

TMDs from Lattice: Ratios of Mellin Moments

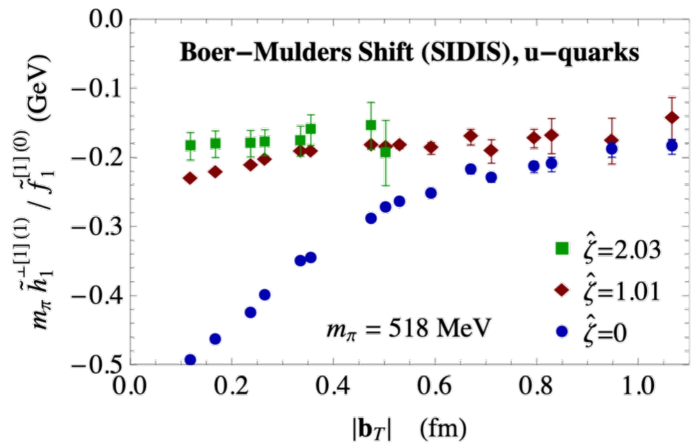
- Through the OPE, TMDs can be converted into local moments that are straightforward to compute on the lattice.
- The ratio eliminates the soft function.
- One can extract the Boer-Mulders shift, Sivers shift, worm-gear shift, etc. from the ratio of lowest order moments:

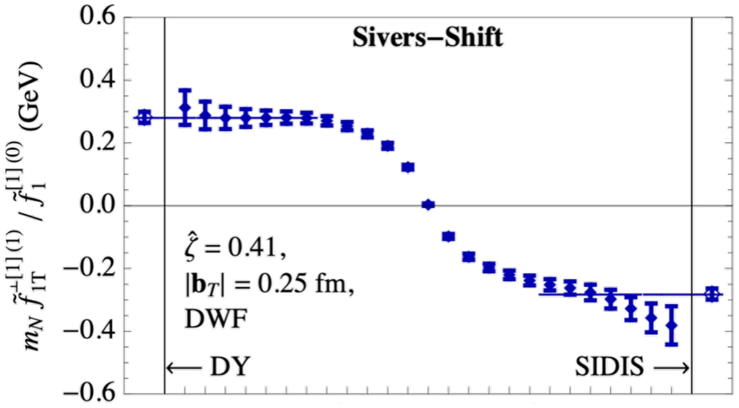
$$\text{Sivers shift} \propto \tilde{f}_{1T}^{[1](1)} / \tilde{f}_{1T}^{[1](0)},$$

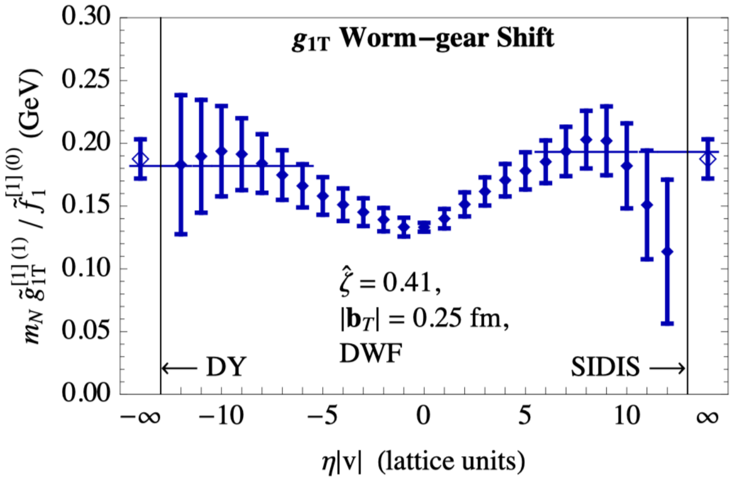
$$\text{BM shift} \propto \tilde{h}_{1}^{[1](1)} / \tilde{f}_{1}^{[1](0)},$$

$$\text{WG shift} \propto \tilde{g}_{1T}^{[1](1)} / \tilde{f}_{1}^{[1](0)}$$

Engelhardt, et.al.. PRD93, 054501 (2016)
Yoon, et.al., PRD96, 094508 (2017)

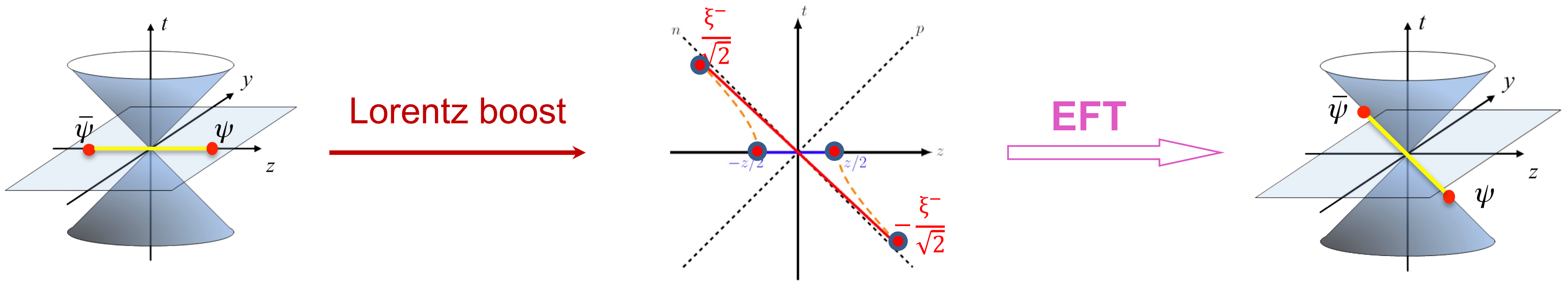






LaMET: A New Horizon for Nucleon Structure from Lattice

Large-momentum effective theory: connecting Euclidean lattice and physical observables



Ji, PRL110(2013), see Rev.Mod.Phys.93, 035005 (2021) for review

- Equal-time distributions can be directly computed in lattice QCD:

$$\tilde{q}(x, P^z) = \int \frac{dz}{2\pi} e^{-izP^z} \langle P^z | \bar{\psi}(z) \Gamma \mathcal{U}(z,0) \psi(0) | P^z \rangle$$

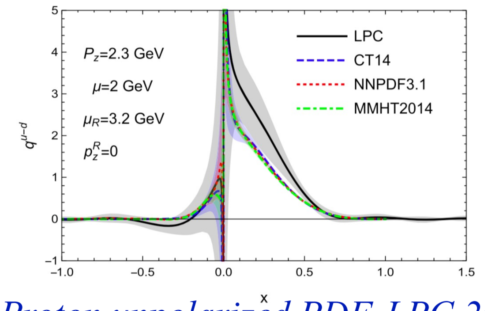
- Recovering light-cone distributions in the large-momentum limit:

$$q(y,\mu) = \int_{-\infty}^{+\infty} \frac{dx}{|x|} C\left(\frac{y}{x}, \frac{\mu}{xP^z}\right) \tilde{q}(x, P^z) + \mathcal{O}\left(\frac{M^2}{(P^z)^2}, \frac{\Lambda_{\rm QCD}^2}{(yP^z, (1-y)P^z)^2}\right)$$

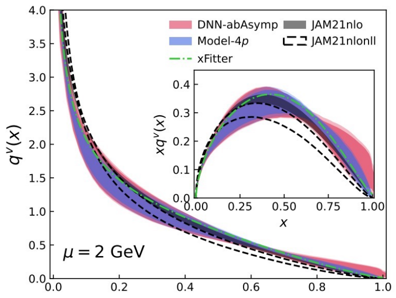
Recent Progresses in LaMET

Achieved great success in the studies of PDFs, LCDAs, GPDs, etc.:

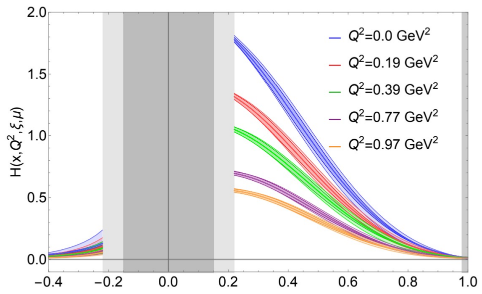
- ✓ Quark unpolarized, helicity, transversity PDFs
- ✓ Gluon PDFs *See Chen's talk*
- ✓ Nucleon GPDs
- ✓ Deutrton PDFs
- ✓ Light meson LCDAs *See Jun's talk*
- ✓ Heavy meson LCDAs *See Hao-Fei's talk*
- ✓ Baryon LCDAs *See Mu-Hua & Hao-Yang's talk*



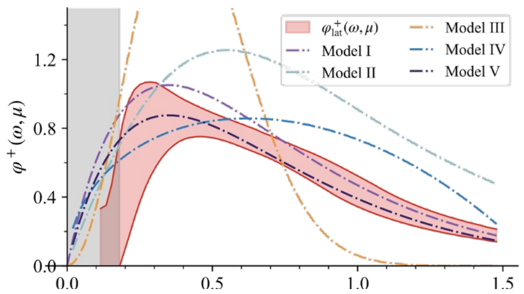
Proton unpolarized PDF, LPC 2020



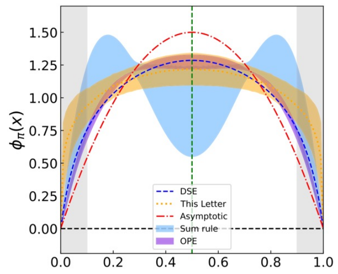
Pion PDF, Gao et.al. 2022



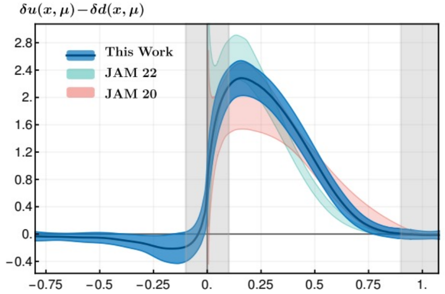
Nucleon GPD, Holligan and Lin 2024



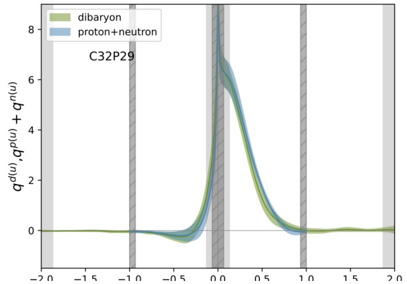
Heavy meson LCDA, LPC 2025



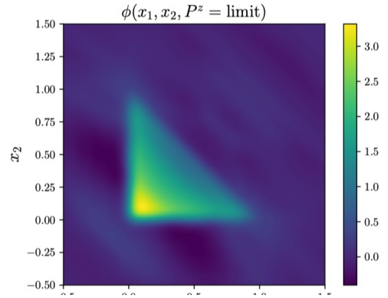
Pion LCDA, LPC 2022



Proton transversity PDF, LPC 2023

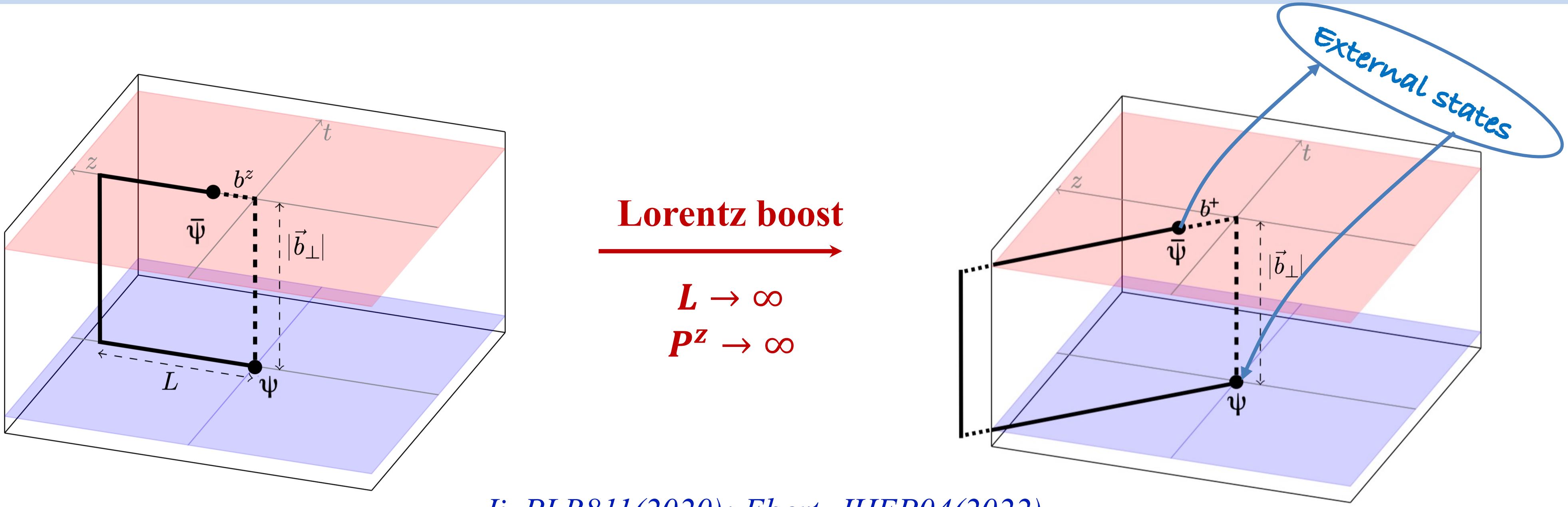


Deuteron PDF, LPC 2025



Baryon LCDA, LPC 2025

TMDs from Lattice QCD within LaMET



Ji, PLB811(2020); Ebert, JHEP04(2022)

