Searching for the critical point of finite density QCD from the heavy quark region

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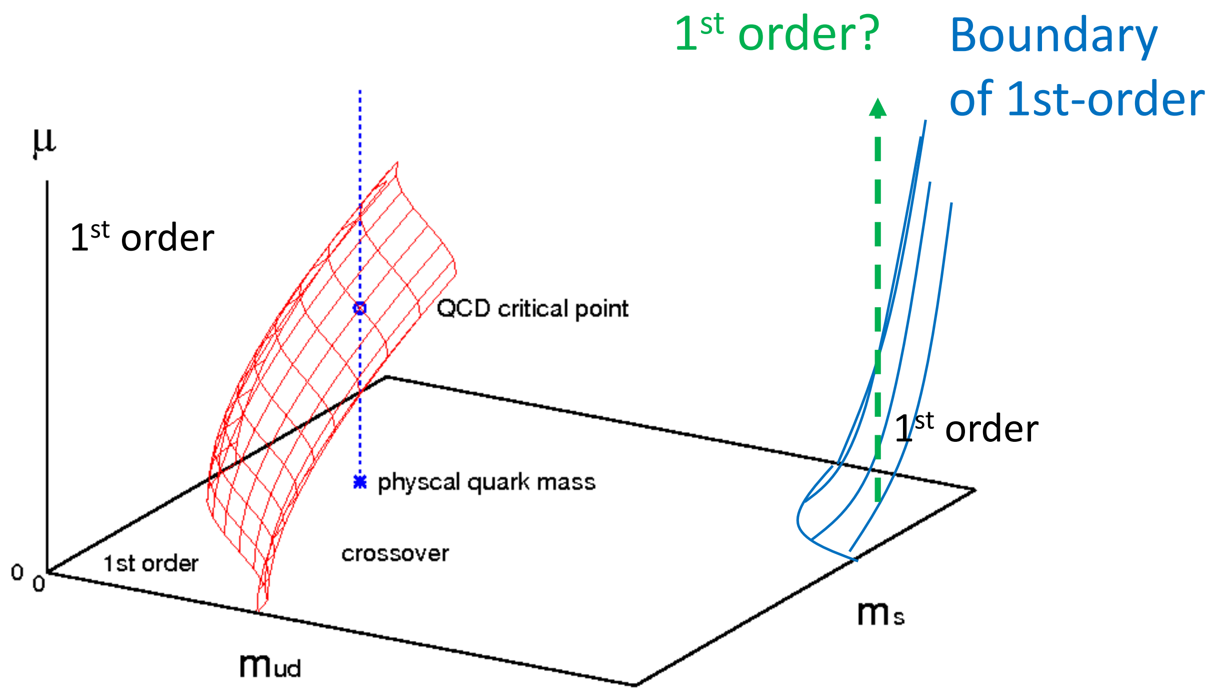
QCD phase diagram with varying quark masses



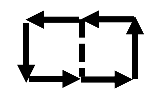
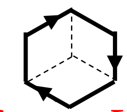
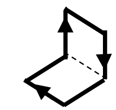
- We expect that the first order phase transition in the light mass region expands with increasing density.
- The first order phase transition region may expand into the heavy quark region.
- We discuss the appearance of first order phase transitions in the heavy and dense region.

Effective theory by hopping parameter expansion

- Expectation value of physical quantity O

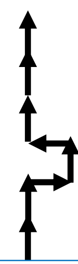
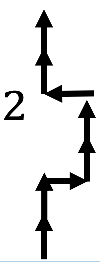
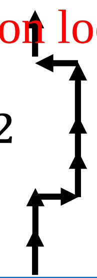
$$\langle O\rangle(\beta,\kappa) = \frac{1}{Z}\int DU\, O[U]\left(\det M(\kappa)\right)^{N_\mathrm{f}} e^{-S_g} = \frac{1}{Z}\int DU\, O[U]\, e^{-S_\mathrm{eff}}$$

- We expand the quark determinant in terms of the hopping parameter κ.

$$\ln\left(\det M(\kappa)\right) = 288 N_\mathrm{site}\kappa^4 P + [768\, N_\mathrm{site}\kappa^6 (3$$  $$+$$  $$+6$$  $$)]+\cdots$$

plaquette

6-step Wilson loop

$$+e^{\frac{\mu}{T}} 6\times 2^{N_t} N_s^3 [\kappa^{N_t}\Omega + 6N_t\kappa^{N_t+2}$$  $$+6N_t\kappa^{N_t+2}$$  $$+3N_t\kappa^{N_t+2}$$  $$] + \cdots$$

Polyakov loop

$$+e^{-\frac{\mu}{T}} 6\times 2^{N_t} N_s^3 [\kappa^{N_t}\Omega^* + \cdots$$

Leading term

Next to leading terms (for N_t=6)

$$\boxed{S_\mathrm{eff} = -6N_\mathrm{site}\beta P - N_\mathrm{f}\ln\det M(\kappa)}$$

$$\ln\det M(\kappa) = N_\mathrm{site}\sum_{n=1}^{\infty}\left(W(n) + \sum_{m=1}^{\infty} e^{m\mu/T} L_m^+(N_t,n) + \sum_{m=1}^{\infty} e^{-m\mu/T} L_m^-(N_t,n)\right)\kappa^n$$

- $L_m^\pm(N_t,n)$: The terms that wind around the periodic boundary in the time direction are important.

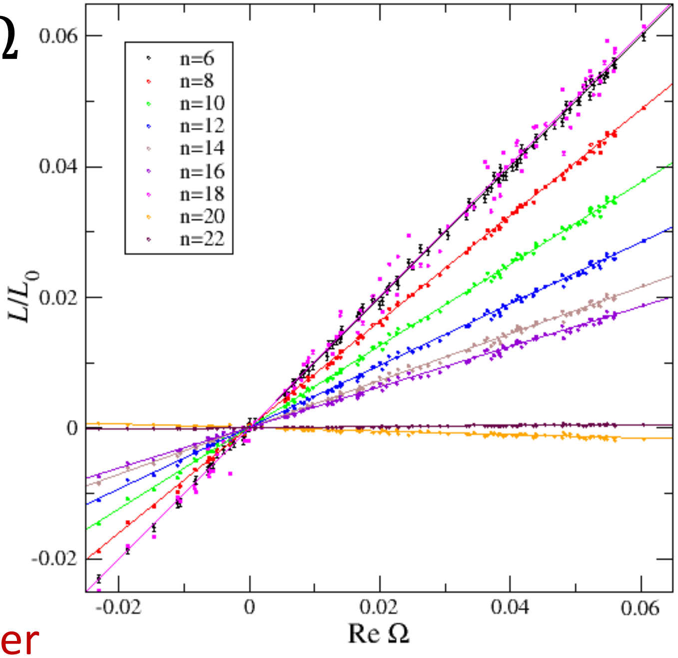
Effective theory based on the hopping parameter expansion

- Higher order expansion terms $L(N_t, n)$ are very strongly correlated with the leading term: Polyakov loop Ω.

$$L(N_t, n) = \sum_m L_m^+(N_t, n) + L_m^-(N_t, n)$$

$$L(N_t, n) \approx L^0(N_t, n) c_n \,\mathrm{Re}\Omega, \quad \mathrm{Arg}\, L_1^+(N_t, n) \approx \mathrm{Arg}\,\Omega$$

(PTEP 2022, 033B05) (arXiv:2311.11508)

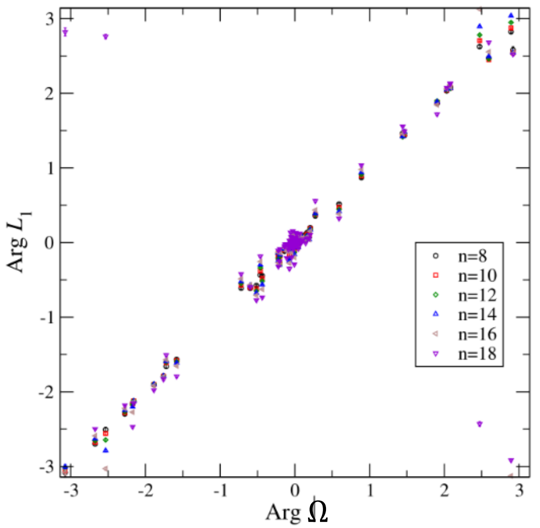


- $L_1(N_t, n)$ is dominant: $L_1(N_t, n) \approx L(N_t, n)$.
- Effective action in the heavy quark region

$$S_{\mathrm{eff}} = -6N_{\mathrm{site}}\beta^* P - \frac{N_s^3 \lambda}{2}\left(e^{\frac{\mu}{T}}\Omega + e^{-\frac{\mu}{T}}\Omega^*\right)$$

$$\lambda = N_{\mathrm{f}} N_t \sum_{n=N_t}^{n_{\max}} L^0(N_t, n)\, c_n \kappa^n \equiv 12 N_{\mathrm{f}}\, 2^{N_t} \underline{\kappa_{\mathrm{eff}}^{N_t}}$$

Effective parameter



- Even if the number of expansion terms increases significantly, the effects of higher-order terms can be incorporated.
- However, as μ increases , the effect of $e^{m\frac{\mu}{T}} L_m(N_t, n)$ $(m \geq 2)$ terms increases relatively. The $e^{m\frac{\mu}{T}}$ term is missing from S_{eff}.

Heavy quark and high-density effective theory

det$M$ is represented by a Polyakov loop (without spatial links) for heavy quarks

$\leftrightarrow$ the effect of spatial link terms can be included by using κ_{eff}.

