Toward Precise ξ Gauge Fixing for the Lattice QCD

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arXiv:2509.09367

ξ-Gauge Fixing on Lattice

- Continuum case:

  Lagrangian $(\partial_\mu A^\mu)^2/(2\xi)$

$$\to \delta\left(\partial_\mu A^\mu(x) - \Lambda(x)\right), P(\Lambda^a(x)) = \frac{1}{\sqrt{2\pi\xi}} \exp\left\{-\frac{1}{2\xi}[\Lambda^a(x)]^2\right\}$$

- Lattice case:

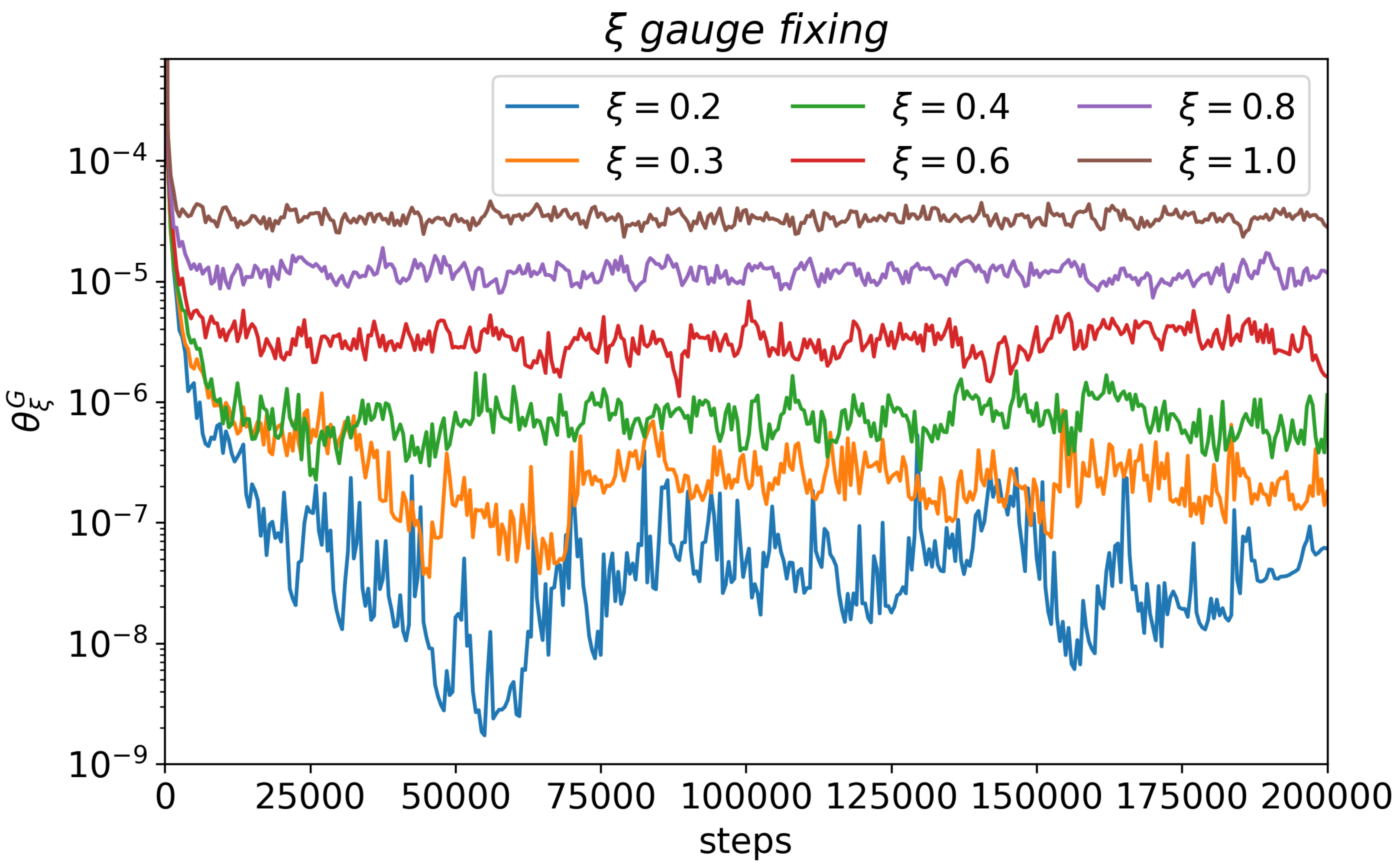
$$\Delta(x) \equiv \sum_{\mu,\eta=\pm} \eta \left[\frac{U_\mu(x+\eta\frac{\hat{\mu}}{2}a) - U_\mu^\dagger(x+\eta\frac{\hat{\mu}}{2}a)}{2ig_0}\right]_{\text{Traceless}} - \Lambda(x)a^2 = 0$$

