



Lattice Parton
Collaboration



A Precise Determination of Collins-Soper Kernel from Lattice QCD

Jin-Xin Tan

Shanghai Jiao Tong University

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In collaboration with:

Hang Liu, Zhi-Chao Gong, Qi-An Zhang, Wei Wang et al.

- **Motivation**
- **Lattice QCD calculation of Collins-Soper Kernel**
 - **Quasi-TMD Wave Function**
 - **Collins-Soper Kernel**
- **Summary and Outlook**

➤ Motivation

➤ Lattice QCD calculation of Collins-Soper Kernel

- Quasi-TMD Wave Function
- Collins-Soper Kernel

➤ Summary and Outlook

Exploration of Nucleon Structure

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Gell-Mann
quark model

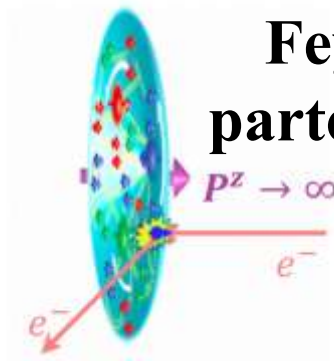


0D: spin and mass

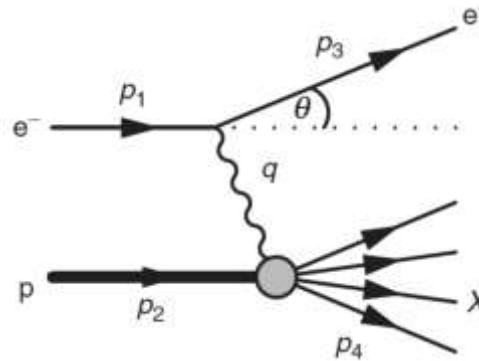


Higgs
mechanism

Feynman
parton model



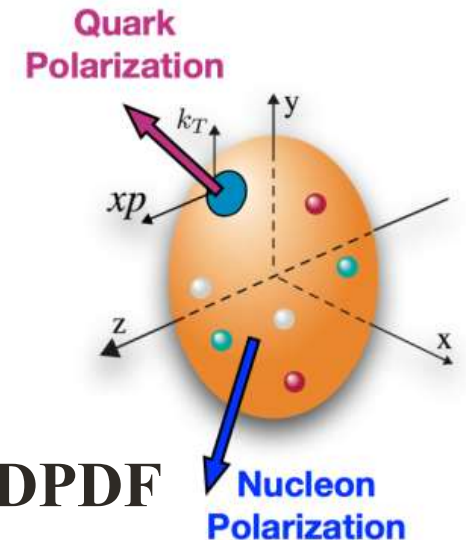
1D: PDF



Deep inelastic
scattering

JLab
COMPASS
JAM
EIC
EicC
.....

3D: TMDs



TMDPDF

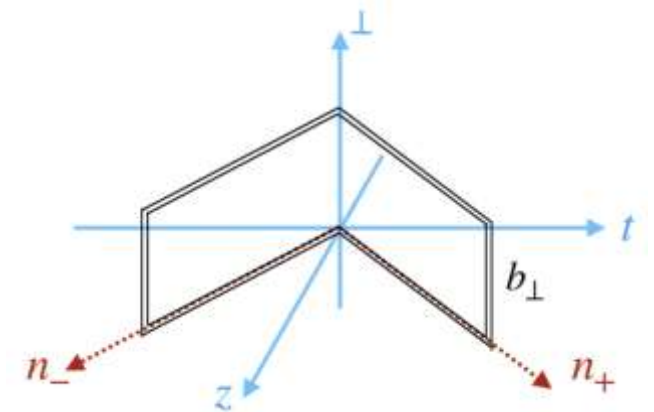
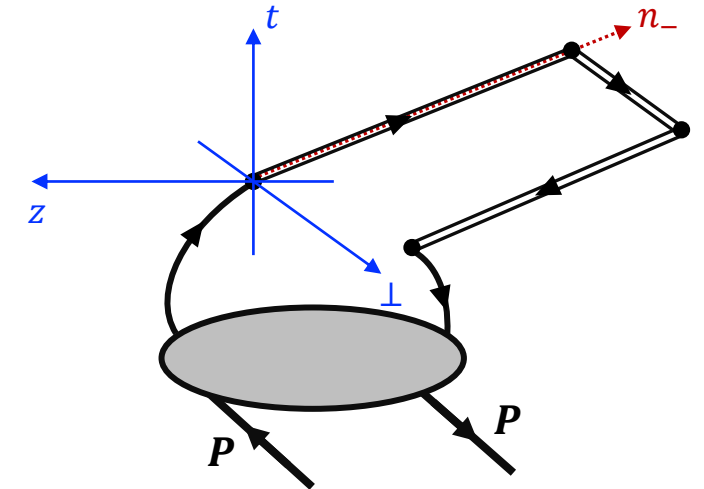
- Rapidity divergence from gluons radiation collinear to lightlike gauge link:

$$I_{\text{div}} \sim \int dk^+ dk^- \frac{f(k^+ k^-)}{(k^+ k^-)^{1+\epsilon}} = \frac{1}{2} \int \frac{d(k^- / k^+)}{k^- / k^+} \int dk^+ dk^- \frac{f(k^+ k^-)}{(k^+ k^-)^{1+\epsilon}}$$

$$y_k = \frac{1}{2} \ln \frac{k^+}{k^-} \quad y_k \rightarrow \pm\infty: \text{rapidity divergence}$$

Rapidity divergence can be canceled by **soft function**,
leaving the **rapidity scale ζ** dependence $\sim e^{-\frac{1}{2} K(b_\perp, \mu) \ln \frac{\mu^2}{\zeta^2}}$

Collins-Soper kernel



➤ Evolution of TMDs:

$$f(x, b_{\perp}, \mu, \zeta) = f(x, b_{\perp}, \mu_0, \zeta_0)$$

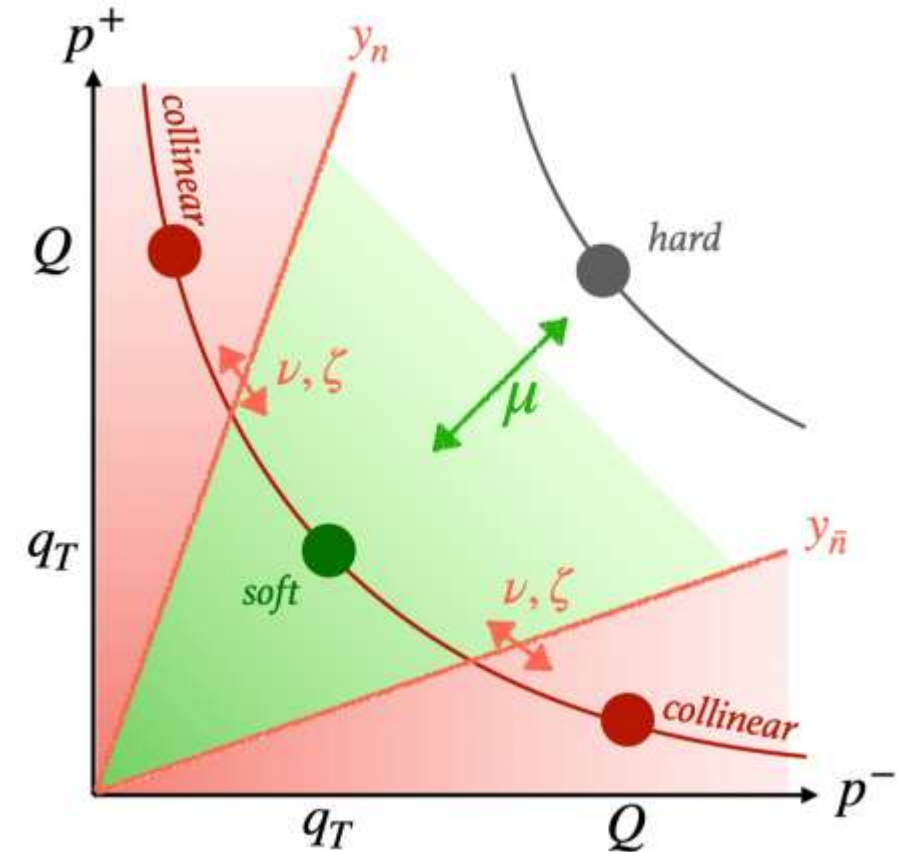
$$\times \exp \left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_{\mu}(\mu', \zeta_0) \right] \exp \left[\frac{1}{2} \underline{K}(\mu, b_{\perp}) \ln \frac{\zeta}{\zeta_0} \right]$$

UV anomalous dimension

Rapidity anomalous dimension
(Collins-Soper kernel)

➤ UV anomalous dimension γ_{μ} is perturbative (NLO, NNLO, ...);

➤ Collins-Soper(CS) kernel $K(\mu, b_{\perp})$ is nonperturbative for $b_{\perp} \gtrsim \Lambda_{\text{QCD}}^{-1} \approx 0.2 \text{ fm}$.



Ji et al., PRD 71, 034005(2005);

Collins, Vol. 32(Cambridge University Press, 2011).

CS Kernel -- An Essential Component for Lattice Calculations of TMDs 7

➤ **LaMET framework:** *Ji, PRL 110, 262002 (2013); Ji, Sci.China 57 (2014)*

Matching kernel

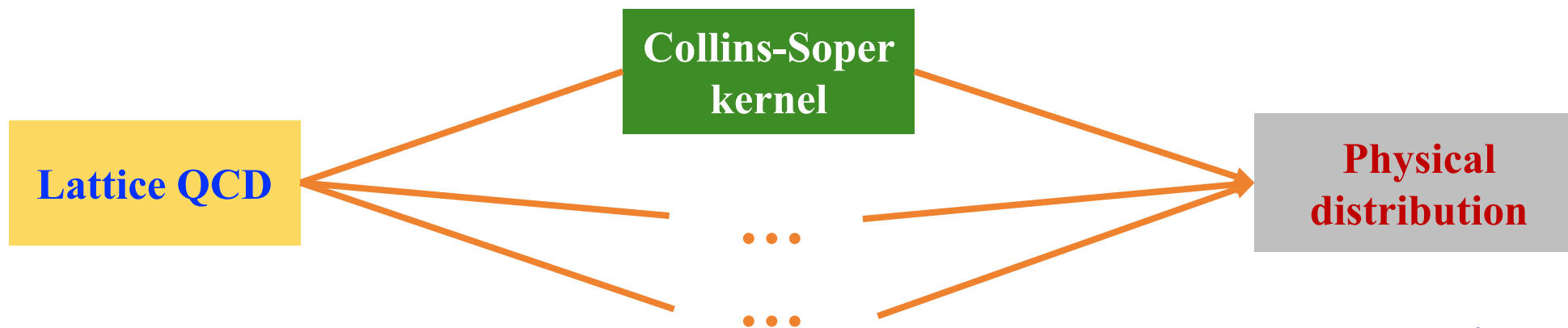
$$\boxed{\tilde{f}_F(x, b_\perp, \mu, \zeta^z)} \boxed{S_F^2(b_\perp, \mu)} = \boxed{H_F(\zeta^z, \mu)} \exp \left[\frac{1}{2} \boxed{K(b_\perp, \mu)} \ln \frac{\zeta^z - i\epsilon}{\zeta} \right] \boxed{f(x, b_\perp, \mu, \zeta)} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{\zeta^z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2 \zeta^z}\right)$$

Quasi TMDs

Intrinsic soft function

Collins-Soper kernel

Light-cone TMDs



*Ji et al., PLB811(2020);
Ebert et al., JHEP 09 (2019);
Ebert et al., JHEP04(2022);*

➤ Motivation

➤ **Lattice QCD calculation of Collins-Soper Kernel**

- **Quasi-TMD Wave Function**
- Collins-Soper Kernel

➤ Summary and Outlook

From quasi TMDs to Collins-Soper kernel

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- At leading power accuracy, CS kernel can be extracted from the P^z dependence of TMDs:

Matching kernel

$$\tilde{f}_F(x, b_\perp, \mu, \zeta^z) S_I^{\frac{1}{2}}(b_\perp, \mu) = H_F(\zeta^z, \mu) \exp \left[\frac{1}{2} K(b_\perp, \mu) \ln \frac{\zeta^z - i\epsilon}{\zeta} \right] f(x, b_\perp, \mu, \zeta) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{\zeta^z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2 \zeta^z}\right)$$

Quasi TMDs

Intrinsic soft function

Collins-Soper kernel

Light-cone TMDs

Taking ratio, S_I and f cancelled each other, then

$$K(b_\perp, \mu) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \left(\frac{H_F(\zeta_2^z, \mu) \tilde{f}_F(x, b_\perp, \zeta_1^z, \mu)}{H_F(\zeta_1^z, \mu) \tilde{f}_F(x, b_\perp, \zeta_2^z, \mu)} \right)$$

$\tilde{f}_F$ could be either unsubtracted **quasi TMD wave function** or **quasi TMDPDF**.

*Ji, PRL 110, 262002 (2013);
Ji, Sci.China 57 (2014)*

From quasi TMDs to Collins-Soper kernel

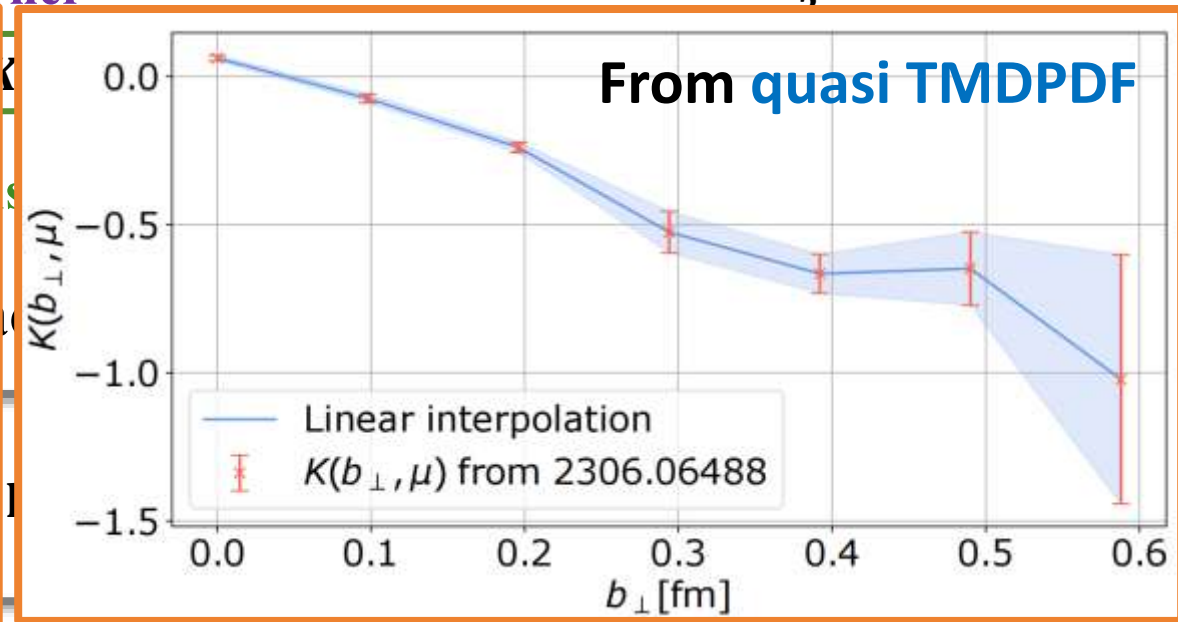
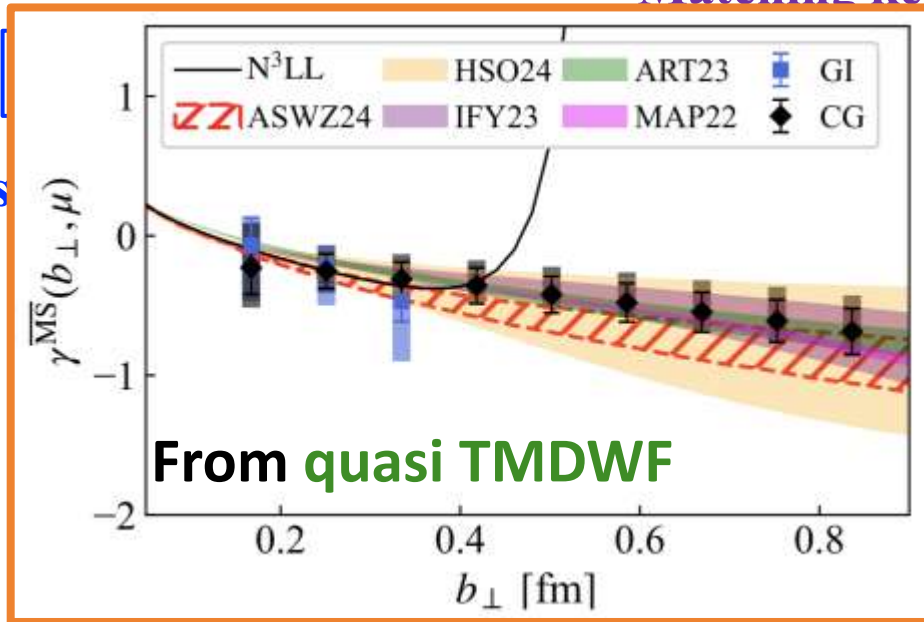
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- At leading power accuracy, CS kernel can be extracted from the P^Z dependence of TMDs:

*Bollweg et al., PLB852, 138617 (2024);
Walter, et al. (LPC), PRD.111.094507(2024).*

Matching kernel

Quas



$\tilde{f}_F$ could be either unsubtracted quasi TMD wave function or quasi TMDPDF.

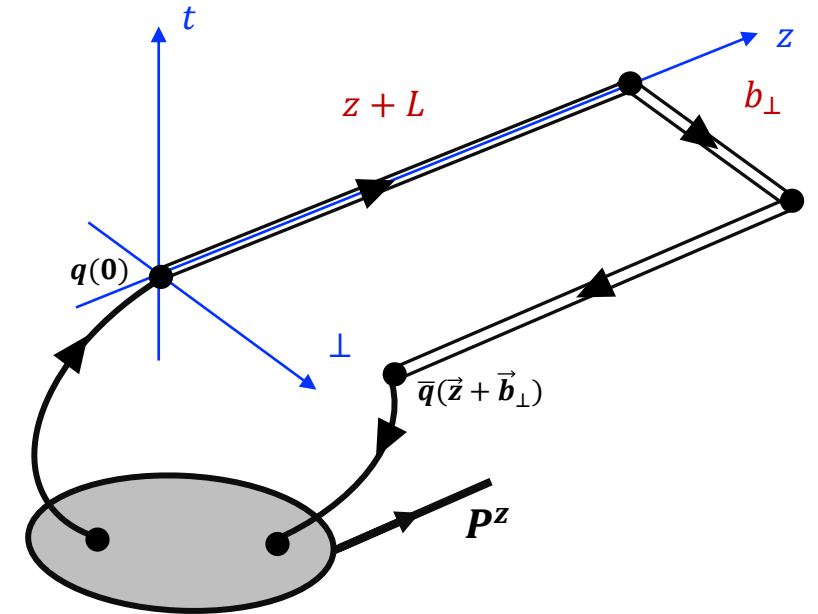
For better signal, the CS kernel is extracted from the pion quasi-TMDWF.