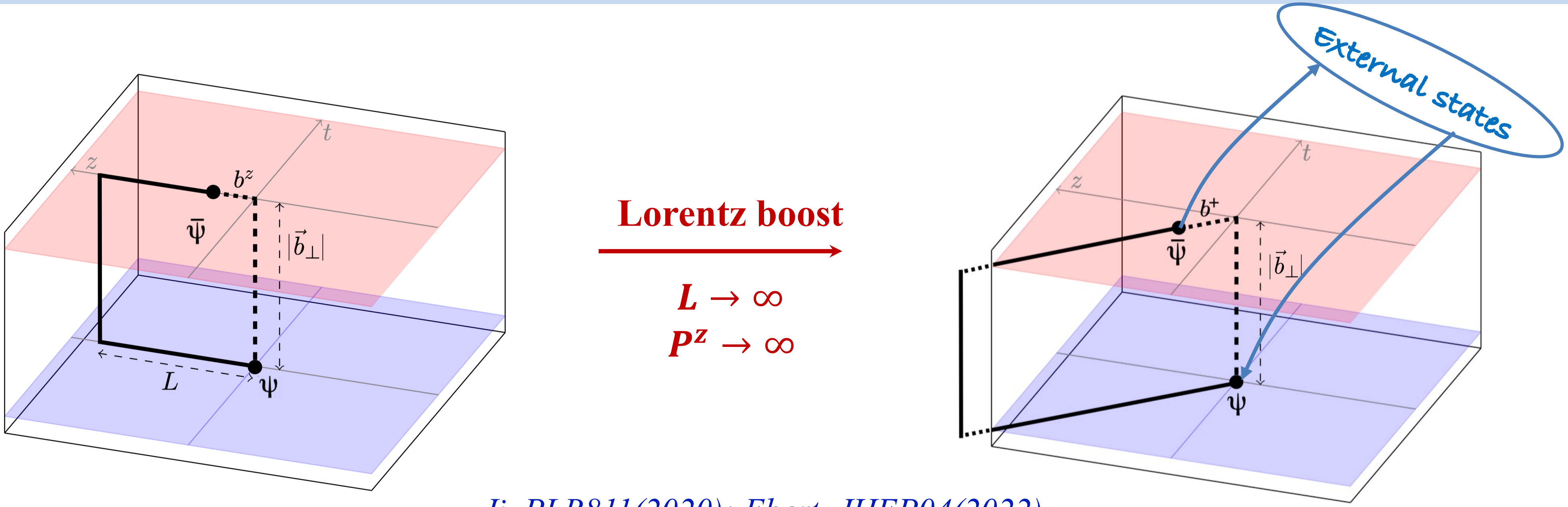
Factorization in LaMET:

✓ Perturbative matching kernel

$$\boxed{\tilde{f}_\Gamma(x, b_\perp, \mu, \zeta^z)}\sqrt{S_I(b_\perp, \mu)} = \boxed{H_\Gamma(\zeta^z, \mu)}\exp\left[\frac{1}{2}K(b_\perp, \mu)\ln\frac{\zeta^z}{\zeta}\right]\boxed{f(x, b_\perp, \mu, \zeta)} + \mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{\zeta^z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2\zeta^z}\right)$$

Quasi TMDs → Light-cone TMDs

TMDs from Lattice QCD within LaMET



Ji, PLB811(2020); Ebert, JHEP04(2022)

Factorization in LaMET:

$$\tilde{f}_\Gamma(x, b_\perp, \mu, \zeta^z)\boxed{\sqrt{S_I(b_\perp, \mu)}} = H_\Gamma(\zeta^z, \mu)\exp\left[\frac{1}{2}\boxed{K(b_\perp, \mu)}\ln\frac{\zeta^z}{\zeta}\right]f(x, b_\perp, \mu, \zeta) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{\zeta^z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2\zeta^z}\right)$$

? Intrinsic soft function ?

? Collins-Soper kernel ?

Rapidity evolution of TMDs: Collins-Soper Kernel

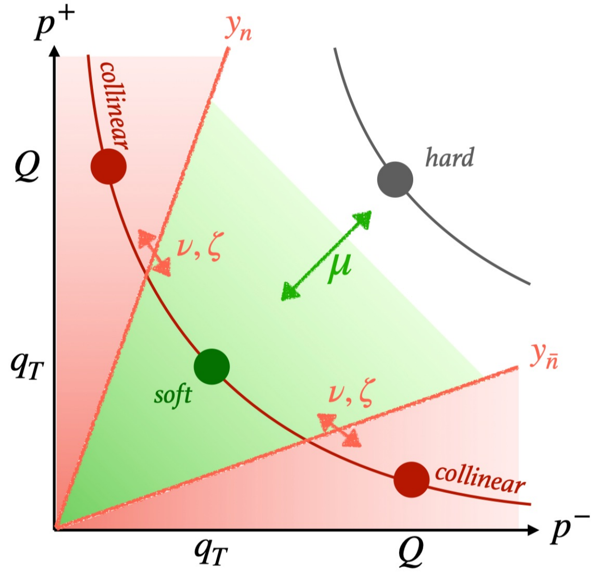
- Rapidity scale evolution of TMDs governed by the Collins-Soper equation:

$$2\zeta\frac{d}{d\zeta}\ln f^{\mathrm{TMD}}(x, b_\perp, \mu, \zeta) \;=\; K(b_\perp, \mu)$$

- By canceling momentum-independent intrinsic soft function S_I and light-cone TMDs f, Collins–Soper kernel can be extracted from the ratio of unsubtracted quasi TMDs:

$$K(x, b_\perp) \;=\; \frac{1}{\ln(P_1^z/P_2^z)}\ln\left(\frac{H(xP_2^z, \mu)\tilde{f}(x, b_\perp, \mu, P_1^z)}{H(xP_1^z, \mu)\tilde{f}(x, b_\perp, \mu, P_2^z)}\right)$$

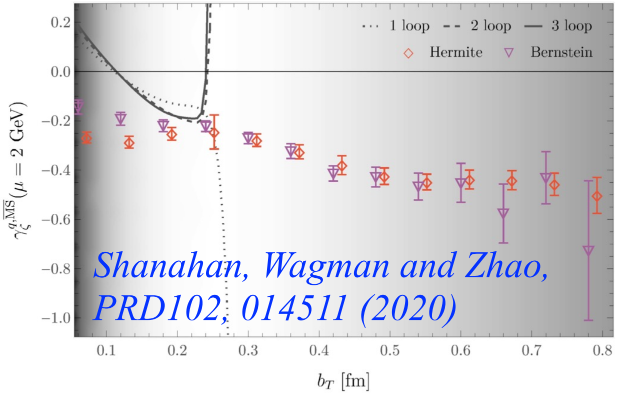
$\tilde{f}$ could be either unsubtracted quasi TMD wave function or quasi TMDPDF (beam function).



Highlights of Lattice QCD Results on the Collins-Soper Kernel

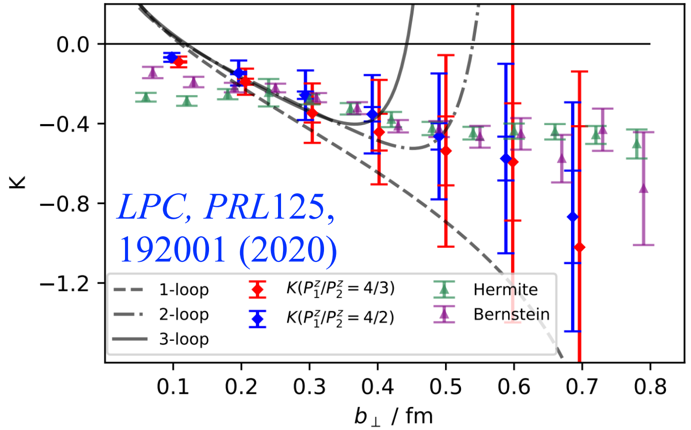
The Origins: Two Routes to the Collins–Soper Kernel

✓ **From quasi beam function**



- Quenched ensemble, single a = 0.06 fm;
- Nonphysical pion mass ~ $m_\pi \simeq$ 1.2 GeV;
- Tree-level matching kernel;
- Largest P^z up to 2.58 GeV.

✓ **From quasi TMDWF**



- 2+1 flavor clover fermion ensemble, a=0.098fm;
- Nonphysical pion mass ~ m_π=547 MeV;
- Tree-level matching kernel;
- Largest P^z up to 2.11 GeV.

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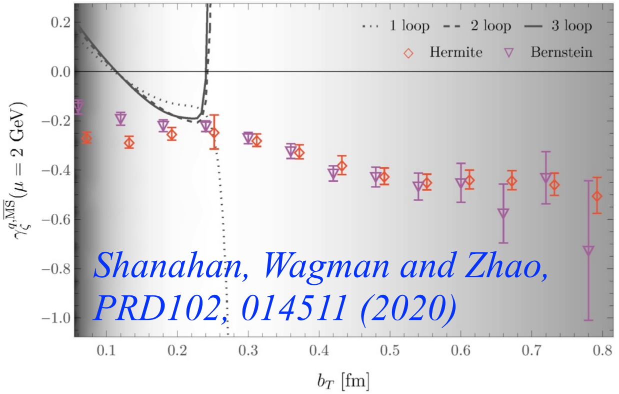

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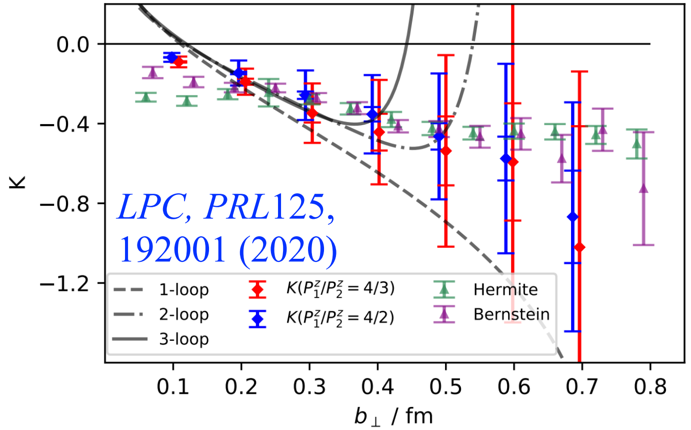
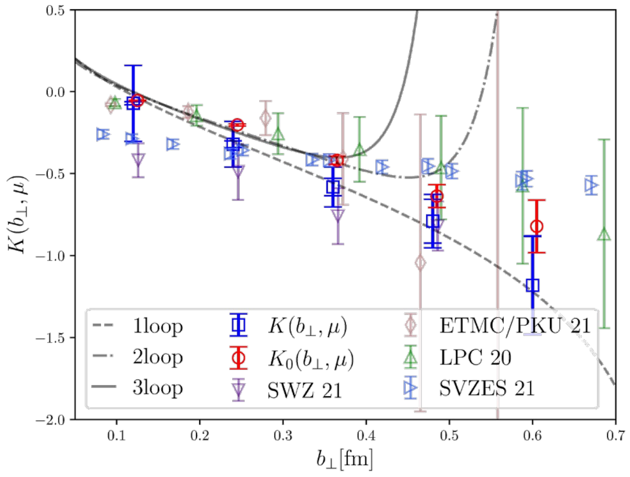


- 2+1 flavor clover fermion ensemble, a=0.098fm;
- Nonphysical pion mass ~ m_π=547 MeV;
- Tree-level matching kernel;
- Largest P^z up to 2.11 GeV.

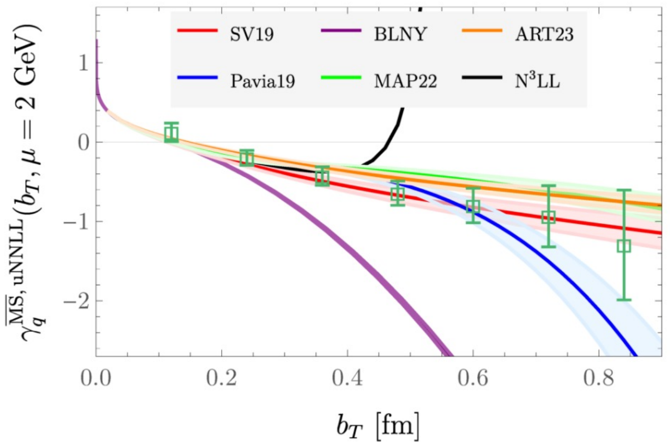
- 2pts have better signals than 3pts on lattice, so with increasing precision, <u>the quasi TMDWF approach is becoming mainstream.</u>

Highlights of Lattice QCD Results on the Collins-Soper Kernel

Continuous Improvements:



LPC, PRD106, 034509 (2022)



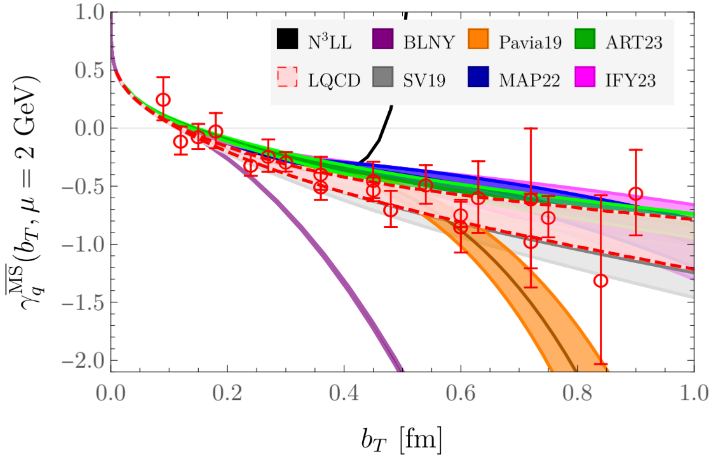
Avkhadiev, Shanahan, Wagman and Zhao, PRD108, 114505 (2023)

- ✓ Better NPR scheme for nonlocal correlators;
- ✓ **Infinite P^z extrapolation**;
- ✓ Systematic uncertainties taken into account;
- ✗ **Unexpected imaginary part from NLO matching kernel.**

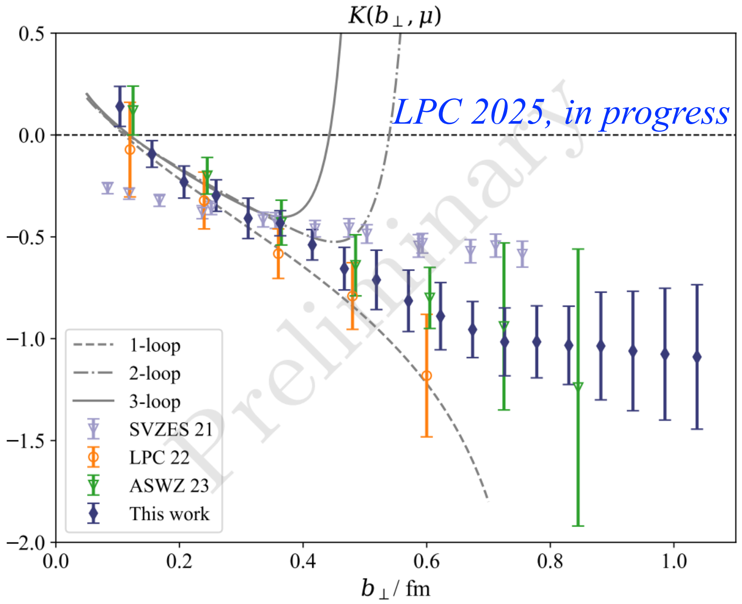
- ✓ $b_\perp$-unexpanded matching coefficient **mitigate the unexpected imaginary part**.
- ✓ Perturbative matching up to NNLL accuracy.

Highlights of Lattice QCD Results on the Collins-Soper Kernel

Continuous Improvements:



Avkhadiev, Shanahan, Wagman and Zhao, PRL132, 231901 (2024)



- ✓ **Physical pion mass;**
- ✓ **Continuum limit** from model constrain;
- ✓ Precision results with **well-controlled systematic uncertainties.**

- ✓ **Physical pion mass;**
- ✓ **Continuum limit;**
- ✓ Largest P^z up to 3GeV;
- ✓ Well-controlled systematic uncertainties;
- ✓ Precision calculation up to **1fm**.