Partition function of QCD $\quad Z_{\mathrm{QCD}} = \int \mathcal{D}U \{\det M(U)\}^{N_{\mathrm{f}}} e^{-S_g(U)}$

Wilson fermion kernel

$$M_{x,y} = \delta_{x,y} - \kappa \sum_{j=1}^{3} [(1-\gamma_j) U_{x,i} \delta_{y,x+\hat{j}} + (1+\gamma_j) U_{y,j}^{\dagger} \delta_{y,x-\hat{j}}]$$
$$-\kappa(1-\gamma_4) U_{x,4} e^{\mu a} \delta_{y,x+\hat{4}} - \kappa(1+\gamma_4) U_{y,4}^{\dagger} e^{-\mu a} \delta_{y,x-\hat{4}}$$

Heavy quark, high-density region $\kappa \to 0, \quad e^{\mu a} \to \infty, \quad \kappa e^{\mu a} \to (\mathrm{finite})$

e.g. Blum, Hetrick, Toussaint, Phys. Rev. Lett. 76, 1019 (1996), Aarts, Stamatescu, JHEP 0809 (2008) 018

Ignore the spatial link terms and the $e^{-\mu a}$ terms

Heavy Dense QCD

$$M_{x,y} = \delta_{x,y} - \kappa_{\mathrm{eff}} (1-\gamma_4) U_{x,4} e^{\mu a} \delta_{y,x+\hat{4}}$$

$$\Rightarrow \det M = \prod_{\vec{x}} \{1 + 3C\Omega(\vec{x}) + 3C^2 \Omega^*(\vec{x}) + C^3\}^2$$

$C = (2\kappa_{\mathrm{eff}})^{N_t} e^{\mu/T}$, local Polyakov loop $\Omega(\vec{x}) = \frac{1}{3} \mathrm{tr}(U_{(\vec{x},1),4} U_{(\vec{x},2),4} \dots U_{(\vec{x},N_t),4})$

(N_t is an even number.)

Symmetries in heavy dense QCD

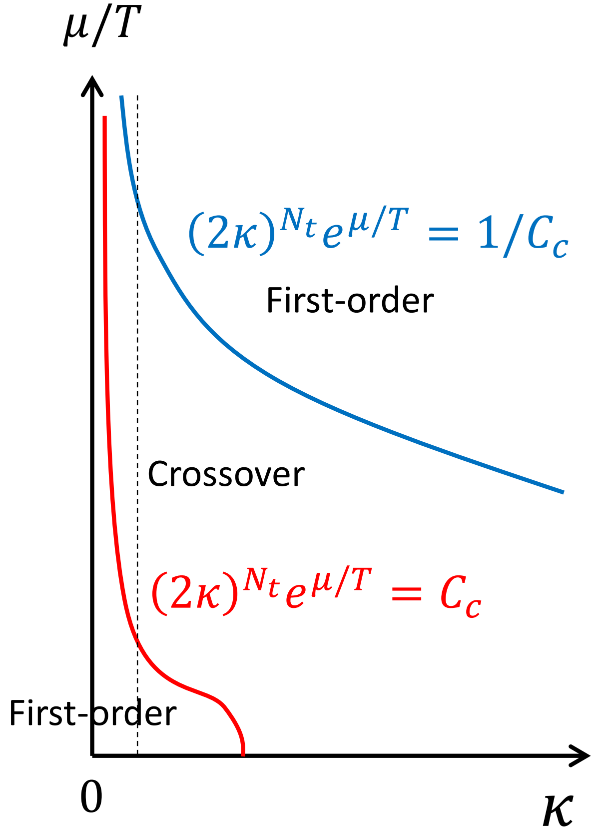
Symmetry

$$\det M = \prod_{\vec{x}} \{1 + 3C\Omega(\vec{x}) + 3C^2\Omega^*(\vec{x}) + C^3\}^2 = C^6 \prod_{\vec{x}} \{1 + 3C^{-1}\Omega^*(\vec{x}) + 3C^{-2}\Omega(\vec{x}) + C^{-3}\}^2$$

$$\Omega(\vec{x}) \to \Omega^*(\vec{x}), \qquad C \to C^{-1}$$

- The theory remains unchanged under this transformation except for constant multiples.
- If κ is kept small, from $C = (2\kappa)^{N_t} e^{\mu/T}$,

  $C \approx 0 \Leftrightarrow \mu/T \approx O(1) \quad C^{-1} \approx 0 \Leftrightarrow \mu/T \approx \infty$
- This is a symmetry under transformations that exchange low density for high density and particles for antiparticles.
- If the critical point is C_c, the boundary of the first-order phase transition is $(2\kappa)^{N_t} e^{\mu/T} = C_c$ and $1/C_c$.



Z3 center symmetry breaking

Universal properties

- If the broken symmetry is the same, the fundamental properties of the phase transition are thought to be the same.

QCD without dynamical fermions

- The Polyakov loop is the order parameter of Z(3) center symmetry.
- The phase transition occurs when Z(3) center symmetry is broken spontaneously.
- The three-state Potts model is a model that breaks Z(3) symmetry.

Explicit symmetry breaking terms

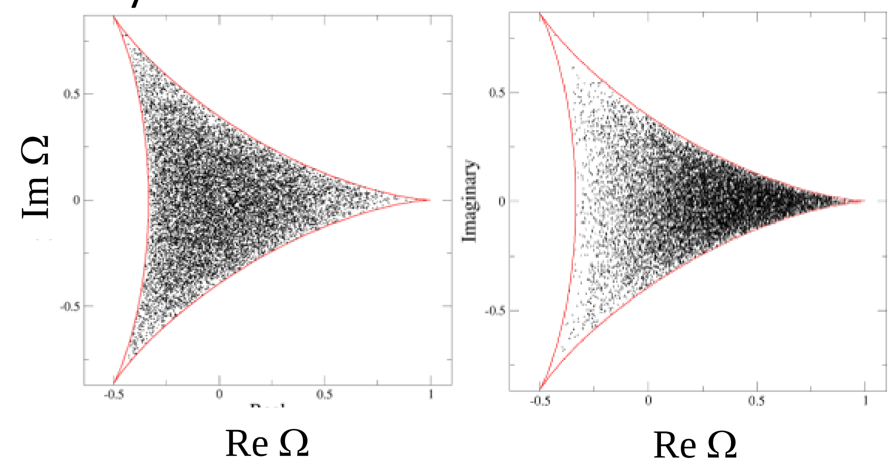
- The quark determinant for QCD.
- The external field term for 3-state Potts model.

We replace the Polyakov loop with a Z(3) spin.

Distribution in one configuration

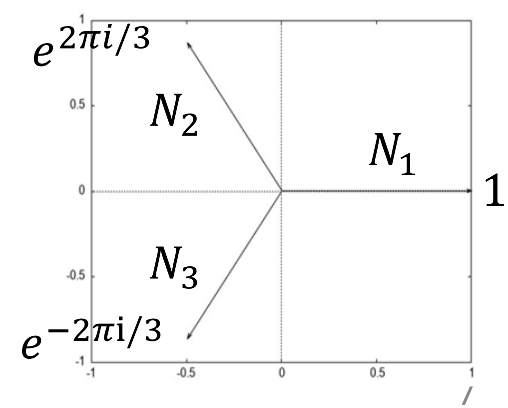
Local Polyakov loop $\Omega(\vec{x})$

Symmetric Phase | Broken Phase



Z(3) spin $s(x)$

Symmetric Phase	Broken Phase
$N_1 \approx N_2 \approx N_3$	$N_1 > N_2, N_3$



Histogram of the order parameter

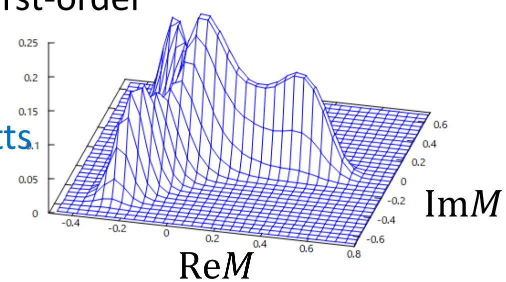
- order parameter: spatial average of $s(x)$ or spatial average of $\Omega(\vec{x})$
- Symmetry breaking terms are added.
- Histograms of the order parameters are very similar.

3-state Potts model: magnetization $M = \frac{1}{V}\sum_x s(x)$ (spatial average)

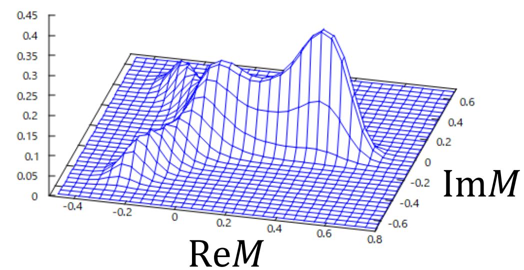
16^3 lattice

3-state Potts model

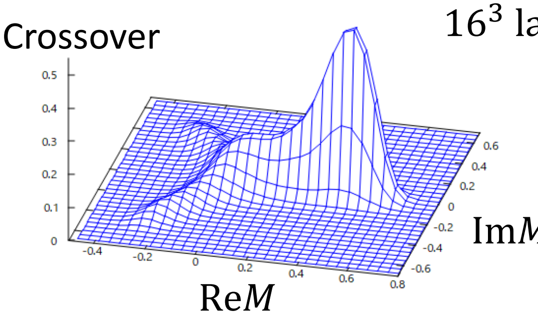
First-order



~ Critical point



Crossover

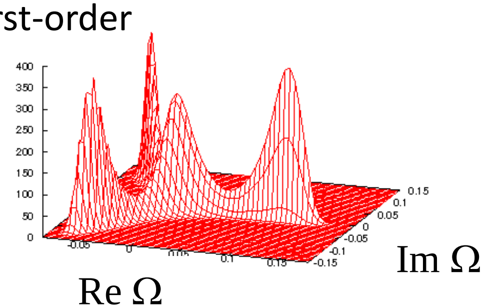


heavy-quark QCD: Polyakov loop $\Omega = \frac{1}{V}\sum_{\vec{x}} \Omega(\vec{x})$ (spatial average)