- The quasi-TMD wave functions can be defined as:

$$\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z) = \lim_{L \rightarrow \infty} \int \frac{dz P^z}{2\pi} e^{ixzP^z} \frac{\tilde{\Phi}^{\pm 0}(z, b_{\perp}, P^z, a, L)}{Z_O(\mu, a) \sqrt{Z_E(2L + z, b_{\perp}, \mu, a)}},$$

where the bare quasi-TMD wave function can be constructed as :

$$\tilde{\Phi}^{\pm 0}(z, b_{\perp}, P^z, a, L) = \langle 0 | \bar{q}(z\hat{n}_z + b_{\perp}\hat{n}_{\perp}) \Gamma U_{\square, \pm}(L, z, b_{\perp}) q(0) | P^z \rangle.$$



*Chu et al.(LPC), PRD109(2024);
Chu et al.(LPC), PRD 106(2022);
Zhang et al., PRL125(2020).*

- CLQCD lattice ensembles with **multiple lattice spacings and physical pion masses**;
- Coulomb gauge fixed wall source propagators;
- **HYP smearing.**

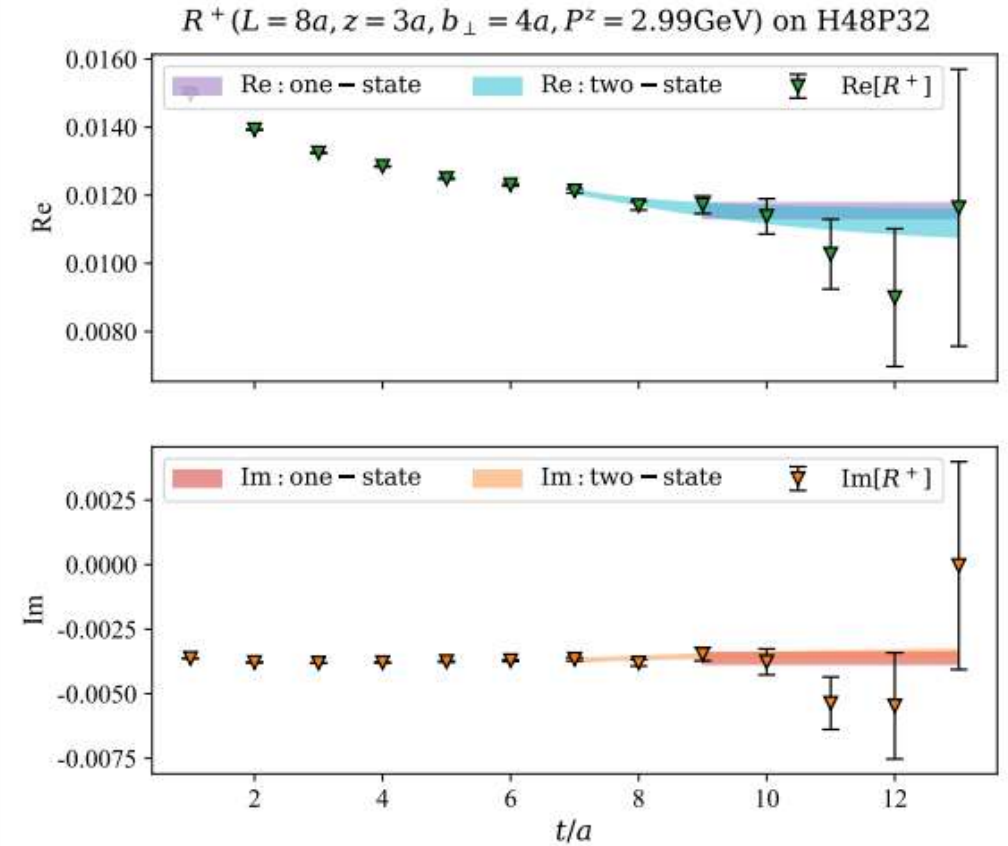
	a/fm	$n_s^3 \times n_t$	m_π/MeV	$n_{c f g}$	P^z/GeV
C32P29	0.10530	$32^3 \times 64$	292.4	984×4	1.47, 1.84, 2,21
C32P23	0.10530	$32^3 \times 64$	228.0	448×10	1.47, 1.84, 2,21
C48P14	0.10530	$48^3 \times 96$	135.5	204×24	0.98, 1.22, 1.47
F32P30	0.07746	$32^3 \times 96$	303.2	1153×6	2.00, 2.50, 3.00
H48P32	0.05187	$48^3 \times 144$	317.2	550×4	1.99, 2.59, 2.99

- To determine the bare quasi-TMDWF, we construct the non-local two point correlation function :

$$C_2^\pm(z, b_\perp, P, p^z, L, t, t') = \frac{1}{n_s^3} \sum_{\vec{x}} e^{i\vec{P} \cdot \vec{x}} \text{Tr} \langle S_w(t'; \vec{x} + L\hat{n}_z + b\hat{n}_x, t; -p^z) \\ \times U_{\square, \pm}(L, z, b_\perp) \Gamma_1 S_w(\vec{x} + L\hat{n}_z + z\hat{n}_z, t; t'; p^z) \Gamma_2 \rangle$$

- The bare matrix elements can be extracted by one-/two-state fit:

$$R^\pm(z, b_\perp, P^z, L, t) = \frac{C_2^\pm(z, b_\perp, P^z, L, t)}{C_2(0, 0, P^z, 0, t)} = \tilde{\Phi}^{\pm 0}(z, b_\perp, P^z, L) e^{-iz\frac{P^z}{2}} \\ \times [1 + c_1(z, b_\perp, P^z, L) e^{-\Delta E t}]$$

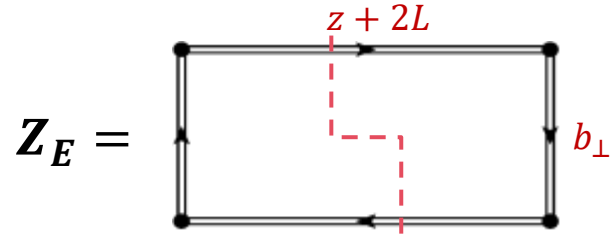


Wilson Loop Renormalization of Bare Quasi-TMDWFs

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➤ Wilson loop definition:

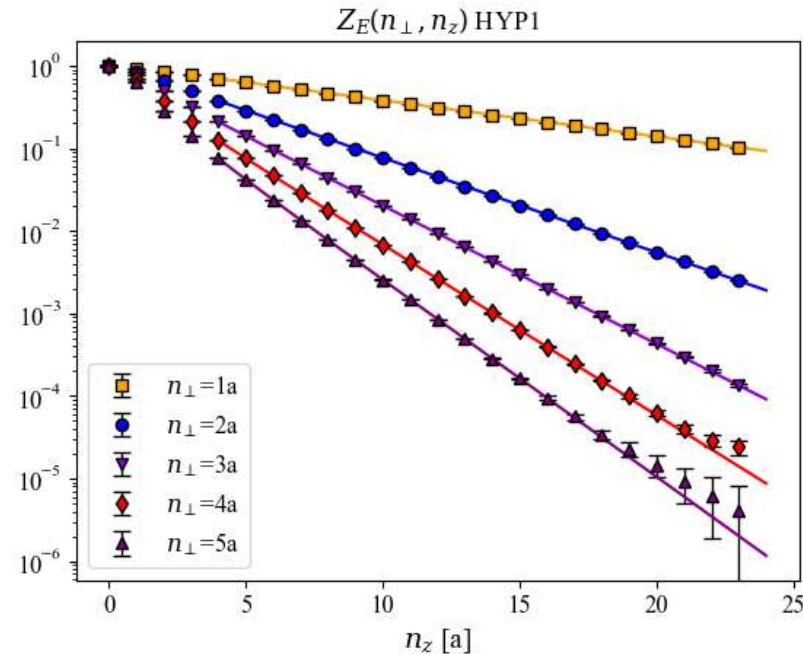
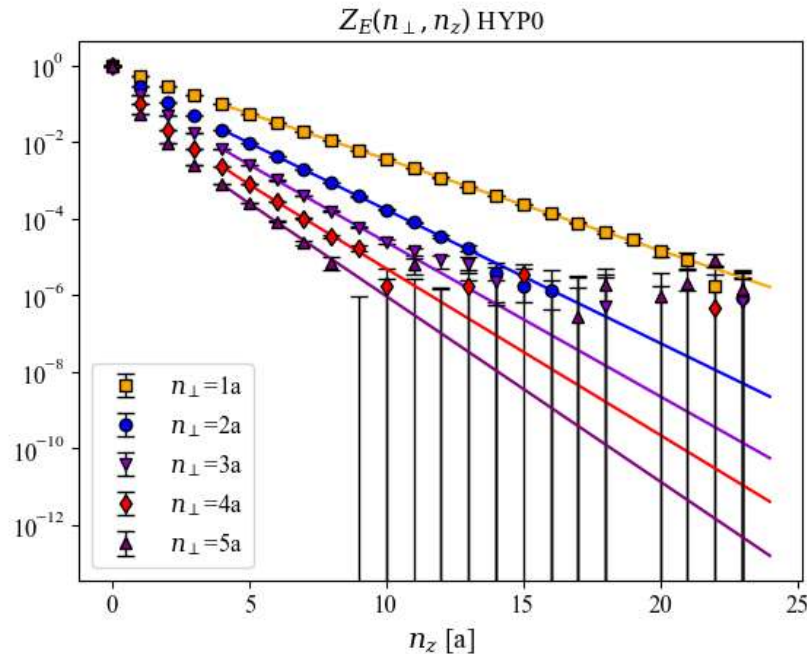
Ji et al., PRL120(11):112001, 2018.



$$Z_E = \frac{1}{N_c} \text{Tr} \langle 0 | U_{\perp}(0; b_{\perp} \hat{n}_{\perp}) U_z(b_{\perp} \hat{n}_{\perp}; (2L + z) \hat{n}_z) | 0 \rangle$$

➤ Parameterization:

$$Z_E(n_{\perp}, n_z) = c_0(n_{\perp}) e^{-E(n_{\perp}) n_z} \left[1 + c_1(n_{\perp}) e^{-\Delta E(n_{\perp}) n_z} \right]$$



LPC, in preparation

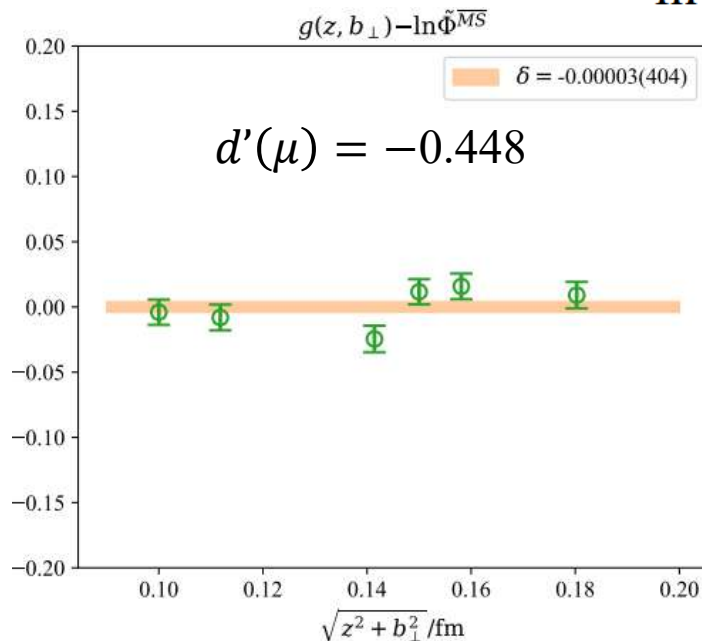
Subtracted matrix element

➤ **Renormalization condition:** $\tilde{h}(a, z, b_\perp) = \frac{\tilde{\Phi}^0(z, b, Pz=0, \mu)}{\sqrt{Z_E(z+2L, b, \mu)}} = Z_O(1/a, \mu) h_0^{\overline{MS}}(\mu, b_\perp, z) + O(\frac{a^2}{b^2})$

➤ **Parametrization:** $\ln \tilde{h}(a, z, b_\perp) = \frac{\gamma_0}{\beta_0} \ln \left[\ln[1/(a\Lambda_{\text{QCD}})] \right] + \frac{c_1}{\ln[1/(a\Lambda_{\text{QCD}})]} + g(z, b_\perp)$

$$\ln Z_O(a, \mu) = \frac{\gamma_0}{\beta_0} \ln \left[\ln[1/(a\Lambda_{\text{QCD}})] \right] + \frac{c_1}{\ln[1/(a\Lambda_{\text{QCD}})]} + d'(\mu)$$

Joint fit result: $c_1=0.020$



LPC, in preparation

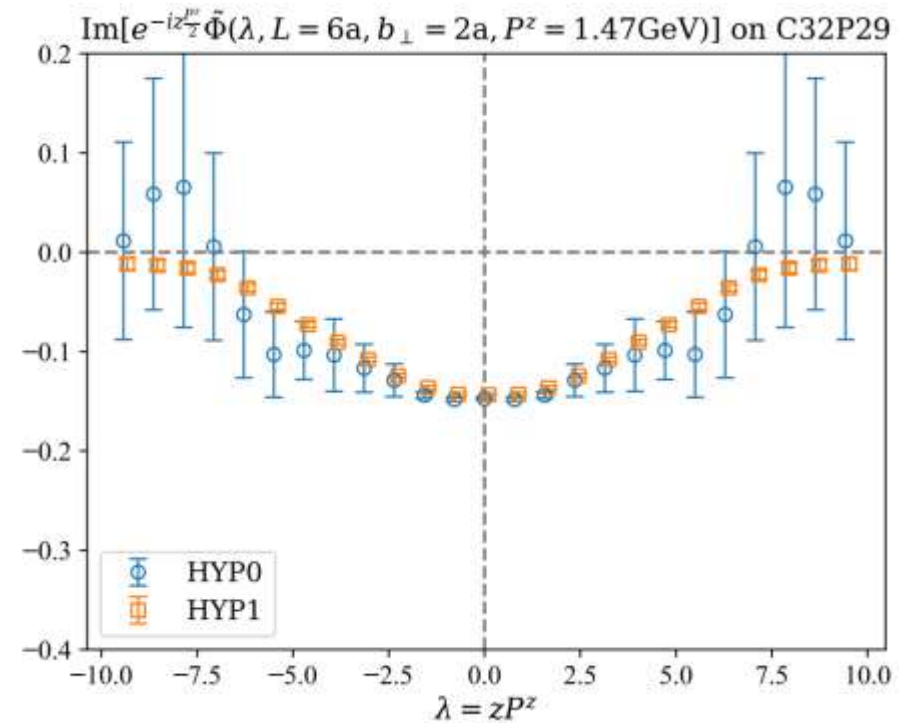
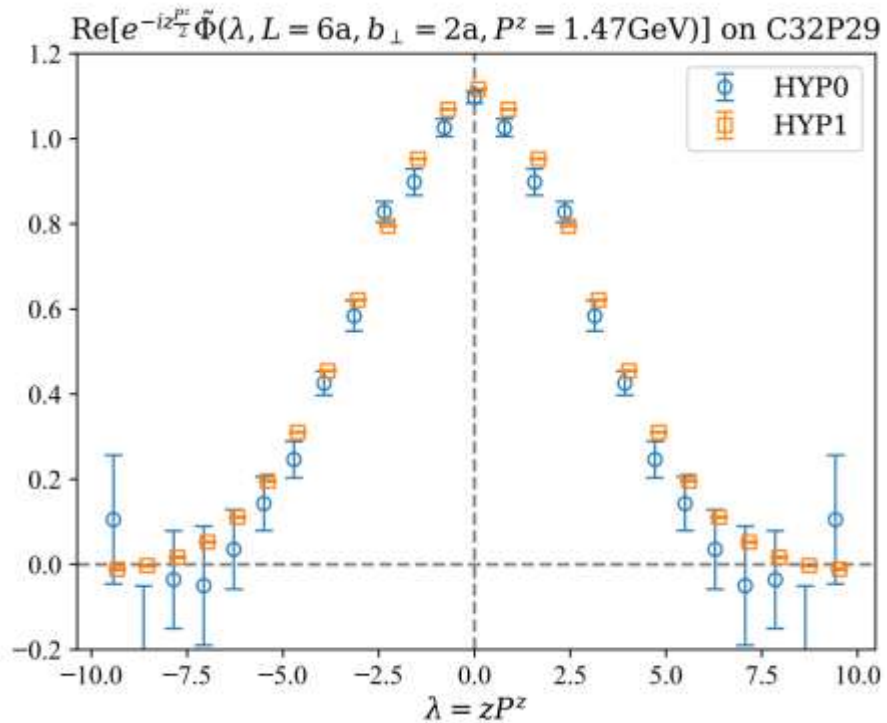
a/fm	0.10530	0.07746	0.05187
HYP0	1.064(21)	1.145(20)	1.241(19)
HYP1	0.92208(62)	0.97526(57)	1.03969(53)

Result for Quasi-TMDWFs in Coordinate Space

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Different HYP
steps

$$\tilde{\Phi}^{\pm}(z, P^z, L, b, \mu) = \frac{\tilde{\Phi}^{\pm 0}(z, b_{\perp}, P^z, a, L)}{Z_O(\mu, a) \sqrt{Z_E(2L + z, b_{\perp}, \mu, a)}}$$



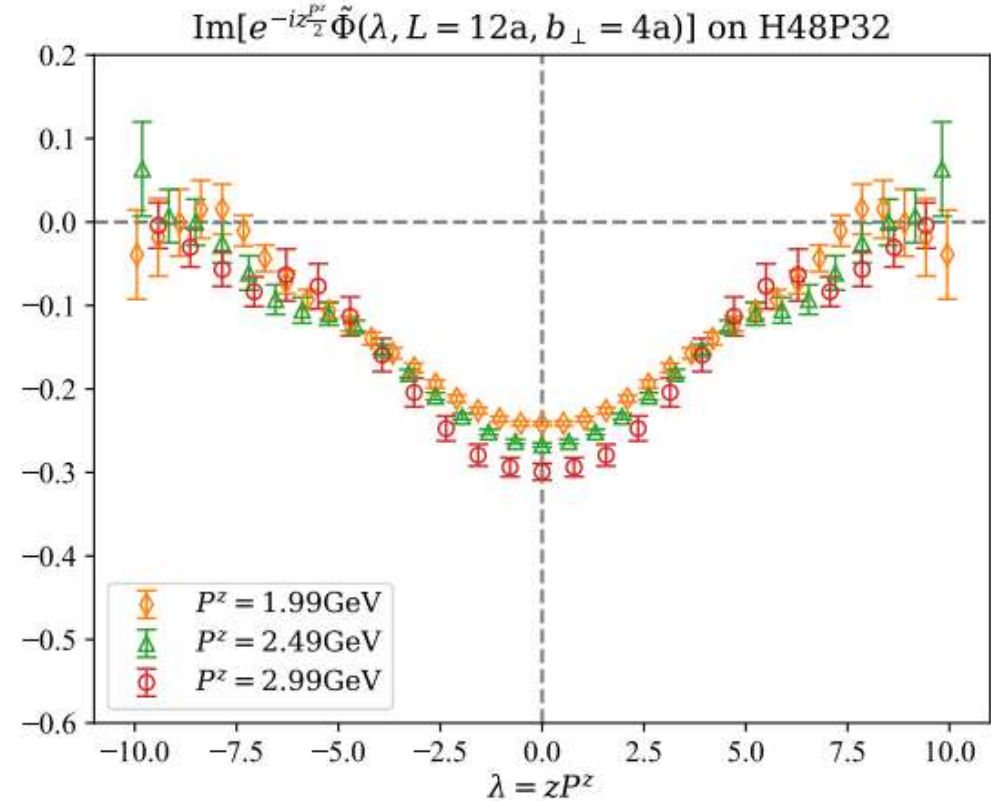
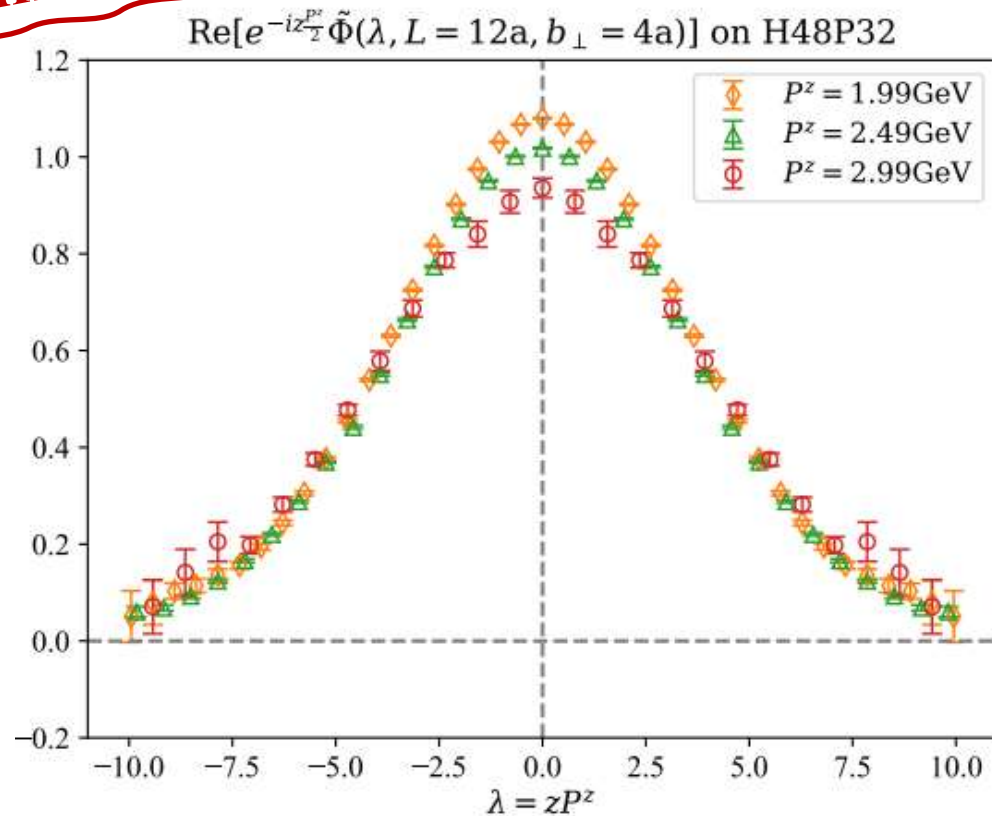
LPC, in preparation