See Jin-Xin's talk

Overview of Collins-Soper Kernel Calculations on the Lattice

Authors/ Collaboration	*Pion mass*	*Continuum*	*Method*	*Matching Kernel*	*Largest P^z*	*Reliable $b_\perp^{\max}$*	*Systematic Uncertainty*	*Highlights*
SWZ PRD102, 2020	1.2 GeV	No	Quasi beam function	LO	2.58 GeV	0.8 fm	No	• First CS kernel extraction via quasi beam function. • Quenched ensemble. • Long-range extrapolation via models.
LPC PRL125, 2020	547 MeV	No	Quasi TMDWF	LO	2.11 GeV	0.4 fm	No	• First extraction via quasi TMDWF. • Dynamical ensemble.
SVZES JHEP08, 2021	422 MeV	No	Mellin moments of quasi TMDPDF	--	2.27 GeV	0.5 fm	Estimated but not included	• Extraction from the ratios of first Mellin moments. • Discussed the sources of systematic uncertainties.
SWZ PRD104, 2021	538 MeV	No	Quasi beam function	NLO	1.5 GeV	0.5 fm	No	• Operator mixing effect in NPR. • Power corrections and higher-twist effects. • Considering NLO matching effects.
PKU/ETMC PRL128, 2022	827 MeV	No	Quasi TMDWF	LO	3.3 GeV	0.3 fm	No	• RI'/MOM renormalization.
LPC PRD106, 2022	670 MeV	No	Quasi TMDWF	NLO	2.58 GeV	0.6 fm	Yes	• Wilson loop renormalization. • Higher-twist effects in operators. • Direction of staple-shaped Wilson link. • Infinite-P^z extrapolation. • Systematic uncertainties are taken into account.
S⁴VWEY PRD108, 2023	686 MeV	No	Mellin moments of quasi TMDPDF	--	1.36 GeV	0.4 fm	No	• Universality of CS kernel from twist-2/3 quasi TMDPDFs of pion and proton

Overview of Collins-Soper Kernel Calculations on the Lattice

Authors/ Collaboration	*Pion mass*	*Continuum*	*Method*	*Matching Kernel*	*Largest P^z*	*Reliable $b_\perp^{max}$*	*Systematic Uncertainty*	*Highlights*
LPC JHEP08, 2023	220 MeV	No	Quasi TMDWF	NLO	3.16 GeV	0.6 fm	Yes	• Consistency check between two different actions. • Multiple lattice spacings and pion masses, but without extrapolation.
ASWZ PRD108, 2023	148.8 MeV	No	Quasi TMDWF	uNNLL	2.15GeV	0.8 fm	Yes	• Physical pion mass. • $b_\perp$-unexpanded matching coefficient up to NNLL. • Mitigate unexpected imaginary part in matching coefficient.
ASWZ PRL132, 2024	148.8 MeV	Model	Quasi TMDWF	uNNLL	2.3GeV	0.8 fm	Yes	• Multiple lattice spacings, and model constrained continuum limit extrapolation. • Calculation directly at physical pion mass.
BGHMZ PRD112, 2025	300 MeV	No	Quasi TMDWF	NLO	4.3 GeV	-- (Large err.)	Yes	• Coulomb gauge fixed correlators.
ABC^{2}S^{3}T 2509.26316	640 MeV	No	Quasi TMDWF	NLO	3.33GeV	0.5 fm	No	• Higher-twist effects in quasi TMDWF.
LPC In preparation	135.5 MeV	Yes	Quasi TMDWF	uNLO	3.0 GeV	1.0 fm	Yes	• Precision calculation at transverse separations beyond the nucleon radius ($b_\perp \approx 1$fm). • Multiple lattice spacings, continuum limit extrapolation. • Multiple pion mass (including physical), and chiral extrapolation.

Adapted from Yong Zhao

Intrinsic Soft Function

- Rapidity divergence from gluons radiation collinear to lightlike gauge link:

$$I_{\rm div} \sim \int dk^+ dk^- \frac{f(k^+k^-)}{(k^+k^-)^{1+\varepsilon}} = \frac{1}{2}\int \boxed{\frac{d(k^-/k^+)}{k^-/k^+}} \int d(k^+k^-)\frac{f(k^+k^-)}{(k^+k^-)^{1+\varepsilon}}$$

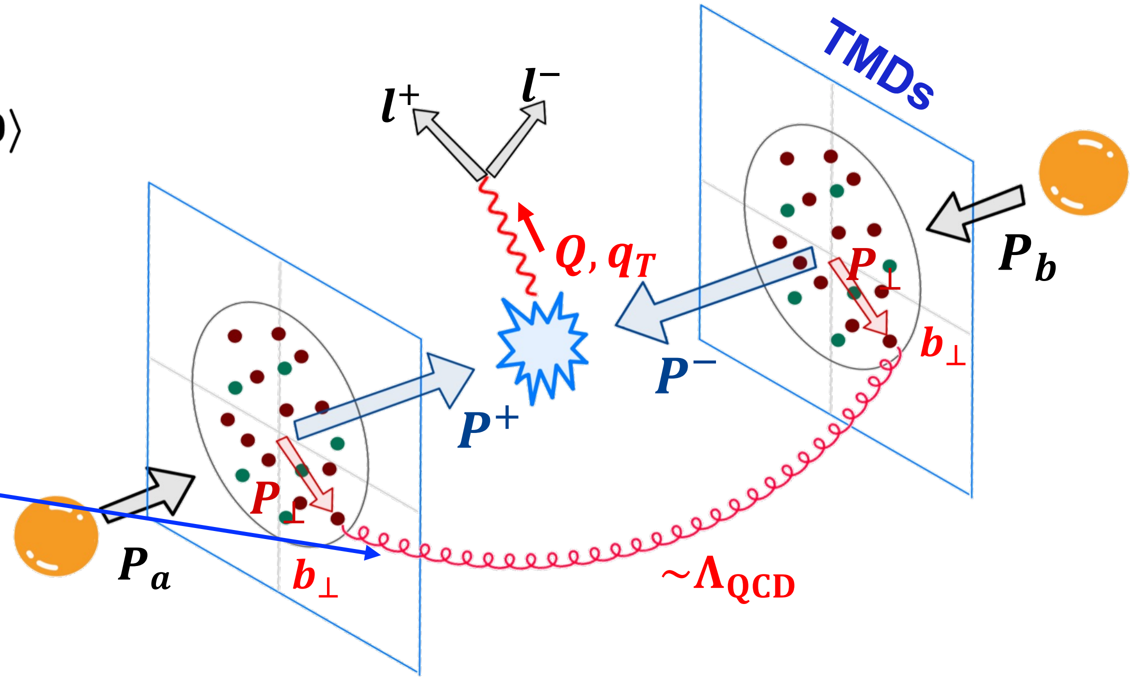
$y_k = \frac{1}{2}\ln\frac{k^+}{k^-}$ $\quad y_k \to \pm\infty$: rapidity divergence

renormalized by soft function:

$$S(b_\perp, \mu, \delta^+, \delta^-) = \frac{1}{N_c}\mathrm{Tr}\langle 0|W_n(\vec{b}_\perp)|_{\delta^+} W_p^\dagger(\vec{b}_\perp)|_{\delta^-}|0\rangle$$

- Physical TMDs are defined as

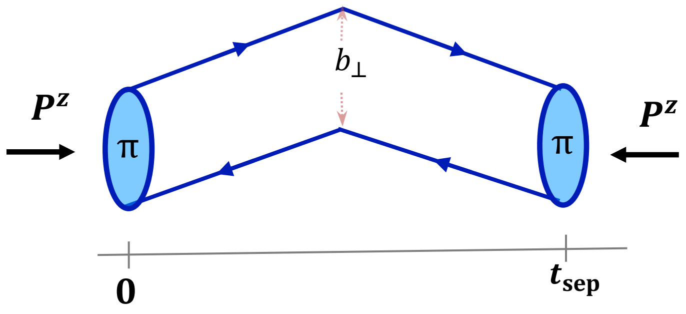
$$f^{\rm TMD}(x, b_\perp, \mu, \zeta) = \lim_{\delta^-\to 0}\frac{f(\lambda, b_\perp, \mu, \delta^-/P^+)}{\boxed{\sqrt{S(b_\perp, \mu, \delta^- e^{2y_n}, \delta^-)}}}$$



Intrinsic Soft Function from Lattice QCD within LaMET

- Four-quark form factor with large P^z:

$$F(b_\perp, P^z) = \langle \pi(P^z) | (\bar{q}_1 \Gamma q_1)(b_\perp)(\bar{q}_2 \Gamma q_2)(0) | \pi(P^z) \rangle$$



- Factorization within LaMET: *Ji, Liu, Liu, Nucl. Phys. B 955 (2020) 115054*

$$F(b_\perp, P^z) = \int dx\, dx'\, H(x, x', P^z) \frac{\tilde{\phi}(x, \bar{Y}, P^z, b_\perp)}{S(Y, \bar{Y}, b_\perp)} \frac{\tilde{\phi}^\dagger(x', \bar{Y}', P^z, b_\perp)}{S(Y', \bar{Y}', b_\perp)} S(Y, Y', b_\perp)$$

perturbative matching kernel

subtracted quasi-TMD wave function

$$\tilde{\phi}(z, P^z, b_\perp) = \lim_{L\to\infty} \frac{\langle 0 | \bar{q}(z\hat{n}_z + b_\perp \hat{n}_\perp)\gamma^t \gamma_5 U_\sqsupset(z\hat{n}_z + b_\perp \hat{n}_\perp, 0; L) q(0) | \pi(P^z) \rangle}{\sqrt{Z_E(2L + z, b_\perp)}}$$

- Intrinsic soft function:

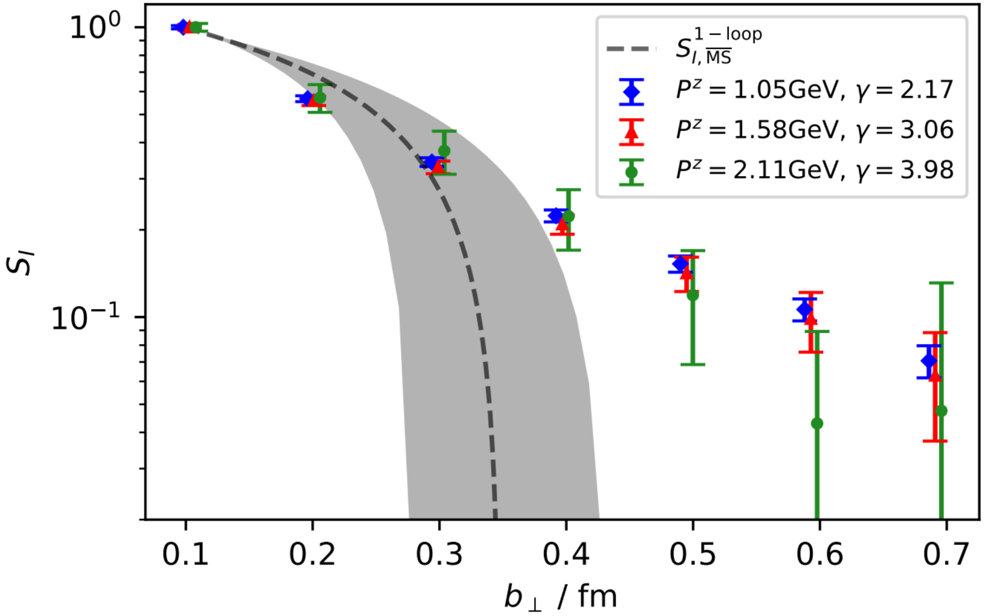
$$S_I(b_\perp, \mu) = \frac{S(b_\perp, \mu, Y, Y')}{S(b_\perp, \mu, Y, 0) S(b_\perp, \mu, 0, Y')}$$

$$= \frac{F(b_\perp, P^z)}{\int dx\, dx'\, H(x, x', P^z) \tilde{\phi}(x, \bar{Y}, P^z, b_\perp) \tilde{\phi}^\dagger(x', \bar{Y}', P^z, b_\perp)}$$

Lattice QCD calculable.

Highlights of Lattice QCD Results on the Intrinsic Soft Function

First Lattice Realization of the Intrinsic Soft Function:

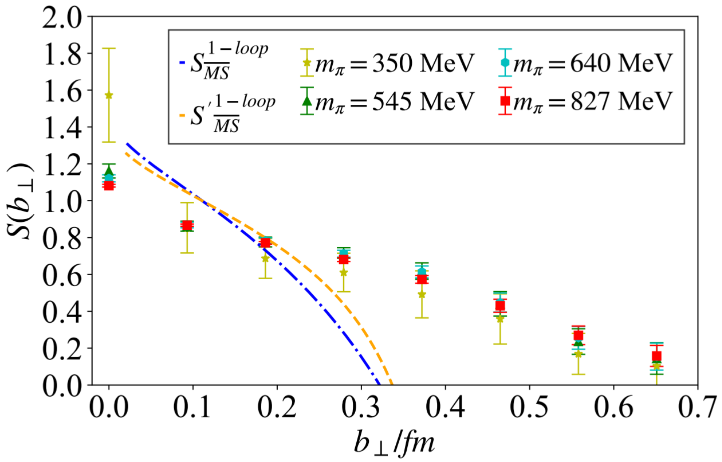


*LPC, PRL*125, 192001 (2020)

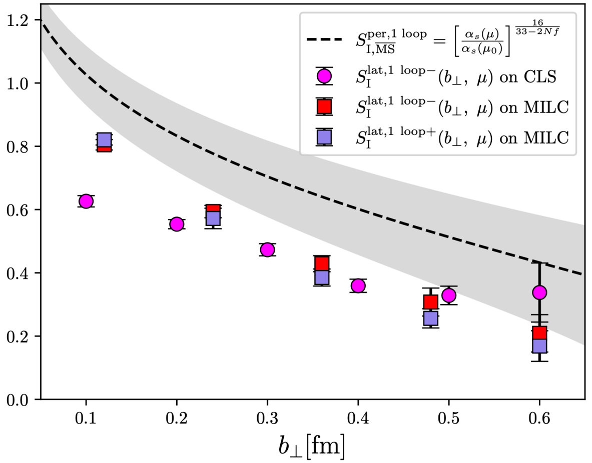
- ✓ 2+1 flavor clover fermion CLS ensemble, a=0.098 fm;
- ✓ Nonphysical pion mass ~ m_π=547 MeV;
- ✓ LO matching kernel;
- ✓ Largest P^z up to 2.11 GeV;
- ✓ Partial systematic uncertainties have been taken into account.