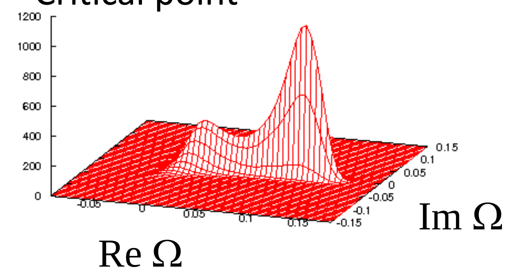
$24^3 \times 4$ lattice

Heavy-quark QCD

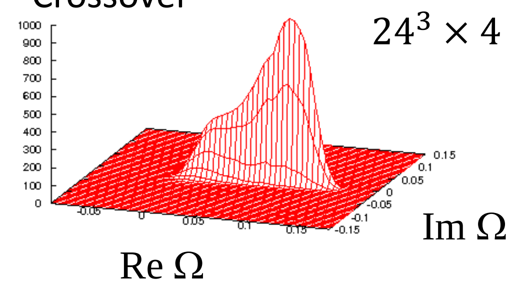
First-order



~ Critical point

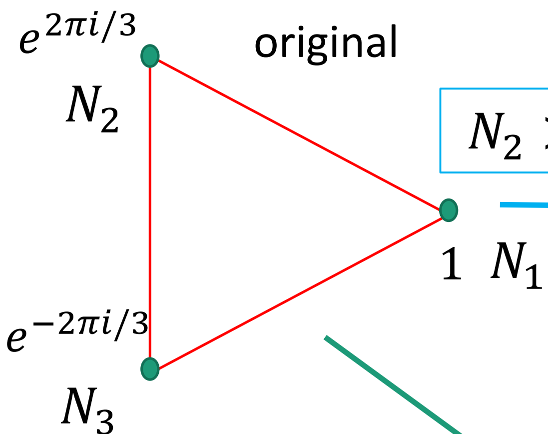


Crossover

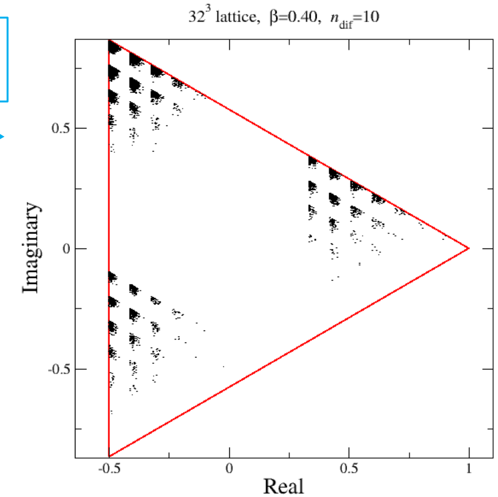


Phys.Rev.D89, 034507(2014)

Distribution of coarse-grained spins

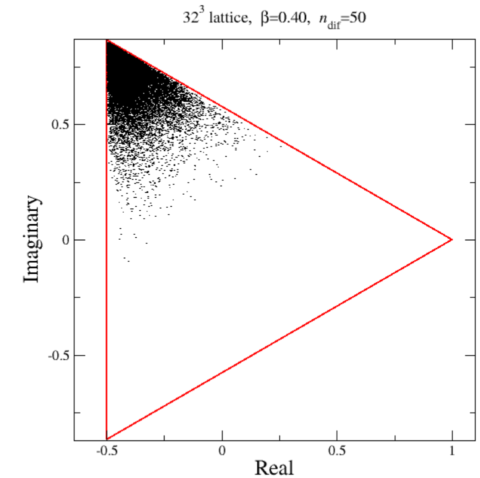


$N_2 > N_1, N_3$



Coase-grained

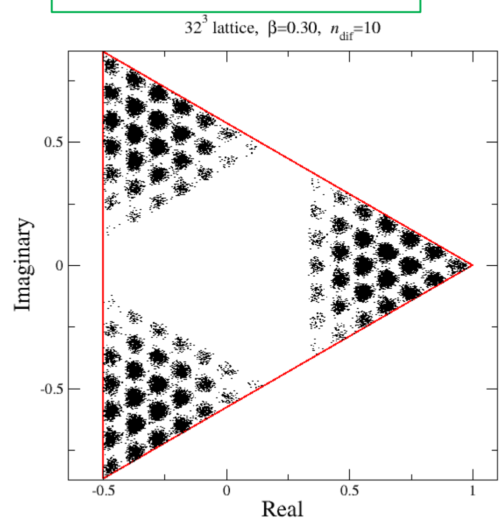
Broken Phase



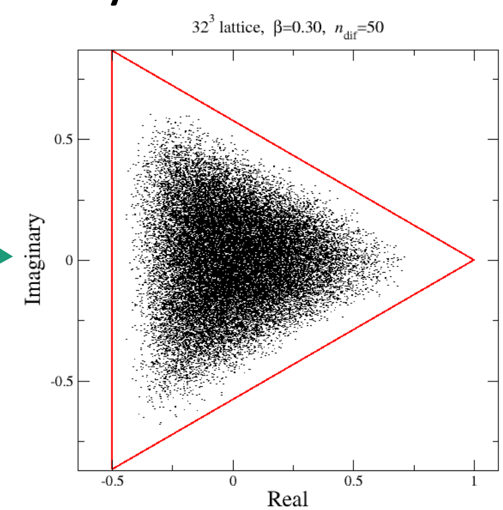
Coarse graining

- Coarse-grained spin $\tilde{s}(x,t)$
- Diffusion equation
$$\frac{\partial}{\partial t}\tilde{s}(x,t) = D\,\frac{\partial^2}{\partial x^2}\tilde{s}(x,t)$$
  - D: diffusion constant
- The magnetization value does not change even when coarse-grained.
- Similar to the distribution of local Polyakov loops $\Omega(\vec{x})$

$N_1 \approx N_2 \approx N_3$



Symmetric Phase



Coarse graining by Gradient flow in quenched QCD

- Flow equation

$$\frac{\partial B_\mu^a}{\partial t}(t,x) = D_\nu G_{\mu\nu}^a(t,x)$$

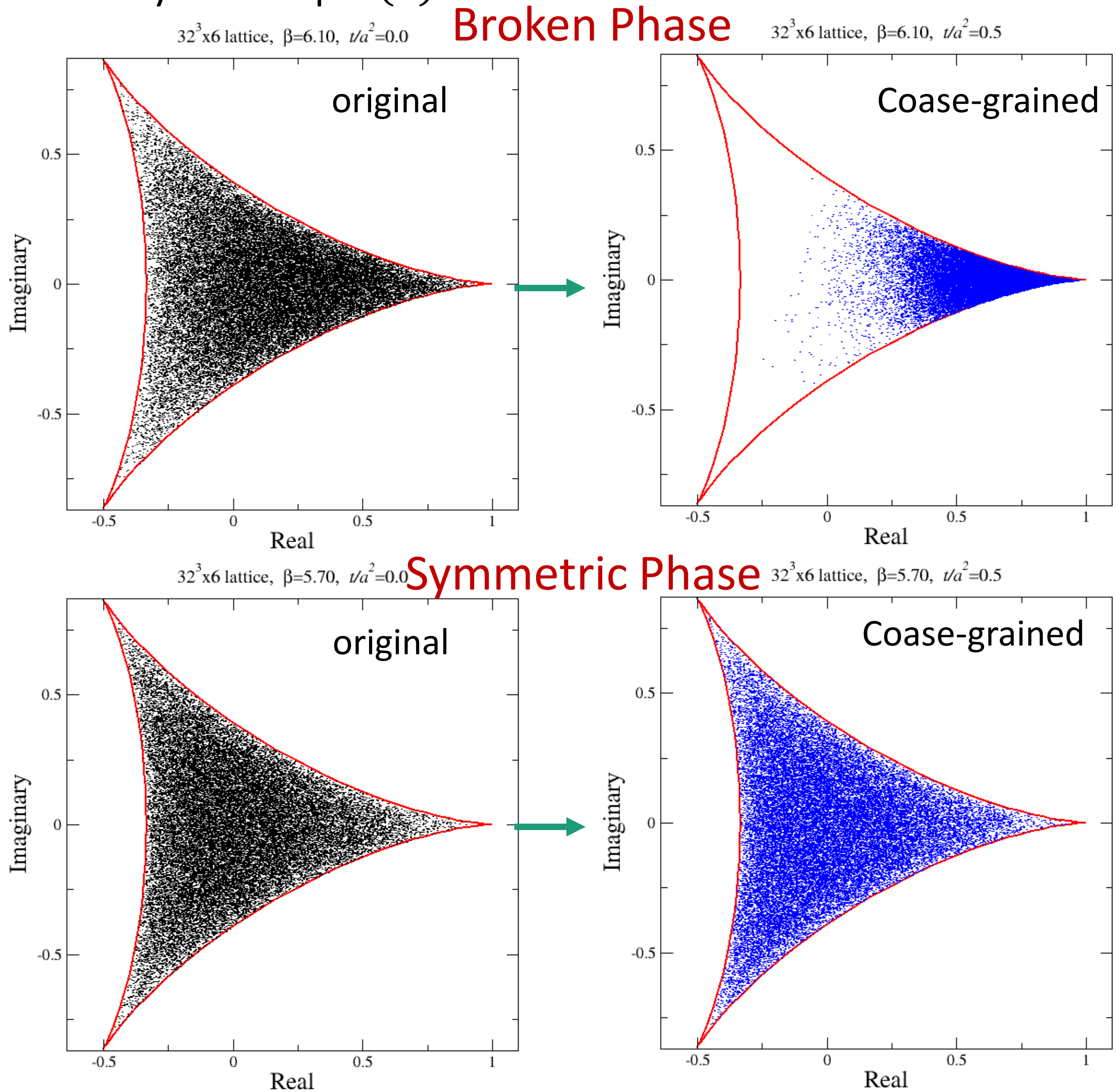
B_μ^a: flowed gauge field

$G_{\mu\nu}^a$: flowed field strength

- The symmetry break is hard to see in the original field, but it is emphasized in the flowed field.