Result for Quasi-TMDWFs in Coordinate Space

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Different P^z

$$\tilde{\Phi}^{\pm}(z, P^z, L, b, \mu) = \frac{\tilde{\Phi}^{\pm 0}(z, b_{\perp}, P^z, a, L)}{Z_O(\mu, a) \sqrt{Z_E(2L + z, b_{\perp}, \mu, a)}}$$



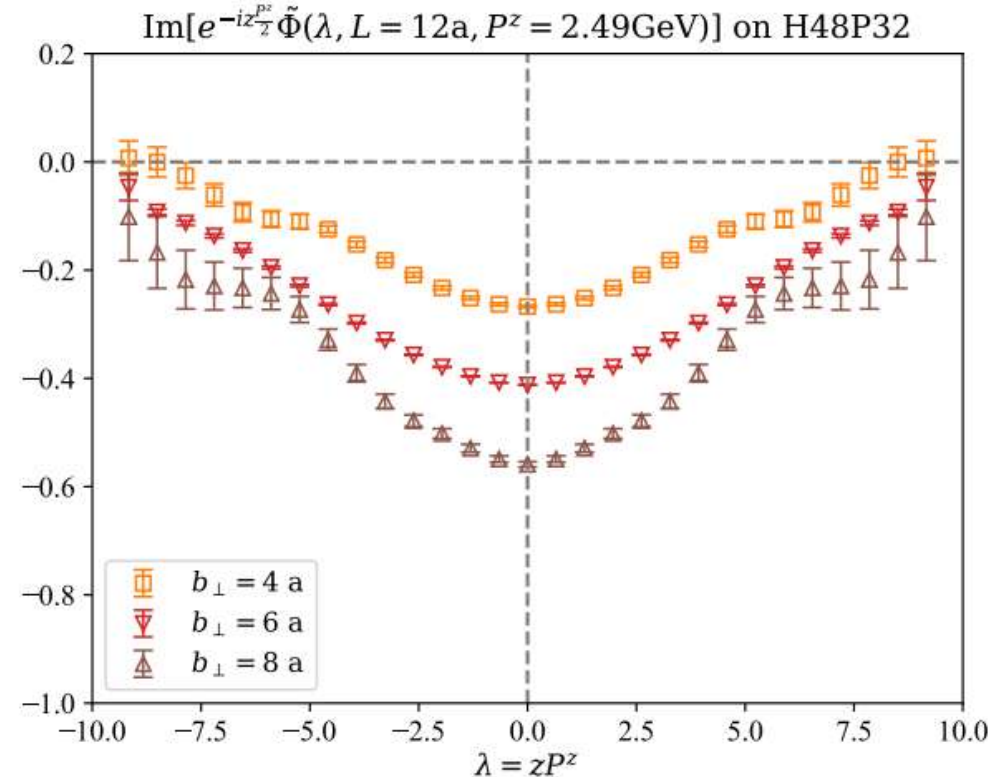
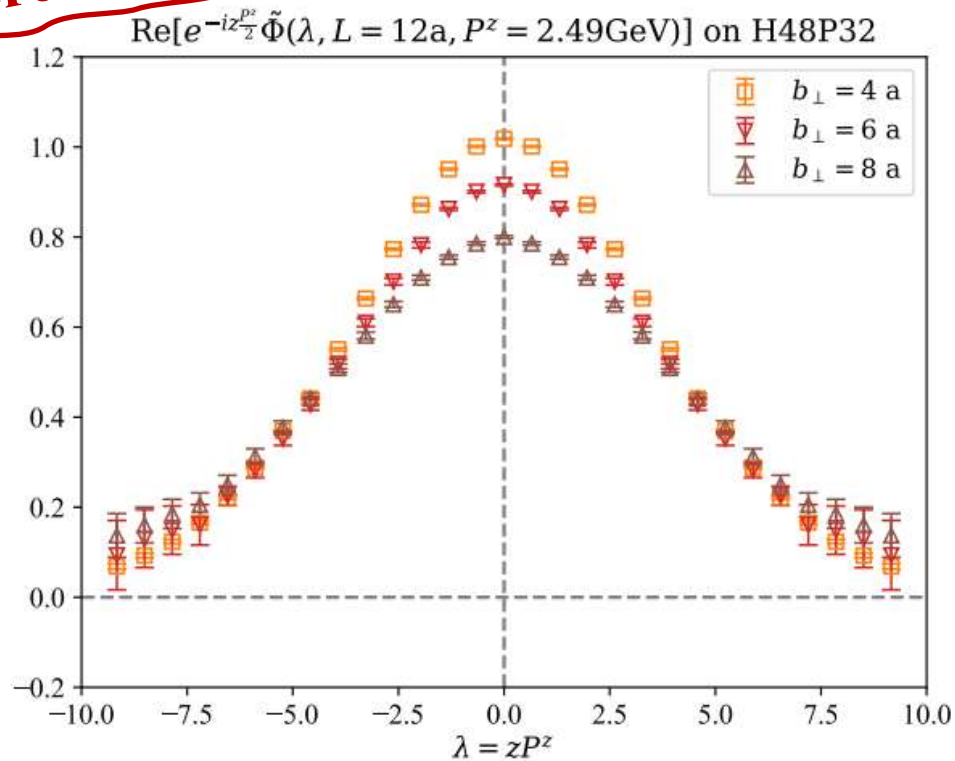
LPC, in preparation

Result for Quasi-TMDWFs in Coordinate Space

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$$\tilde{\Phi}^{\pm}(z, P^z, L, b, \mu) = \frac{\tilde{\Phi}^{\pm 0}(z, b_{\perp}, P^z, a, L)}{Z_O(\mu, a) \sqrt{Z_E(2L + z, b_{\perp}, \mu, a)}}$$

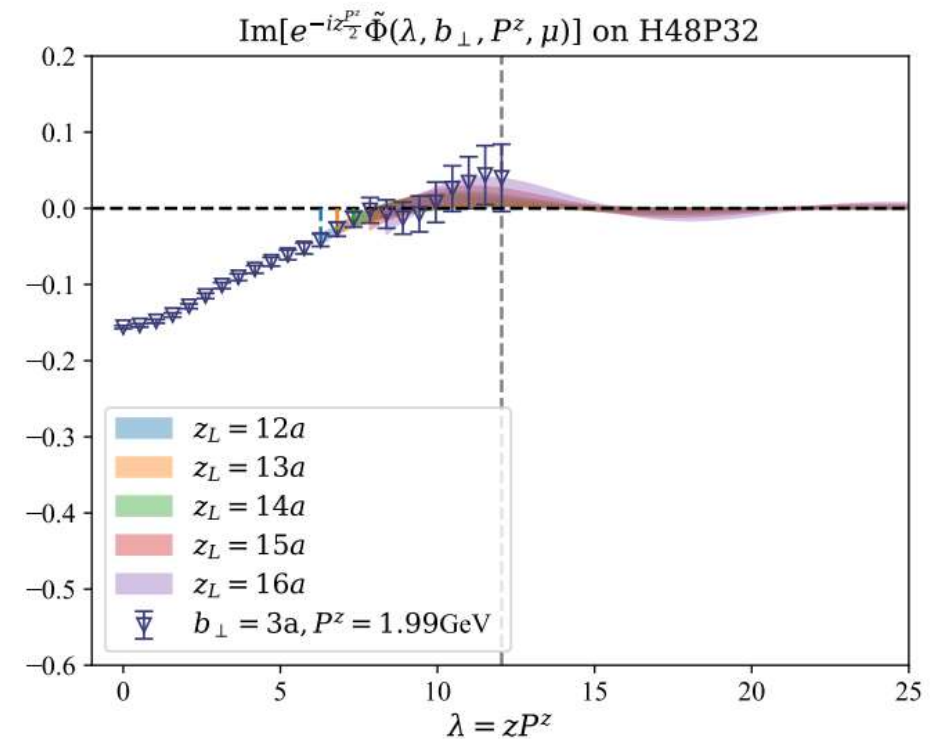
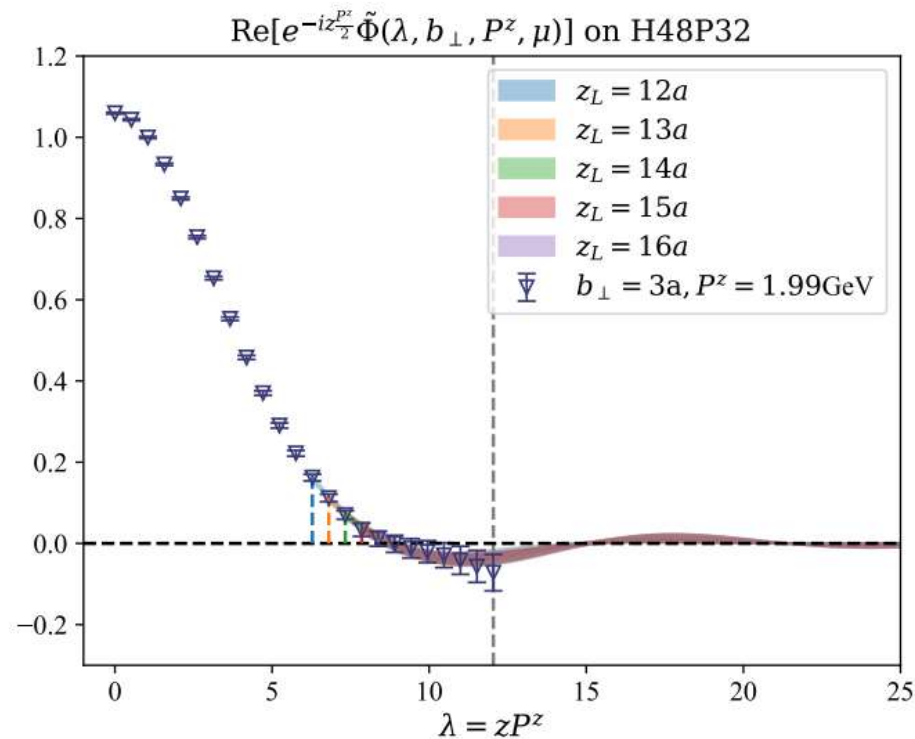
Different $b_{\perp}$



LPC, in preparation

- In large λ area, a feasible form of asymptotic behavior can be written as:

$$\lim_{\lambda \rightarrow \infty} \tilde{\Phi}(z, b_{\perp}, P^z, a, L) = [f_1(b_{\perp}) + if_2(b_{\perp})] \left[\frac{c_1}{(-i\lambda)} + e^{i\lambda} \frac{c_2}{(i\lambda)^d} \right] e^{\frac{\lambda}{\lambda_0}}$$

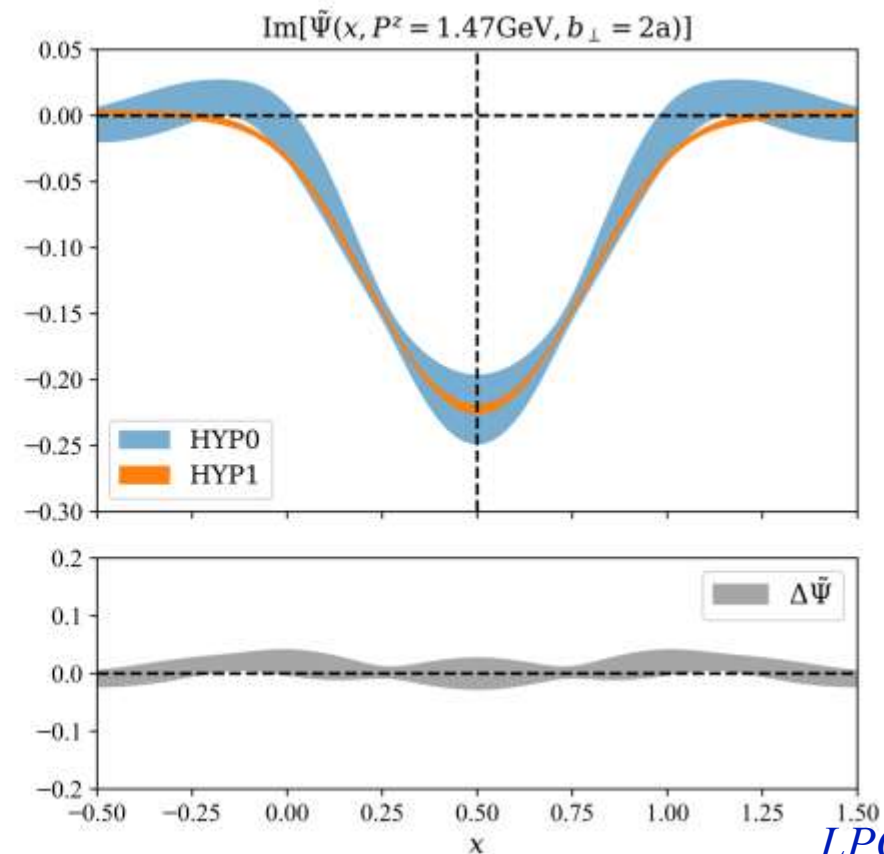
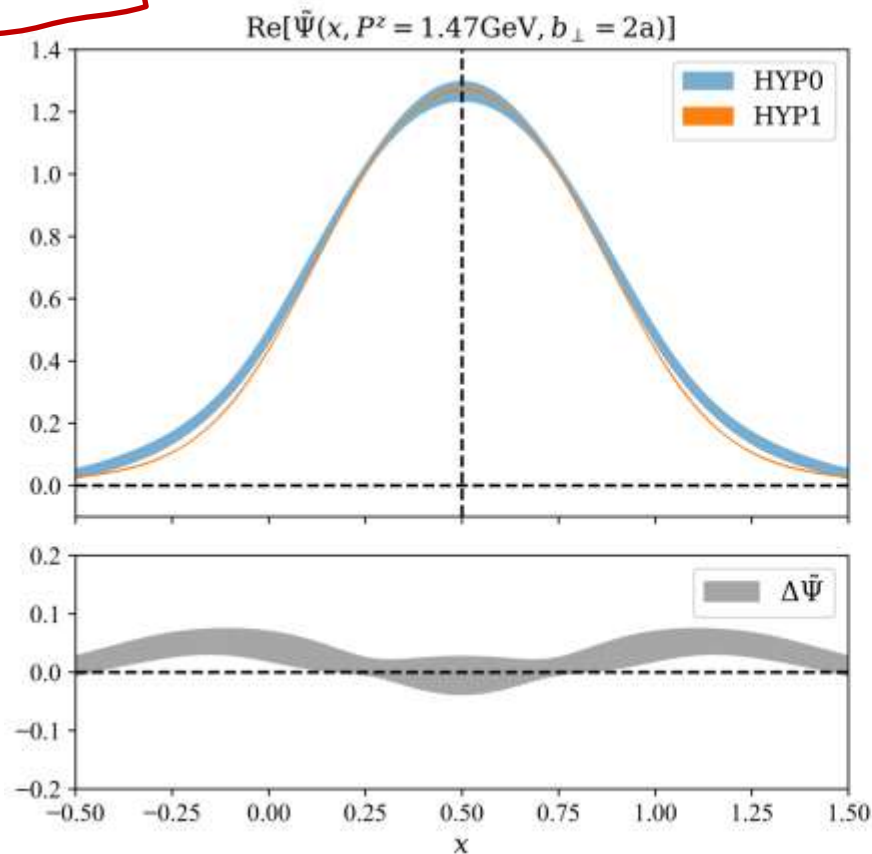


Result for Quasi-TMDWFs in Momentum Space

20

Different HYP
steps

$$\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z) = \lim_{L \rightarrow \infty} \int \frac{dz}{2\pi} e^{ixzP^z} \tilde{\Phi}^{\pm}(z, P^z, L, b, \mu)$$