Depends on g_0, $\xi^{\text{phy}} = \xi^{\text{lattice}}/g_0^2$

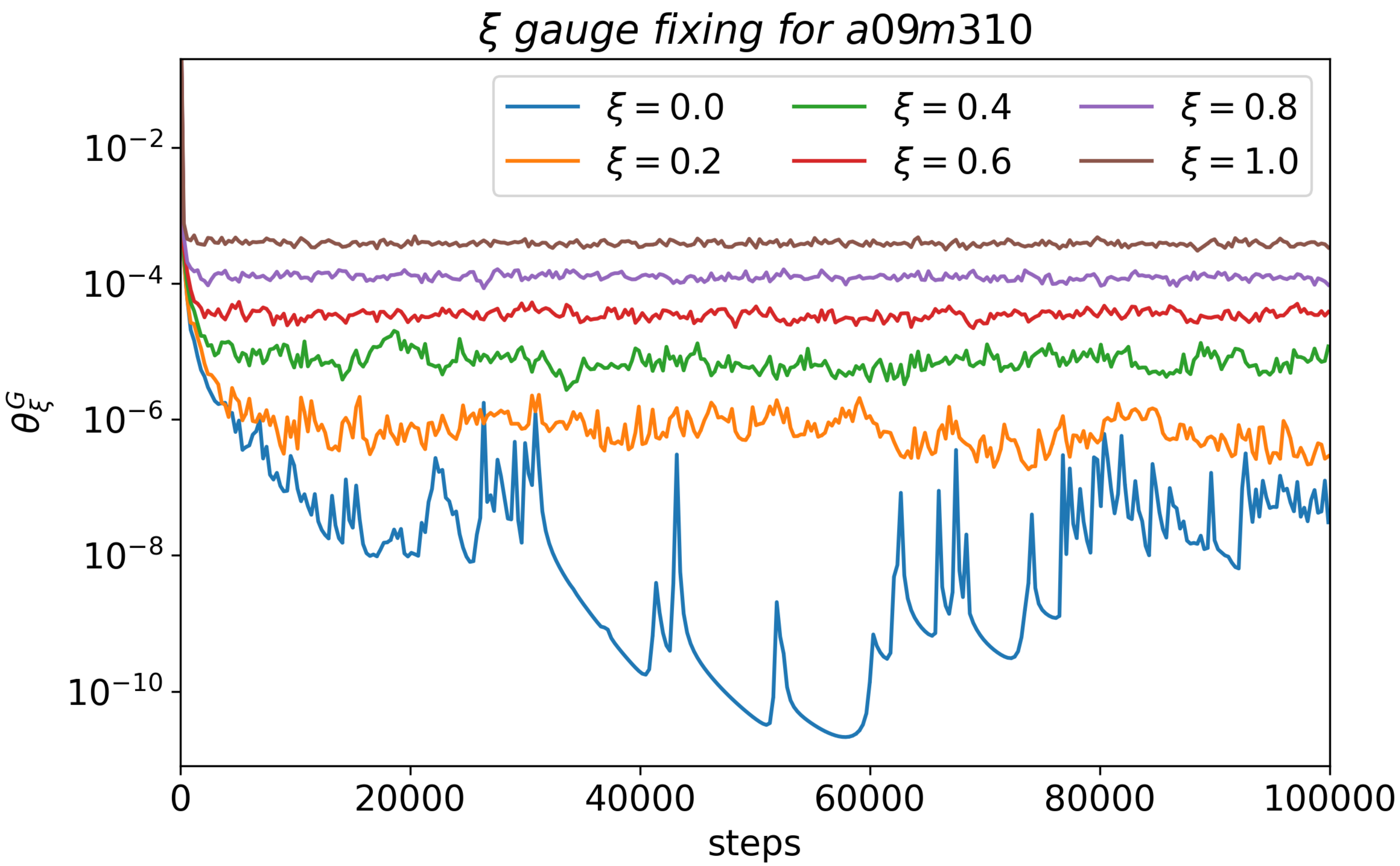
Fixing Criterion

- $\theta = \frac{1}{N_c V} \sum_x \mathrm{Tr}\left[\Delta(x)\Delta^\dagger(x)\right]$

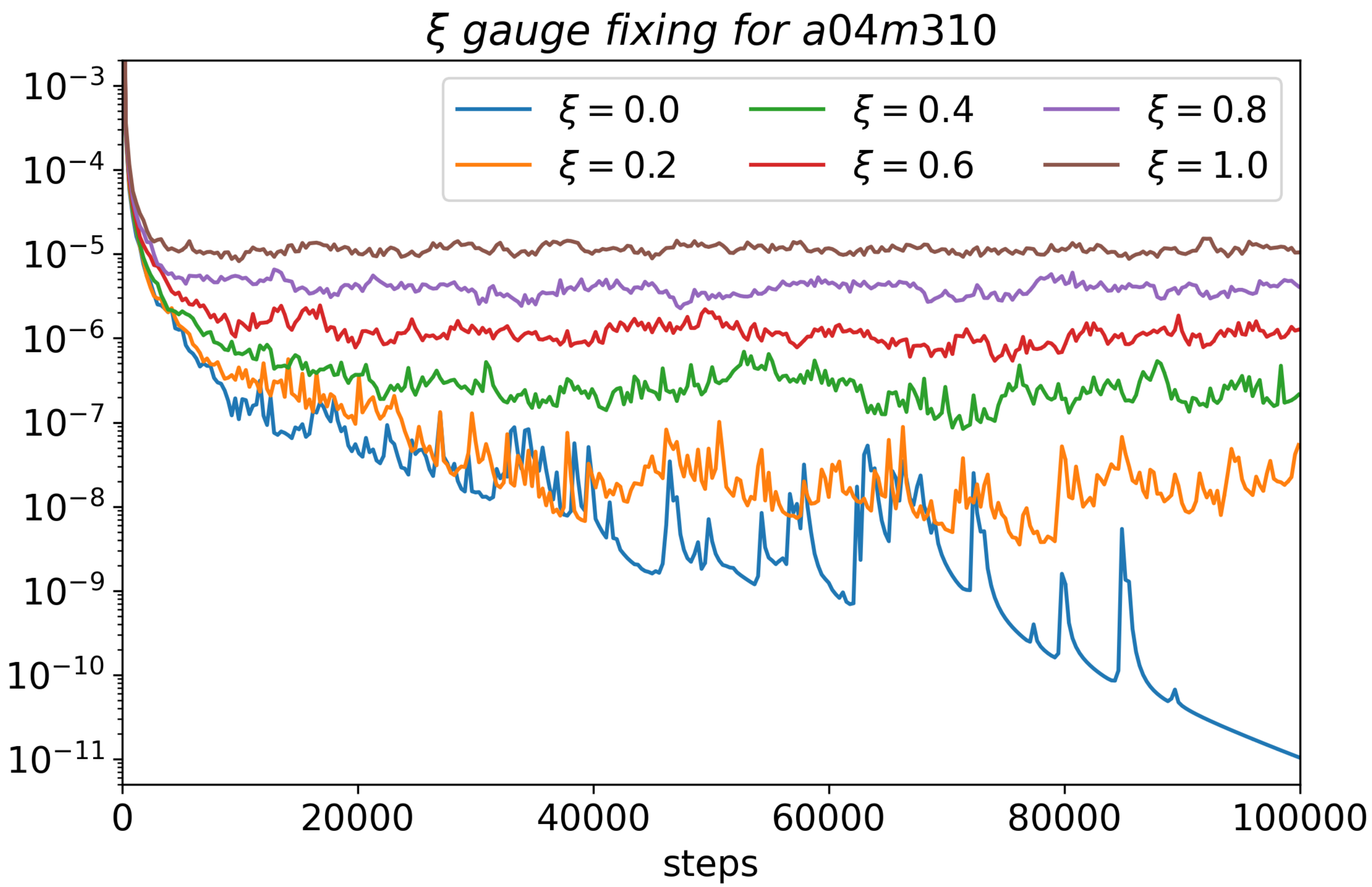
- Lower bound exists
- "Plateau" increases on ξ