Highlights of Lattice QCD Results on the Intrinsic Soft Function

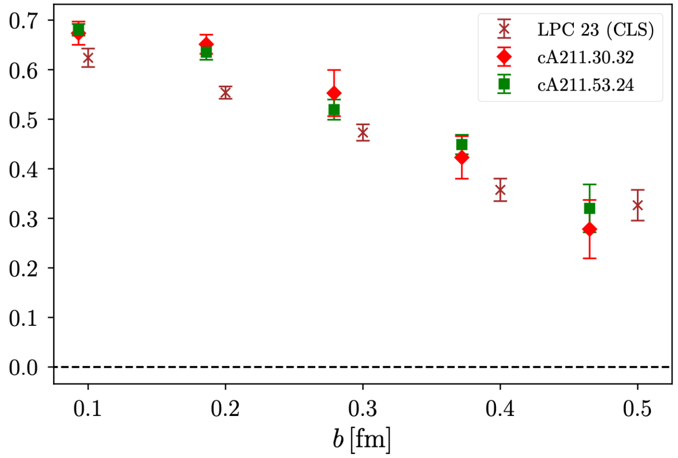
Continuous Improvements:



Li et.al, PRL128, 062002 (2022)



LPC, JHEP08, 172 (2023)



Alexandrou et.al, arXiv:2509.26316

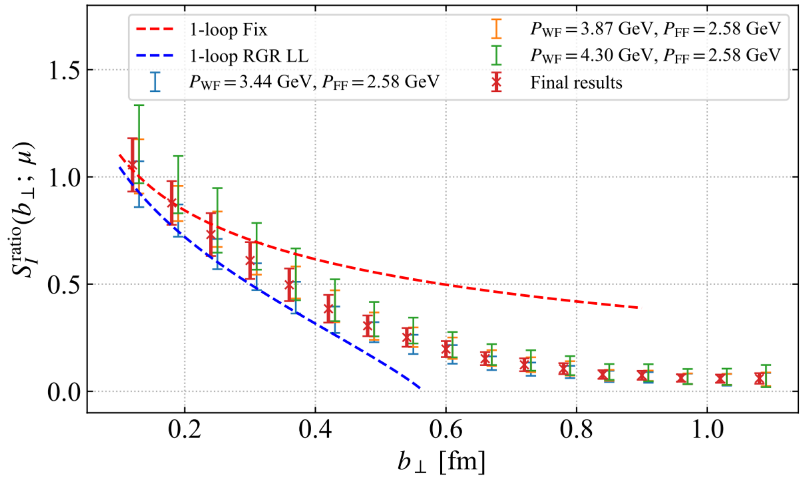
- ✓ Eliminated higher-twist contaminations in form factors from combinations of different Dirac structures;
- ✓ Verified that the intrinsic soft function is essentially independent of the meson mass.
- ✓ Verified the robustness of the results from different ensembles.
- ✓ Considering higher-twist effects in quasi TMDWF.

🤔 The key obstacle lies in improving the signal-to-noise ratio of the form factor.

Highlights of Lattice QCD Results on the Intrinsic Soft Function

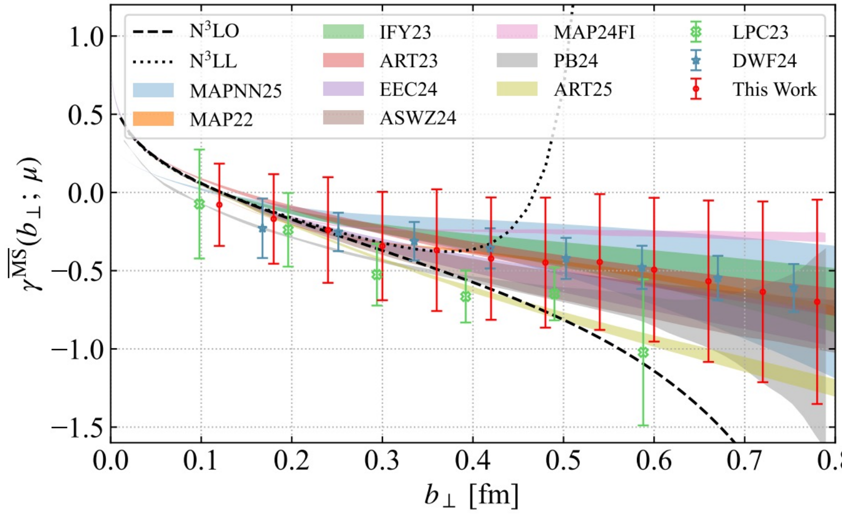
Continuous Improvements:

- Under Coulomb gauge fixing, Wilson lines vanish, signal-to-noise ratio can be significantly improved. *Yong Zhao, Phys.Rev.Lett. 133 , 241904 (2024)*



- Preliminary results for the Collins-Soper kernel and intrinsic soft function have been performed.

Bollweg, Gao, He, Mukherjee, Zhao, PRD 112, 034501 (2025).

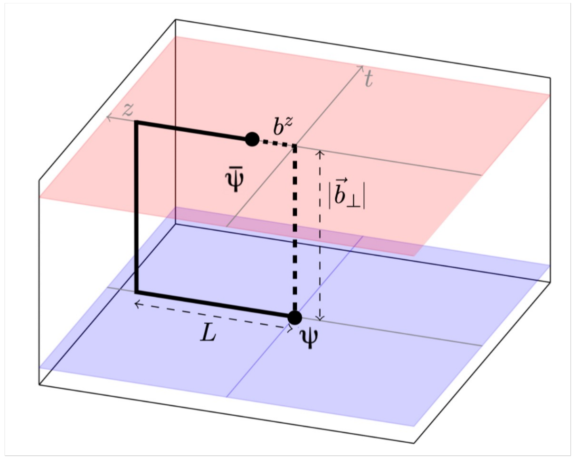


- The reliability of this approach still requires further verification.

Overview of Intrinsic Soft Function Calculations on the Lattice

Authors/ Collaboration	*Pion mass*	*Continuum*	*Matching Kernel*	*Largest P^z*	*Reliable $b_\perp^{max}$*	*Systematic Uncertainty*	*Highlights*
LPC PRL125, 2020	547 MeV	No	LO	2.11 GeV	0.7 fm	No	• First lattice QCD calculation. • Operator mixing effects in NPR.
PKU/ETMC PRL128, 2022	350-827 MeV	No	LO	3.3 GeV	0.7 fm	No	• Higher-twist effects in form factors. • Verified the independent of meson mass.
LPC JHEP08, 2023	670 MeV	No	NLO	2.58 GeV	0.6 fm	No	• Consistency check between different actions.
BGHMZ PRD112, 2025	300 MeV	No	NLO	2.58 GeV for FF, 4.3 GeV for qWF	1.1 fm	No	• Coulomb gauge fixed correlators. • Greatly improved the signal-to-noise ratio problem.
ABC^2S^3T 2509.26316	640 MeV	No	NLO	3.33 GeV	0.5 fm	No	• Higher-twist effects in quasi TMDWF.
LPC In progress	200-300 MeV	No	NLO	3GeV	1 fm	Yes	

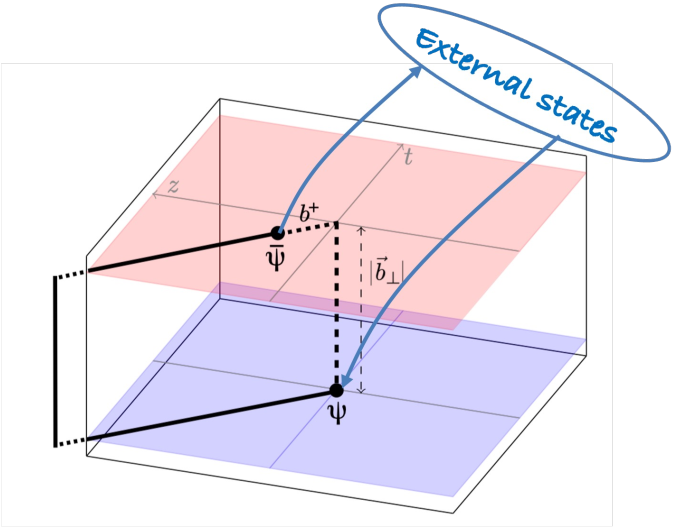
TMDs from Lattice QCD within LaMET



Lorentz boost

$L \to \infty$

$P^z \to \infty$



Factorization in LaMET:

Ji, PLB811(2020); Ebert, JHEP04(2022)

✓ Perturbative matching kernel

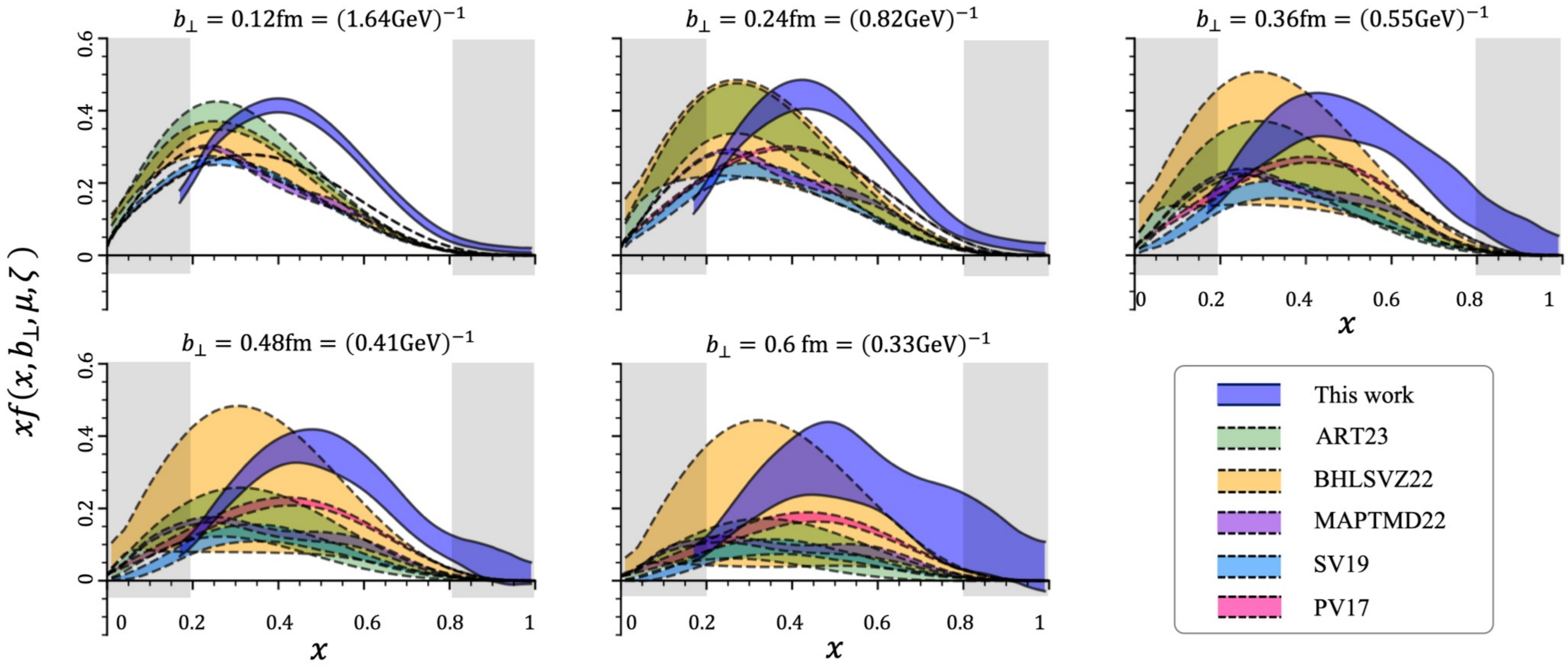
$$\tilde{f}_\Gamma(x, b_\perp, \mu, \zeta^z)\boxed{\sqrt{S_I(b_\perp, \mu)}} = \boxed{H_\Gamma(\zeta^z, \mu)}\exp\left[\frac{1}{2}\boxed{K(b_\perp, \mu)}\ln\frac{\zeta^z}{\zeta}\right]f(x, b_\perp, \mu, \zeta) + \mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{\zeta^z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2\zeta^z}\right)$$

✓ Intrinsic soft function ✓ Collins-Soper kernel

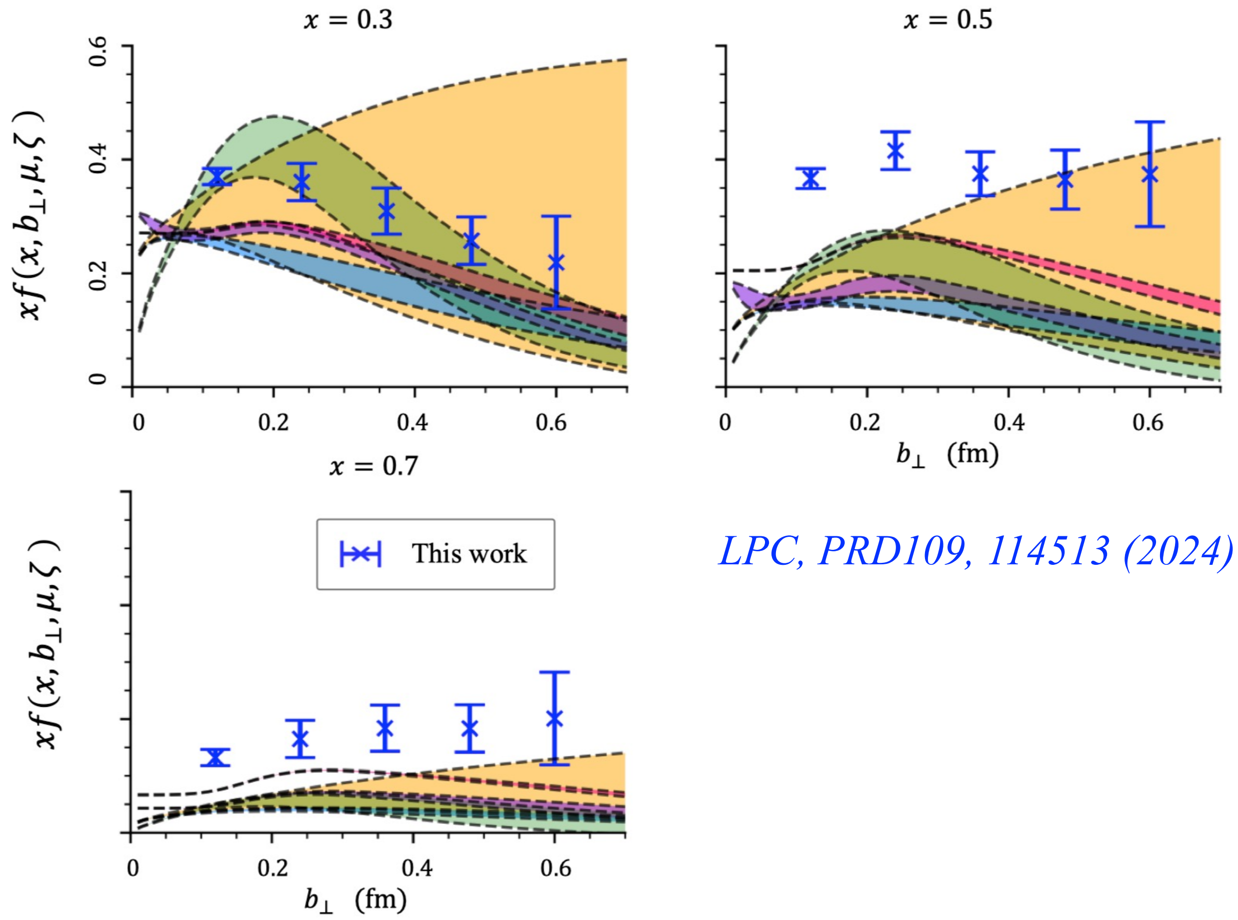
Everything is ready……

Nucleon Unpolarized TMDPDF

- **The first lattice QCD calculation of nucleon unpolarized TMDPDF:** *LPC, PRD109, 114513 (2024)*