LPC, in preparation

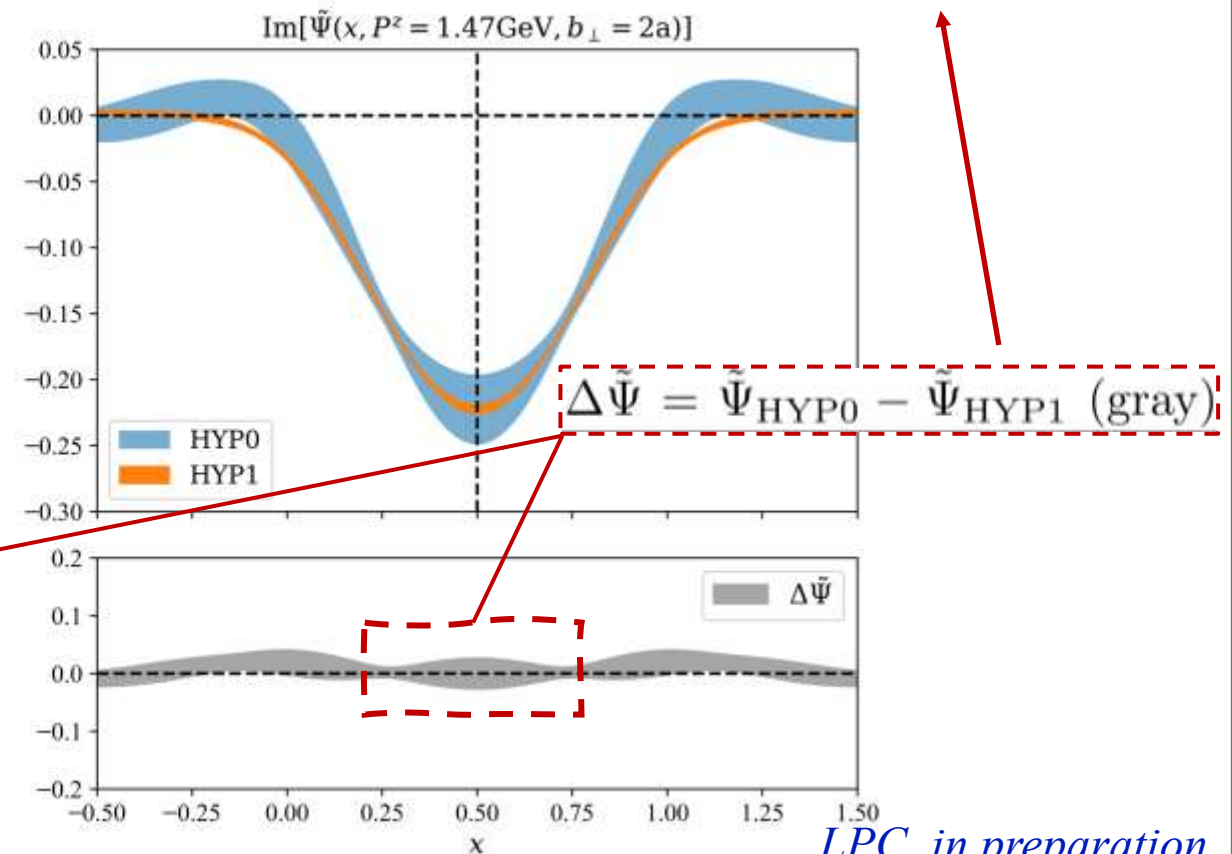
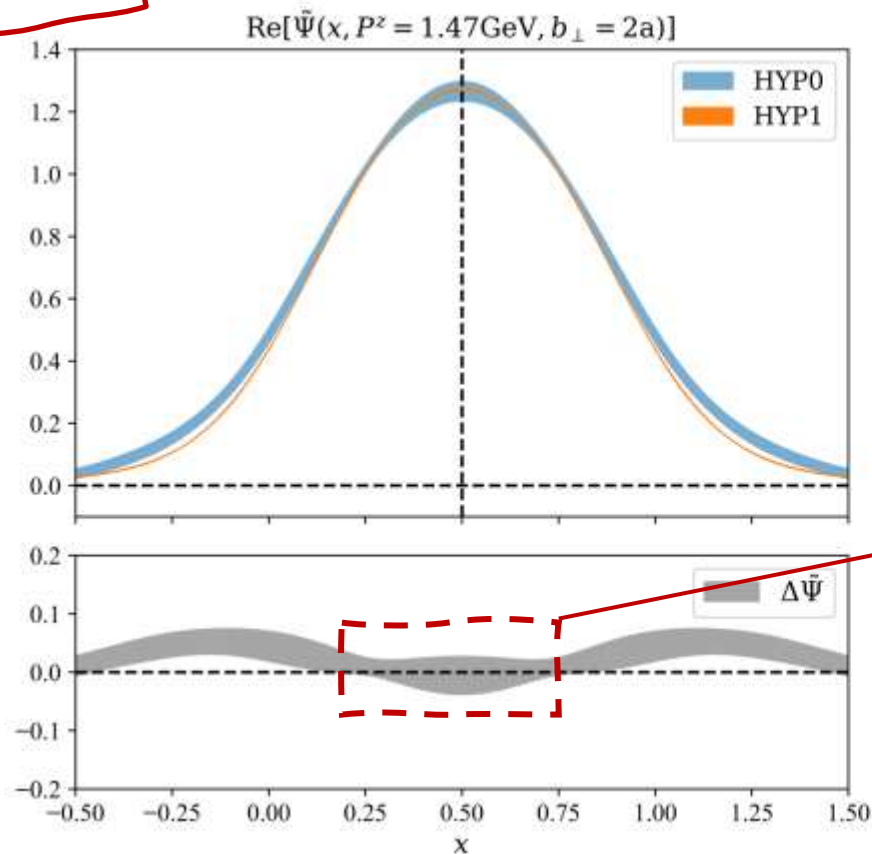
Result for Quasi-TMDWFs in Momentum Space

21

Different HYP
steps

$$\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z) = \lim_{L \rightarrow \infty} \int \frac{dz}{2\pi} e^{ixzP^z} \tilde{\Phi}^{\pm}(z, P^z, L, b, \mu)$$

HYP smearing:
✓ Better signal;
✓ Same physics.



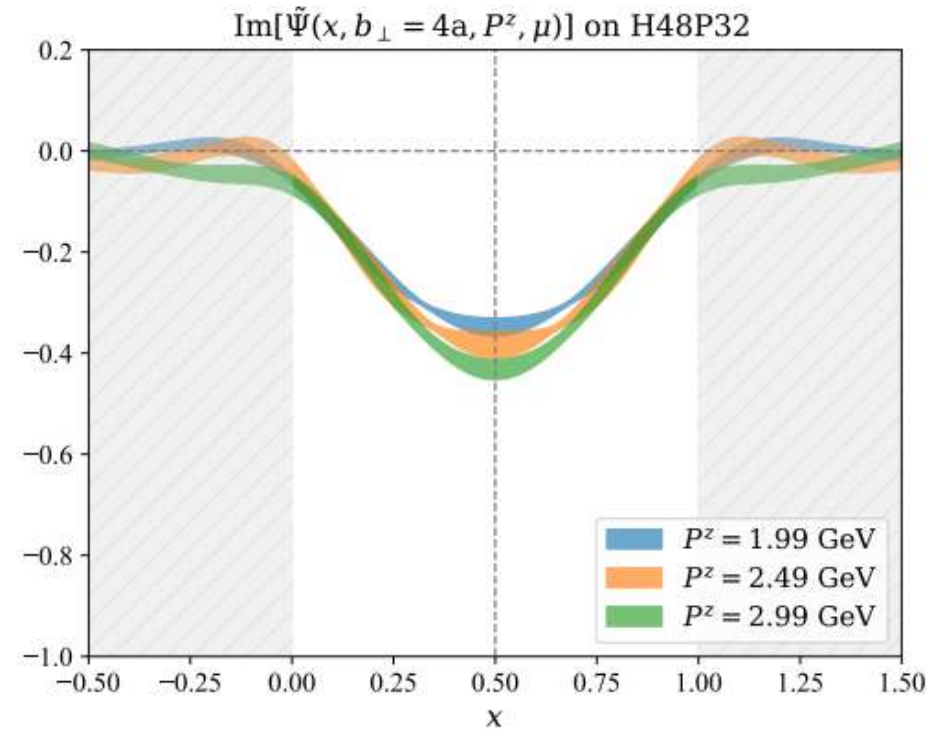
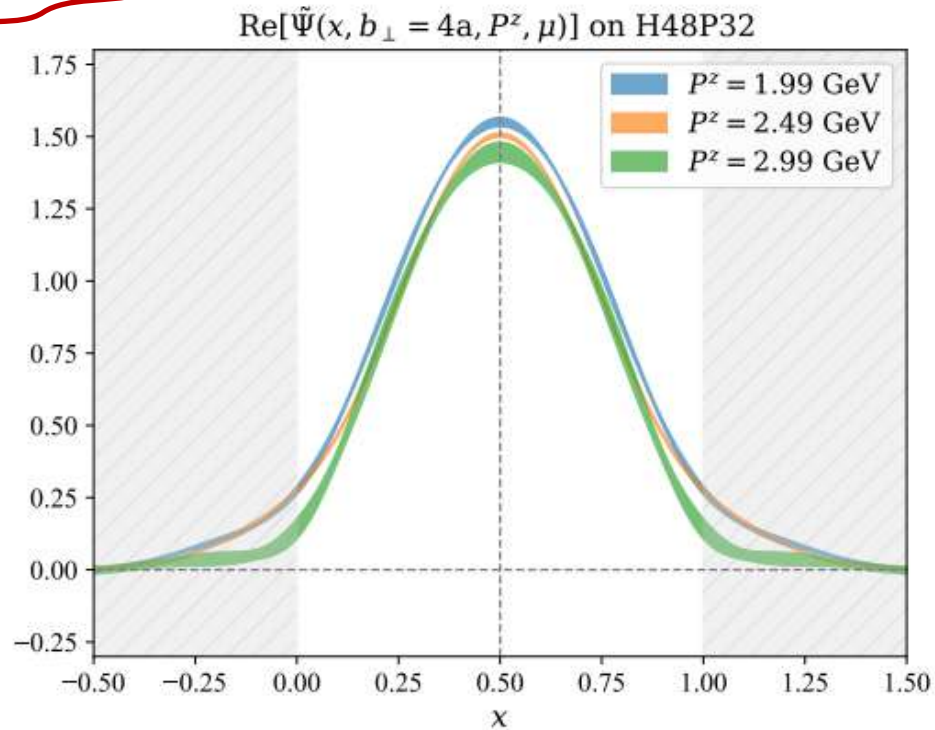
LPC, in preparation

Result for Quasi-TMDWFs in Momentum Space

22

$$\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z) = \lim_{L \rightarrow \infty} \int \frac{dz}{2\pi} e^{ixzP^z} \tilde{\Phi}^{\pm}(z, P^z, L, b, \mu)$$

Different P^z

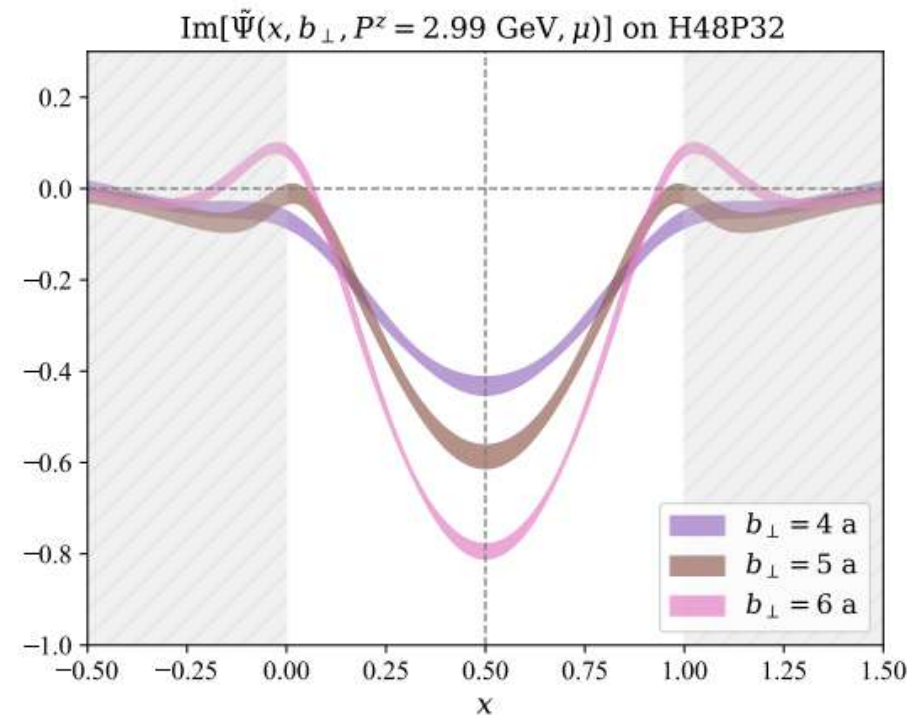
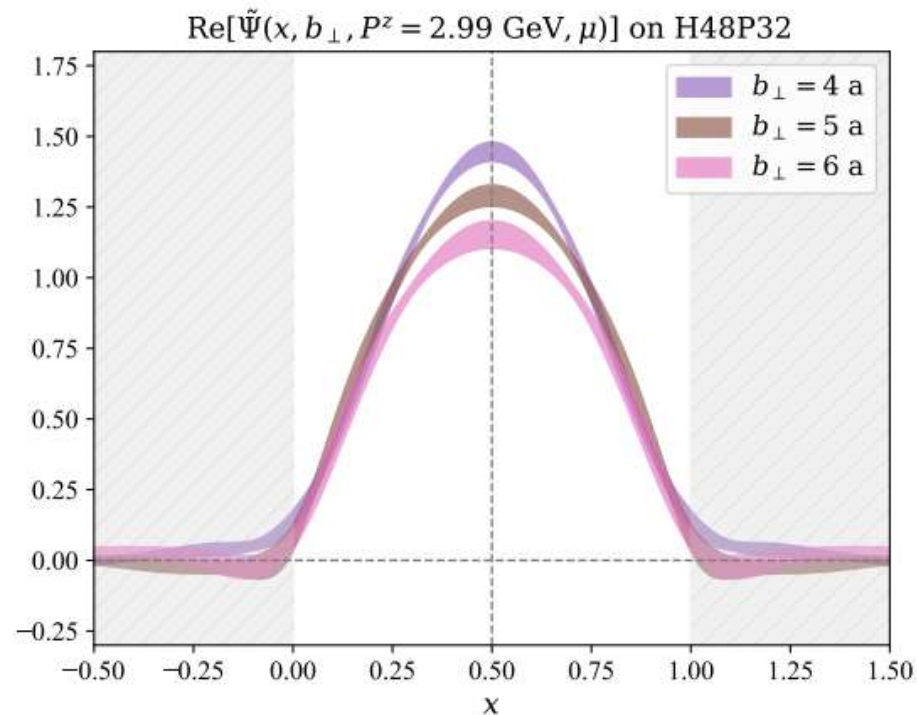


Result for Quasi-TMDWFs in Momentum Space

23

$$\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z) = \lim_{L \rightarrow \infty} \int \frac{dz}{2\pi} e^{ixzP^z} \tilde{\Phi}^{\pm}(z, P^z, L, b, \mu)$$

Different $b_{\perp}$



➤ Motivation

➤ **Lattice QCD calculation of Collins-Soper Kernel**

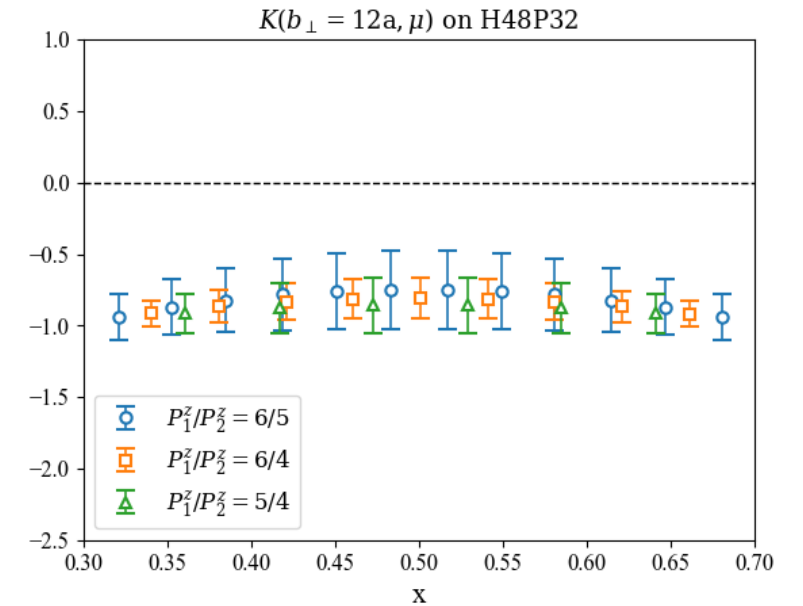
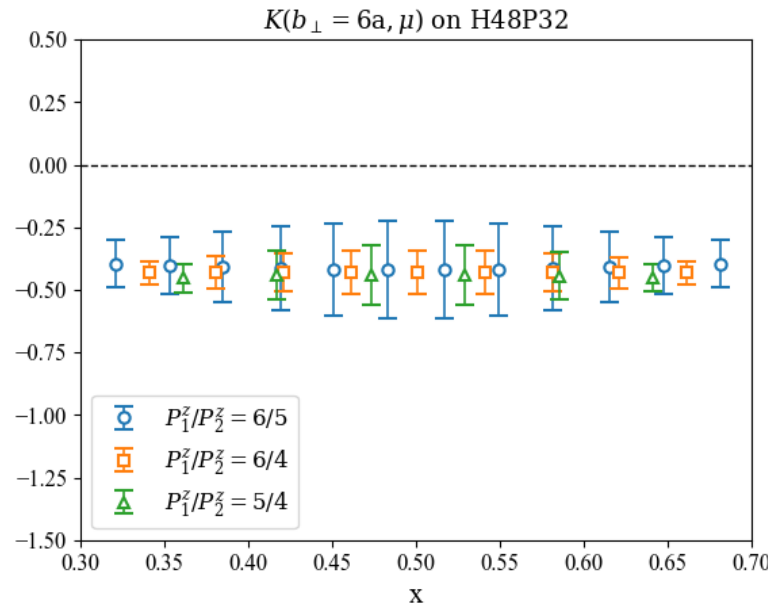
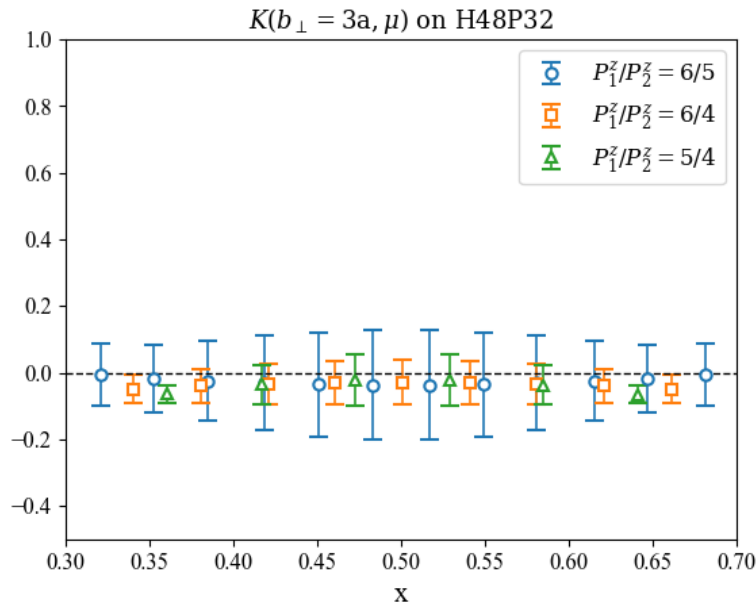
- Quasi-TMD Wave Function
- **Collins-Soper Kernel**

➤ Summary and Outlook

quasi-TMDWFs

$$K(b_{\perp}, \mu; a, x, P_1^z, P_2^z) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \left(\frac{h(xP_2^z, \mu) \tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, P_1^z)}{h(xP_1^z, \mu) \tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, P_2^z)} \right)$$

matching kernel



(For better visualization, we present a subset of points selected from the 200 data points in each of the 3 cases.)