Some Good News



$a = 0.09\mathrm{fm}$



$a = 0.04\mathrm{fm}$

- "Plateau" for same ξ decrease while lattice spacing goes to continuum limit

Precision Extrapolation

Local Operators

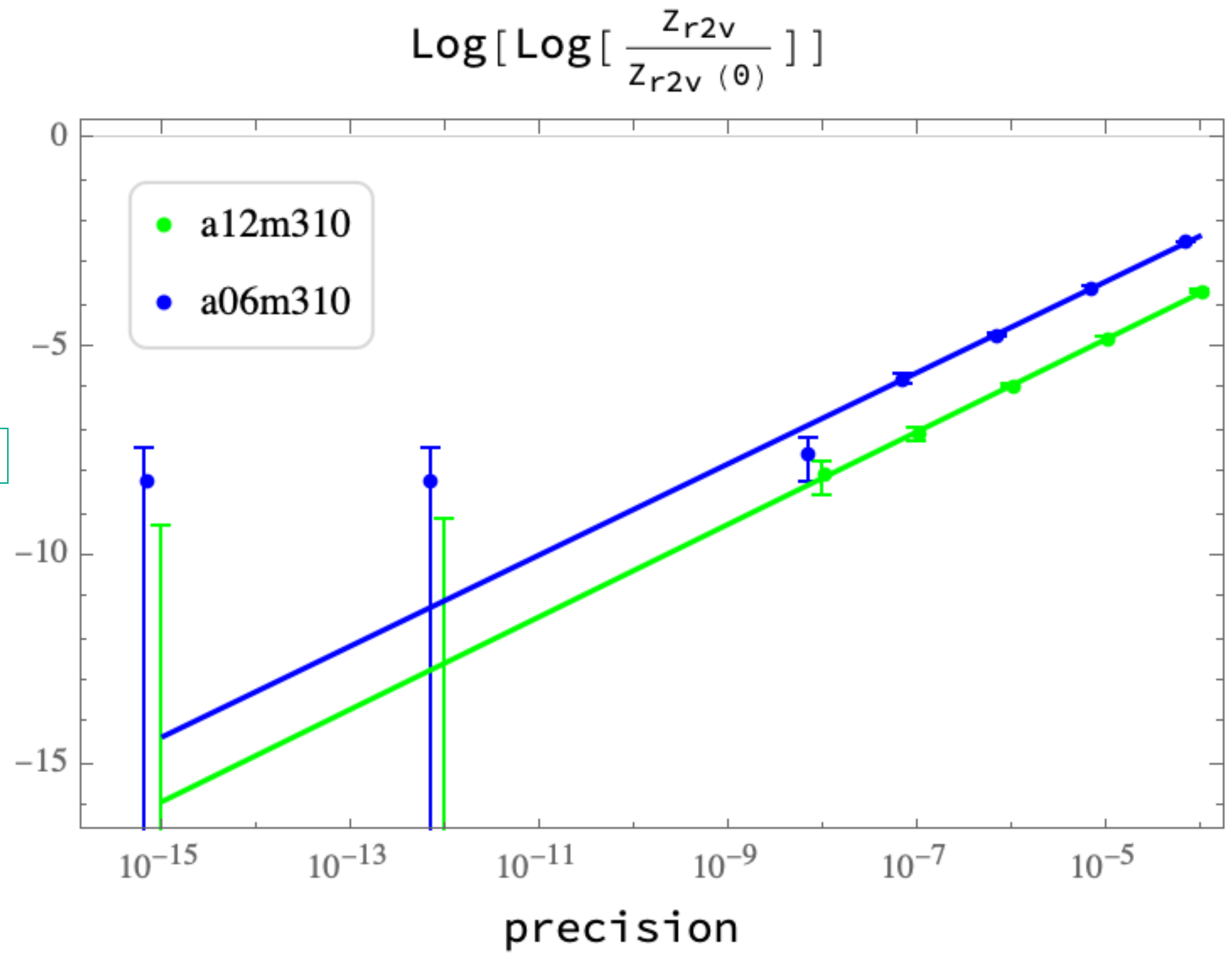
- Precision Extrapolation

$$X(\theta) = X(0)e^{-c(X)\theta^{n(X)}}$$

K. Zhang et al, Phys. Rev. D 110, 074505 (2024), arXiv:2405.14097 [hep-lat]

- Renormalization constants of quark energy moment tensor operator

$$\frac{Z_{r2v}}{Z_V} = \frac{\frac{1}{48}\mathrm{Tr}[S^{-1}(p)\langle\bar{\psi}\gamma_\mu\psi\rangle S^{-1}(p)\gamma_\mu]}{\frac{1}{\mathrm{Tr}[\Gamma_{tree}\Gamma_{tree}]}\mathrm{Tr}[S^{-1}(p)\langle\bar{\psi}\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}\psi\rangle S^{-1}(p)\Gamma_{tree}]}$$