- Coarse graining does not change the nature of the phase transition.

Local Polyakov loop $\Omega(\vec{x})$



3-state Potts model as an effective model of dense QCD

$$Z_{\rm QCD} = \int \mathcal{D}U\, e^{-S_g(U)} \prod_{\vec{x}} \{1 + 3C\Omega(\vec{x}) + 3C^2\Omega^*(\vec{x}) + C^3\}^{2N_f} \qquad C = (2\kappa)^{N_t} e^{\mu/T}$$

$$\boxed{3\Omega(\vec{x}) \to s(x)}$$

The spin has three states. $s(x) = \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$

$$\prod_x [1 + Cs(x) + C^2 s^*(x) + C^3]^{2N_f} = \prod_x F(x) = \exp[N_1 A_1 + N_2(A_2 + i\theta) + N_3(A_2 - i\theta)]$$

$$F(x) = \begin{cases} (1 + C + C^2 + C^3)^{2N_f} & \text{for } s(x) = 1 \\ \left(1 + C\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + C^2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + C^3\right)^{2N_f} & \text{for } s(x) = e^{2\pi i/3} \\ \left(1 + C\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + C^2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + C^3\right)^{2N_f} & \text{for } s(x) = e^{-2\pi i/3} \end{cases} \equiv \begin{cases} e^{A_1} & \text{for } s(x) = 1 \\ e^{A_2 + i\theta} & \text{for } s(x) = e^{2\pi i/3} \\ e^{A_2 - i\theta} & \text{for } s(x) = e^{2\pi i/3} \end{cases}$$

$$\sum_x \mathrm{Re}(s(x)) = N_1 - \frac{1}{2}(N_2 + N_3) = \frac{3}{2}N_1 - \frac{1}{2}N_{\rm site}, \qquad \sum_x \mathrm{Im}(s(x)) = \frac{\sqrt{3}}{2}(N_2 - N_3)$$

$$\Longrightarrow \quad Z_{\rm potts} = \sum_{\{s\}} \exp\left\{\beta \sum_{x,j} \mathrm{Re}(s(x)s^*(x+j)) + h\sum_x \mathrm{Re}(s(x)) + iq\sum_x \mathrm{Im}(s(x))\right\}$$

$$h(C) = \frac{4}{3}N_f \ln(1 + C + C^2 + C^3) - \frac{2}{3}N_f \ln\left\{\left(1 - \frac{1}{2}C - \frac{1}{2}C^2 + C^3\right)^2 + \frac{3}{4}(C - C^2)^2\right\}$$

$$q(C) = \frac{4}{\sqrt{3}}N_f \arctan\left\{\frac{\frac{\sqrt{3}}{2}(C - C^2)}{1 - \frac{C}{2} - \frac{C^2}{2} + C^3}\right\}$$

complex external magnetic field terms with real parameters *h* and *q*.

3-D 3-state Potts model with a complex external magnetic field

$$C = (2\kappa)^{N_t} e^{\mu/T}$$

$$s(x) = \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$$

$$Z_{\mathrm{QCD}} = \int \mathcal{D}U \, e^{-S_g(U)} \prod_{\vec{x}} \{1 + 3C\Omega(\vec{x}) + 3C^2\Omega^*(\vec{x}) + C^3\}^{2N_f}$$

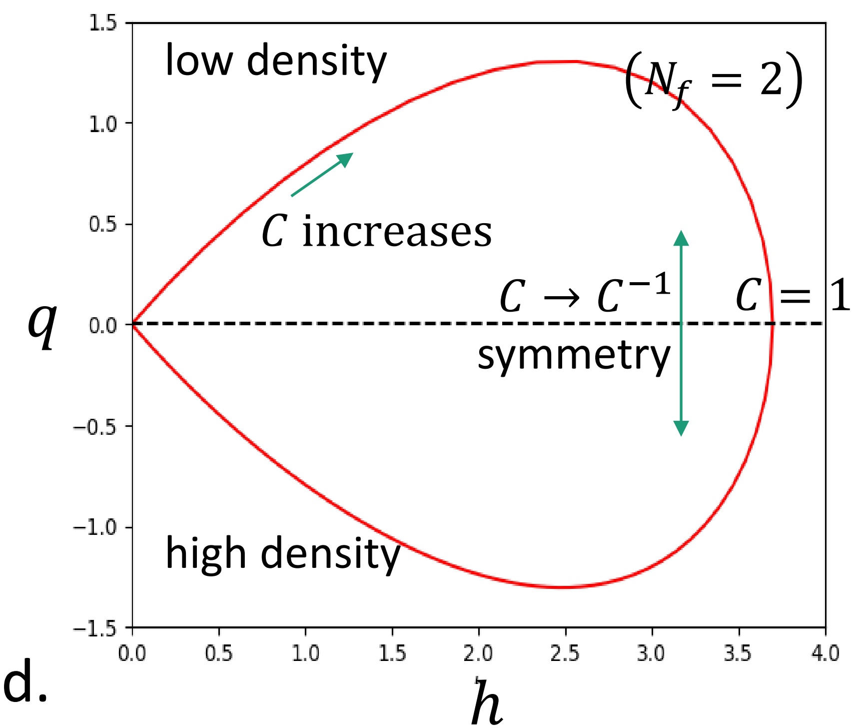
$$Z_{\mathrm{potts}} = \sum_{\{s\}} \exp\left\{\beta \sum_{x,j} \mathrm{Re}(s(x)s^*(x+j)) + h\sum_x \mathrm{Re}(s(x)) + iq\sum_x \mathrm{Im}(s(x))\right\}$$

$$h(C) = \frac{4}{3}N_f \ln(1 + C + C^2 + C^3) - \frac{2}{3}N_f \ln\left\{\left(1 - \frac{1}{2}C - \frac{1}{2}C^2 + C^3\right)^2 + \frac{3}{4}(C - C^2)^2\right\}$$

$$q(C) = \frac{4}{\sqrt{3}}N_f \arctan\left\{\frac{\frac{\sqrt{3}}{2}(C - C^2)}{1 - \frac{C}{2} - \frac{C^2}{2} + C^3}\right\}$$

For real μ, simply h and q are real parameters.
The red curve is the parameter corresponding to HDQCD.

This argument can be extended to a complex chemical potential.
Roberge-Weiss singularity can also be discussed.



Phase structure of heavy dense QCD

The relationship between heavy quark QCD with μ=0 and the standard three-state Potts model (q=0) has been well studied.

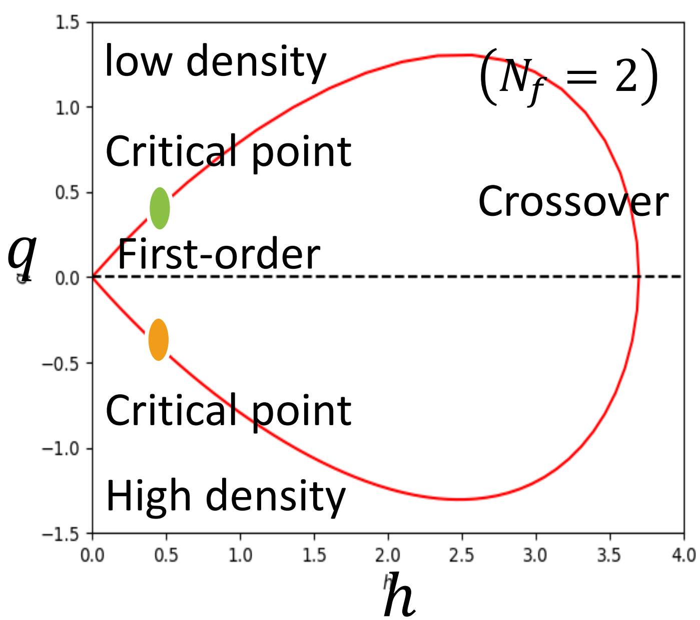
When q=0, there is a first-order phase transition at $h < h_c \approx 0.0005$ and crossover at $h > h_c$

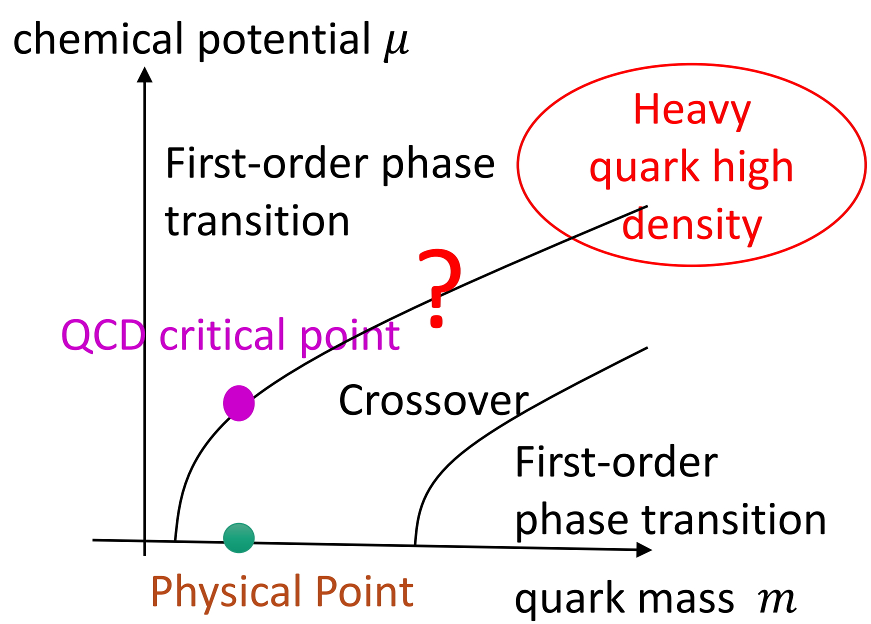
Due to the symmetry $s \to s^*$, if (h_c, q_c) is a critical point, then $(h_c, -q_c)$ is also a critical point.