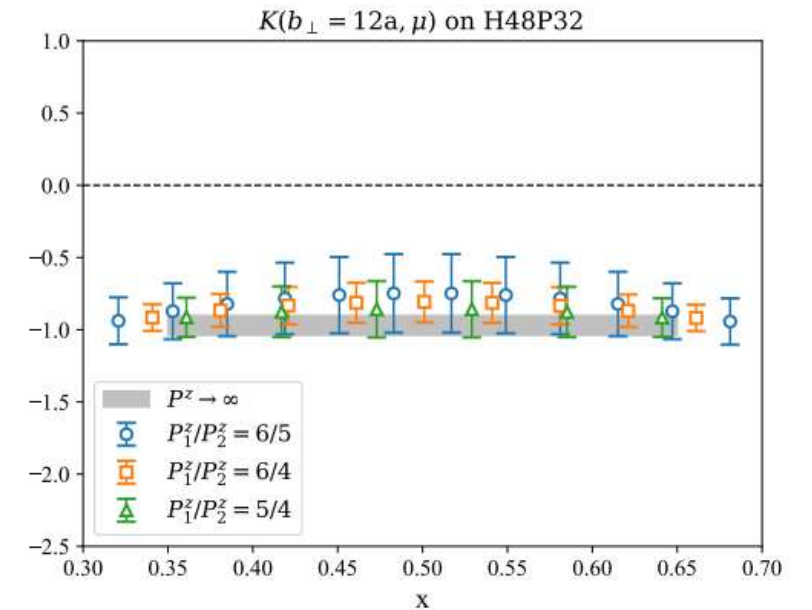
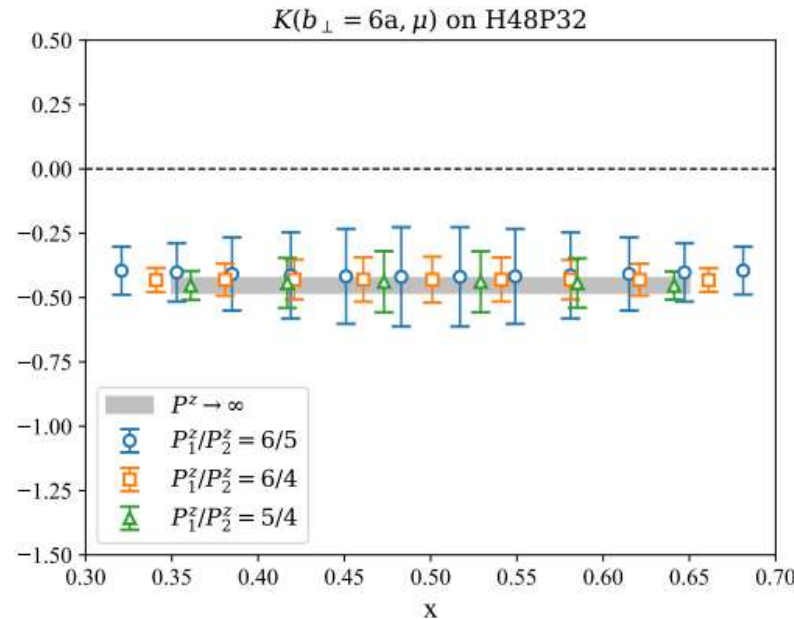
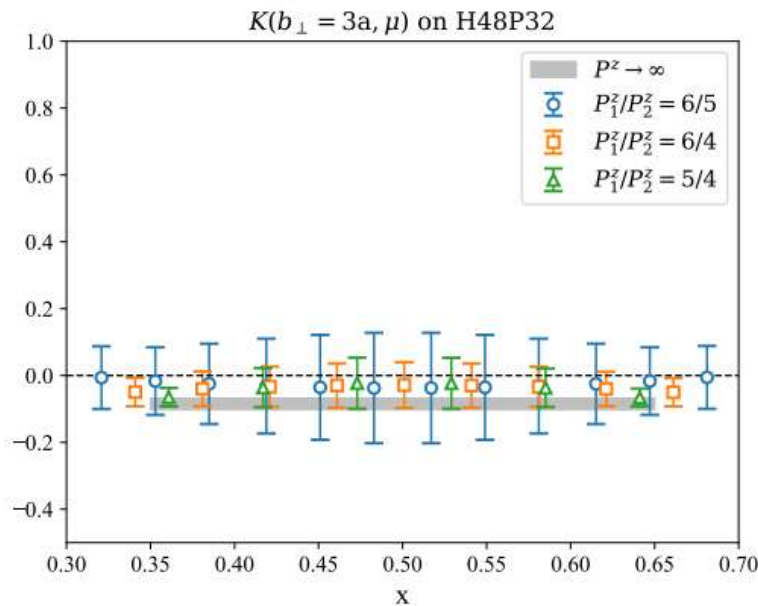
LPC, in preparation

Avkhadiev, et al., PRD 108, 114505 (2023).

Large Momentum Limit Extrapolation

26

$$K(b_{\perp}, \mu; a, x, P_1^z, P_2^z) = K(b_{\perp}, \mu; a) + \left[\frac{1}{(P_1^z)^2} - \frac{1}{(P_2^z)^2} \right] \frac{1}{x^2(1-x)^2} A(b_{\perp}, x, \mu; a)$$



(For better visualization, we present a subset of points selected from the 200 data points in each of the 3 cases.)

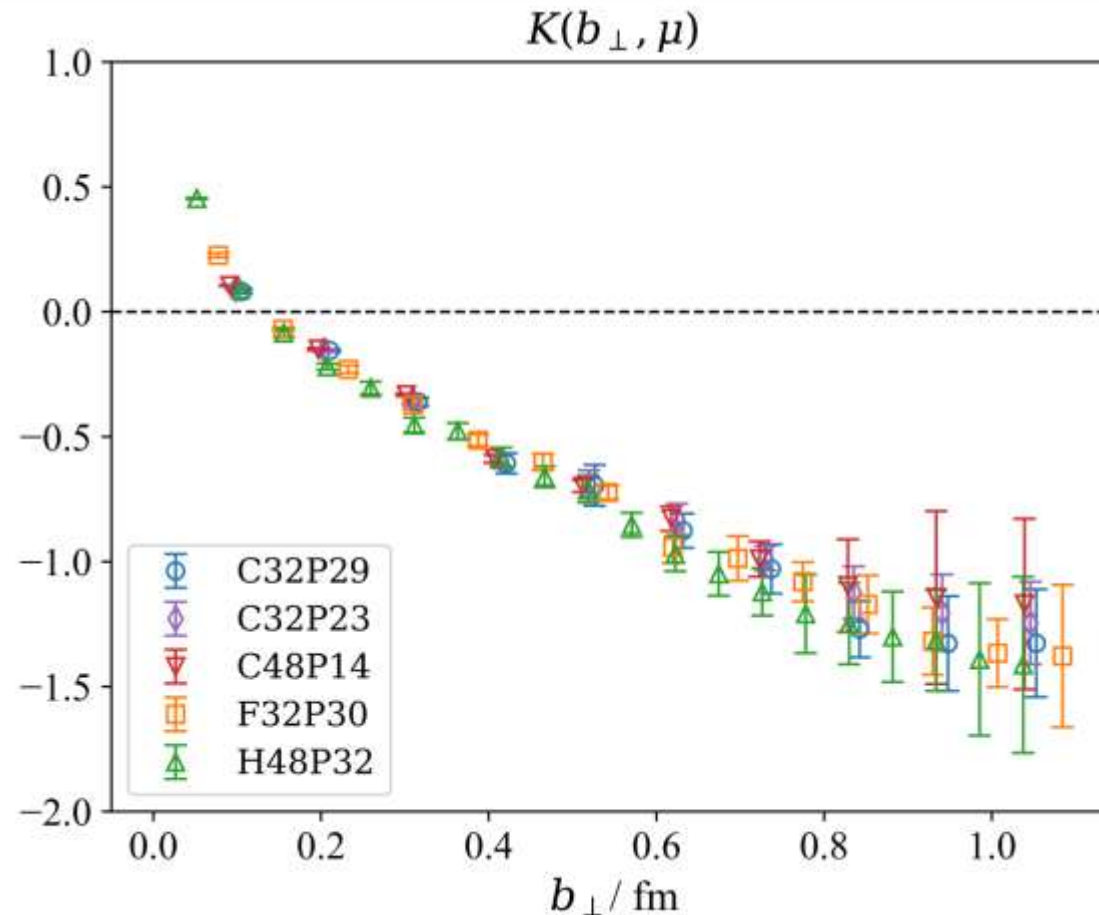
LPC, in preparation

Large Momentum Limit Extrapolation

27

$$K(b_{\perp}, \mu; a, x, P_1^z, P_2^z) = K(b_{\perp}, \mu; a) + \left[\frac{1}{(P_1^z)^2} - \frac{1}{(P_2^z)^2} \right] \frac{1}{x^2(1-x)^2} A(b_{\perp}, x, \mu; a)$$

... repeat for each $b_{\perp}$
and each ensemble ...

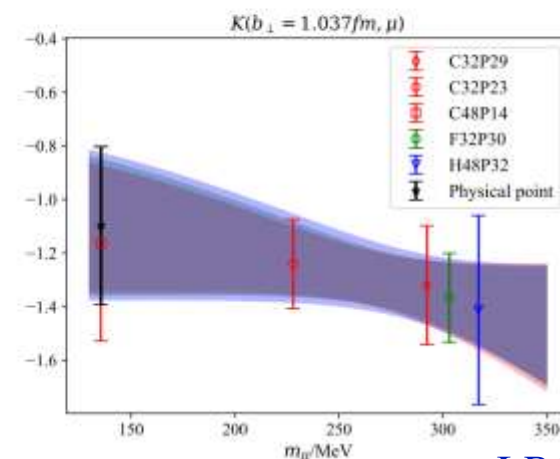
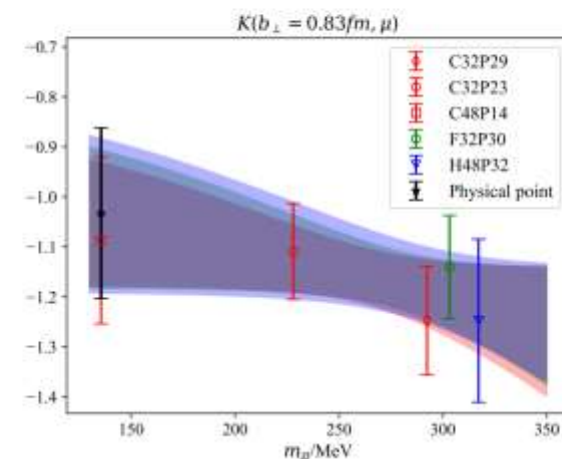
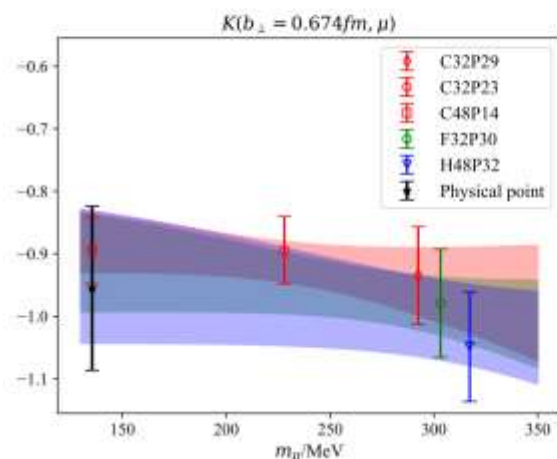
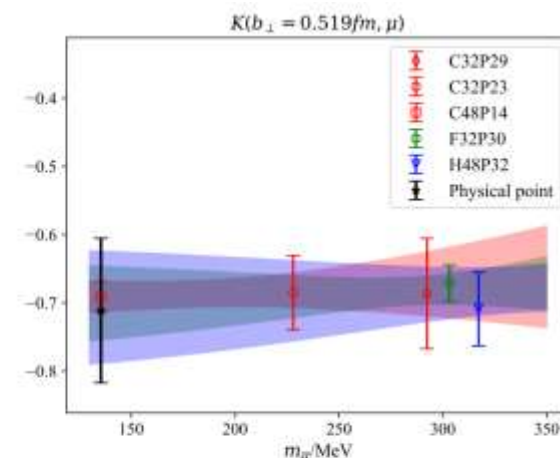
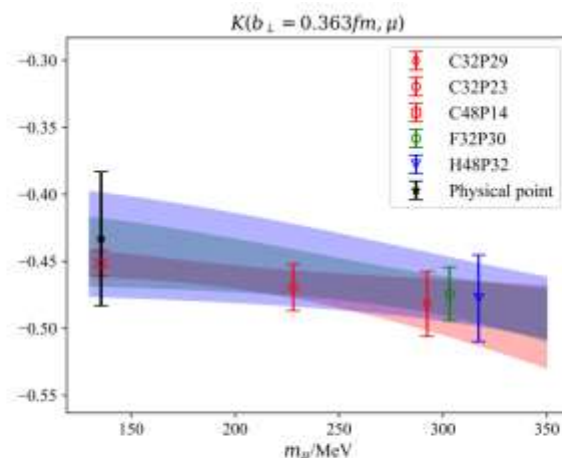
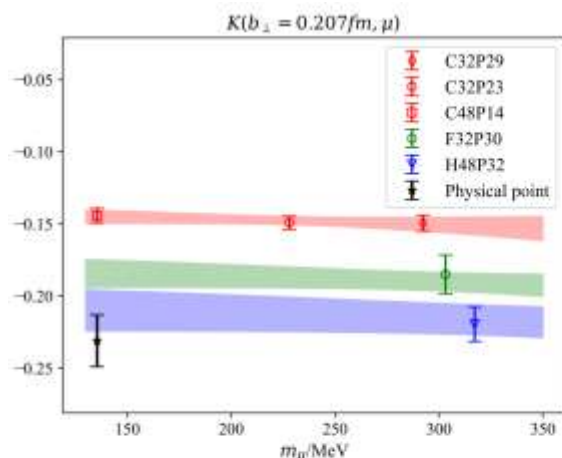


LPC, in preparation

Continuum Limit and Physical Mass Extrapolation

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$$K(b_{\perp}, \mu; a, m_{\pi}) = K(b_{\perp}, \mu) + a^2 B(b_{\perp}, \mu) + (m_{\pi}^2 - m_{\pi,phy}^2) C(b_{\perp}, \mu)$$

*LPC, in preparation*

➤ We have considered four sources of systematic uncertainties:

1.the unphysical imaginary part,

2.the large- λ extrapolation,

3.the large-momentum extrapolation,

4.the continuum and physical mass extrapolations.

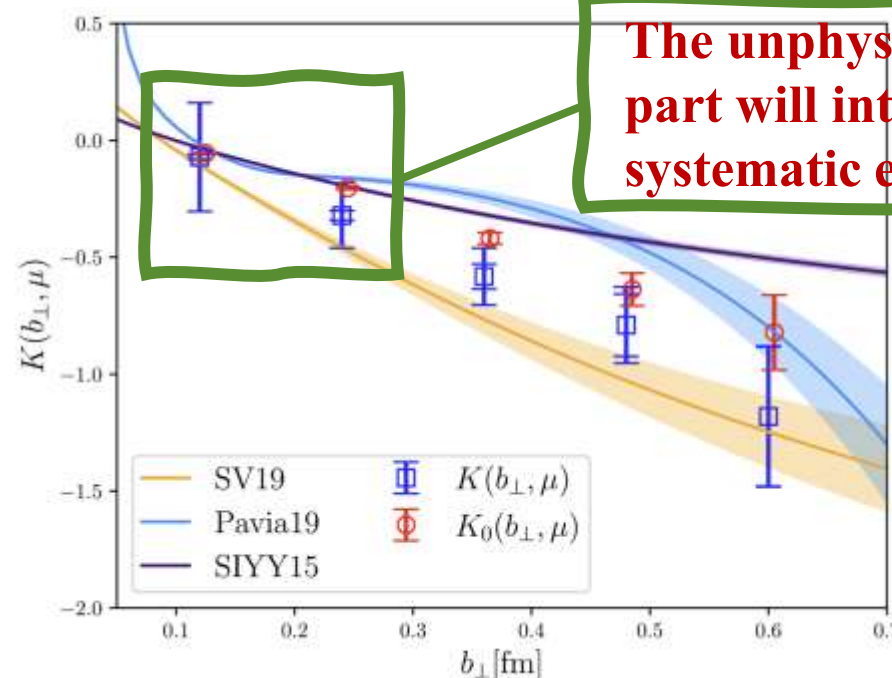
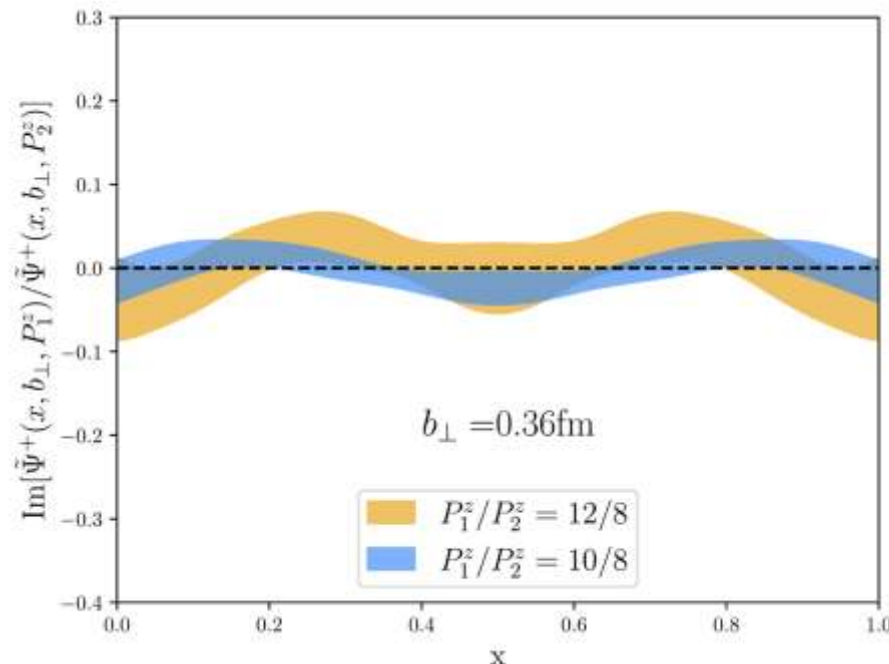
“Imaginary part” of the CS kernel

30

- Systematic uncertainty from the imaginary part of the CS kernel:

$$\sigma_{\text{sys}}^{\text{Im}} = \sqrt{K(b_{\perp}, \mu)^2 + \text{Im}[K(b_{\perp}, \mu)^2] - |K(b_{\perp}, \mu)|}$$

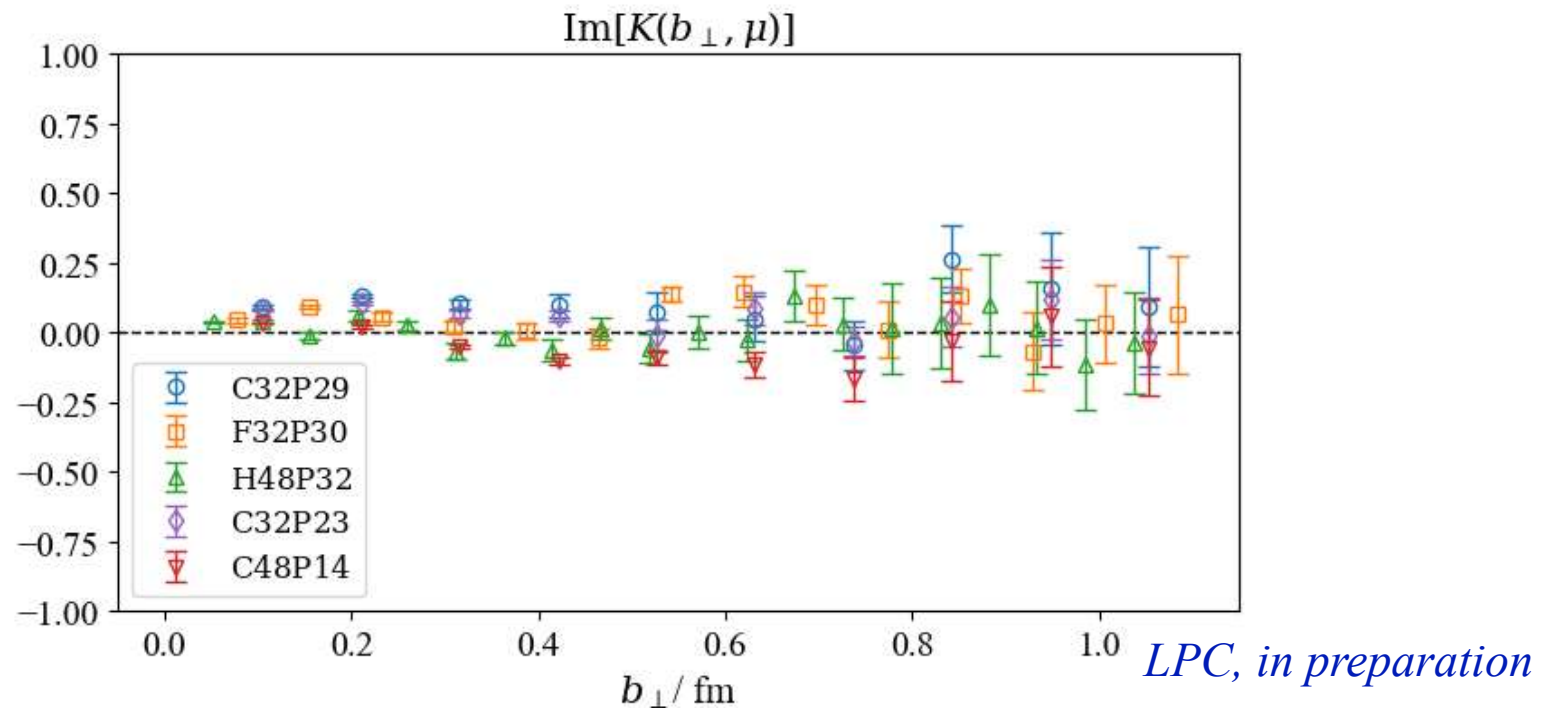
- Previous studies have shown that the imaginary part of the CS kernel primarily originates from the matching kernel.