Renormalization Constants of Quark Bilinear Operators

RI/MOM Scheme

$\mathcal{O}(0)$ $\mathcal{O}(x) \equiv \bar{\psi}(x)\Gamma\psi(x)$

- Definition

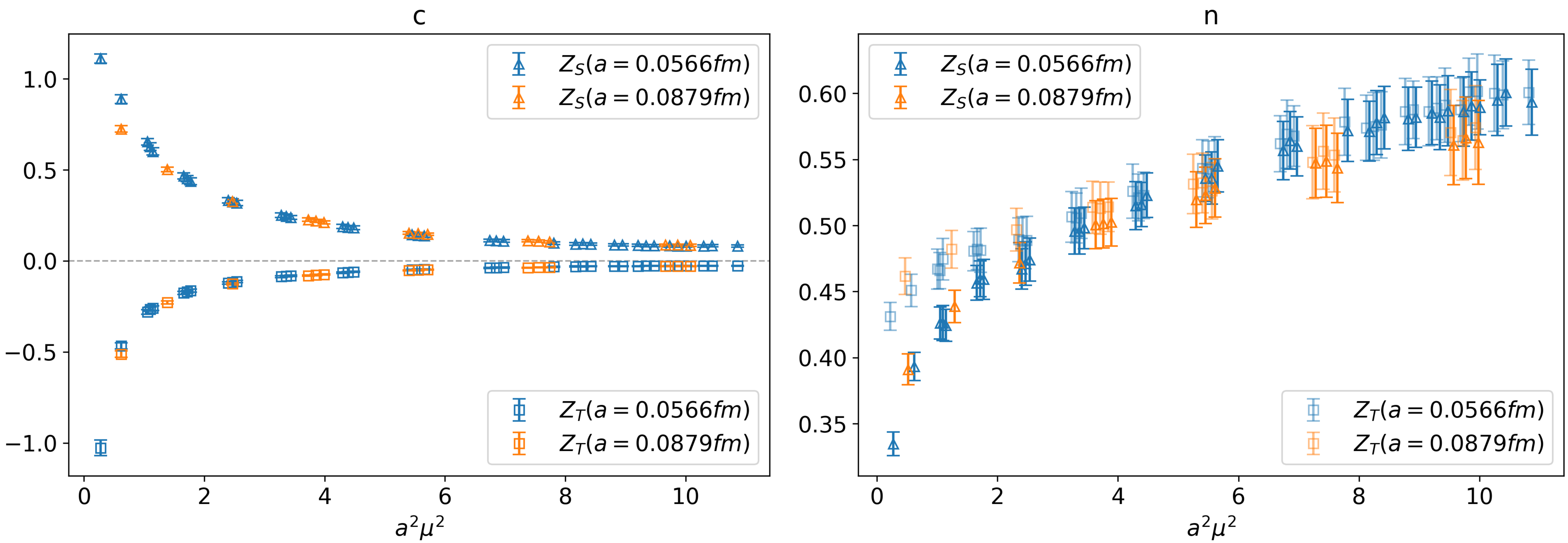
$$G_{\mathcal{O}}\left(p,p\right) = \sum_{x,y} e^{-ip\cdot(x-y)}\langle\psi(x)\left(\bar{\psi}(0)\Gamma\psi(0)\right)\bar{\psi}(y)\rangle$$

$$Z_{\mathcal{O}_\Gamma}(\mu) \equiv \frac{Z_q(\mu)}{\frac{1}{12}\mathrm{Tr}[S^{-1}(p)G_{\mathcal{O}}(p,p)S^{-1}(p)\Gamma]}\Big|_{\mu^2=p^2}$$

$$Z_q(\mu) \equiv \frac{\mathrm{Tr}[\not{p}S^{-1}(p)]}{12p^2}\Big|_{\mu^2=p^2}$$

LSZ reduction

- Evaluate $Z_{\mathcal{O}}/Z_V$ to avoid Z_q



- Shows similar behavior on $a^2\mu^2$ and insensitive to the lattice spacing a
- $n \approx 0.5$ in large range of momenta, while decreases to roughly 0.4 at lower momenta

Applications on ξ Gauge

- Fewer data points

ξ	0.0	0.2	0.4	0.6	0.8	1.0
# of θ	21	11	9	8	7	6

- Criterion for precise ξ-gauge fixing

$$Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(\mu_0, a^2\mu^2) = \frac{Z_{\mathcal{O}}^{\overline{\mathrm{MS}},\mathrm{pert}}(\mu_0)}{Z_{\mathcal{O}}^{\mathrm{RI/MOM,pert}}(\xi,\mu)} Z_{\mathcal{O}}^{\mathrm{RI/MOM}}(\xi, \mu, a^2\mu^2)$$

Perturbative calculation

Lattice calculation

ξ independent, with discretization error

J. A. Gracey, Nucl. Phys. B662, 247 (2003), arXiv:hep-ph/0304113 [hep-ph]

Discretization error:

- $a^2\mu^2$
- ξ

$$R \equiv Z_{\mathcal{O},\xi}^{\overline{\mathrm{MS}}}(\mu_0, a^2\mu^2) / Z_{\mathcal{O},\xi=0}^{\overline{\mathrm{MS}}}(\mu_0, a^2\mu^2)$$

Discretization error:

- ξ

Ambiguity of g_0 Definition

- Recall that $\xi^{\text{phy}} = \xi^{\text{lattice}}/g_0^2$
- Definitions bare coupling g_0
  - $g_0^{(a)} = \sqrt{6/\beta}$: naive definition
  - $g_0^{(b)} = \sqrt{6/\beta}/u_0$: tadpole improvement, with $u_0 = \langle \mathrm{ReTr} \sum_{x,\mu<\nu} \mathscr{P}^U_{\mu\nu}(x)/(6N_cV)\rangle^{1/4}$
  - $g_0^{(c)} = \sqrt{-\frac{16\pi \ln u_0}{3.0684} u_0}$: approximate from u_0 only via $\alpha_s \simeq -\frac{4 \ln u_0}{3.0684}$