Nucleon Unpolarized TMDPDF

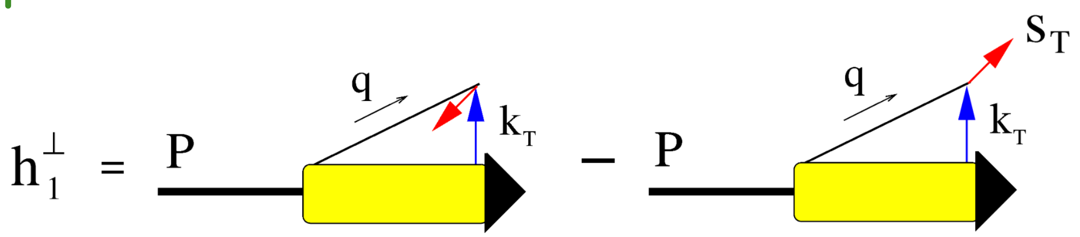


LPC, PRD109, 114513 (2024)

- ✓ MILC ensemble, $a = 0.12$ fm;
- ✓ $m_\pi = (220, 310)$ MeV, and **extrapolate to physical pion mass**;
- ✓ Largest P^z up to 2.58 GeV, and **infinite momentum extrapolation;**
- ✓ NNLO + RGR matching kernel;
- ✓ **Systematic uncertainties.**

Boer-Mulders Functions

- The Boer-Mulders function is related to the probability of finding a transversely polarized quark inside an unpolarized proton:



- The Boer-Mulders function is time-reversal odd (T-odd):

$$\mathcal{T}: \quad h_1^\perp(x, k_\perp^2) \;\to\; -h_1^\perp(x, k_\perp^2)$$

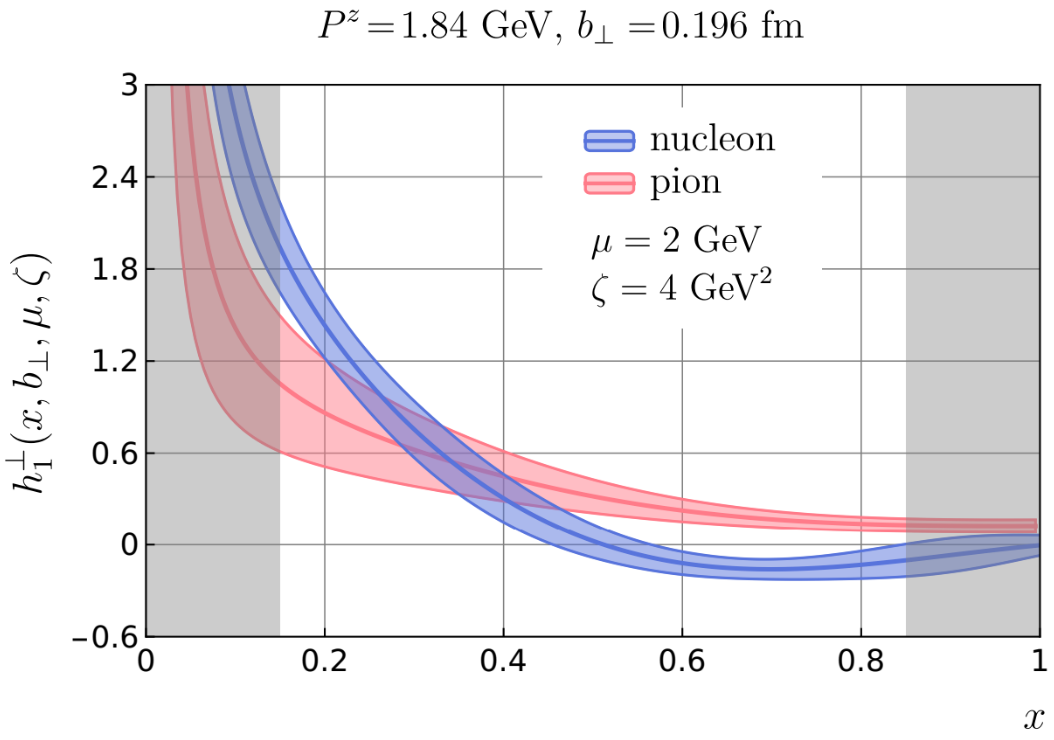
- The Boer-Mulders function inbeds the correlation between the quark spin and its transverse momentum.

Boer-Mulders Functions

- **Boer-Mulders function from nucleon and pion:**

Pion: LPC, PRD111, 094507 (2025)

Nucleon: LPC, JHEP 08, 086 (2025)

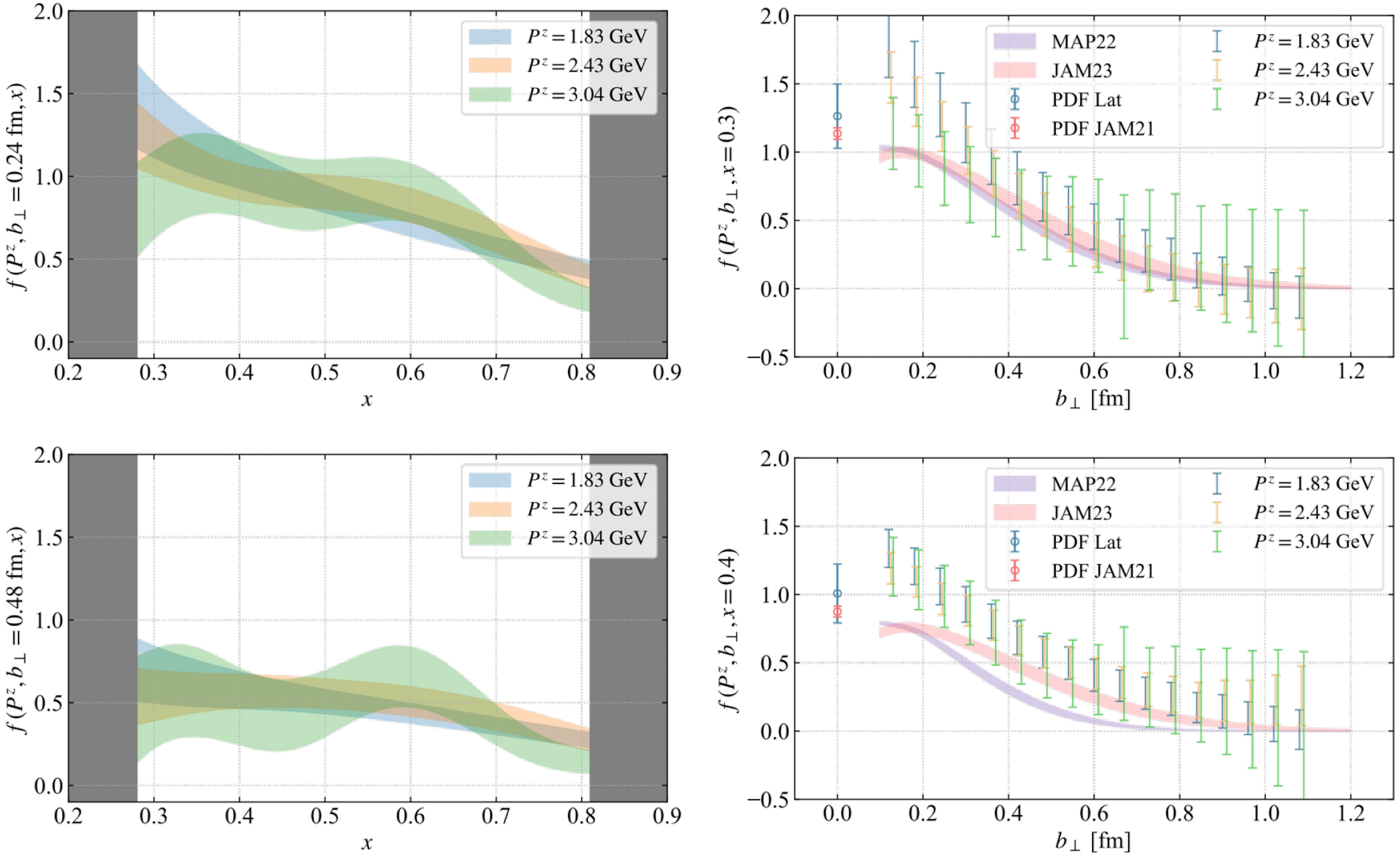


- ✓ CLS ensemble, a = 0.098 fm;
- ✓ m_π = 338 MeV;
- ✓ Largest P^z up to 2.11 GeV;
- ✓ NNLO + RGR matching kernel.

More details see Lingquan's talk

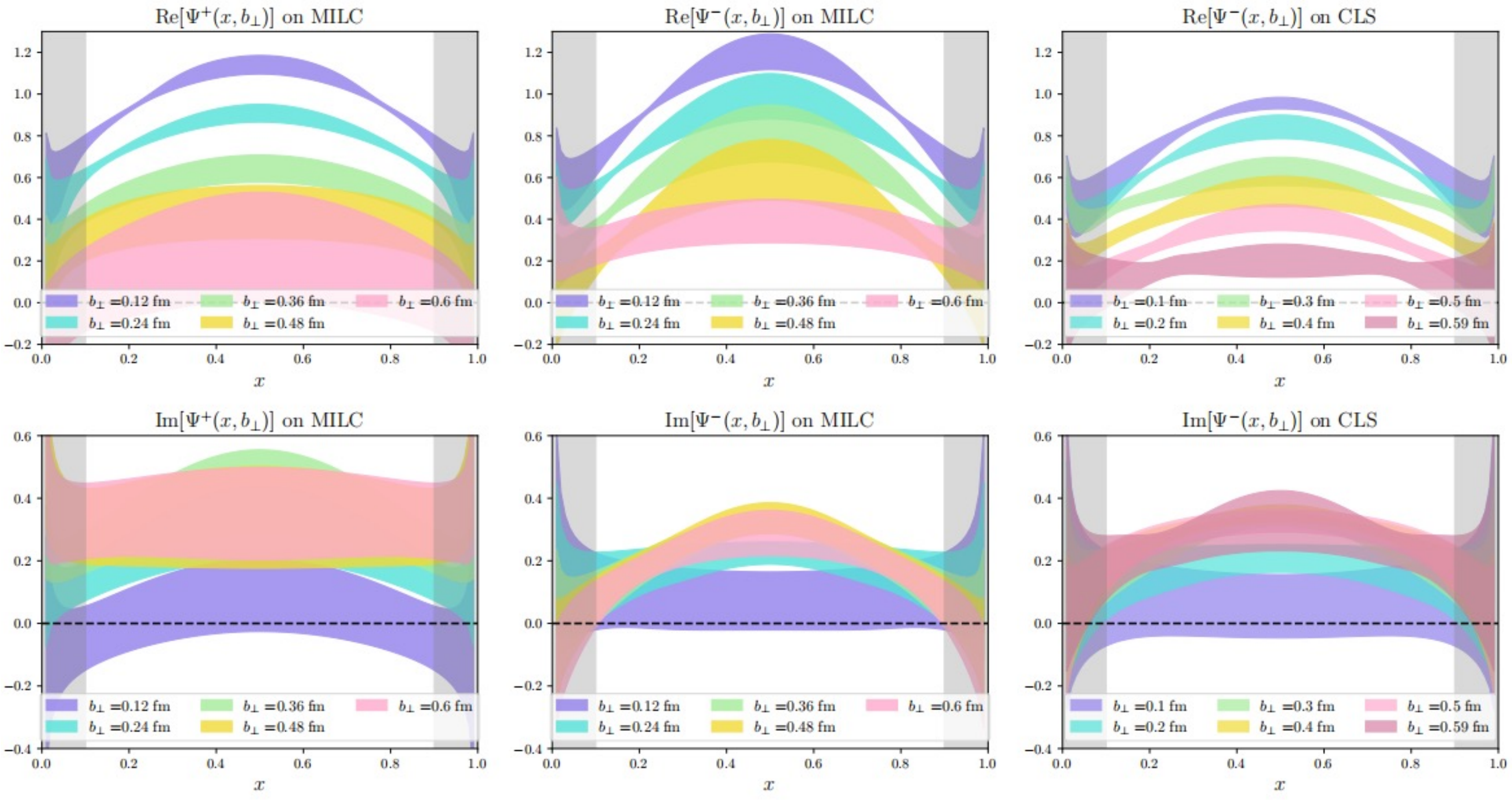
Pion Unpolarized TMDPDF

Bollweg, Gao, He, Mukherjee, Zhao, PRD 112, 034501 (2025).



- HotQCD ensemble, a = 0.06 fm, $m_\pi \simeq 300$ MeV;
- Largest P^z up to 3.04 GeV;
- **Coulomb gauge fixed correlators.**

Pion TMD Wave Function



LPC, PRD 109, L091503 (2024)

- CLS, a = 0.098 fm, and MILC, a = 0.12 fm.
- $m_\pi \simeq$ 670 MeV;
- Largest P^z up to 2.64 GeV;

Overview of Intrinsic Soft Function Calculations on the Lattice

	Authors/ Collaboration	*Pion mass*	*Continuum*	*Matching Kernel*	*Largest P^z*	*Systematic Uncertainty*
Nucleon Unpolarized TMDPDF	**LPC** PRD109, 2024	310, 220 MeV (extra to continuum)	No	NNLO	2.58 GeV	Yes
Nucleon Boer-Mulders Function	**LPC** JHEP08, 2025	338 MeV	No	NNLO	2.21 GeV	No
Pion Unpolarized TMDPDF	**BGHMZ** PRD112, 2025	300 MeV	No	NLO	3.04 GeV	No
Pion Boer-Mulders Function	**LPC** PRD111, 2025	338 MeV	No	NNLO	2.21 GeV	No
Pion TMD Wave Function	**LPC** PRD109, 2024	670 MeV	No	NLO	2.64 GeV	No

The Grand Feast of Lattice TMDs Has Just Begun……

Lattice QCD + LaMET provides a new perspective on exploring the 3-D structure of hadrons.