Our conclusions

- The phase transition is first order in the low-density region, changes to a crossover at the critical point, and then becomes first order again.
- The critical point belongs to the 3-D Ising universality class.
- As h increases, the changes in physical quantities become milder.

Parameters corresponding to HDQCD

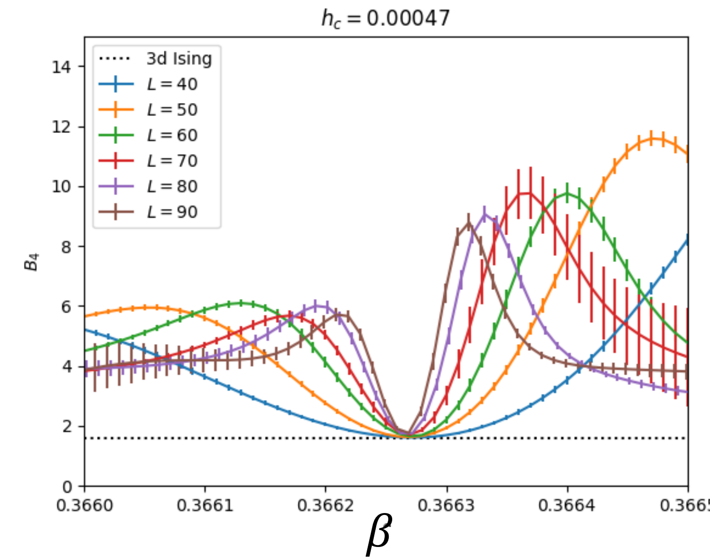




Critical points where first-order phase transition end

- Binder cumulant: $B_4 = \frac{\langle (M-\langle M\rangle)^4\rangle}{\langle (M-\langle M\rangle)^2\rangle^2}$
- No volume dependence at the critical point

$$B_4(h-h_c, L^{-1}) = \frac{f_4\left((h-h_c)L^{1/\nu}\right)}{\left(f_2\left((h-h_c)L^{1/\nu}\right)\right)^2} + 3 \qquad (V = L^3)$$

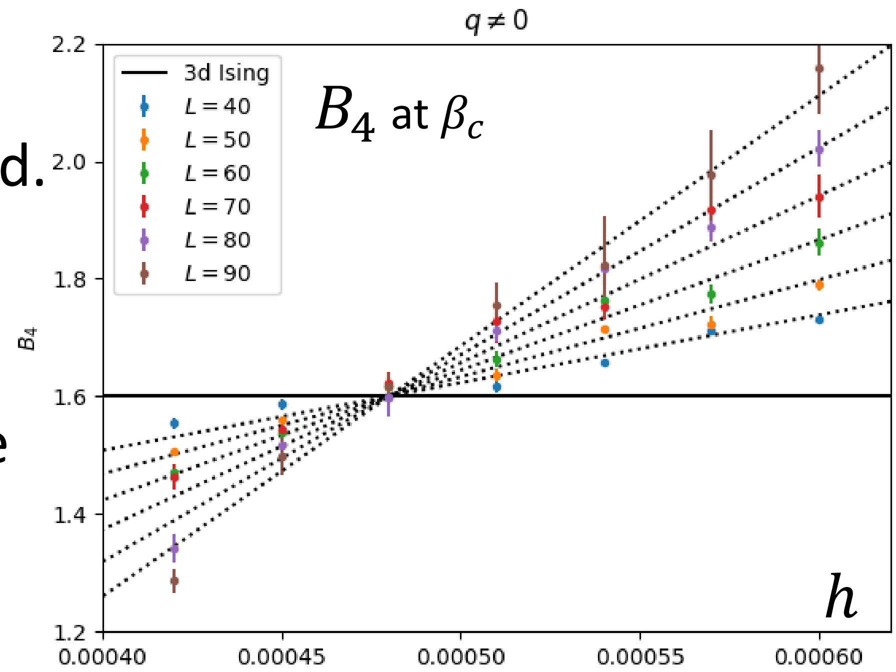


- With $q = 0$, the results were consistent with those by [Karsh et al, Phys.Lett.B 488(2000)].

$$h_c = 0.000510(8) \quad (q = 0)$$

- Near h_c, q is small. Using the reweighting method, complex terms can be incorporated.
- Dotted lines: Fitted results

$$B_4 = B_{4c} + A(h-h_c)L^{1/\nu}$$



- h_c becomes smaller due to the effect of the complex external field.

$$\underline{h_c = 0.000470(2) \quad (q \neq 0)}$$

- The universality class is the same as the 3D Ising model. ($B_{4c} \approx 1.60$, $\nu \approx 0.63$)
- $C_c = 0.0001195(7)$ $(= 1/C_c)$

$B_{4c} = 1.601(6)$, A=3.01(1),
$h_c = 0.000473(3)$, $\nu = 0.624(3)$,
$q_c = 0.000478(3)$, $\chi^2/\mathrm{dof} = 1.805$

Scaling law of the susceptibility peak

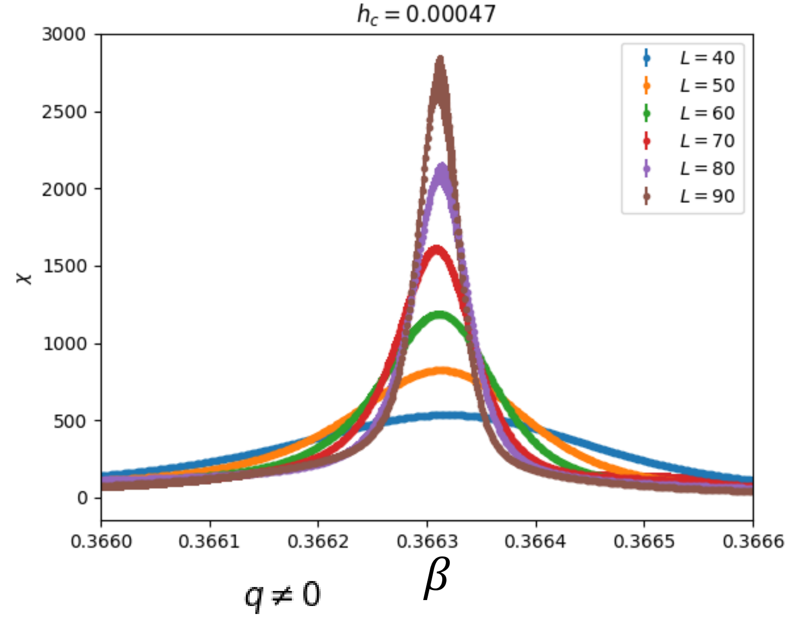
- Susceptibility: $\chi = V\langle (M - \langle M \rangle)^2 \rangle$ $\qquad V = L^3$
- Susceptibility peaks become higher as the volume increases.

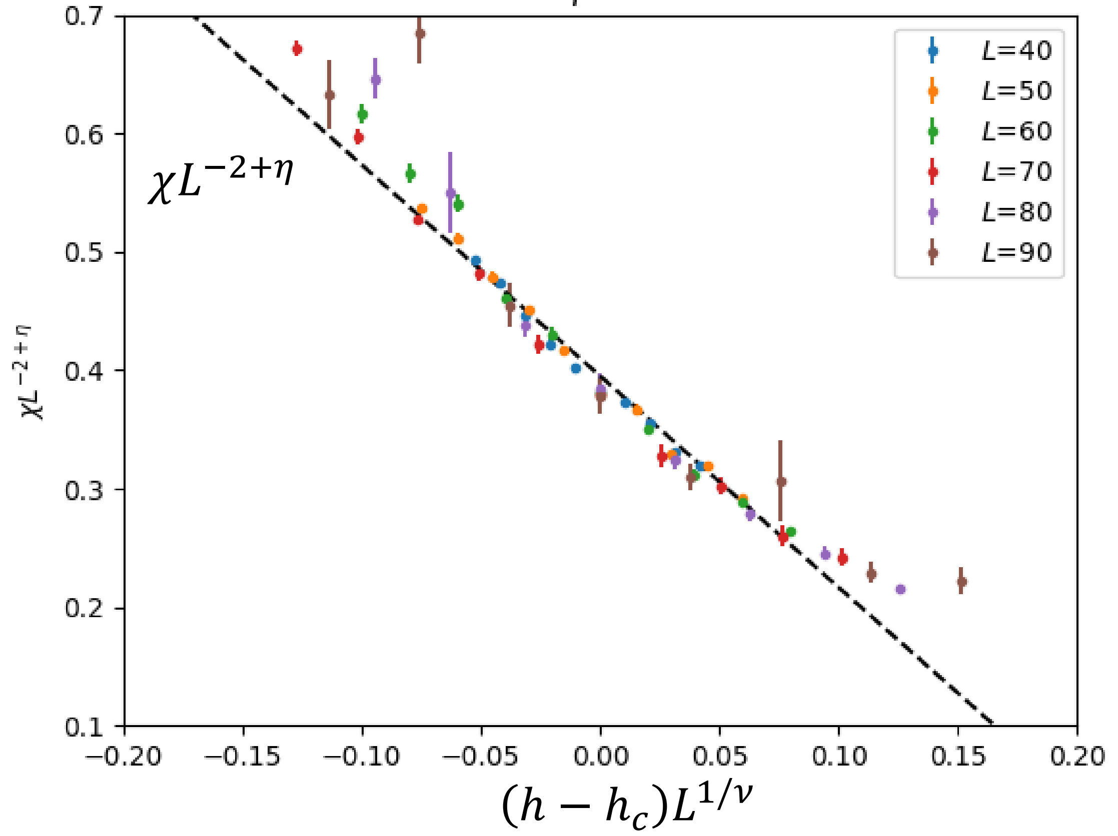
$$\chi(h - h_c, L^{-1}) = L^{2-\eta} f_2\left((h - h_c)L^{1/\nu}\right)$$

