Mass dimension one fermions and spinor classifications

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Prelude

The Standard Model of particle physics (SM) has been successful in describing the properties of elementary particles. But we know that it is incomplete. One of the most convincing evidence is **dark matter**.

There are many theories of dark matter but almost all of them have one thing in common. The particles are described in the framework of quantum field theory (QFT).

- Dirac/Majorana fermions.
- Scalar bosons.

There are many theories of dark matter but almost all of them have one thing in common. The particles are described in the framework of quantum field theory (QFT).

- Dirac/Majorana fermions.
- Scalar bosons.

Are there other possibilities within the QFT framework? This is not just a question about dark matter but it also concerns the structure of QFT.

Lounesto spinor classification

Lounesto, P., 2001. *Clifford algebras and spinors*. Lond. Math. Soc. Lect. Note Ser. 286, 1-338.

Dirac spinors

1. $\Omega_1 \neq 0$, $\Omega_2 \neq 0$.
2. $\Omega_1 \neq 0$, $\Omega_2 = 0$.
3. $\Omega_1 = 0$, $\Omega_2 \neq 0$.

Singular spinors

4. $\Omega_1 = 0$, $\Omega_2 = 0$, $K^\mu \neq 0$, $S^{\mu\nu} \neq 0$.
5. $\Omega_1 = 0$, $\Omega_2 = 0$, $K^\mu = 0$, $S^{\mu\nu} \neq 0$.

Weyl spinors

6. $\Omega_1 = 0$, $\Omega_2 = 0$, $K^\mu \neq 0$, $S^{\mu\nu} = 0$.

$$\begin{array}{ll} \Omega_1 = \overline{\psi}\psi, & \Omega_2 = \overline{\psi}\gamma^5\psi, \\ K^\mu = \overline{\psi}\gamma^5\gamma^\mu\psi, & S^{\mu\nu} = \overline{\psi}\gamma^\mu\gamma^\nu\psi \end{array}$$

Do these **singular spinors** and their quantum field operators have any applications in particle physics?

Elko and mass dimension one fermions

The theory of **Elko** and **mass dimension one fermions** was an unexpected discovery made by **Dharam Vir Ahluwalia** and **Daniel Grumiller**

D. V. Ahluwalia and D. Grumiller, *Spin-half fermions with mass dimension one: theory, phenomenology, and dark matter*, JCAP 0507 (2005) 012. arXiv: hep-th/0412080

D. V. Ahluwalia and D. Grumiller, *Dark matter: A spin one-half fermion field with mass dimension one?* Phys. Rev. D 72, 067701 (2005). arXiv: hep-th/0410192

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Elko is a **flag-pole spinor** belonging to the **4th class** of the Lounesto classification.

R. da Rocha and W. A. Rodrigues Jr., *Where are ELKO spinor fields in Lounesto spinor field classification?* , Mod. Phys. Lett. A21 (2006) 65-74

It was an unexpected discovery because the fermionic field satisfy the **Klein-Gordon** but not the Dirac equation. It has **mass dimension one** instead of three-half.

It was an unexpected discovery because the fermionic field satisfy the **Klein-Gordon** but not the Dirac equation. It has **mass dimension one** instead of three-half.

These fermions have renormalisable **quartic self-interaction**, a desirable property for dark matter.

CYL. *Consistent interactions for mass dimension one fermions.*
17th October 2019.

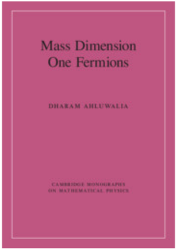
The theory has been studied in many disciplines

- Cosmology
- Mathematical physics
- Quantum field theory

For a comprehensive review and references therein, please see

D. V. Ahluwalia, *The theory of local mass dimension one fermions of spin one half*, Adv. Appl. Clifford Algebras 27 (2017) no.3, 2247-2285. arXiv: 1601.03188

Cambridge Monograph on Mathematical Physics



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Dharam Ahluwalia

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What is Elko?