Collins-Soper Kernel:

- ✓ Continuum limit
- ✓ Physical pion mass
- ✓ Power corrections
- ✓ Systematic uncertainties
- ✓ Reliable predictions ~ 1fm

Intrinsic Soft Function:

- X Continuum limit
- X Physical pion mass
- X Power corrections
- X Systematic uncertainties
- X Reliable predictions

TMD distributions:

- X Continuum limit
- ⍻ Physical pion mass
- ⍻ Power corrections
- ⍻ Systematic uncertainties
- X Reliable predictions ~ 1fm

- ☐ Achieve high-precision calculations (extrapolation to physical point, well-controlled systematic uncertainties, b~1fm);
- ☐ Enable computations of more polarized TMDPDFs;
- ☐ New methods, new observables, ……

Thank you for your attention!

BACKUP SLIDES

Overview of Collins-Soper Kernel Calculations on the Lattice

Authors/ Collaboration	*Pion mass*	*Continuum*	*Method*	*Matching Kernel*	*Largest P^z*	*Reliable $b_\perp^{\max}$*	*Systematic Uncertainty*	*Highlights*
SWZ PRD102, 2020	1.2 GeV	No	Quasi beam function	LO	2.58 GeV	0.8 fm	No	• First CS kernel extraction via quasi beam function. • Quenched ensemble. • Long-range extrapolation via models.
LPC PRL125, 2020	547 MeV	No	Quasi TMDWF	LO	2.11 GeV	0.4 fm	No	• First extraction via quasi TMDWF. • Dynamical ensemble.
SVZES JHEP08, 2021	422 MeV	No	Mellin moments of quasi TMDPDF	--	2.27 GeV	0.5 fm	Estimated but not included	• Extraction from the ratios of first Mellin moments. • Discussed the sources of systematic uncertainties.
SWZ PRD104, 2021	538 MeV	No	Quasi beam function	NLO	1.5 GeV	0.5 fm	No	• Operator mixing effect in NPR. • Power corrections and higher-twist effects. • Considering NLO matching effects.
PKU/ETMC PRL128, 2022	827 MeV	No	Quasi TMDWF	LO	3.3 GeV	0.3 fm	No	• RI'/MOM renormalization.
LPC PRD106, 2022	670 MeV	No	Quasi TMDWF	NLO	2.58 GeV	0.6 fm	Yes	• Wilson loop renormalization. • Higher-twist effects in operators. • Direction of staple-shaped Wilson link. • Infinite-P^z extrapolation. • Systematic uncertainties are taken into account.
S⁴VWEY PRD108, 2023	686 MeV	No	Mellin moments of quasi TMDPDF	--	1.36 GeV	0.4 fm	No	• Universality of CS kernel from twist-2/3 quasi TMDPDFs of pion and proton

Overview of Collins-Soper Kernel Calculations on the Lattice

Authors/ Collaboration	*Pion mass*	*Continuum*	*Method*	*Matching Kernel*	*Largest P^z*	*Reliable $b_\perp^{\max}$*	*Systematic Uncertainty*	*Highlights*
LPC JHEP08, 2023	220 MeV	No	Quasi TMDWF	NLO	3.16 GeV	0.6 fm	Yes	• Consistency check between two different actions. • Multiple lattice spacings and pion masses, but without extrapolation.
ASWZ PRD108, 2023	148.8 MeV	No	Quasi TMDWF	uNNLL	2.15GeV	0.8 fm	Yes	• Physical pion mass. • $b_\perp$-unexpanded matching coefficient up to NNLL. • Mitigate unexpected imaginary part in matching coefficient.
ASWZ PRL132, 2024	148.8 MeV	Model	Quasi TMDWF	uNNLL	2.3GeV	0.8 fm	Yes	• Multiple lattice spacings, and model constrained continuum limit extrapolation. • Calculation directly at physical pion mass.
BGHMZ PRD112, 2025	300 MeV	No	Quasi TMDWF	NLO	4.3 GeV	-- (Large err.)	Yes	• Coulomb gauge fixed correlators.
ABC^{2}S^{3}T 2509.26316	640 MeV	No	Quasi TMDWF	NLO	3.33GeV	0.5 fm	No	• Higher-twist effects in quasi TMDWF.
LPC In preparation	135.5 MeV	Yes	Quasi TMDWF	uNLO	3.0 GeV	1.0 fm	Yes	• High-precision lattice calculation at transverse separations beyond the nucleon radius ($b_\perp \approx 1$fm). • Multiple lattice spacings, continuum limit extrapolation. • Multiple pion mass (including physical), and chiral extrapolation.

Overview of Intrinsic Soft Function Calculations on the Lattice

Authors/ Collaboration	*Pion mass*	*Continuum*	*Matching Kernel*	*Largest P^z*	*Reliable $b_\perp^{max}$*	*Systematic Uncertainty*	*Highlights*
LPC PRL125, 2020	547 MeV	No	LO	2.11 GeV	0.7 fm	No	• First lattice QCD calculation. • Operator mixing effects in NPR.
PKU/ETMC PRL128, 2022	350-827 MeV	No	LO	3.3 GeV	0.7 fm	No	• Higher-twist effects in form factors. • Verified the independent of meson mass.
LPC JHEP08, 2023	670 MeV	No	NLO	2.58 GeV	0.6 fm	No	• Consistency check between different actions.
BGHMZ PRD112, 2025	300 MeV	No	NLO	2.58 GeV for FF, 4.3 GeV for qWF	1.1 fm	No	• Coulomb gauge fixed correlators. • Greatly improved the signal-to-noise ratio problem.
ABC^{2}S^{3}T 2509.26316	640 MeV	No	NLO	3.33 GeV	0.5 fm	No	• Higher-twist effects in quasi TMDWF.
LPC In progress	200-300 MeV	No	NLO	3GeV	1 fm	Yes	

Overview of Intrinsic Soft Function Calculations on the Lattice

	Authors/ Collaboration	*Pion mass*	*Continuum*	*Matching Kernel*	*Largest P^z*	*Systematic Uncertainty*
Nucleon Unpolarized TMDPDF	**LPC** PRD109, 2024	310, 220 MeV (extra to continuum)	No	NNLO	2.58 GeV	Yes
Nucleon Boer-Mulders Function	**LPC** JHEP08, 2025	338 MeV	No	NNLO	2.21 GeV	No
Pion Unpolarized TMDPDF	**BGHMZ** PRD112, 2025	300 MeV	No	NLO	3.04 GeV	No
Pion Boer-Mulders Function	**LPC** PRD111, 2025	338 MeV	No	NNLO	2.21 GeV	No
Pion TMD Wave Function	**LPC** PRD109, 2024	670 MeV	No	NLO	2.64 GeV	No

References of Collins-Soper Kernel:

- P. Shanahan, M. Wagman and Y. Zhao (SWZ), Phys. Rev. D 102, no.1, 014511 (2020).
- QAZ, *et al.* (LPC), Phys. Rev. Lett. 125, no.19, 192001 (2020).
- M. Schlemmer, A. Vladimirov, C. Zimmermann, M. Engelhardt and A. Schafer (SVZES), JHEP 08, 004 (2021).
- P. Shanahan, M. Wagman and Y. Zhao (SWZ), Phys. Rev. D 104, no.11, 114502 (2021).
- Y. Li, *et al.* (PKU/ETMC), Phys. Rev. Lett. 128, no.6, 062002 (2022).
- M. H. Chu, *et al.* (LPC), Phys. Rev. D 106, no.3, 034509 (2022).
- M. H. Chu, *et al.* (LPC), JHEP 08, 172 (2023).
- H. T. Shu, M. Schlemmer, T. Sizmann, A. Vladimirov, L. Walter, M. Engelhardt, A. Schafer and Y. B. Yang (S^4VWEY), Phys. Rev. D 108, no.7, 074519 (2023).
- A. Avkhadiev, P. Shanahan, M. Wagman and Y. Zhao (ASWZ), Phys. Rev. D 108, no.11, 114505 (2023).
- A. Avkhadiev, P. Shanahan, M. Wagman and Y. Zhao (ASWZ), Phys. Rev. Lett. 132, no.23, 231901 (2024).
- C. Alexandrou, S. Bacchio, K. Cichy, M. Constantinou, A. Sen, G. Spanoudes, F. Steffens and J. Tarello (ABC^2S^3T), arXiv:2509.26316 [hep-lat].
- J. X. Tan, *et al.* (LPC), in preparition.

References of Intrinsic Soft Function:

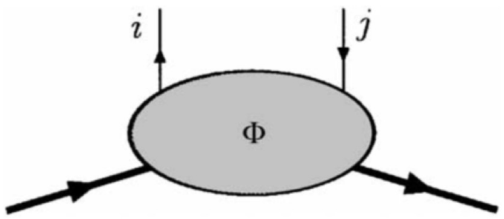
- QAZ, *et al.* (LPC), Phys. Rev. Lett. 125, no.19, 192001 (2020).
- Y. Li, *et al.* (PKU/ETMC), Phys. Rev. Lett. 128, no.6, 062002 (2022).
- M. H. Chu, *et al.* (LPC), JHEP 08, 172 (2023).
- D. Bollweg, X. Gao, J. He, S. Mukherjee and Y. Zhao (BGHMZ), Phys. Rev. D 112, no.3, 034501 (2025).
- C. Alexandrou, S. Bacchio, K. Cichy, M. Constantinou, A. Sen, G. Spanoudes, F. Steffens and J. Tarello (ABC^{2}S^{3}T), arXiv:2509.26316 [hep-lat].

References of TMDs:

- J. He, *et al.* (LPC), Phys. Rev. D. 109, no.11, 114513 (2024).
- L. Ma, *et al.* (LPC), JHEP 08, 086 (2025).
- D. Bollweg, X. Gao, J. He, S. Mukherjee and Y. Zhao (BGHMZ), Phys. Rev. D 112, no.3, 034501 (2025).
- L. Walter, *et al.* (LPC), Phys. Rev. D 111, no.9, 094507 (2025).
- M. H. Chu, *et al.* (LPC), Phys. Rev. D 109, no.9, L091503 (2024).

Leading Power Spin Dependent Quark TMDPDFs

- **The quark distribution functions are obtained by applying certain projection to the quark-quark correlation matrix:**



$$\Phi_{ij}(k,P,s) = \int d^4\xi\, e^{ik\cdot\xi} \langle PS|\bar{\psi}_j(0)W_c(0,\xi)\psi_i(\xi)|PS\rangle$$
$$\mathrm{Tr}[\Gamma\Phi] = \int d^4\xi\, e^{ik\cdot\xi} \langle PS|\bar{\psi}_j(0)\Gamma W_c(0,\xi)\psi_i(\xi)|PS\rangle$$

- **The correlation matrix satisfies following symmetries from hemiticity, parity and time-reversal invariance:**

$$\Phi^\dagger(k,P,S) = \gamma^0\Phi(k,P,S)\gamma^0 \qquad \text{Hemiticity}$$
$$\Phi(k,P,S) = \gamma^0\Phi(\tilde{k},\tilde{P},-\tilde{S})\gamma^0 \qquad \text{Parity}$$
$$\Phi^*(k,P,S) = \gamma_5 C\Phi(\tilde{k},\tilde{P},\tilde{S})C^\dagger\gamma_5 \qquad \text{Time reversal}$$

- **For staple-shaped gauge link (TMD correlations) the last constraint can be relaxed.**

Leading Power Spin Dependent Quark TMDPDFs

- **At leading twist accuracy:**

$$
\begin{aligned}
V^{\mu} &= A_1 P^{\mu} + \frac{1}{M} A_1' \varepsilon^{\mu\nu\rho\sigma} P_{\nu} k_{\perp\rho} S_{\perp\sigma} \\
A^{\mu} &= \lambda_N A_2 P^{\mu} + \frac{1}{M} \tilde{A}_1 k_{\perp} \cdot S_{\perp} P^{\mu} \\
T^{\mu\nu} &= A_3 P^{[\mu} S_{\perp}^{\nu]} + \frac{\lambda_N}{M} \tilde{A}_2 P^{[\mu} k_{\perp}^{\nu]} + \frac{1}{M} \tilde{A}_3 k_{\perp} \cdot S_{\perp} P^{[\mu} k_{\perp}^{\nu]} + \frac{1}{M} A_2' \varepsilon^{\mu\nu\rho\sigma} P_{\rho} k_{\perp\sigma}
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(k, P, S) = \frac{1}{2} \Big\{ & A_1 \not{P} + A_2 \lambda_N \gamma_5 \not{P} + A_3 \not{P} \gamma_5 \not{S}_{\perp} + \tilde{A}_1 k_{\perp} \cdot S_{\perp} \gamma_5 \not{P} + \frac{1}{M} A_1' \varepsilon^{\mu\nu\rho\sigma} \gamma_{\mu} P_{\nu} k_{\perp\rho} S_{\perp\sigma} \\
& + \frac{i}{2M} A_2' \varepsilon^{\mu\nu\rho\sigma} \sigma_{\mu\nu} \gamma_5 P_{\rho} k_{\perp\sigma} + \tilde{A}_2 \frac{\lambda_N}{M} \not{P} \gamma_5 \not{k}_{\perp} + \frac{1}{M^2} \tilde{A}_3 k_{\perp} \cdot S_{\perp} \not{P} \gamma_5 \not{k}_{\perp} \Big\}
\end{aligned}
$$

red terms denote the constrain on time-reversal is relaxed.