- The reweighting method is used.
- We use $\eta = 0.0366, \nu = 0.630$ and $h_c = 0.00047$
- Dashed line: Fitted results

$$\chi L^{-2+\eta} = A(h - h_c)L^{1/\nu} + B$$

(Fit parameters A, B)





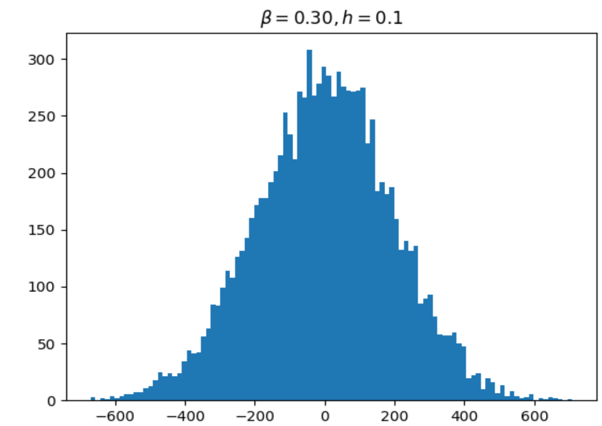
The universality class is the same as the 3D Ising model.

physical quantities in the crossover region

$$Z_{\text{potts}} = \sum_{\{s\}} \exp\left\{ \underbrace{\beta \sum_{x,j} \mathrm{Re}(s(x)s^*(x+j))}_{=E} + h\sum_{x} \mathrm{Re}(s(x)) + iq\sum_{x} \mathrm{Im}(s(x)) \right\}$$

- We compute physical quantities in the crossover region by the reweighting method.
- When q is large and the volume is large, the sign problem is severe.
- The imaginary terms are expressed as the number of spins

$$q\sum_{j} \mathrm{Im}(s_j) = q\frac{\sqrt{3}}{2}(N_2 - N_3)$$



$N_2 - N_3 \ (V = 32^3)$

$N_2 - N_3$ is Gaussian distributed.

- Using the reweighting method, the energy incorporating the complex external field is

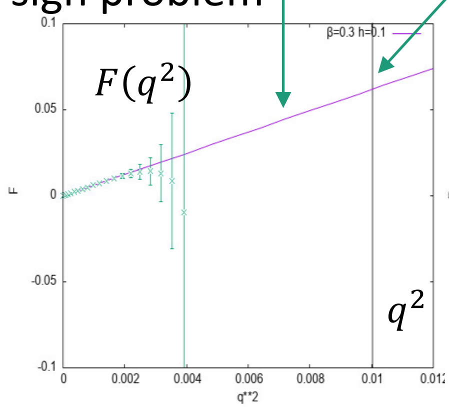
$$\langle E\rangle_q = \frac{\left\langle E\cos\left\{q\frac{\sqrt{3}}{2}(N_2 - N_3)\right\}\right\rangle_0}{\left\langle \cos\left\{q\frac{\sqrt{3}}{2}(N_2 - N_3)\right\}\right\rangle_0} \equiv \langle E\rangle_0 e^{F(q^2)}$$

- Assuming a Gaussian distribution, $F(q^2) \approx Aq^2$
- We extrapolate $F(q^2)$ and get $\langle E\rangle_q$

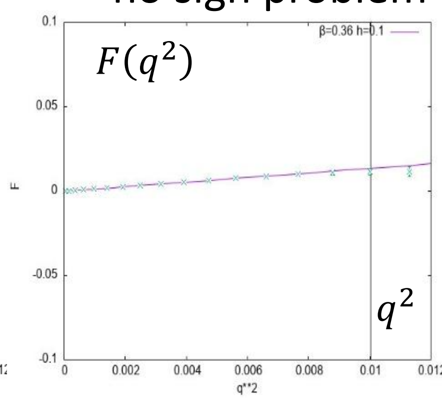
Fit line

Target q

sign problem

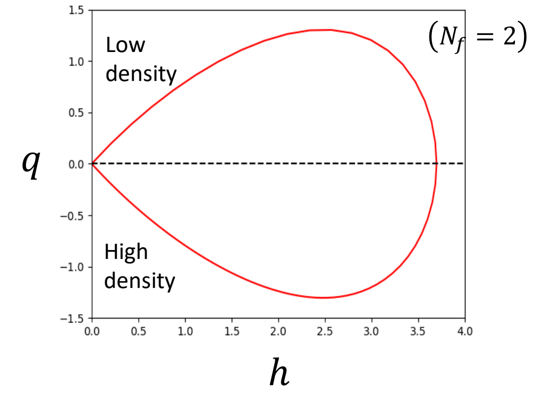


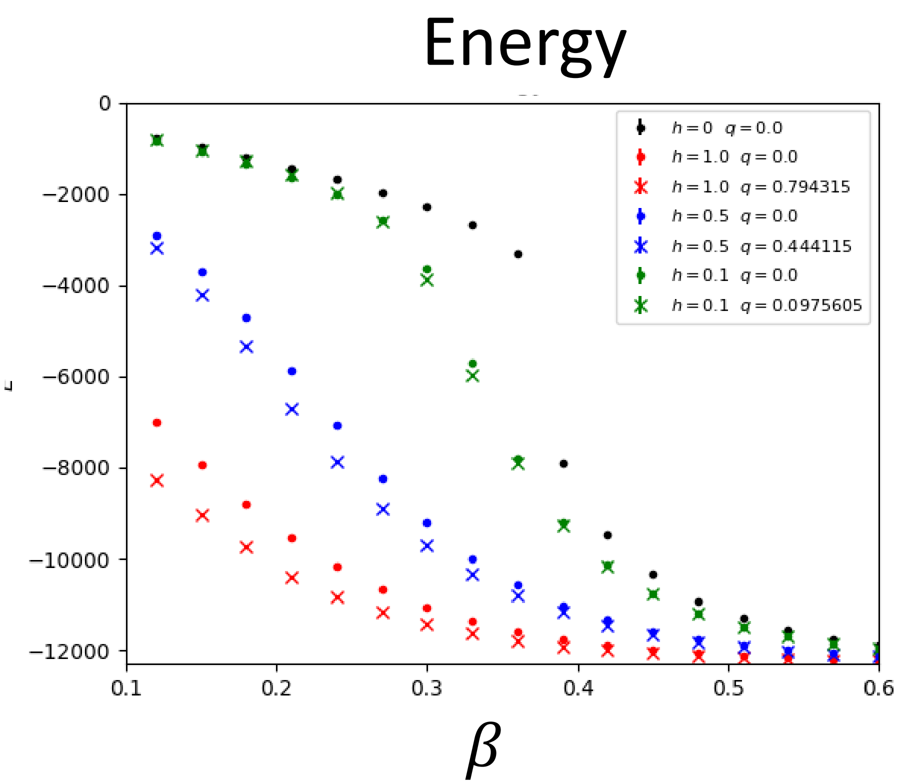
no sign problem

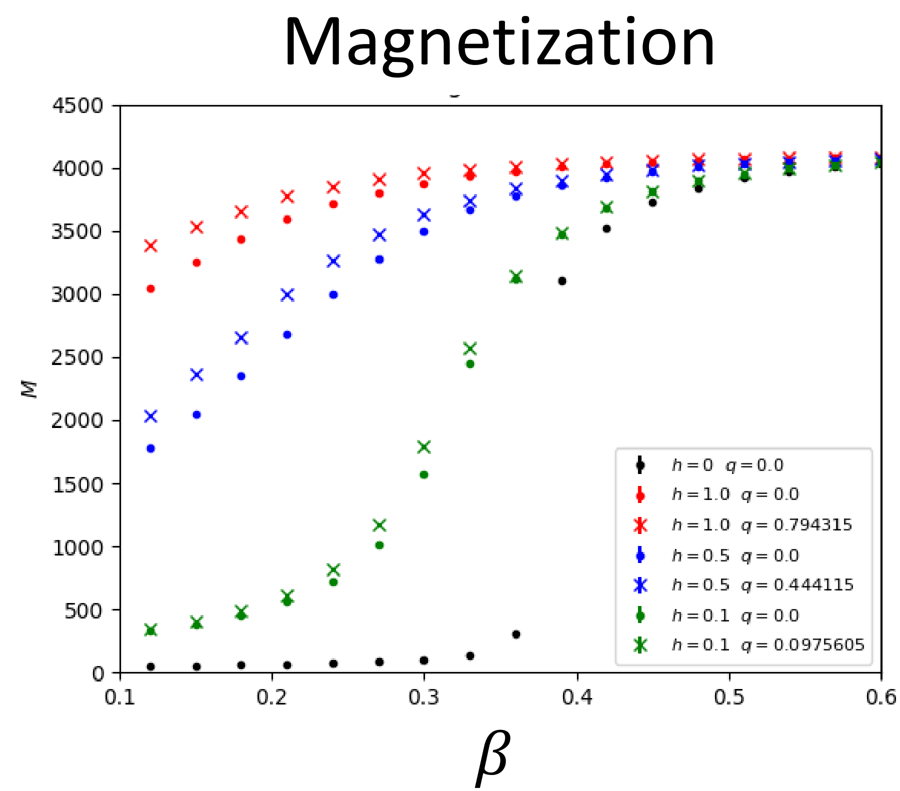


Energy and Magnetization by reweighting method

- Results by reweighting method
- The volume dependence is small, when h is large.
- As h increases, the change becomes milder.
- When h is large, the phase transition is a crossover.