E*igenspinoren des* **L***adungs***k***onjugations***o***perators*

Elko is a complete set of eigenspinors of the charge-conjugation operator of the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation

The charge-conjugation operator

In the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ representation, we can define the charge-conjugation operator as

$$\mathcal{C} = \begin{pmatrix} O & -i\Theta^{-1} \\ -i\Theta & O \end{pmatrix} K, \qquad \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where K is the complex-conjugation operator

$$\Theta \boldsymbol{\sigma} \Theta^{-1} = -\boldsymbol{\sigma}^*$$

Dirac spinors

$$\mathcal{C}u(\mathbf{p},\sigma) = iv(\mathbf{p},\sigma), \qquad \mathcal{C}v(\mathbf{p},\sigma) = iu(\mathbf{p},\sigma)$$

Majorana spinors

$$\mathcal{C}\psi_M(\mathbf{p},\sigma) = \psi_M(\mathbf{p},\sigma)$$

Dirac spinors

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Majorana spinors

$$\mathcal{C}\psi_M(\mathbf{p},\sigma) = \psi_M(\mathbf{p},\sigma)$$

Elko (a complete set of Majorana spinors)

$$\mathcal{C}\xi(\mathbf{p},\sigma) = \xi(\mathbf{p},\sigma), \qquad \mathcal{C}\zeta(\mathbf{p},\sigma) = -\zeta(\mathbf{p},\sigma)$$

The Elko construct

Let $\phi(\boldsymbol{\epsilon}, \sigma)$ be a left-handed Weyl spinor at rest

$$\left(\frac{1}{2}\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}\right) \phi(\boldsymbol{\epsilon}, \sigma) = \sigma\phi(\boldsymbol{\epsilon}, \sigma), \qquad \boldsymbol{\epsilon} = \lim_{|\boldsymbol{p}| \to 0} \hat{\mathbf{p}}.$$

$$\mathbf{p} = |\mathbf{p}|(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta).$$

The Elko construct

The spinor of arbitrary momentum is obtained by

$$\phi(\mathbf{p}, \sigma) = \exp\left(-\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right)\phi(\boldsymbol{\epsilon}, \sigma)$$

The Elko construct

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$$\phi(\mathbf{p},\sigma) = \exp\left(-\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right)\phi(\boldsymbol{\epsilon},\sigma)$$

Because $\phi(\mathbf{p},\sigma)$ is a left-handed Weyl spinor, it follows that $\vartheta\Theta\phi^*(\mathbf{p},\sigma)$ transforms as a right-handed Weyl spinor

$$\vartheta\Theta\phi^*(\mathbf{p},\sigma) = \exp\left(\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right)\left[\vartheta\Theta\phi^*(\boldsymbol{\epsilon},\sigma)\right].$$

The Elko construct

We can construct a four-component spinor

$$\chi(\mathbf{p},\sigma) = \begin{bmatrix} \vartheta\Theta\phi^*(\mathbf{p},\sigma) \\ \phi(\mathbf{p},\sigma) \end{bmatrix}$$

It becomes Elko with eigenvalues ± 1

$$\boxed{\mathcal{C}\chi(\mathbf{p},\sigma)|_{\vartheta=\pm i} = \pm\chi(\mathbf{p},\sigma)|_{\vartheta=\pm i}}$$

The Elko construct

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$$\boxed{\mathcal{C}\chi(\mathbf{p},\sigma)|_{\vartheta=\pm i} = \pm\chi(\mathbf{p},\sigma)|_{\vartheta=\pm i}}$$

They are denoted as

$$\boxed{\mathcal{C}\xi(\mathbf{p},\sigma) = \xi(\mathbf{p},\sigma)} \qquad \boxed{\mathcal{C}\zeta(\mathbf{p},\sigma) = -\zeta(\mathbf{p},\sigma)}$$

Elko **is not** a Majorana field. The latter takes the form

$$\psi_M(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2E}} \sum_\sigma \left[e^{-ip\cdot x} u(\mathbf{p}, \sigma) c(\mathbf{p}, \sigma) + e^{ip\cdot x} v(\mathbf{p}, \sigma) c^\dagger(\mathbf{p}, \sigma) \right]$$

It can be put into the form

$$\psi_M(x) = \begin{bmatrix} -i\sigma_2 \psi_L^*(x) \\ \psi_L(x) \end{bmatrix}$$

but this is a **field operator**. The field equation is

$$\boxed{(i\partial\!\!\!/ - m)\psi_M(x) = 0 \to \bar{\sigma}^\mu \partial_\mu \psi_L = -m\sigma_2 \psi_L^*}$$