Leading Power Spin Dependent Quark TMDPDFs

- **The projections define the eight leading-twist quark TMDPDFs:**

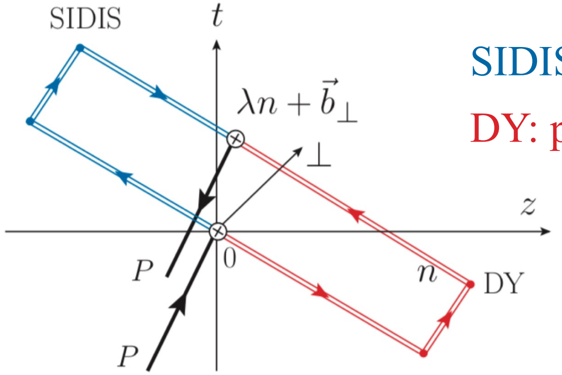
$$\Phi^{[\Gamma]} \equiv \frac{1}{2}\int \frac{dk^+dk^-}{(2\pi)^4}\,\mathrm{Tr}[\Gamma\Phi]\delta(k^+ - xP^+)$$
$$= \int \frac{d\xi^- d^2\xi_\perp}{2(2\pi)^3}\, e^{i(xP^+\xi^- - k_\perp\cdot\xi_\perp)}\langle PS|\bar\psi(0)\Gamma W_c(0,\xi)\psi(\xi)|PS\rangle \qquad \xi = (0,\xi^-,\xi_\perp)$$

then

$$\Phi^{[\gamma^+]} = f_1(x,k_\perp^2) - \frac{\varepsilon_\perp^{ij}k_{\perp i}S_{\perp j}}{M} f_{1T}^\perp(x,k_\perp^2)$$
$$\Phi^{[\gamma^+\gamma_5]} = \lambda_N g_{1L}(x,k_\perp^2) - \frac{k_\perp\cdot S_\perp}{M} g_{1T}^\perp(x,k_\perp^2)$$
$$\Phi^{[i\sigma^{i+}\gamma_5]} = S_\perp^i h_1(x,k_\perp^2) + \frac{\lambda_N}{M}k_\perp^i h_{1L}^\perp(x,k_\perp^2) + \frac{k_\perp^2}{M^2}\Big(\frac{1}{2}g_\perp^{ij} - k_\perp^i k_\perp^j\Big)S_{\perp j}h_{1T}^\perp(x,k_\perp^2)$$
$$- \frac{\varepsilon_\perp^{ij}k_\perp^j}{M} h_1^\perp(x,k_\perp^2)$$

Theoretical Definition of TMDPDFs

Theoretical Complexity of Quark TMDPDF



SIDIS: future-pointing

DY: past-pointing

- **Process-dependence of Wilson lines:**
- **Rapidity divergence from gluons radiation collinear to lightlike gauge link:**

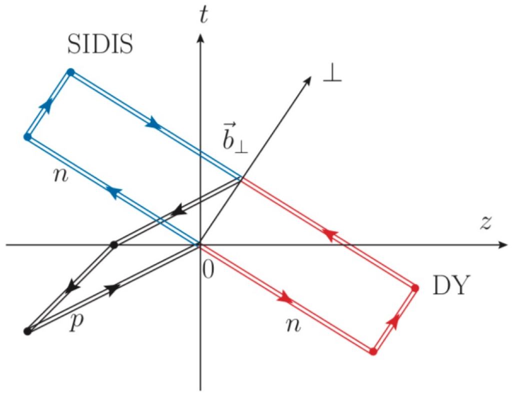
$$I_{\mathrm{div}} \sim \int dk^+ dk^- \frac{f(k^+k^-)}{(k^+k^-)^{1+\varepsilon}} = \frac{1}{2}\int \frac{d(k^-/k^+)}{k^-/k^+}\int d(k^+k^-)\frac{f(k^+k^-)}{(k^+k^-)^{1+\varepsilon}}$$

$$y_k = \frac{1}{2}\ln\frac{k^+}{k^-}$$

$y_k \to \pm\infty$: **rapidity divergence**

Rapidity divergence renormalized by soft function:

$$S(b_\perp, \mu, \delta^+, \delta^-) = \frac{1}{N_c}\mathrm{Tr}\langle 0|W_n(\vec{b}_\perp)|_{\delta^+} W_p^\dagger(\vec{b}_\perp)|_{\delta^-}|0\rangle$$



δ as the rapidity regulator.

Theoretical Definition of TMDPDFs

- **Physical TMDPDF is defined as:** ***e. g. unpolarized***

$$f(\lambda, b_\perp, \mu, \delta^-/P^+) = \langle P|\bar{\psi}(\lambda n/2 + b_\perp)\not{n}\, W_n(\lambda n/2 + b_\perp)|_{\delta^-}\psi(-\lambda n/2)|P\rangle \quad \Rightarrow \text{ Unsubtracted TMDPDF}$$

$$f^{\mathrm{TMD}}(x, b_\perp, \mu, \zeta) = \boxed{\lim_{\delta^-\to 0} \frac{f(\lambda, b_\perp, \mu, \delta^-/P^+)}{\sqrt{S(b_\perp, \mu, \delta^- e^{2y_n}, \delta^-)}}}$$

Rapidity regulator cancelled in the ratio, leaving the rapidity scale ζ dependence $\sim e^{-\frac{1}{2}K(b_\perp,\mu)\ln\frac{\mu^2}{\zeta^2}}$

- **Rapidity scale evolution controlled by the Collins-Soper equation:**

$$2\zeta\frac{d}{d\zeta}\ln f^{\mathrm{TMD}}(x, b_\perp, \mu, \zeta) = K(b_\perp, \mu)$$

Scale evolution controlled by RGE

$$\mu^2\frac{d}{d\mu^2}\ln f^{\mathrm{TMD}}(x, b_\perp, \mu, \zeta) = \frac{1}{2}\Gamma_{\mathrm{cusp}}(\alpha_s)\ln\frac{\mu^2}{\zeta} - \gamma_H(\alpha_s) = \gamma_\mu(\mu, \zeta)$$

Theoretical Definition of TMDPDFs

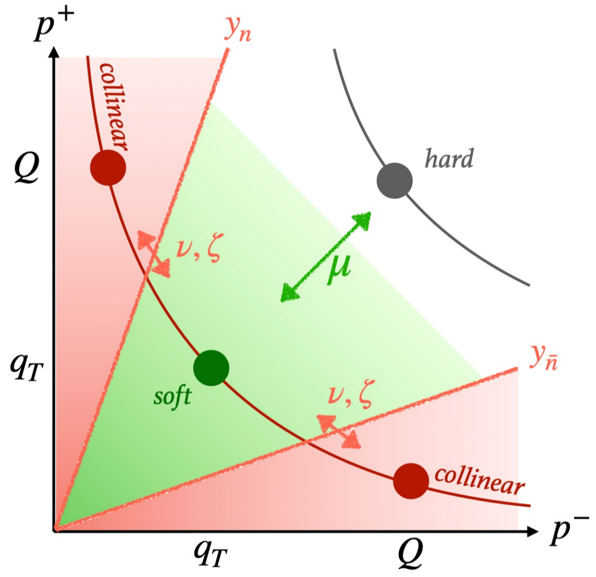
- **The two evolution kernels following:**

$$\mu^2\frac{d}{d\mu^2}K(b_\perp,\mu) = -2\zeta\frac{d}{d\zeta}\ \gamma_\mu(\mu,\zeta) = -\Gamma_{\text{cusp}}(\alpha_s(\mu))$$

So the physical TMDPDF defined this way is independent of rapidity scheme:

$$f^{\text{TMD}}(x,b_\perp,\mu,\zeta) \ = \ f^{\text{TMD}}(x,b_\perp,\mu_0,\zeta_0)\exp\Big[\int_{\mu_0}^{\mu}\frac{d\mu'}{m'}\gamma_\mu(\mu',\zeta_0)\Big]\exp\Big[\frac{1}{2}K(b_\perp,\mu)\ln\frac{\zeta}{\zeta_0}\Big]$$

- **The system of differential equations gives the TMD evolution.**

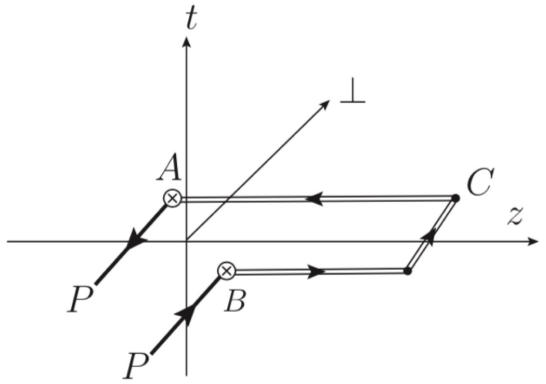


Collins-Soper Kernel, Intrinsic Soft function, and TMDPDFs from Lattice QCD

- **Quasi TMDPDF is defined as**

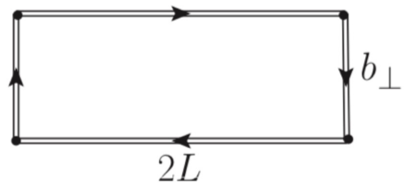
$$\tilde{f}(\lambda, b_\perp, \mu, \zeta_z) = \lim_{L\to\infty} \frac{\left\langle P \middle| \bar{\psi}(\frac{\lambda n_z}{2} + b_\perp)\gamma^z W_z(\frac{\lambda n_z}{2} + b_\perp, -\frac{\lambda n_z}{2}; L)\psi(-\frac{\lambda n_z}{2}) \middle| P \right\rangle}{\sqrt{Z_E(2L, b_\perp, \mu)}}$$

$$Z_E(2L, b_\perp, \mu) = \frac{1}{N_c}\mathrm{Tr}\langle 0|W_\perp W_z(b_\perp; 2L)|0\rangle$$



- $\tilde{f}$ follows the **RGE**:

$$\mu^2 \frac{d}{d\mu^2} \ln \tilde{f}(x, b_\perp, \mu, \zeta_z) = \gamma_F(\alpha_s(\mu))$$



γ_F is the anomalous dimension of quark-Wilson line vertices, as $\tilde{f}$ contains logarithmic divergence from quark-Wilson line vertices only.

This log divergence can be removed by dividing a renormalization factor extracted from the quasi matrix element in the rest frame at perturbative z and $b_\perp$.

Collins-Soper Kernel, Intrinsic Soft function, and TMDPDFs from Lattice QCD

- $\tilde{f}$ also satisfies the rapidity evolution equation:

$$P^z \frac{d}{dP^z} \ln \tilde{f}(x, b_\perp, \mu, \zeta_z) = K(b_\perp, \mu) + \underline{\mathcal{G}\left(\frac{(P^z)^2}{\mu^2}\right)} \quad \text{perturbative}$$

  One can extract the Collins-Soper kernel from the P^z dependence of $\tilde{f}$.

- $\tilde{f}$ still contains rapidity scheme dependence, which can be removed by further dividing the so-called intrinsic/reduced soft function capturing this scheme dependence in $\tilde{f}$.

  The intrinsic/reduced soft function is defined from

$$S_{\mathrm{DY}}(b_\perp, \mu, Y, Y') = e^{(Y+Y')K(b_\perp,\mu)+\mathcal{D}(b_\perp,\mu)} + \cdots$$

  as

$$S_I(b_\perp, \mu) \equiv e^{\mathcal{D}(b_\perp,\mu)}$$

  One can extract the intrinsic/reduced soft function from TMD factorization of a meson form factor.

Collins-Soper Kernel, Intrinsic Soft function, and TMDPDFs from Lattice QCD

- **Factorization between quasi and physical TMDPDFs in LaMET:**

$$\tilde{f}(x, b_\perp, \mu, \zeta_z)/\sqrt{S_I(b_\perp, \mu)} = H\Big(\frac{\zeta_z}{\mu^2}\Big)\, e^{\frac{1}{2}K(b_\perp,\mu)\ln\frac{\zeta_z}{\zeta}}\, f^{\rm TMD}(x, b_\perp, \mu, \zeta) + \mathcal{O}\Big(\frac{\Lambda^2_{\rm QCD}}{\zeta_z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2\zeta_z}\Big)$$

- **Polarized TMDPDFs:**

  - **Decomposition of leading-twist TMDPDFs**

$$\begin{aligned}
f_{q/h_S}^{[\not{n}]}(x, b_\perp) &= f_1(x, b_\perp) - \frac{\varepsilon_{\rho\sigma} b_\perp^\rho S_\perp^\sigma}{b_\perp} f_{1T}^\perp(x, b_\perp) \\
f_{q/h_S}^{[\not{n}\gamma_5]}(x, b_\perp) &= S_L g_{1L}(x, b_\perp) - \frac{b_\perp \cdot S_\perp}{b_\perp} g_{1T}^\perp(x, b_\perp) \\
f_{q/h_S}^{[i\bar{n}_\beta \sigma^{\alpha\beta}\gamma_5]}(x, b_\perp) &= S_\perp^\alpha h_1(x, b_\perp) + \frac{S_L b_\perp^\alpha}{b_\perp} h_{1L}^\perp(x, b_\perp) \\
&\quad + \Big(\frac{1}{2} g_\perp^{\alpha\rho} - \frac{b_\perp^\alpha b_\perp^\rho}{b_\perp^2}\Big) S_{\perp\rho} h_{1T}^\perp(x, b_\perp) \\
&\quad - \frac{\varepsilon^{\alpha\rho} b_{\perp\rho}}{b_\perp} h_1^\perp(x, b_\perp)
\end{aligned}$$

  - **Analogue for quasi TMDPDFs**

$$\begin{aligned}
\tilde{f}_{q/h_S}^{[\gamma^\lambda]}(x, b_\perp) &= \tilde{f}_1^\lambda(x, b_\perp) - \frac{\varepsilon_{\rho\sigma} b_\perp^\rho S_\perp^\sigma}{b_\perp} \tilde{f}_{1T}^{\lambda\perp}(x, b_\perp) \\
\tilde{f}_{q/h_S}^{[\gamma^\lambda\gamma_5]}(x, b_\perp) &= S_L \tilde{g}_{1L}^\lambda(x, b_\perp) - \frac{b_\perp \cdot S_\perp}{b_\perp} \tilde{g}_{1T}^{\lambda\perp}(x, b_\perp) \\
\tilde{f}_{q/h_S}^{[i\sigma^{\alpha\lambda}\gamma_5]}(x, b_\perp) &= S_\perp^\alpha \tilde{h}_1^\lambda(x, b_\perp) + \frac{S_L b_\perp^\alpha}{b_\perp} \tilde{h}_{1L}^{\lambda\perp}(x, b_\perp) \\
&\quad + \Big(\frac{1}{2} g_\perp^{\alpha\rho} - \frac{b_\perp^\alpha b_\perp^\rho}{b_\perp^2}\Big) S_{\perp\rho} \tilde{h}_{1T}^{\lambda\perp}(x, b_\perp) \\
&\quad - \frac{\varepsilon^{\alpha\rho} b_{\perp\rho}}{b_\perp} \tilde{h}_1^{\lambda\perp}(x, b_\perp)
\end{aligned}$$

Collins-Soper Kernel from Lattice QCD

- **At leading power accuracy, CS kernel can be extracted from the P^z dependence of TMDs:**

$$\tilde{f}(x, b_\perp, \mu, \zeta_z)/\sqrt{S_I(b_\perp, \mu)} = H\left(\frac{\zeta_z}{\mu^2}\right) e^{\frac{1}{2}K(b_\perp,\mu)\ln\frac{\zeta_z}{\zeta}} f^{\mathrm{TMD}}(x, b_\perp, \mu, \zeta)$$

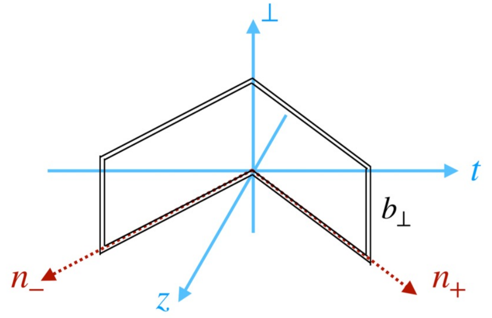
Taking ratio, S_I and f^{TMD} cancelled each other, then

$$K(x, b_\perp) = \frac{1}{\ln(P_1^z/P_2^z)} \ln\left(\frac{H(xP_2^z, \mu)\tilde{f}(x, b_\perp, \mu, P_1^z)}{H(xP_1^z, \mu)\tilde{f}(x, b_\perp, \mu, P_2^z)}\right)$$

f^{TMD} could be either unsubtracted quasi TMD wave function or quasi TMDPDF (beam function).