“Imaginary part” of the CS kernel

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- The quasi-TMDWF factorization formula requires $xP^z b_\perp \gg 1$, at small $b_\perp$ region, remain the **dependence on $b_\perp$** when computing the matching kernel.



Imaginary parts of the CS kernel are all very close to zero!

The Systematic Uncertainty

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➤ We have considered four sources of systematic uncertainties:

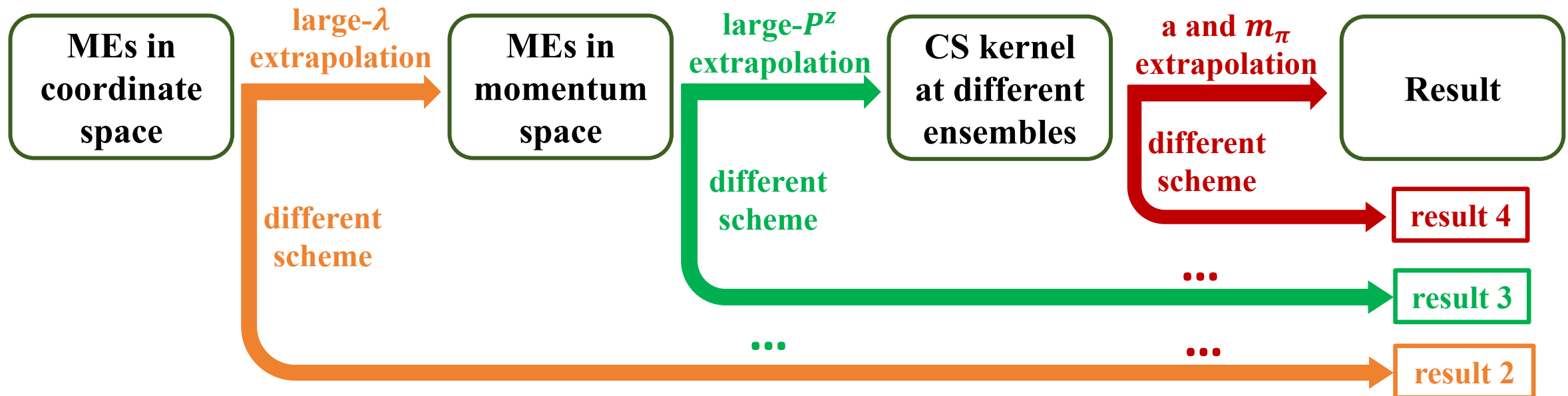
1.the unphysical imaginary part,

2.the large- λ extrapolation,

3.the large-momentum extrapolation,

4.the continuum and physical mass extrapolations.

➤ For these extrapolations(2, 3, 4), the difference of the result with different extrapolation schemes denotes the systematic error.



The Systematic Uncertainty

33

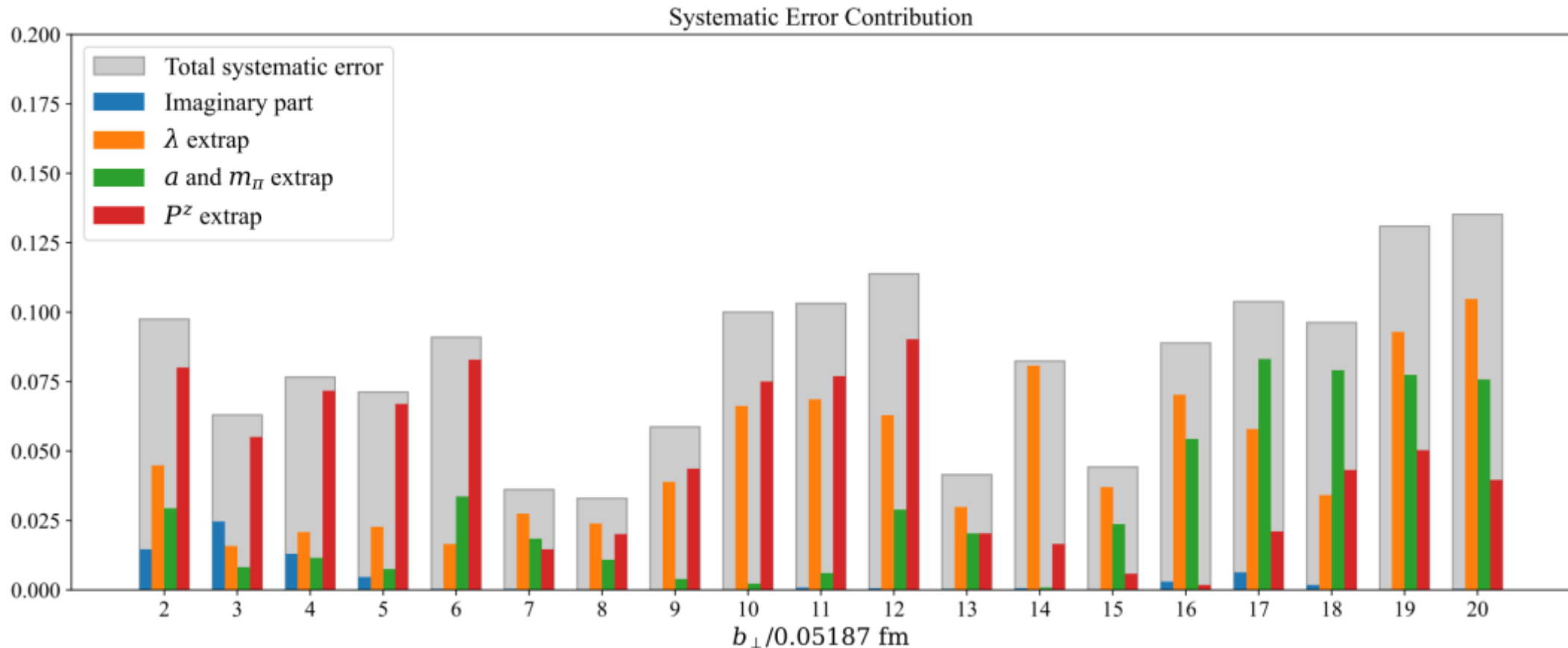
➤ We have considered four sources of systematic uncertainties:

1.the unphysical imaginary part,

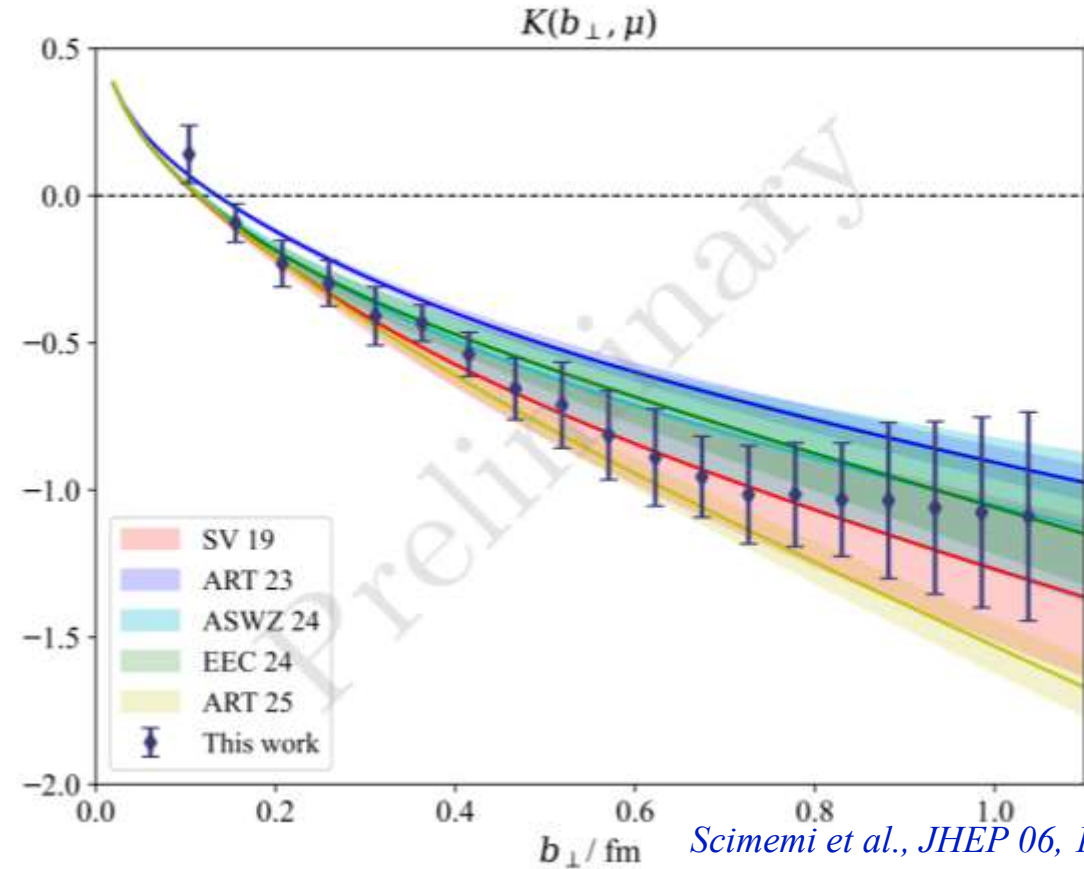
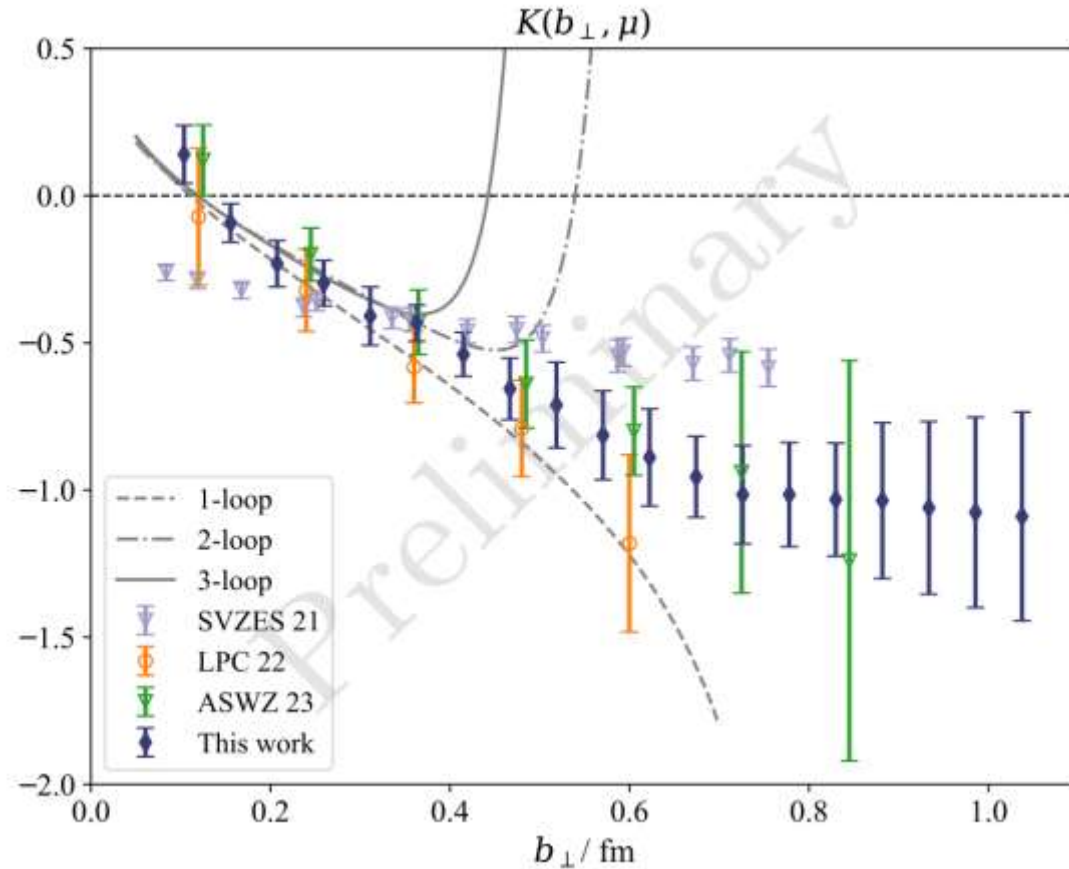
2.the large- λ extrapolation,

3.the large-momentum extrapolation,

4.the continuum and physical mass extrapolations.



✓ Continuum limit; ✓ Physical mass; ✓ $b_{\perp} = 1\text{fm}$.



Schlemmer et al., JHEP 08, 004 (2021);
Chu et al. (LPC), PRD 106, 034509 (2022);
Avkhadiev, et al., PRD 108, 114505 (2023).

Scimemi et al., JHEP 06, 137 (2020);
Moos et al., JHEP 05, 036 (2024);
Avkhadiev et al., PRL. 132, 231901 (2024);
Kang et al., arXiv:2410.21435;
Moos et al., arXiv:2503.11201.

Review of CS Kernel Calculation

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	Pion mass	Renormalization	Fourier transform	Matching	x-plateau	Continuum	$b_{\perp}$ separation (*)
SWZ20 PRD 102, 014511 (2020) Quenched	1.2 GeV	Yes	Yes	LO	Yes	No	0.8fm
LPC20 PRL 125,192001 (2020)	547 MeV	N/A	N/A	LO	N/A	No	0.3fm
SVZES JHEP08 (2021), 2302.06502	422 MeV	N/A	N/A	NLO	N/A	No	0.5fm
PKU/ETMC 21 PRL128, 062002 (2022)	827 MeV	N/A	N/A	LO	N/A	No	0.3fm
SWZ21 PRD 104,114502 (2022)	580 MeV	Yes	Yes	NLO	Yes	No	0.5fm
LPC22 PRD 106, 034509 (2022)	670 MeV	Yes	Yes	NLO	Yes	No	0.6fm
LPC23 JHEP 08,172 (2023)	220 MeV	Yes	Yes	NLO	Yes	No	0.6fm
ASWZ23 PRD108, 114505 (2023)	148.8 MeV	Yes	Yes	uNNLL	Yes	No	0.8fm
ASWZ24 PRL132, 231901 (2024)	148.8 MeV	Yes	Yes	uNNLL	Yes	Model (0.15,0.12,0.09) fm	0.9fm
LPC 25 (preliminary)	135.5 MeV	Yes	Yes	uNLO	Yes	Yes	~1.0fm

Table adapted/updated from Yong Zhao's collection

(*) Only the results that deviate from 0 by more than 2σ are considered.

- Motivation
- Lattice QCD calculation of Collins-Soper Kernel
 - Quasi-TMD Wave Function
 - Collins-Soper Kernel
- **Summary and Outlook**

We present the lattice QCD calculation of CS kernel:

- ✓ **Systematic uncertainty** have been carefully addressed;
- ✓ Physical extrapolation include large momentum, **continuum limit** and **physical mass**;
- ✓ The CS kernel has been extracted in the long-distance region ($b_{\perp} = 1\text{fm}$).

In the future, we will:

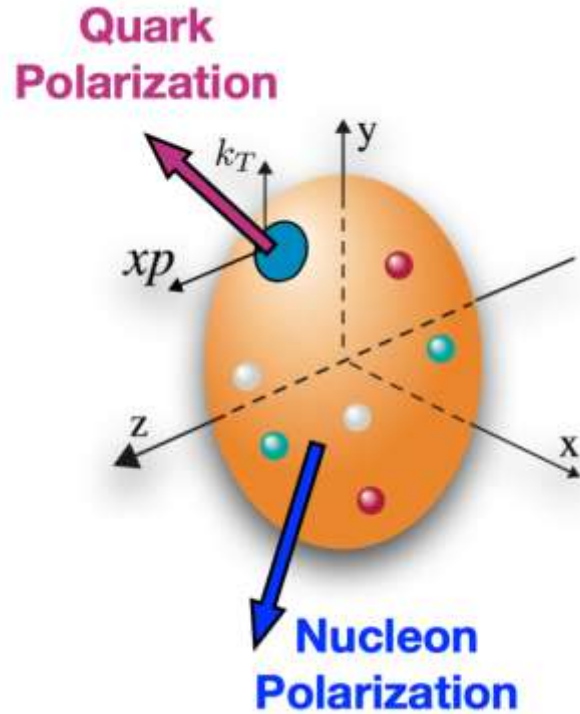
- Better control of uncertainties;
- Polarized TMDPDFs calculation;
-

Thank you for your attention!