K. Orginos and D. Toussaint (MILC), Phys. Rev. D 59,014501 (1999), arXiv:hep-lat/9805009

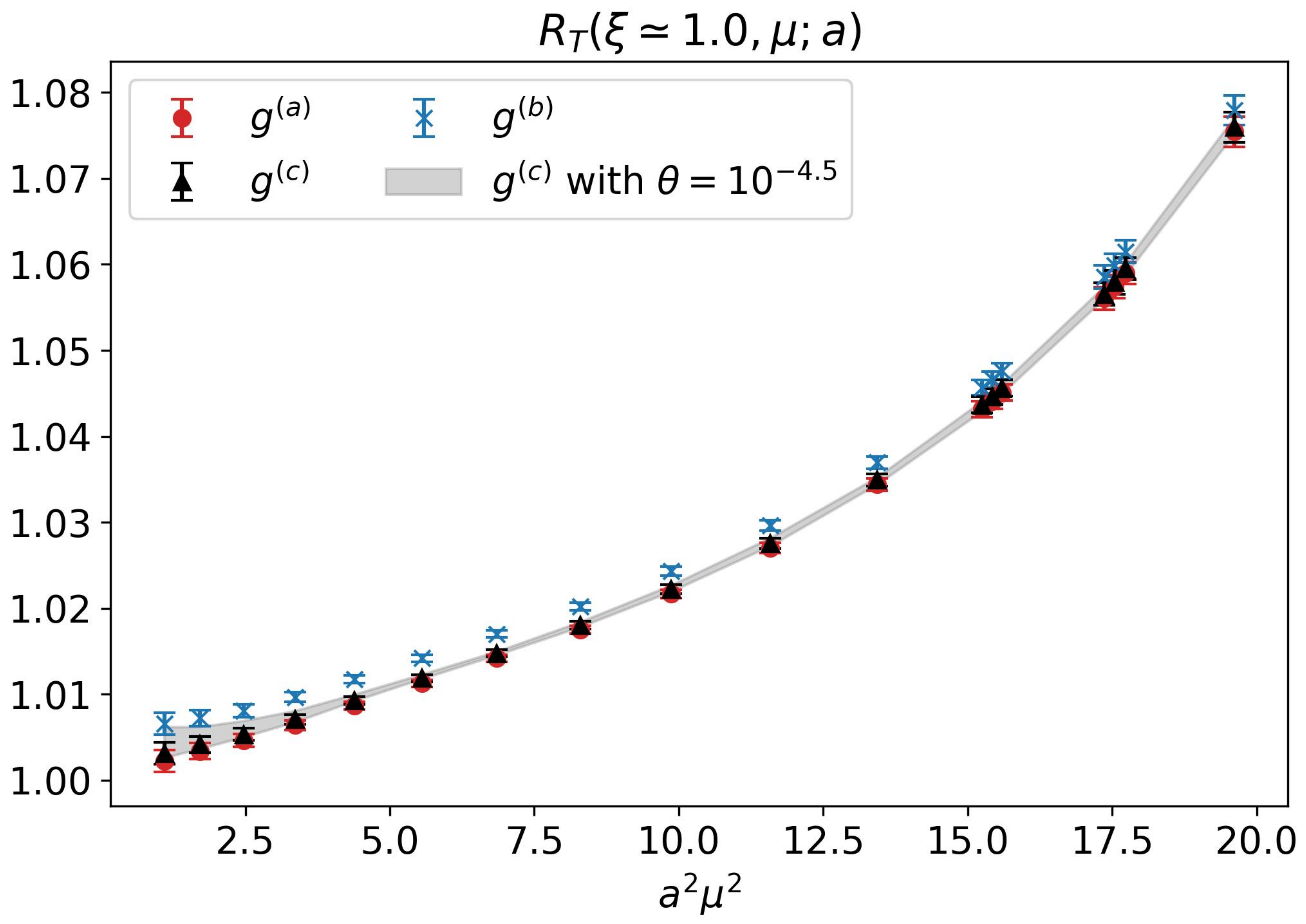
Impact of g_0 Choice

- Ratio

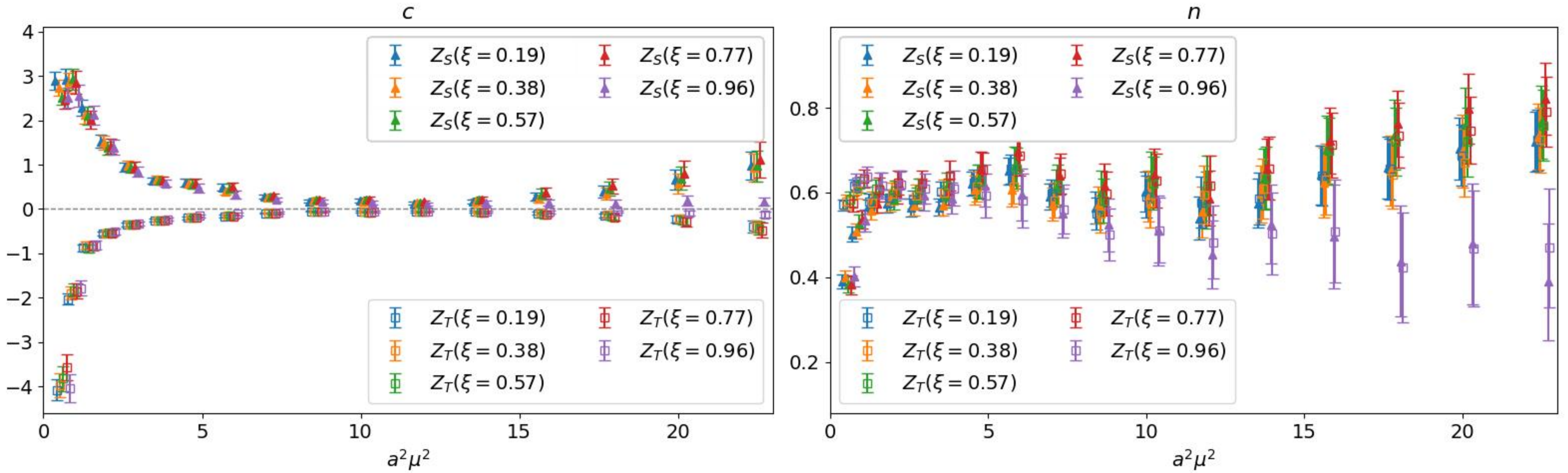
$$R_X(\xi, \mu; a) = \frac{Z_X^{\rm RI}(\xi, \mu; a)}{Z_X^{\rm RI}(0,\mu; a)} \frac{Z_X^{\rm RI,pert}(0,\mu)}{Z_X^{\rm RI,pert}(\xi, \mu)}$$