Intrinsic Soft Function from Lattice QCD

- **TMD soft function is defined by two conjugate light-like Wilson lines:**

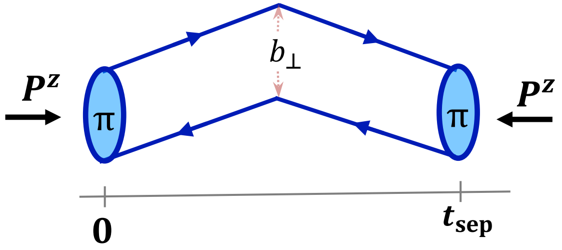


$$S(b_\perp,\mu,Y,Y') = \frac{1}{N_c}\mathrm{Tr}\langle 0|\bar{\mathcal{T}}[U^\dagger_{n_+}(-\infty,b_\perp)_{Y'}U^\dagger_{n_-}(\pm\infty,b_\perp)_Y]|0\rangle$$
$$\mathcal{T}[U_{n_-}(\pm\infty,0)_Y U_{n_+}(-\infty,0)_{Y'}]|0\rangle$$

Intrinsic soft function reflects the rapidity independent part:

$$S(b_\perp,\mu,Y,Y') = e^{(Y+Y')K(b_\perp,\mu)+\mathcal{D}(b_\perp,\mu)}$$

- **Four-quark form factor with large P^z:**



$$F(b_\perp,P^z) = \langle \pi(P^z)|(\bar{q}_1\Gamma q_1)(b_\perp)(\bar{q}_2\Gamma q_2)(0)|\pi(P^z)\rangle$$

Factorization of form factor in the framework of LaMET:

$$F(b_\perp,P^z) = \int dx\,dx'\,H(x,x',P^z)\frac{\tilde{\phi}(x,\bar{Y},P^z,b_\perp)}{S(Y,\bar{Y},b_\perp)}\frac{\tilde{\phi}^\dagger(x',\bar{Y}',P^z,b_\perp)}{S(Y',\bar{Y}',b_\perp)}S(Y,Y',b_\perp)$$

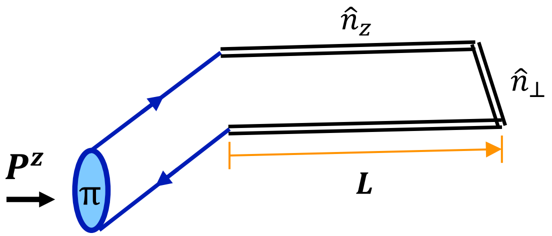
perturbative matching kernel

subtracted quasi-TMD wave function

Intrinsic Soft Function from Lattice QCD

- **Subtracted quasi-TMD wave function:**

$$\tilde{\phi}(z,P^z,b_\perp) = \lim_{L\to\infty}\frac{\langle 0|\,\bar{q}(z\hat{n}_z+b_\perp\hat{n}_\perp)\gamma^t\gamma_5 U_\sqsupset(z\hat{n}_z+b_\perp\hat{n}_\perp,0;L)q(0)\,|\pi(P^z)\rangle}{\sqrt{Z_E(2L+z,b_\perp)}}$$



- **Intrinsic soft function:**

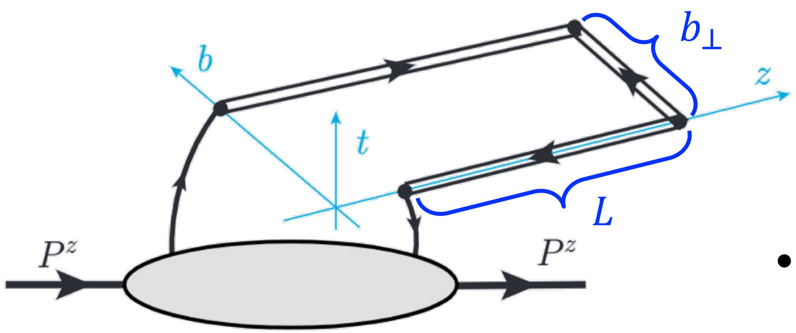
$$\begin{aligned} S_I(b_\perp,\mu) &= \frac{S(b_\perp,\mu,Y,Y')}{S(b_\perp,\mu,Y,0)S(b_\perp,\mu,0,Y')} \\ &= \frac{F(b_\perp,P^z)}{\int dx\,dx'\,H(x,x',P^z)\tilde{\phi}(x,\bar{Y},P^z,b_\perp)\tilde{\phi}^\dagger(x',\bar{Y}',P^z,b_\perp)} \end{aligned}$$

leading order matching:

$$S_I(b_\perp) = 2N_c\frac{F(b_\perp,P^z)}{|\,\tilde{\phi}(0,P^z,b_\perp)|^2}+\mathcal{O}\Big(\alpha_s,\frac{1}{(P^z)^2}\Big)$$

Lattice QCD calculable.

➢ Matching from quasi TMDs to TMDs:



- **Hadronic matrix element from equal-time correlators:**

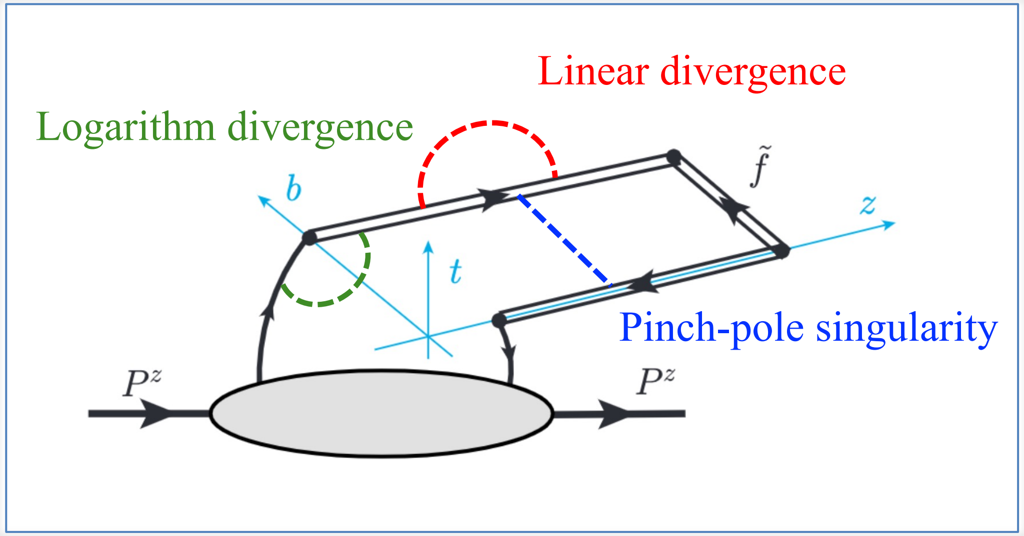
$$\tilde{h}^0_\Gamma(z, P^z, b_\perp) = \lim_{L\to\infty} \langle P^z | \bar{\psi}(b_\perp \hat{n}_\perp) \Gamma \times U_\sqsupset(b_\perp \hat{n}_\perp \leftarrow b_\perp \hat{n}_\perp + L\hat{n}_z; b_\perp \hat{n}_\perp + L\hat{n}_z \leftarrow L\hat{n}_z; L\hat{n}_z \leftarrow z\hat{n}_z) \psi(z\hat{n}_z) | P^z \rangle$$

- **Subtracted quasi TMDPDFs:**

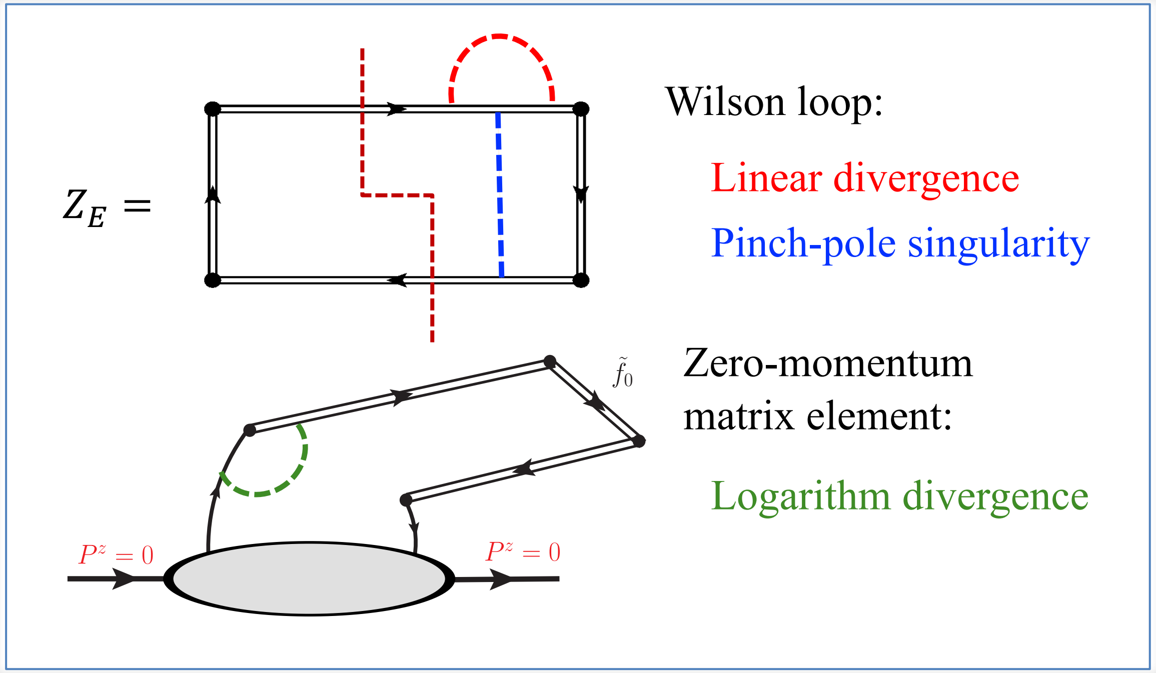
$$\tilde{f}_\Gamma(x, P^z, b_\perp, \mu) = \lim_{\substack{a\to 0 \\ L\to\infty}} \int \frac{dz}{2\pi} e^{-iz(xP^z)} \frac{\tilde{h}^0_\Gamma(z, P^z, b_\perp)}{\sqrt{Z_E(2L+z, b_\perp, a)} Z_O(1/a, \mu, \Gamma)}$$

TMDPDFs from Lattice QCD

➢ **Divergences in bare quasi TMDPDF**

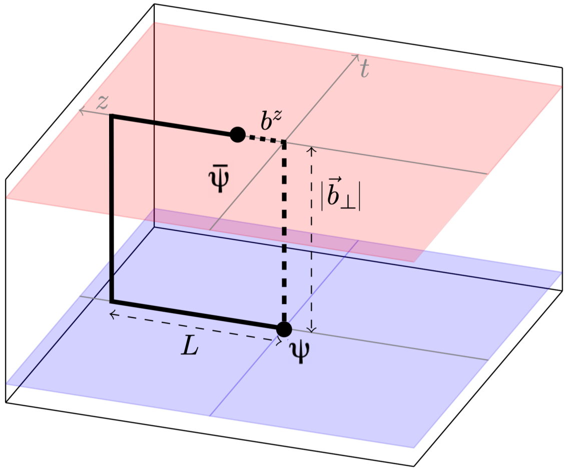


➢ **Renormalization**



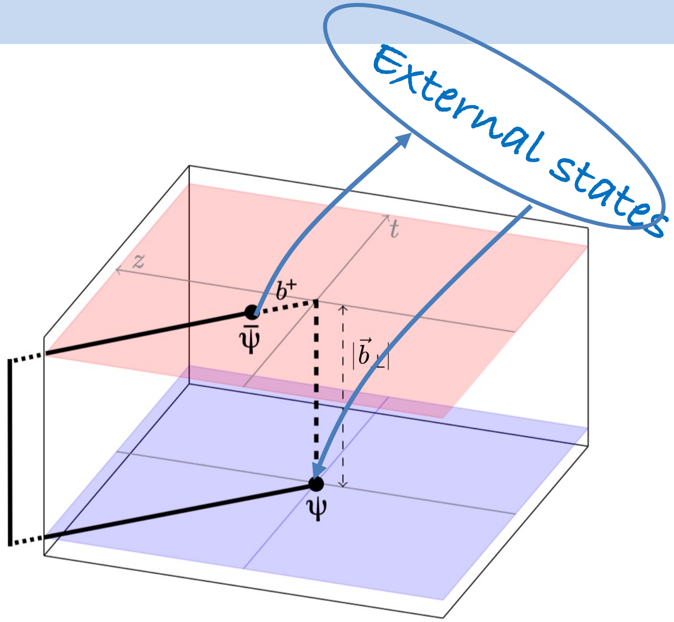
Ji, PRL120(2018), NPB964(2021), PLB257(1991); Zhang, PRD95(2017), NPB939(2019); Ishikawa, PRD96(2017); Green, PRL121(2018); Huo, NPB969(2021); Chen, NPB915(2017); Musch, PRD83(2011); ……

TMDPDFs from Lattice QCD



Lorentz boost

$L \to \infty$



Equal-time correlators, directly calculable on lattice

Space-like correlators, Physical TMDs

- **Connected at large-momentum limit**

Ji, PLB811(2020); Ebert, JHEP04(2022)

Matching kernel

$$\boxed{\tilde{f}_\Gamma(x, b_\perp, \zeta_z, \mu)}\sqrt{\boxed{S_I(b_\perp, \mu)}} = \boxed{H_\Gamma\left(\frac{\zeta_z}{\mu^2}\right)} e^{\frac{1}{2}\ln\left(\frac{\zeta_z}{\zeta}\right)\boxed{K(b_\perp,\mu)}} \boxed{f(x, b_\perp, \mu, \zeta)} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{\zeta_z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2 \zeta_z}\right)$$

Quasi TMDPDF — Intrinsic soft function — Collins-Soper kernel — Physical TMDs