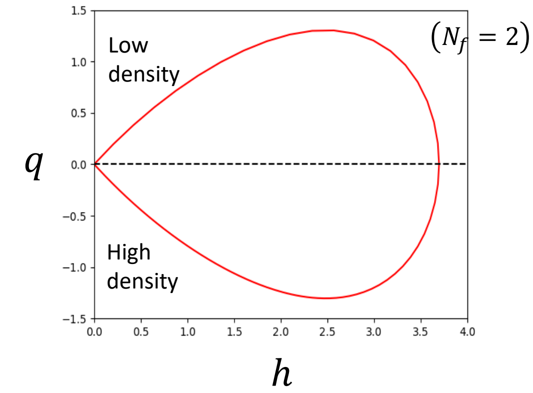


○ Parameters ignoring the complex phase, $q = 0$.

× Parameters corresponding to HDQCD

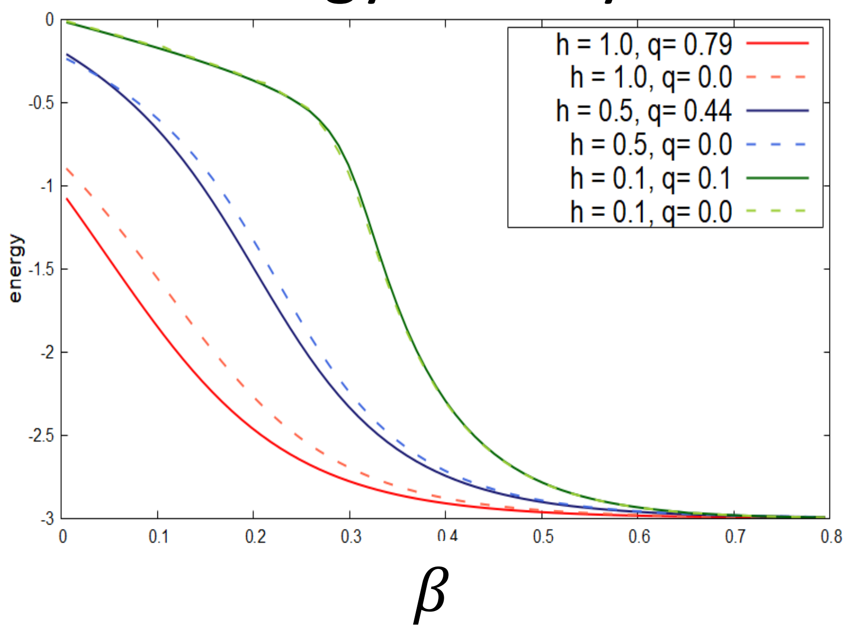
Volume: 16^3

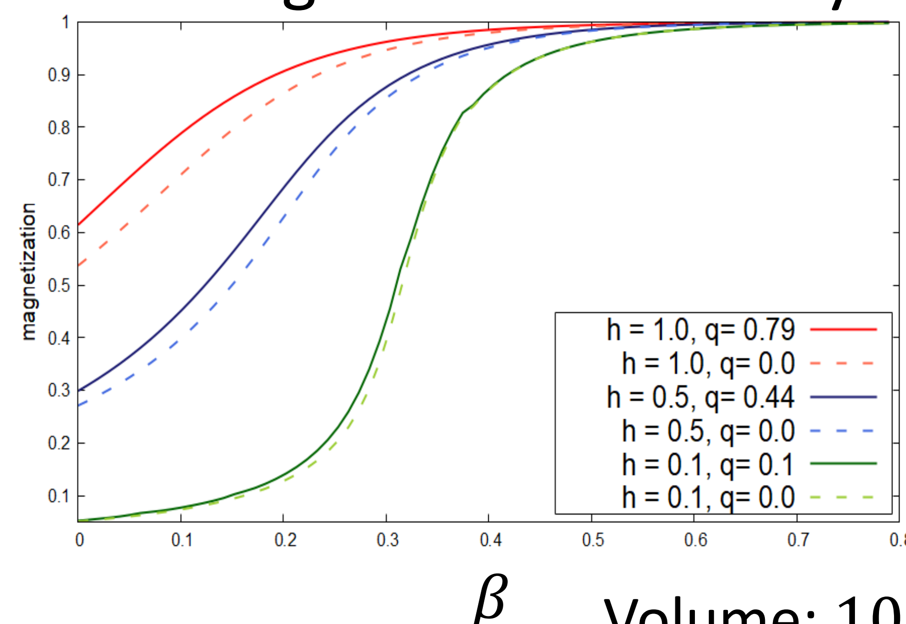
Tensor renormalization group analysis



- 3-D 3-state Potts model with a complex external field
- TRG has nothing to do with the sign problem.
- Higher Order Tensor Renormalization Group (HOTRG)
- The truncation error of the singular value decomposition is small in the crossover region.

Energy density HOTRG (Dcut=8) Magnetization density





Volume: 1024^3

Solid line: Parameters corresponding to HDQCD

Dash line: Parameters ignoring the complex phase

Same result as the reweighting method. The larger h, the milder the change.

Phase quenched QCD (Ignore complex phase) (isospin μ)

$(\Omega = \Omega_{\mathrm{R}} + i\Omega_{\mathrm{I}})$

- Effective action for heavy quarks

$$S_{\mathrm{eff}} = -6N_{\mathrm{site}}\beta^* P - N_s^3 \lambda \left(\cosh\frac{\mu}{T}\,\Omega_{\mathrm{R}} + \cancel{i\sinh\frac{\mu}{T}\,\Omega_{\mathrm{I}}}\right)$$

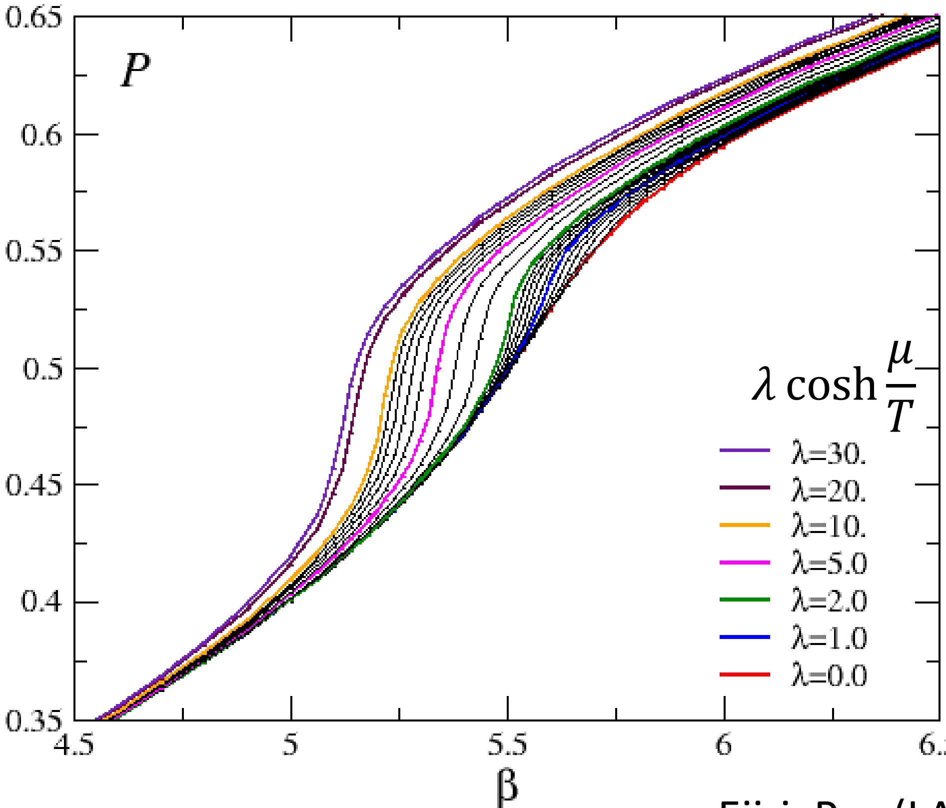
- The $e^{m\mu/T}$ terms ($\cosh(m\mu/T)$ terms) are missing from S_{eff}. $(m \geq 2)$
- However, $e^{m\mu/T}$ terms only contribute when the density is extremely high.
- If we ignore the complex phase, Monte-Carlo simulation is possible.
- For $N_{\mathrm{f}} = 2$, this is QCD with an isospin chemical potential.
- Pion condensation is expected in the high-density region.
- The effect of the complex phase must be added, but there may be other phase structures present.