- Carries only discretization errors from ξ, shall tends to 1 at lower momenta

- Take $g^{(c)}$ in this work

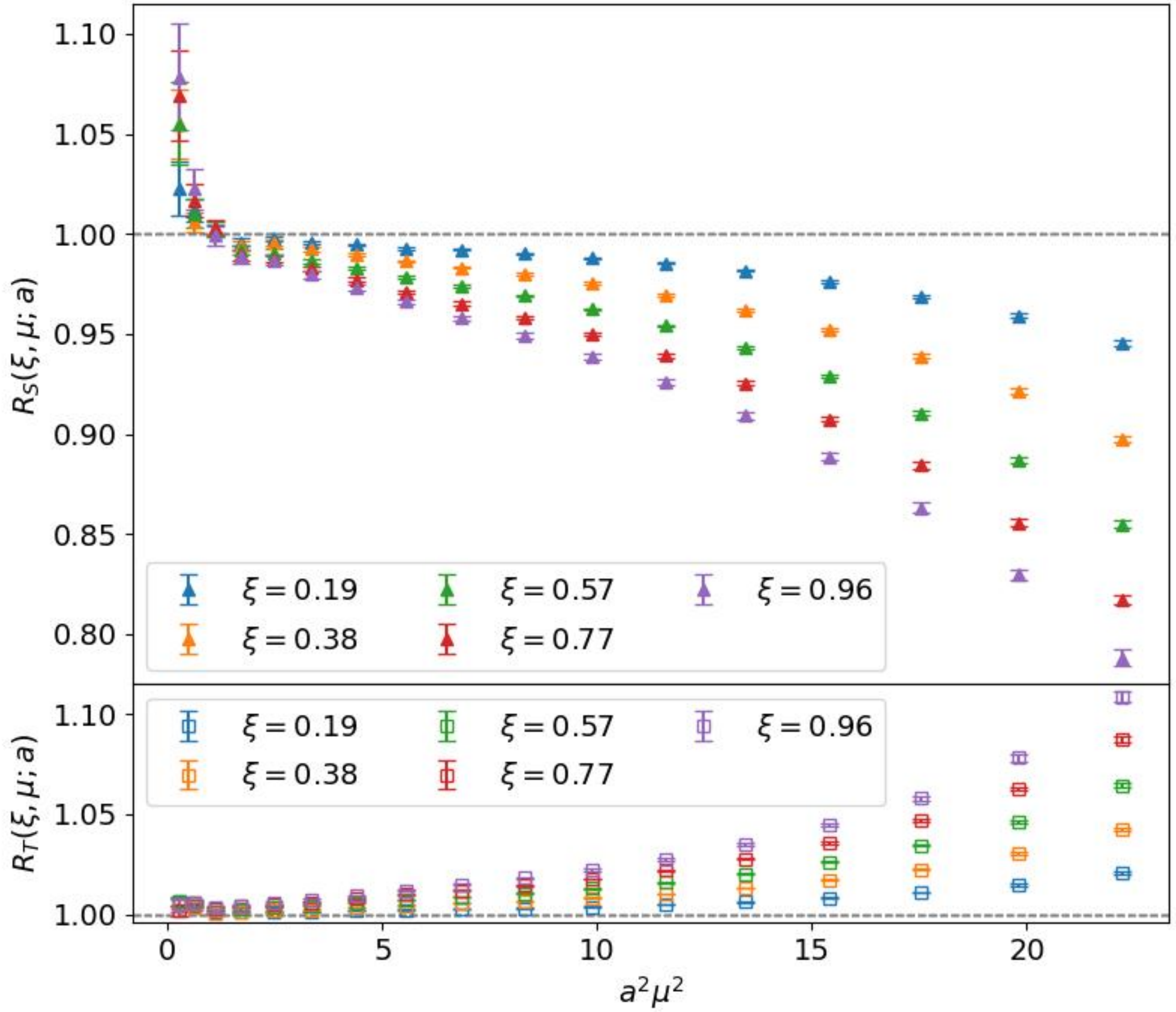


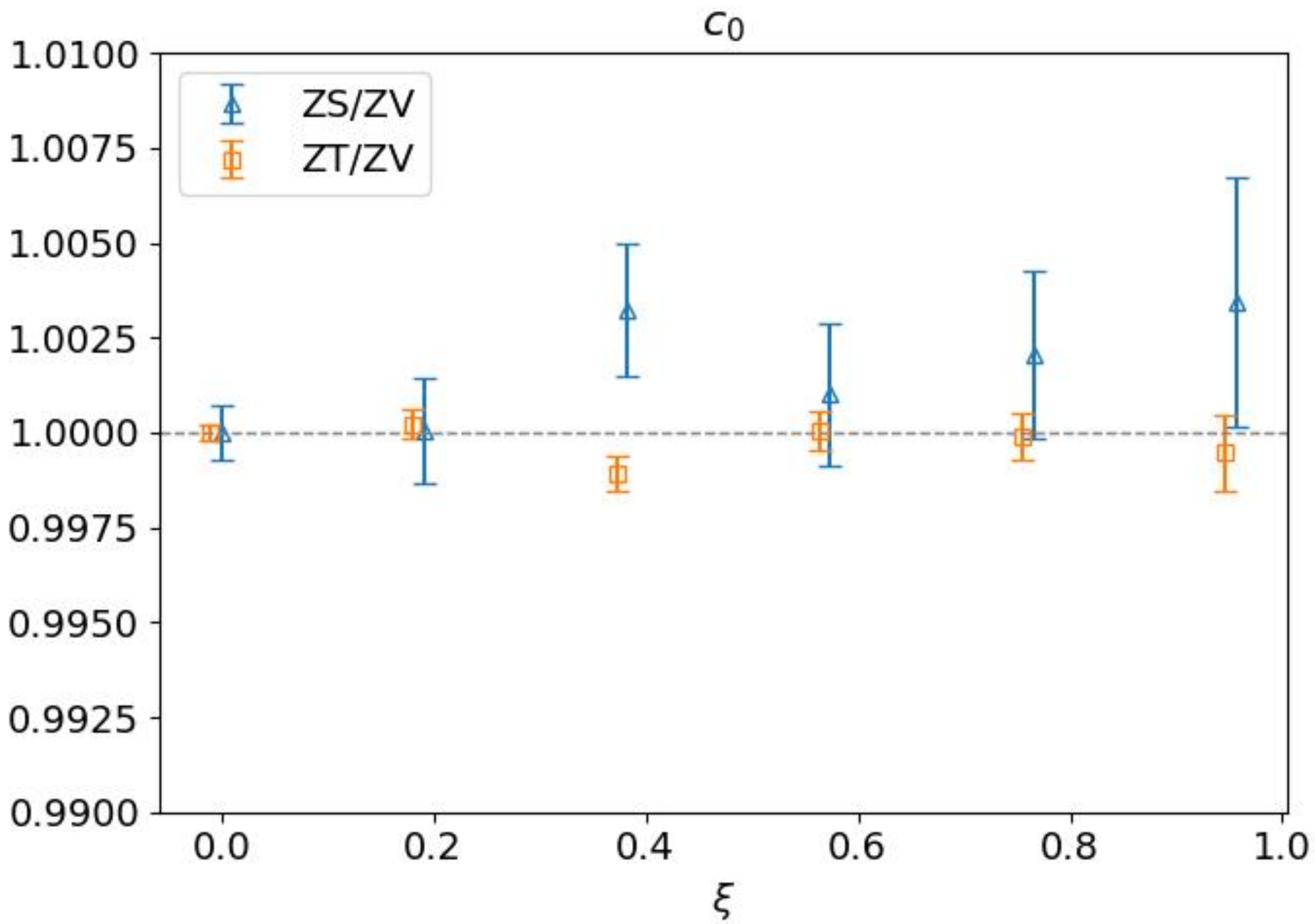
ξ Gauge Extrapolation



- $n \approx 0.5$ regardless of ξ and μ