Backup

TMDPDFs: 3D tomography of the nucleon

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Leading Quark TMDPDFs



		Quark Polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Nucleon Polarization	U	$f_1 = \text{Unpolarized}$		$h_1^\perp = \text{Boer-Mulders}$
	L		$g_{1L} = \text{Helicity}$	$h_{1L}^\perp = \text{Worm-gear}$
	T	$f_{1T}^\perp = \text{Sivers}$	$g_{1T}^\perp = \text{Worm-gear}$	$h_1 = \text{Transversity}$ $h_{1T}^\perp = \text{Pretzelosity}$

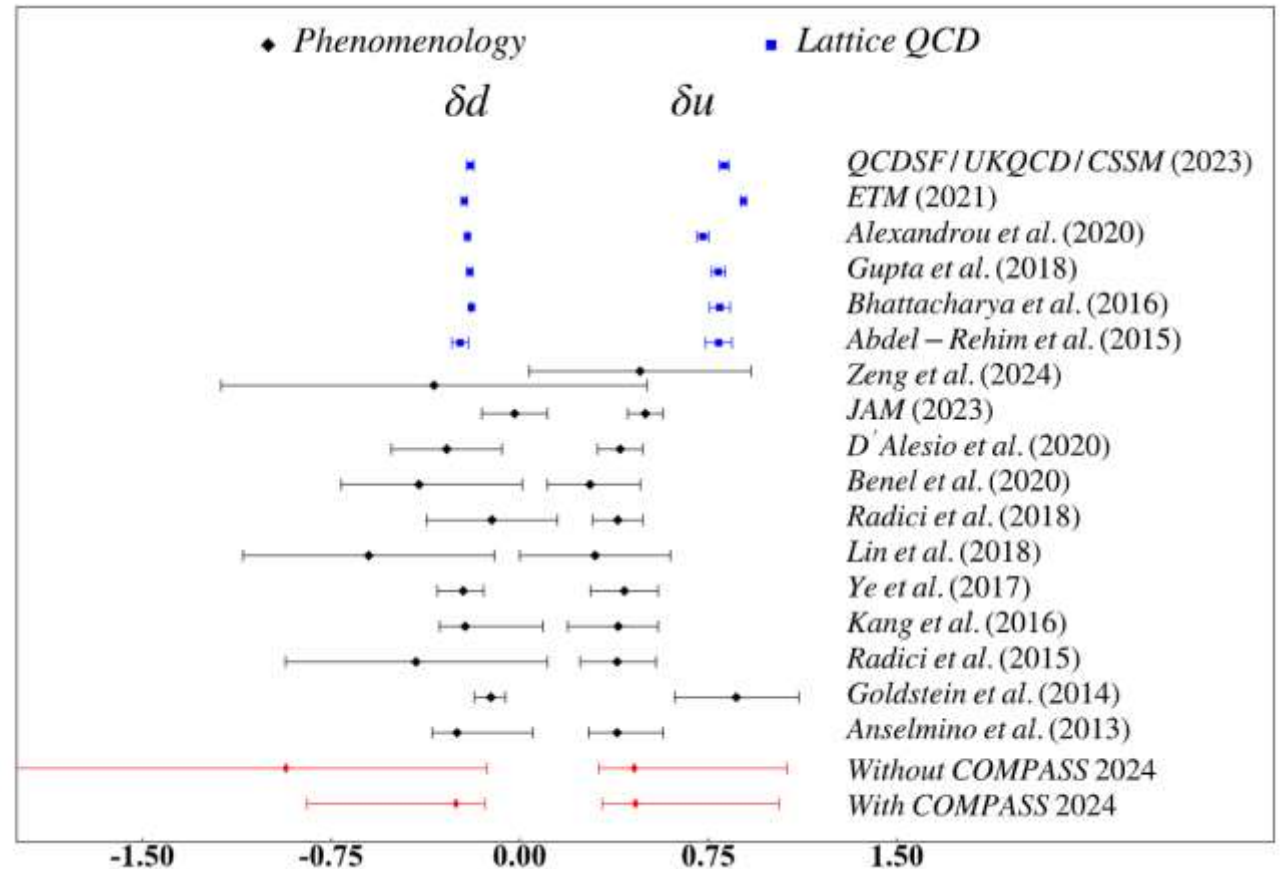
TMDPDFs are indispensable and multifaceted.

- The tensor charge δq is the first moment of the quark transversity distribution function $h_1(x)$:

$$\delta q = \int_0^1 dx [h_1^q(x) - h_1^{\bar{q}}(x)]$$

- There exists a **noticeable difference** between lattice QCD and phenomenological results obtained from global fits.

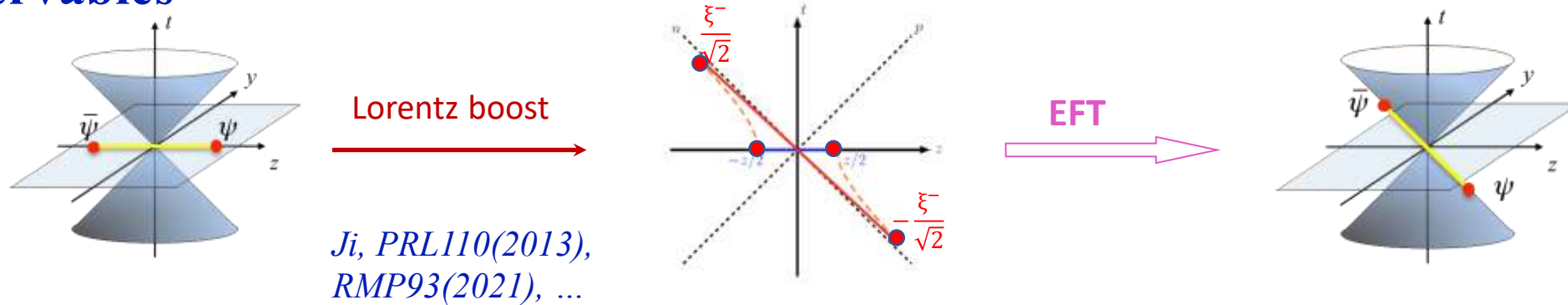
Most recent global data fit-Tensor Charge



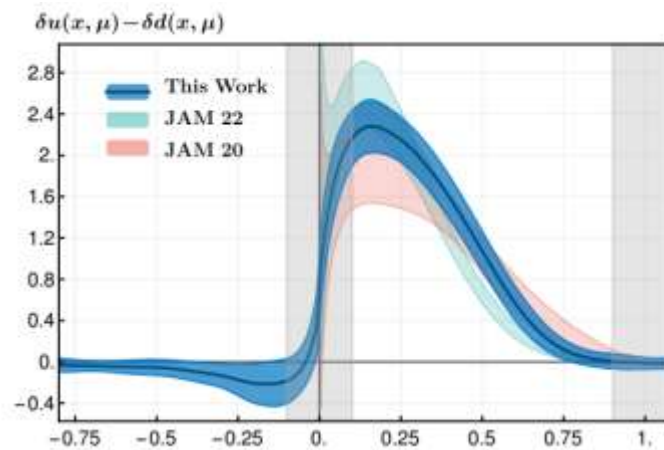
Zeng et al., arXiv:2412.18324

High-precision calculations of TMDs on the lattice are urgently needed.

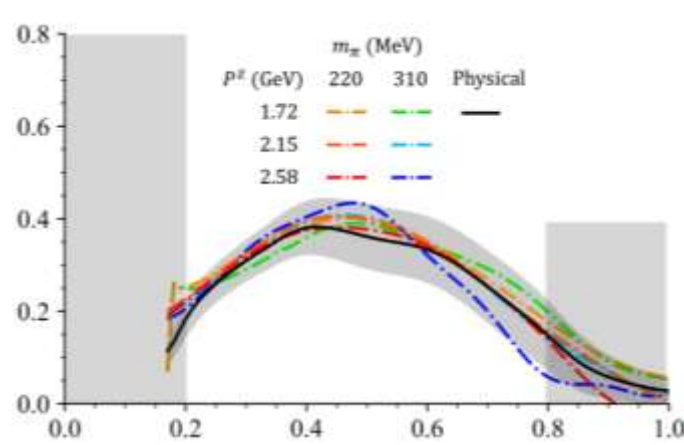
- Large-momentum effective theory: connecting Euclidean lattice and physical observables



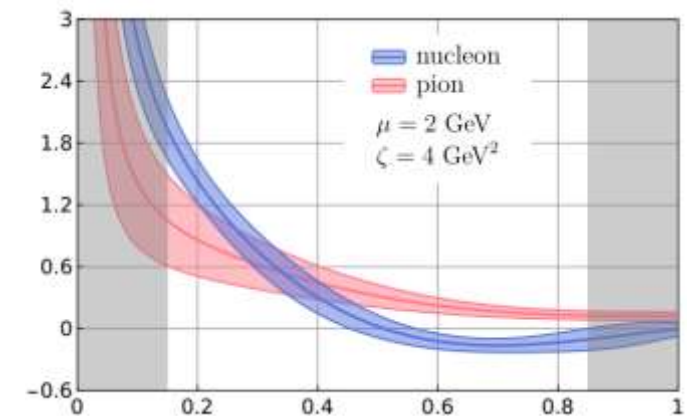
- Achieved great success in the studies of TMDPDFs:



Proton transversity PDF, PRL131(2023)



Unpolarized nucleon TMDPDF, PRD109 (2024)

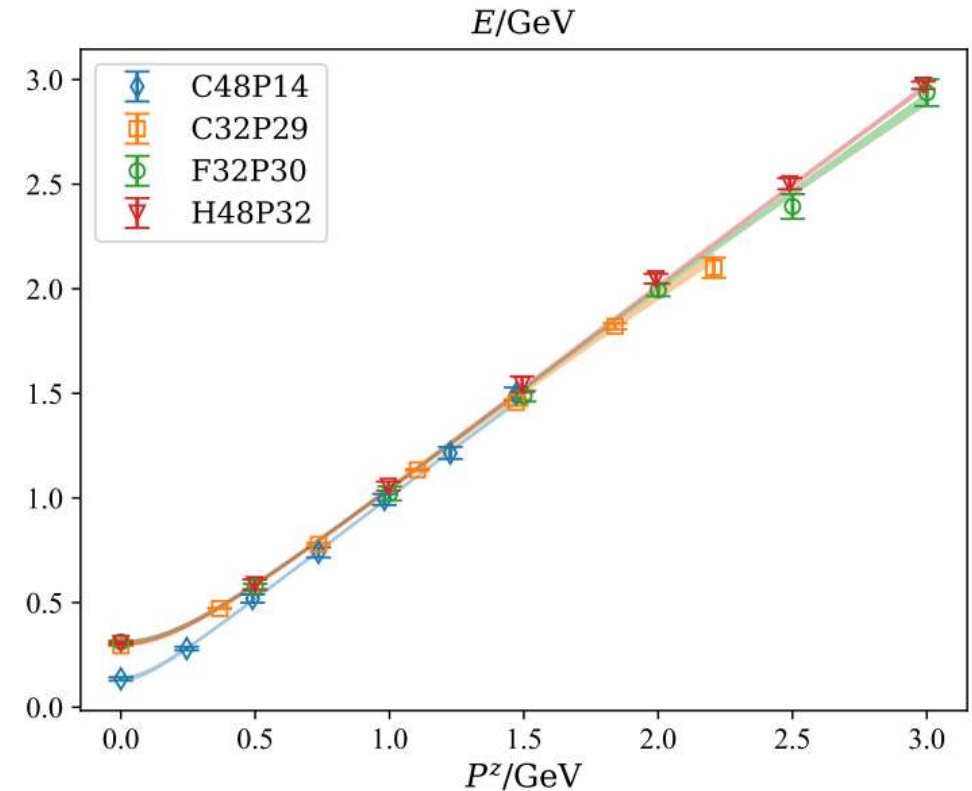


Boer-Mulders Function, [arXiv:2502.11807](https://arxiv.org/abs/2502.11807)

- We use the following fitting formula to fit the dispersion relation of pion,

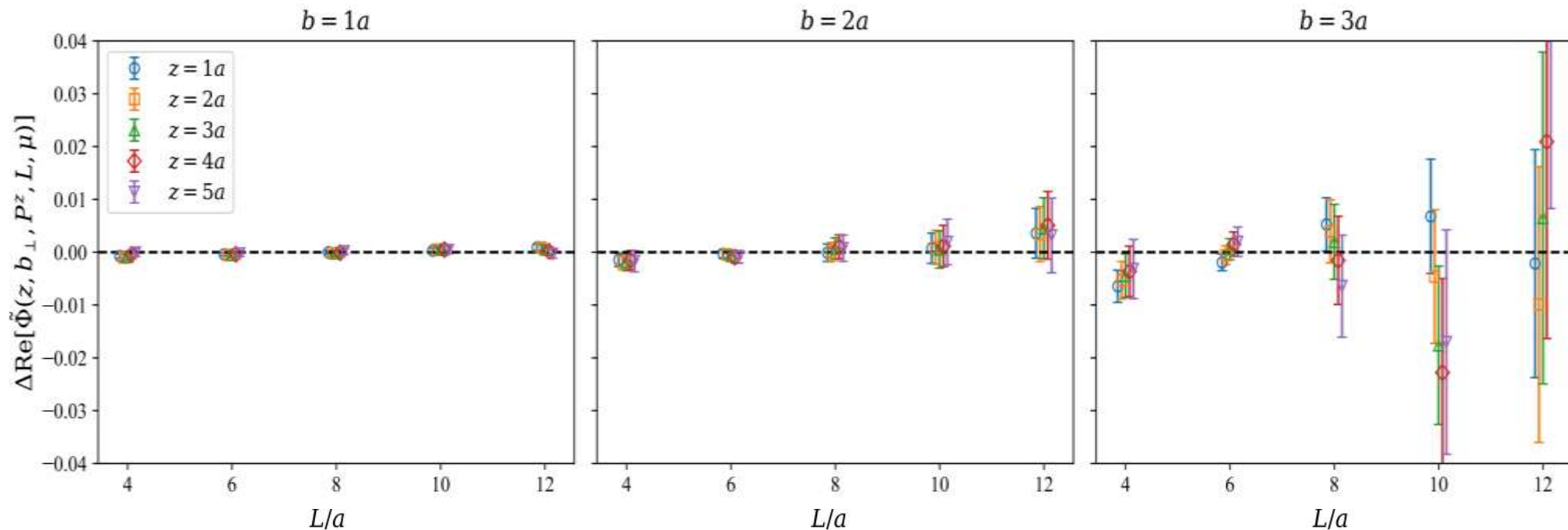
$$E(P^z) = \sqrt{m^2 + b_1(P^z)^2 + b_2(P^z)^4} a^2$$

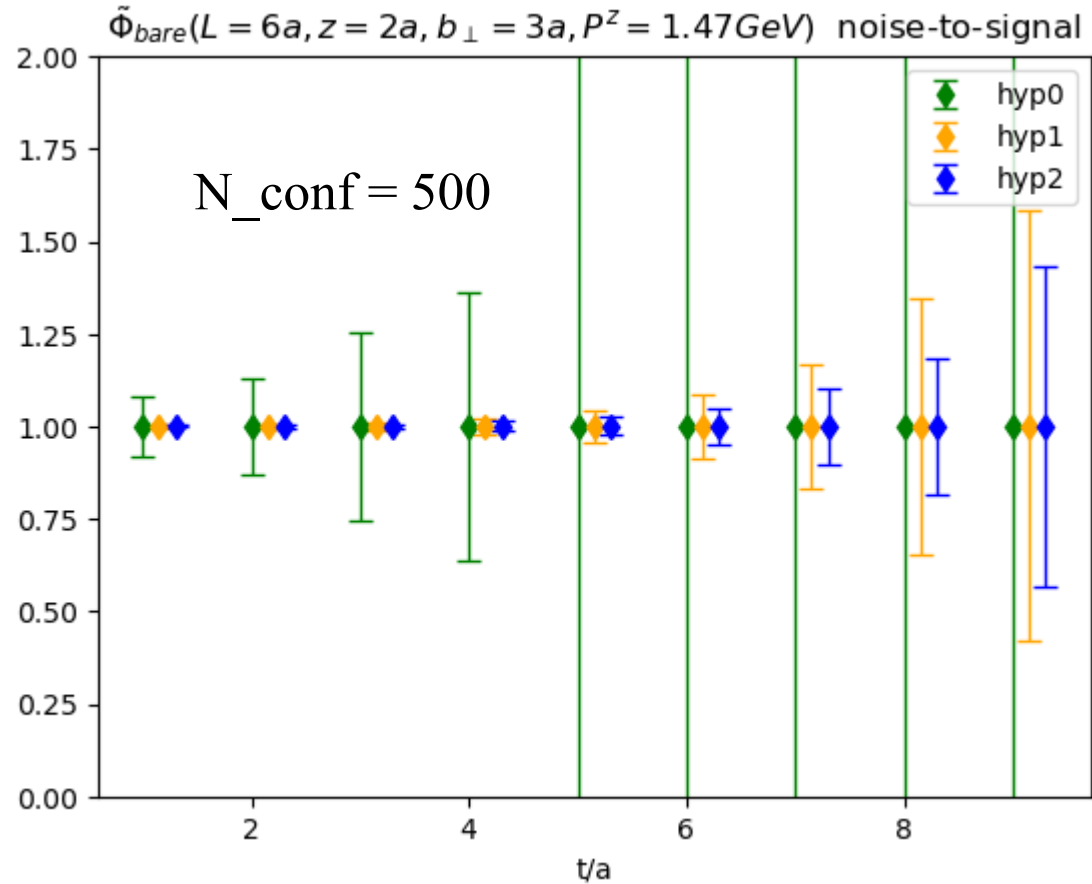
where the fit result is $b_1 = 1.0048$ (90),
 $b_2 = -0.055$ (16).



- The extraction of $\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z)$ requires L to be large enough. When **$L \geq 6a \cong 0.6\text{fm}$** , the matrix elements remain invariant.

$$\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z) = \lim_{L \rightarrow \infty} \int \frac{dz P^z}{2\pi} e^{ixzP^z} \frac{\langle 0 | \bar{q}(z\hat{n}_z + b_{\perp}\hat{n}_{\perp}) \gamma^t \gamma_5 U_{c\pm} q(0) | \pi(P^z) \rangle}{\sqrt{Z_E(2L + |z|, b_{\perp}, \mu)} Z_O(1/a, \mu)}$$





$N_{\text{conf}} = 500$, fix timeslice = $4a$, $P^z = 1.47\text{GeV}$,
the noise-to-signal ratio of bare-TMDWF

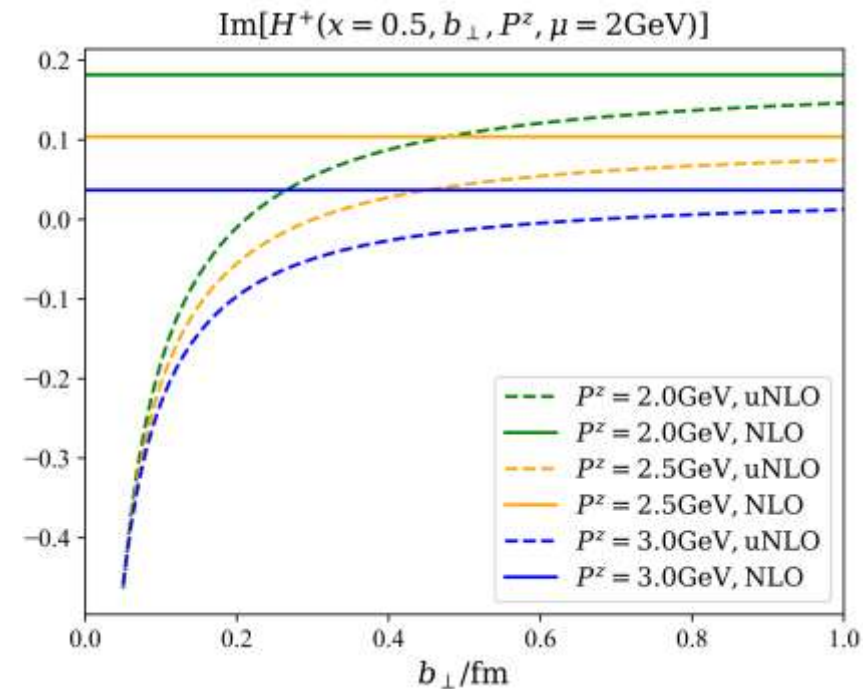
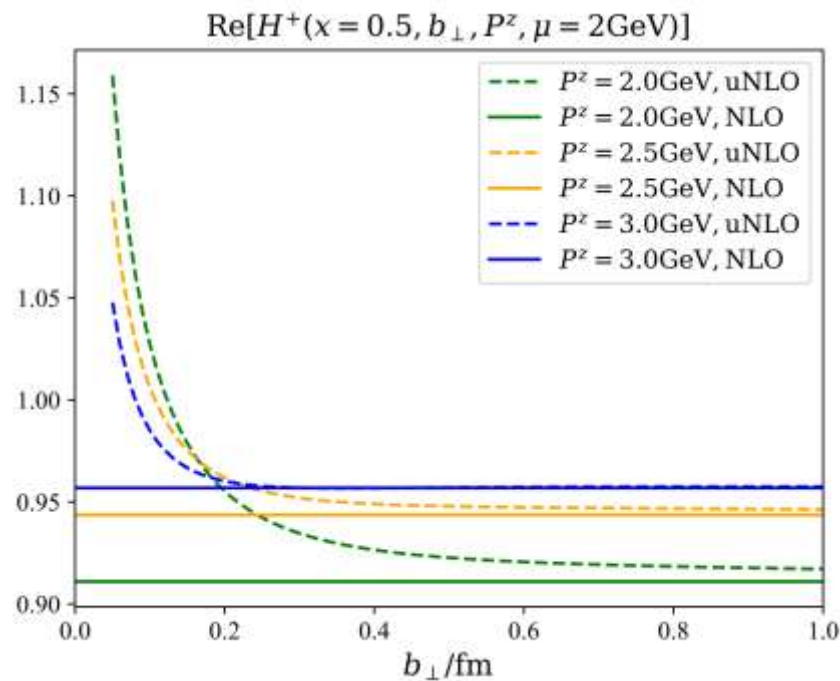
$(b_{\perp}, z)$	HYP 0	HYP 1	HYP 2
(4,0)	0.5509	0.0187	0.0112
(4,3)	0.8883	0.1231	0.0706
(4,6)	2.0127	0.4808	0.1474
(4,9)	30.4924	0.5352	1.2179

- HYP smearing significantly suppress the noise-to-signal ratio of bare-TMDWF
- The noise-to-signal ratio of HYP smearing 1 times is lower than that of without HYP smearing.