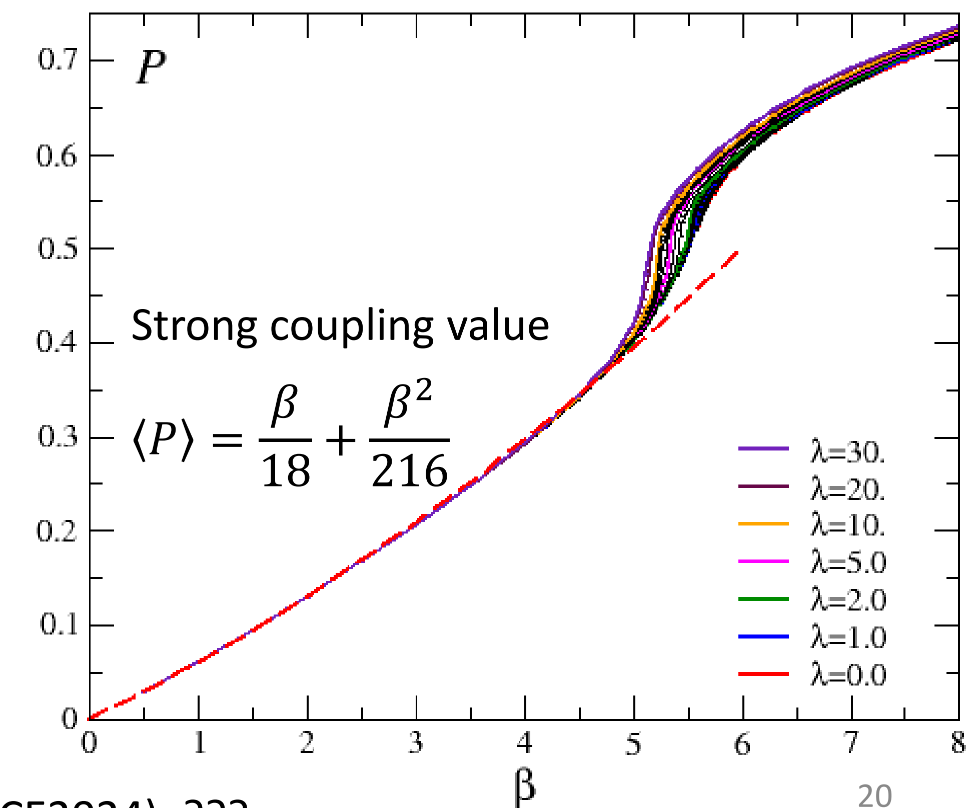
- There are no singularity in the large h region of 3-state Potts model.
- For large h, Z(3) center symmetry is completely broken. The phase transition is irrelevant to the symmetry.
- We do not conclude that there is no further singularity in HDQCD.

Plaquette in phase quenched QCD (isospin QCD)

- As $\cosh(\mu/T)$ increases, the change in plaquette becomes steeper.
- This behavior is consistent with the existence of a pion-condensed phase at large μ.
- The strong coupling expansion of $\langle P \rangle$ does not depend on $\lambda \cosh(\mu/T)$.
- Confinement phase: $\langle P \rangle$ is consistent with the strong coupling expansion.
- The $\lambda \cosh(\mu/T)$ term forces the deconfinement phase.

Lattice size: $30^3 \times 6$





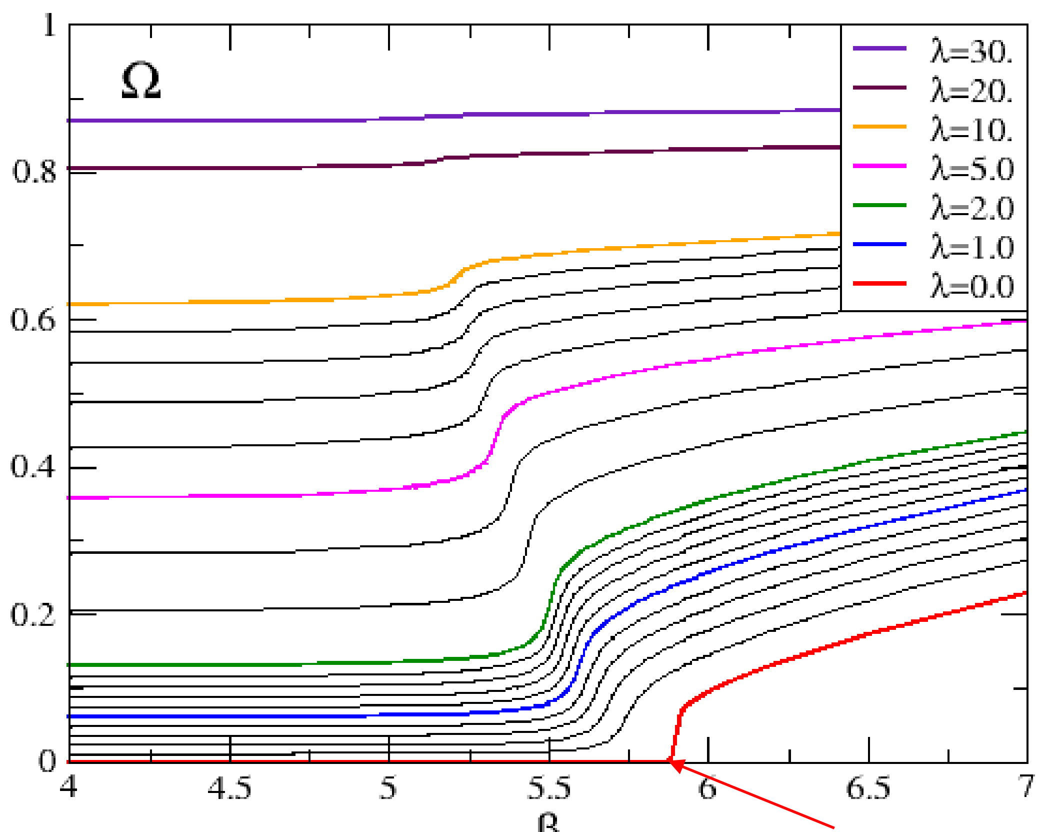
Ejiri, Pos (LATTICE2024), ???

Polyakov loop in phase quenched QCD (isospin QCD)

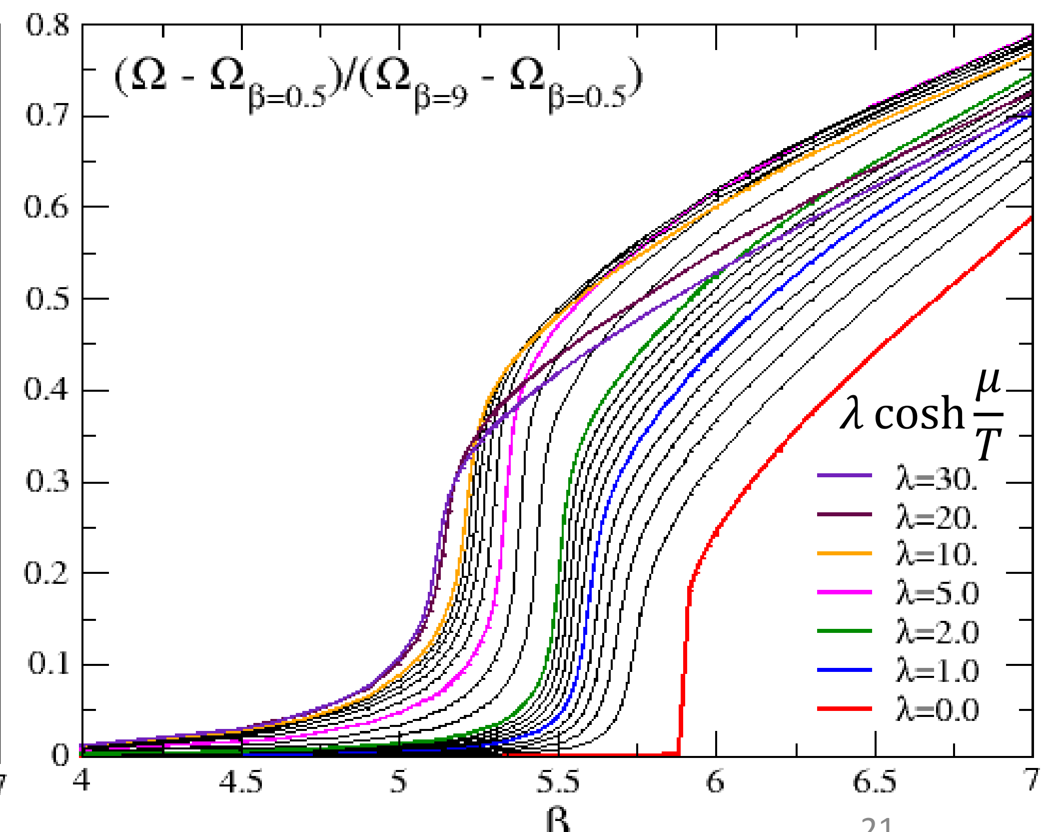
- First order phase transition at $\lambda = 0$
- Regarding the Polyakov loop $\langle \Omega_{\mathrm{R}} \rangle$, if we look closely at the changing part, the change becomes steeper as μ increases.
- Effect from the complex phase may be important. [Ejiri, PoS (LATTICE2024), 162]

Discontinuity at large μ? → first-order transition?

$$\frac{\langle \Omega_{\mathrm{R}} \rangle - \langle \Omega_{\mathrm{R}} \rangle_{\beta=0.5}}{\langle \Omega_{\mathrm{R}} \rangle_{\beta=9} - \langle \Omega_{\mathrm{R}} \rangle_{\beta=0.5}}$$



first-order transition



Summary

Phase structure of heavy dense QCD is discussed through the effective theory of the 3-state Potts model.

- The phase transition is first order in the low-density region, changes to a crossover at the critical point, and then becomes first order again.
- The critical point belongs to the 3-D Ising universality class.
- As h increases, the changes in physical quantities become milder.
- However, we do not conclude that there is no further singularity in HDQCD.