- $|c|$ becomes larger at both ends of the $a^2\mu^2$ range

Discretization Error from ξ





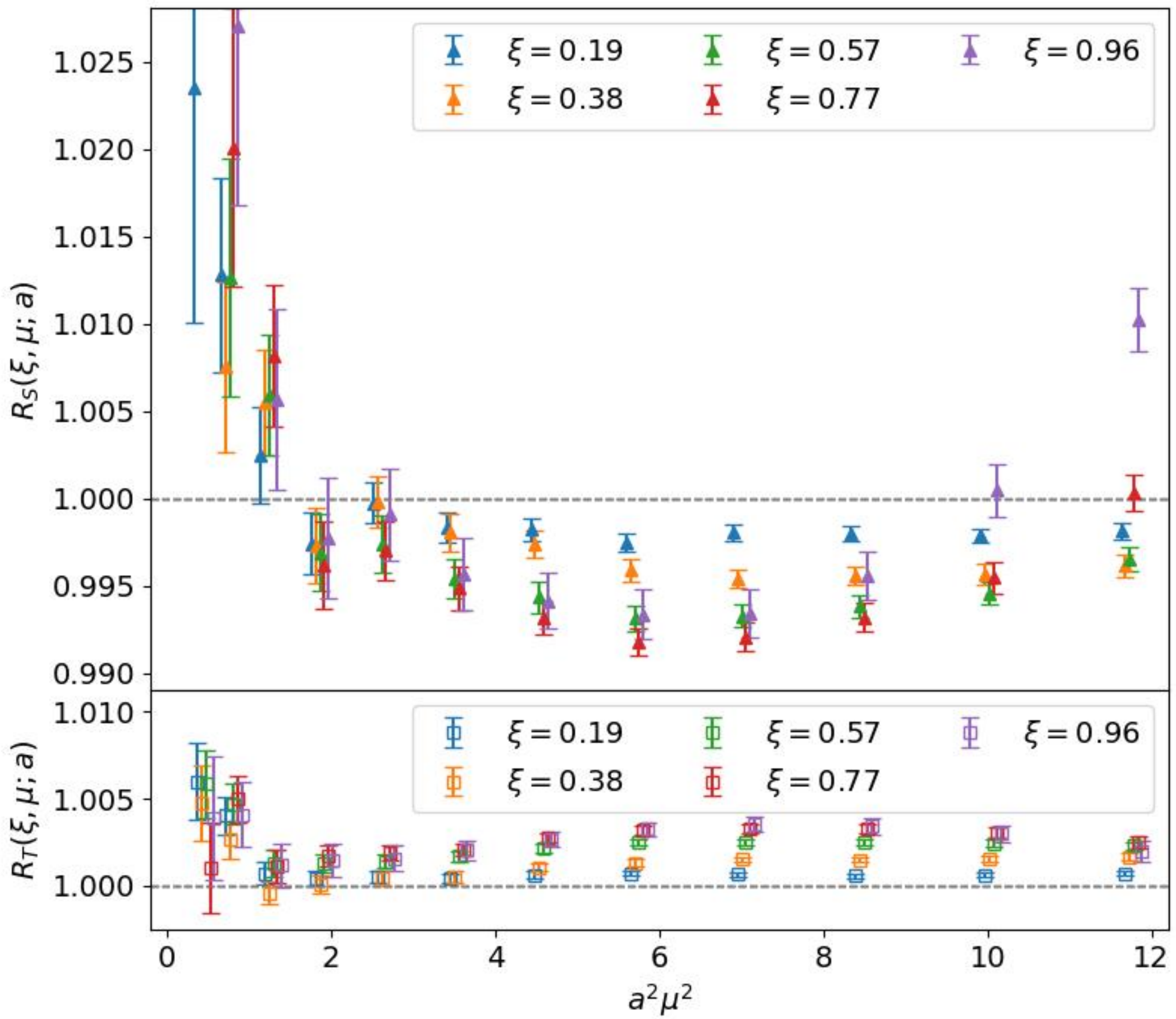
$$R_X(\xi, \mu; a) = c_{0,X}(\xi)\left(1 + \sum_{i=1}^{3} c_{i,X}(\xi)\left(a^2\mu^2\right)^i\right)$$