Fermionic fields from Elko

We can construct two fermionic fields

$$\Lambda(x) = (2\pi)^{-3/2} \int \frac{d^3 p}{\sqrt{2mE_{\mathrm{p}}}} \sum_{\sigma} [e^{-ip\cdot x}\xi(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + e^{ip\cdot x}\zeta(\mathbf{p},\sigma)b^{\ddagger}(\mathbf{p},\sigma)]$$

$$\lambda(x) = \Lambda(x)|_{b^{\ddagger}=a^{\ddagger}}$$

The annihilation and creation operators satisfy

$$\{a(\mathbf{p}',\sigma'), a^{\ddagger}(\mathbf{p},\sigma)\} = \delta_{\sigma'\sigma}\delta^3(\mathbf{p}'-\mathbf{p}),$$

$$\{b(\mathbf{p}',\sigma'), b^{\ddagger}(\mathbf{p},\sigma)\} = \delta_{\sigma'\sigma}\delta^3(\mathbf{p}'-\mathbf{p}).$$

Solutions for Elko

There are subtleties involved in choosing the labellings and phases for Elko. They directly affect the locality structure of the fermionic fields.

D. V. Ahluwalia, C.-Y. Lee and D. Schritt, *Elko as self-interacting fermionic dark matter with axis of locality*, Phys. Lett. B 687 (2010). arXiv: 0804.1854

D.V. Ahluwalia, C.-Y. Lee and D. Schritt, *Self-interacting Elko dark matter with an axis of locality*, Phys. Rev. D 83, 065017 (2011). arXiv: 0911.2947

C.-Y. Lee, *Self-interacting mass-dimension one fields for any spin*, Int.J. Mod. Phys, A30, 1550048 (2015). arXiv:1210.7916

The dual space

What are the kinematics for
mass dimension one fermions?

The dual space

What are the kinematics for mass dimension one fermions?

In order to derive the Lagrangian for the fermionic fields, we need to need to construct scalar invariants. Therefore, we need to define an appropriate dual space.

Dirac spinors satisfy the orthonormal relation

$$\bar{u}(\mathbf{p},\sigma')u(\mathbf{p},\sigma) = -\bar{v}(\mathbf{p},\sigma')v(\mathbf{p},\sigma) = 2m\delta_{\sigma\sigma'}$$

The dual is defined as

$$\bar{u}(\mathbf{p},\sigma) = u^\dagger(\mathbf{p},\sigma)\Gamma, \qquad \bar{v}(\mathbf{p},\sigma) = v^\dagger(\mathbf{p},\sigma)\Gamma$$

$$\Gamma = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$$

Spin-sums

$$\sum_{\sigma} \psi(\mathbf{p}, \sigma)\bar{\psi}(\mathbf{p}, \sigma) = \not{p} \pm mI$$

Field equation

$$(\not{p} \pm mI)\psi(\mathbf{p}, \sigma) = 0$$

Completeness

$$\frac{1}{2m}\sum_{\sigma}[u(\mathbf{p}, \sigma)\bar{u}(\mathbf{p}, \sigma) - v(\mathbf{p}, \sigma)\bar{v}(\mathbf{p}, \sigma)] = I$$

The dual space

The Dirac dual is inadequate for Elko

$$\overline{\chi}(\mathbf{p},\sigma)\chi(\mathbf{p},\sigma) = 0,$$

$$\overline{\chi}(\mathbf{p},\sigma) = \chi^{\dagger}(\mathbf{p},\sigma)\Gamma, \quad \Gamma = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$$

Recall the Lounesto classification:

Singular spinors

4. $\Omega_1 = 0$, $\Omega_2 = 0$, $K^\mu \neq 0$, $S^{\mu\nu} \neq 0$.
5. $\Omega_1 = 0$, $\Omega_2 = 0$, $K^\mu = 0$, $S^{\mu\nu} \neq 0$.

$$\begin{aligned} \Omega_1 &= \overline{\psi}\psi, & \Omega_2 &= \overline{\psi}\gamma^5\psi, \\ K^\mu &= \overline{\psi}\gamma^5\gamma^\mu\psi, & S^{\mu\nu} &= \overline{\psi}\gamma^\mu\gamma^\nu\psi \end{aligned}$$

The Elko dual

A new dual is required for Elko

$$\widetilde{\xi}(\mathbf{p},\sigma) = [\Xi(\mathbf{p})\xi(\mathbf{p},\sigma)]^\dagger\Gamma, \qquad \widetilde{\zeta}(\mathbf{p},\sigma) = [\Xi(\mathbf{p})\zeta(\mathbf{p},\sigma)]^\dagger\Gamma$$

$$\Xi(\mathbf{p}) = \frac{1}{2m}\sum_\sigma \left[\xi(\mathbf{p},\sigma)\overline{\xi}(\mathbf{p},\sigma) - \zeta(\mathbf{p},\sigma)\overline{\zeta}(\mathbf{p},\sigma)\right]$$