“Imaginary part” of the CS kernel

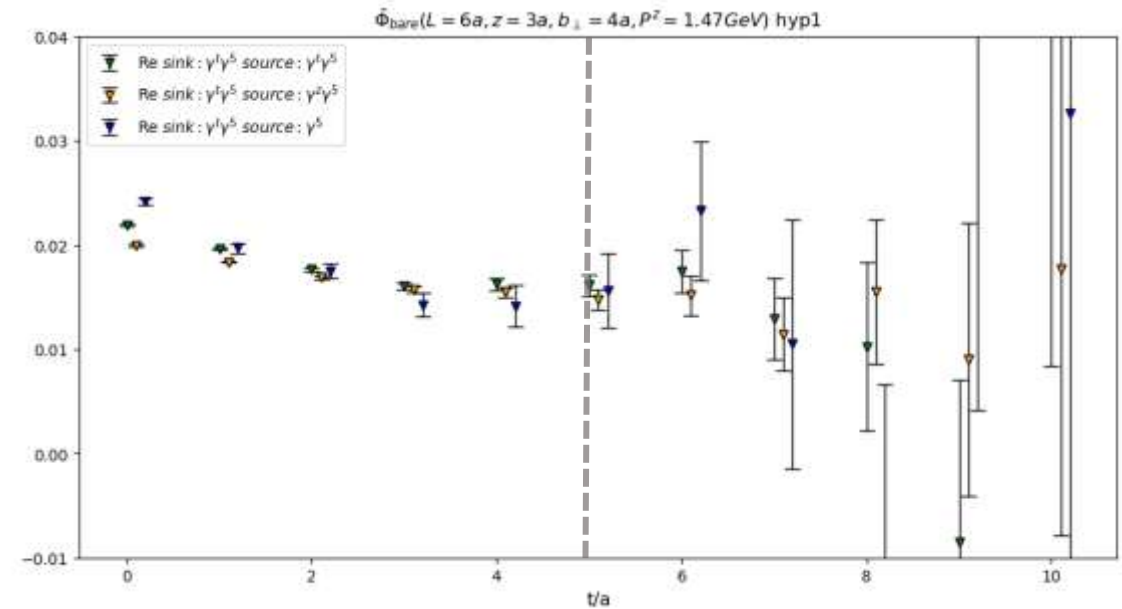
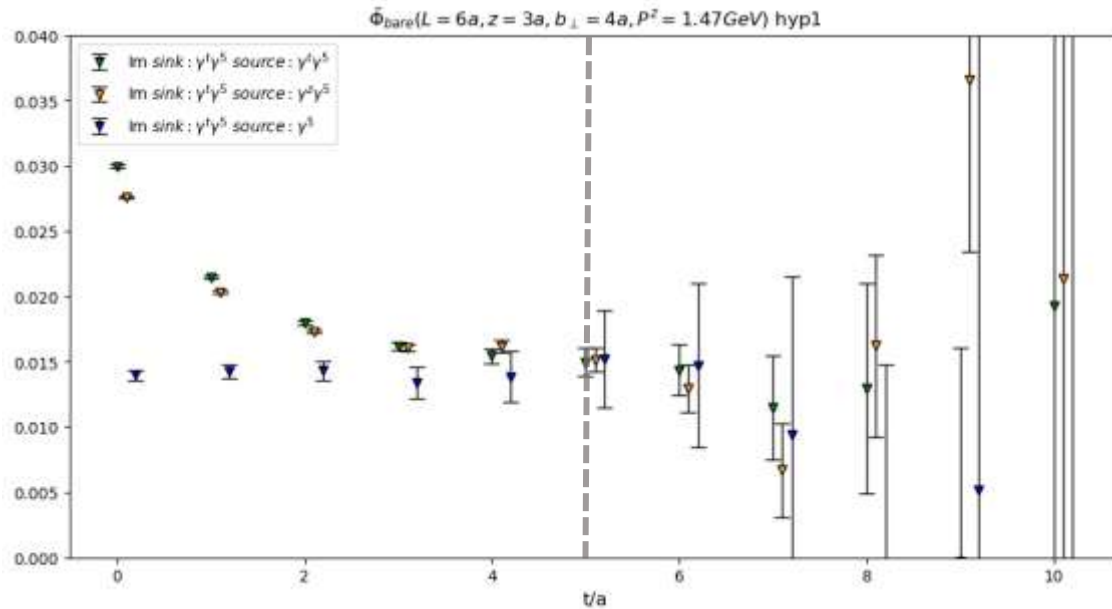
45

- The quasi-TMDWF factorization formula requires $xP^z b_\perp \gg 1, (1-x)P^z b_\perp \gg 1$, at small $b_\perp$ region, remain the **dependence on $b_\perp$** when computing the matching kernel.



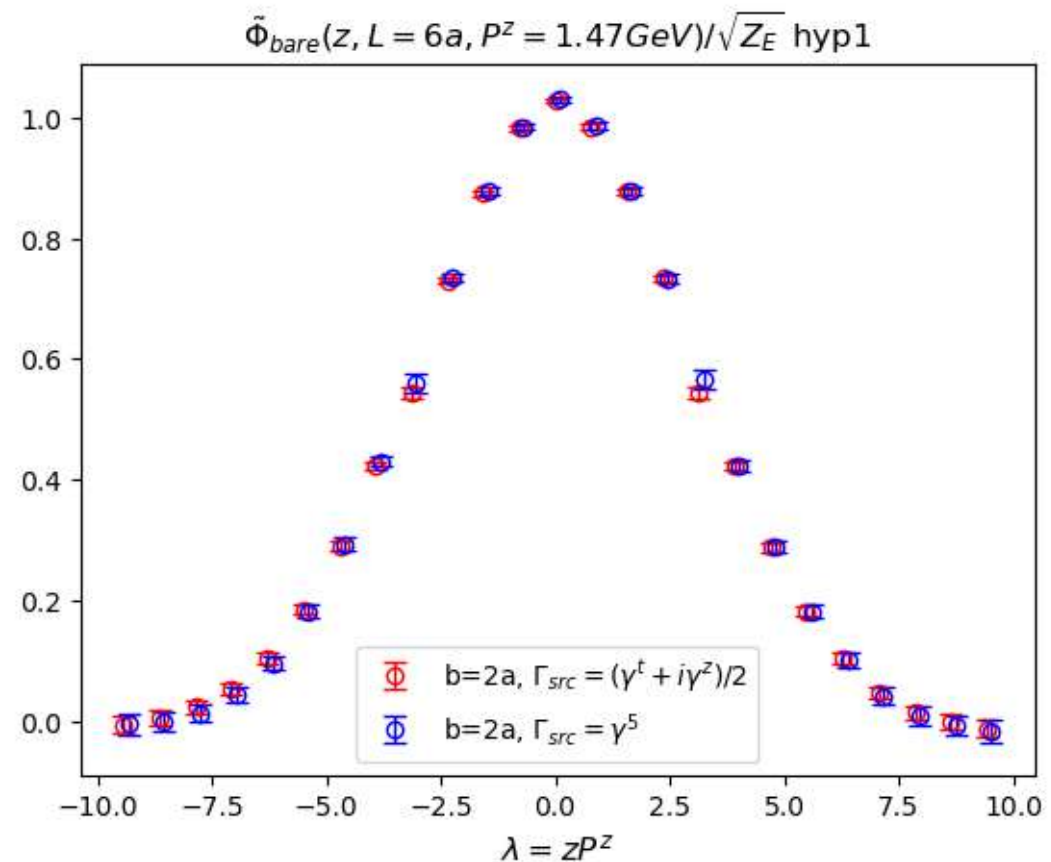
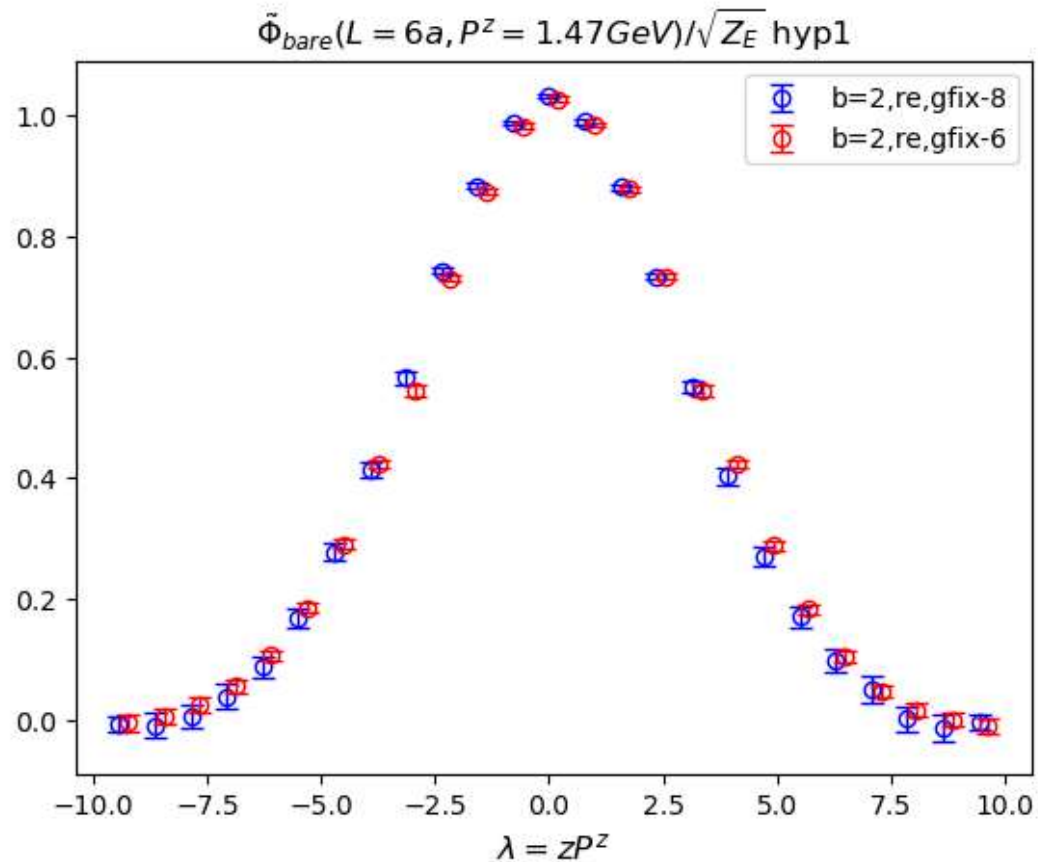
When $P^Z = 1.47\text{GeV}$

- $\Gamma_{src} = \gamma^t \gamma_5, \gamma^z \gamma_5, \gamma_5$ are consistent when the timeslice is large;
- The signal-to-noise ratio of $\gamma^t \gamma_5, \gamma^z \gamma_5$ is greater than that of γ_5 .



Gauge Fixing Accuracy and Lorentz Structure

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Left panel :

- the red: gauge fixing accuracy 10^{-6}
- the blue: gauge fixing accuracy 10^{-8}

Right panel : We fix $\Gamma_{sink} = \gamma^t \gamma_5$,

- the red: $\Gamma_{src} = (\gamma^t \gamma_5 + i \gamma^z \gamma_5)/2$
- the blue: $\Gamma_{src} = \gamma_5$

Subtracted matrix element

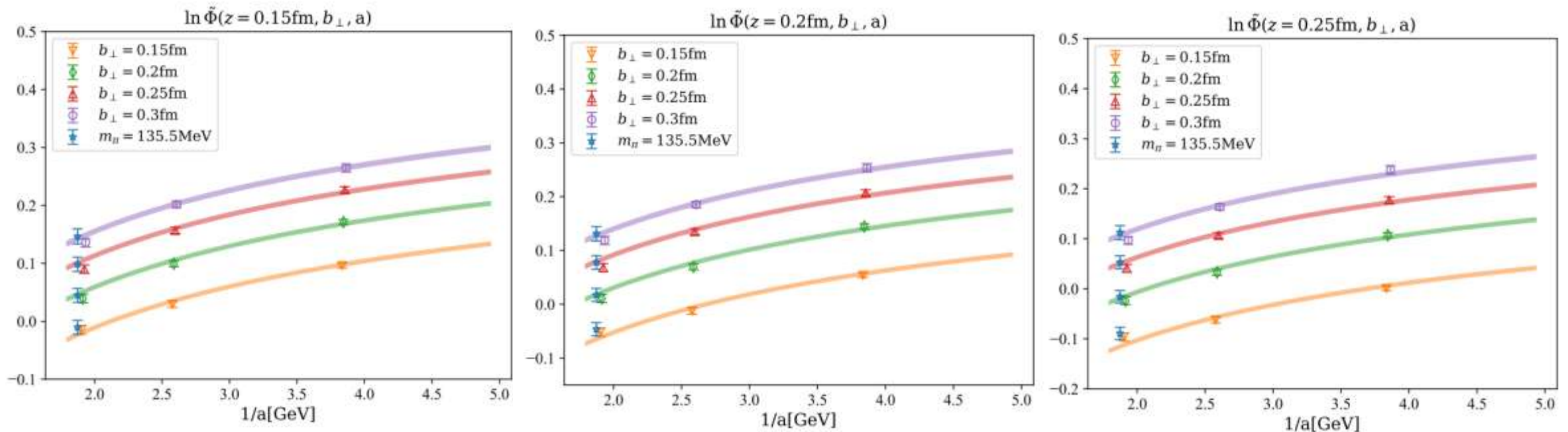
➤ **Renormalization condition:** $\tilde{h}(a, z, b_\perp) = \frac{\tilde{\Phi}^0(z, b, Pz=0, \mu)}{\sqrt{Z_E(z+2L, b, \mu)}} = Z_O(1/a, \mu) h_0^{\overline{MS}}(\mu, b_\perp, z) + O(\frac{a^2}{b^2})$

➤ **Parametrization:**

$$\ln \tilde{h}(a, z, b_\perp) = \frac{\gamma_0}{\beta_0} \ln \left[\ln[1/(a\Lambda_{\text{QCD}})] \right] + \frac{c_1}{\ln[1/(a\Lambda_{\text{QCD}})]} + g(z, b_\perp)$$
$$\ln Z_O(a, \mu) = \frac{\gamma_0}{\beta_0} \ln \left[\ln[1/(a\Lambda_{\text{QCD}})] \right] + \frac{c_1}{\ln[1/(a\Lambda_{\text{QCD}})]} + d'(\mu)$$

Subtracted matrix element

- Renormalization condition: $\tilde{h}(a, z, b_\perp) = \frac{\tilde{\Phi}^0(z, b, Pz=0, \mu)}{\sqrt{Z_E(z+2L, b, \mu)}} = Z_O(1/a, \mu) h_0^{\overline{MS}}(\mu, b_\perp, z) + O(\frac{a^2}{b^2})$
- Parametrization: $\ln \tilde{h}(a, z, b_\perp) = \frac{\gamma_0}{\beta_0} \ln \left[\ln[1/(a\Lambda_{\text{QCD}})] \right] + \frac{c_1}{\ln[1/(a\Lambda_{\text{QCD}})]} + g(z, b_\perp)$
- Joint fit result: $c_1 = 0.020$



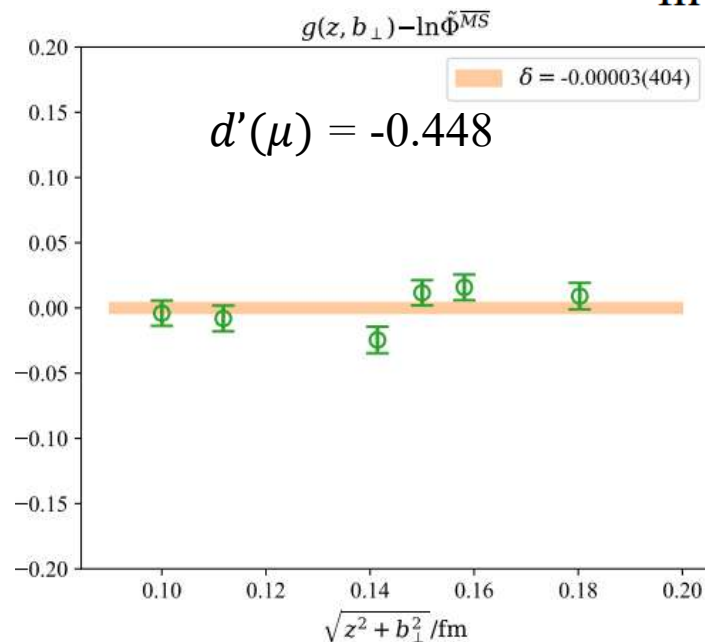
Subtracted matrix element

➤ **Renormalization condition:** $\tilde{h}(a, z, b_\perp) = \frac{\tilde{\Phi}^0(z, b, Pz=0, \mu)}{\sqrt{Z_E(z+2L, b, \mu)}} = Z_O(1/a, \mu) h_0^{\overline{MS}}(\mu, b_\perp, z) + O(\frac{a^2}{b^2})$

➤ **Parametrization:** $\ln \tilde{h}(a, z, b_\perp) = \frac{\gamma_0}{\beta_0} \ln \left[\ln[1/(a\Lambda_{\text{QCD}})] \right] + \frac{c_1}{\ln[1/(a\Lambda_{\text{QCD}})]} + g(z, b_\perp)$

$$\ln Z_O(a, \mu) = \frac{\gamma_0}{\beta_0} \ln \left[\ln[1/(a\Lambda_{\text{QCD}})] \right] + \frac{c_1}{\ln[1/(a\Lambda_{\text{QCD}})]} + d'(\mu)$$

Joint fit result: $c_1=0.020$



a/fm	0.10530	0.07746	0.05187
HYP0	1.064(21)	1.145(20)	1.241(19)
HYP1	0.92208(62)	0.97526(57)	1.03969(53)