Another Trial for Elimination of Discretization Error

- Define effective $\tilde{\xi}$

$$\tilde{\xi} = \xi a^2\mu^2/(4\sin^2(a\mu/2)) = \xi(1 + \frac{1}{12}a^2\mu^2 + \mathcal{O}(a^4\mu^4))$$

- Suppress the $a^2\mu^2$ dependence in $R_{X=S,T}$



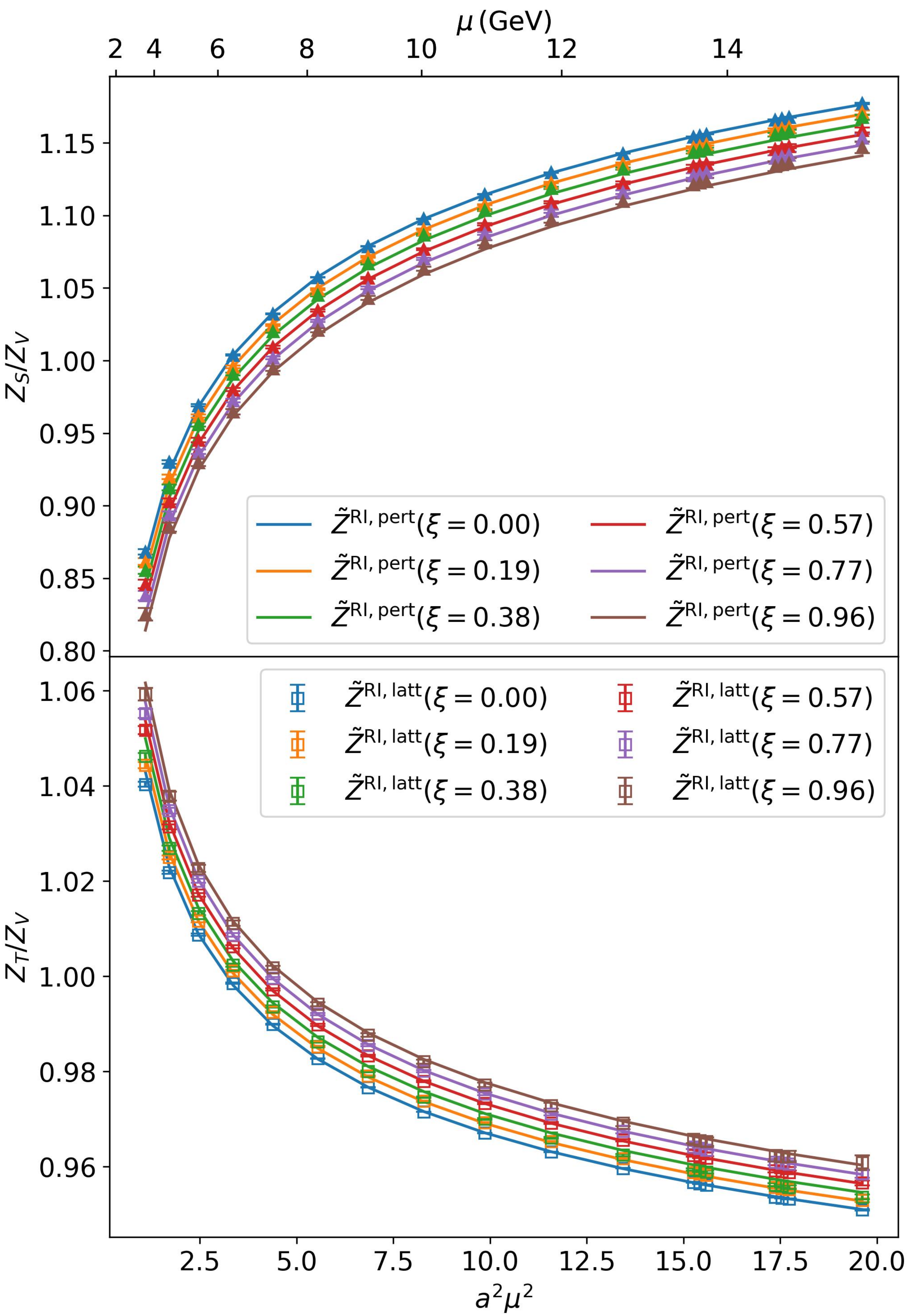
Compare Lattice Result to Perturbative Result

$$Z_{\mathcal{O}}^{\overline{\text{MS}}}(\mu_0)\frac{Z_{\mathcal{O}}^{\text{RI/MOM,pert}}(\xi,\mu)}{Z_{\mathcal{O}}^{\overline{\text{MS}},\text{pert}}(\mu_0)}$$

$$Z_{\mathcal{O}}^{\overline{\text{MS}}}(\mu_0, a^2\mu^2) = Z_{\mathcal{O}}^{\overline{\text{MS}}}(\mu_0)\text{Err}(a^2\mu^2)$$

$$\frac{Z_{\mathcal{O}}^{\text{RI/MOM}}(\xi,\mu,a^2\mu^2)}{\text{Err}(a^2\mu^2)}$$

- Lines are perturbation results, using $Z_{\mathcal{O}}$ to reconstruct $\tilde{Z}^{\text{RI,pert}}$
- Data points are lattice results with discretization errors removed



Summary and Outlook

- The validity of empirical precision extrapolation formula
- Achieve ξ-gauge through precision extrapolation

- TODO: Gauge dependent parton (quark, gluon) propagators

Thanks for your attention!

Simulation Setups

Valence clover/overlap fermions on (2+1+1) flavor MILC ensembles.

Action	Symbol	6/g^2	L^3*T	a(fm)	m_π(MeV)
HISQ+S	a12m310	3.60	24^3*64	0.1222	310
HISQ+S	a09m310	3.78	32^3*96	0.0879	310
HISQ+S	a06m310	4.03	48^3*144	0.0566	310