Equivalently

$$\boxed{\widetilde{\xi}(\mathbf{p},\sigma) = i(-1)^{1/2+\sigma}\overline{\xi}(\mathbf{p},-\sigma), \quad \widetilde{\zeta}(\mathbf{p},\sigma) = i(-1)^{1/2+\sigma}\overline{\zeta}(\mathbf{p},-\sigma)}$$

L. D. Sperança, *An identification of the Dirac operator with the parity operator, Int.J.Mod.Phys. D23 (2014) 14, 1444003. arXiv: 1304.4794*

Orthonormal and completeness relations

Elko is orthonormal under the new dual

$$\widetilde{\xi}(\mathbf{p},\sigma)\xi(\mathbf{p},\sigma') = -\widetilde{\zeta}(\mathbf{p},\sigma)\zeta(\mathbf{p},\sigma') = 2m\delta_{\sigma\sigma'}$$

They also satisfy the completeness relation

$$\frac{1}{2m}\sum_{\sigma}\left[\xi(\mathbf{p},\sigma)\widetilde{\xi}(\mathbf{p},\sigma) - \zeta(\mathbf{p},\sigma)\widetilde{\zeta}(\mathbf{p},\sigma)\right] = I$$

Spin-sums

This is one of the most important features of the theory

$$\sum_\sigma \xi(\mathbf{p},\sigma)\widetilde{\xi}(\mathbf{p},\sigma) = m[\mathcal{G}(\phi) + I]$$

$$\sum_\sigma \zeta(\mathbf{p},\sigma)\widetilde{\zeta}(\mathbf{p},\sigma) = m[\mathcal{G}(\phi) - I]$$

$$\mathcal{G}(\phi) = i\begin{pmatrix} 0 & 0 & 0 & -e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & -e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \end{pmatrix}$$

Field equation/identities

Elko does not satisfy the Dirac equation

$$\not{p}\chi(\mathbf{p},\sigma) = \pm i m \chi(\mathbf{p},-\sigma), \quad (p^\mu p_\mu - m^2)\chi(\mathbf{p},\sigma) = 0$$

Elko satisfies the following identities

$$\boxed{\mathcal{G}(\phi)\xi(\mathbf{p},\sigma) = \xi(\mathbf{p},\sigma)} \qquad \boxed{\mathcal{G}(\phi)\zeta(\mathbf{p},\sigma) = -\zeta(\mathbf{p},\sigma)}$$

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There has been attempt to formulate the theory with the Dirac dual. We may rewrite the equations as

$$\not{p}\chi_1(\mathbf{p},\sigma) = \pm m\chi_2(\mathbf{p},\sigma)$$

Introduce two fermionic fields $\Lambda_1(x)$ and $\Lambda_2(x)$ with $\xi_i(\mathbf{p}, \sigma)$ and $\zeta_i(\mathbf{p}, \sigma)$ as expansion coefficients. Then

$$i\partial\!\!\!/ \Lambda_k = m\Lambda_\ell, \quad k \neq \ell$$

Introduce two fermionic fields $\Lambda_1(x)$ and $\Lambda_2(x)$ with $\xi_i(\mathbf{p},\sigma)$ and $\zeta_i(\mathbf{p},\sigma)$ as expansion coefficients. Then

$$i\not\partial\Lambda_k = m\Lambda_\ell, \quad k \neq \ell$$

We can write down two possible Lagrangians

$$\begin{aligned}
\mathscr{L}_1 &= \sum_{k\neq\ell}\bar{\Lambda}_k i\gamma^\mu\partial_\mu\Lambda_\ell - m\sum_k\bar{\Lambda}_k\Lambda_k, \\
\mathscr{L}_2 &= \sum_{k}\bar{\Lambda}_k i\gamma^\mu\partial_\mu\Lambda_k - m\sum_{k\neq\ell}\bar{\Lambda}_k\Lambda_\ell.
\end{aligned}$$

They have **non-local** field-conjugate momentum anti-commutators at equal-time and the free Hamiltonians are **not positive-definite**.

Propagator

The propagator is obtained from the time-ordered product:

$$\begin{aligned} S(x,y) &= \langle\, |T[\Lambda(x)\widetilde{\Lambda}(y)]|\, \rangle \\ &= i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \left[\frac{I+\mathcal{G}(\phi)}{p^2-m^2+i\epsilon}\right] \end{aligned}$$

$$(\partial^\mu\partial_\mu + m^2)S(x,y) = -i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}[I+\mathcal{G}(\phi)]$$

There exists a **preferred axis**. When $x-y$ is aligned to the 3-axis

$$\boxed{\int \frac{d^4p}{(2\pi)^4} e^{-ip_3(x-y)_3}\mathcal{G}(\phi) = O}$$

Canonical quantisation

We take the free Lagrangian to be Klein-Gordon:

$$\mathscr{L}_0 = \partial^\mu \widetilde{\Lambda} \partial_\mu \Lambda - m^2 \widetilde{\Lambda} \Lambda$$

It has a **positive-definite** Hamiltonian. Equal-time commutators

$$\{\Lambda(t, \mathbf{x}), \Lambda(t, \mathbf{y})\} = \{\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})\} = O$$

$$\{\Lambda(t, \mathbf{x}), \Pi(t, \mathbf{y})\} = i \int \frac{d^3 p}{(2\pi)^3} \, e^{i \boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{y})} \, [I + \mathcal{G}(\phi)]$$

Lorentz violation

The spin-sums are not Lorentz-covariant. Therefore, the fermions described by $\Lambda(x)$ and $\lambda(x)$ is **Lorentz-violating**. However, it is premature to discard the theory

- The theory has a rich structure.
- Energy-momentum relation: $E^2 = p^2 + m^2$.
- What are the physics of singular spinors?
- Do dark matter satisfy Lorentz symmetry?

A Lorentz-invariant τ-deformation

The matrices $\pm I + \mathcal{G}(\phi)$ is non-invertible. But we observe that

$$\boxed{\left[\pm I + \tau\mathcal{G}(\phi)\right]^{-1} = \frac{\pm I - \tau\mathcal{G}(\phi)}{1 - \tau^2}}$$

Perform an infinitesimal τ-deformation away from $\tau = 1$ to the spin-sums:

$$\boxed{\mathcal{G}(\phi) \to \tau\mathcal{G}(\phi)}$$

We define the following new duals

$$\boxed{\overset{\neg}{\xi}(\mathbf{p},\sigma) \equiv \widetilde{\xi}(\mathbf{p},\sigma)\left[\frac{+I-\tau\mathcal{G}(\phi)}{1-\tau^2}\right], \quad \overset{\neg}{\zeta}(\mathbf{p},\sigma) \equiv \widetilde{\zeta}(\mathbf{p},\sigma)\left[\frac{-I-\tau\mathcal{G}(\phi)}{1-\tau^2}\right]}$$

such that their spin-sums are **Lorentz-invariant**. For example,

$$\begin{aligned}
\sum_{\sigma}\xi(\mathbf{p},\sigma)\,\overset{\neg}{\xi}(\mathbf{p},\sigma) &\equiv \left[\sum_{\sigma}\xi(\mathbf{p},\sigma)\widetilde{\xi}(\mathbf{p},\sigma)\right]^{(\tau)}\left[\frac{I-\tau\mathcal{G}(\phi)}{1-\tau^2}\right] \\
&= \left[\frac{I+\tau\mathcal{G}(\phi)}{1-\tau^2}\right]\left[\frac{I-\tau\mathcal{G}(\phi)}{1-\tau^2}\right] \\
&= mI
\end{aligned}$$

The orthonormal and completeness relations remain unchanged. The **order of operation** is important.

D. V. Ahluwalia, *The theory of local mass dimension one fermions of spin one half*, Adv. Appl. Clifford Algebras 27 (2017) no.3, 2247-2285. arXiv:1304.4794

By implementing the τ-deformation, we obtain

$$\mathscr{L}_0 = \partial^\mu \bar{\Lambda} \partial_\mu \Lambda - m^2 \bar{\Lambda} \Lambda$$

Propagator:

$$S(x, y) = i \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{I}{p^2 - m^2 + i\epsilon}$$

Equal-time commutators:

$$\{\Lambda(t, \mathbf{x}), \Lambda(t, \mathbf{y})\} = \{\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})\} = O$$

$$\{\Lambda(t, \mathbf{x}), \Pi(t, \mathbf{y})\} = i\delta^3(\mathbf{x} - \mathbf{y}) I$$

Generalisations

We have discussed the construct of Elko. It is a **flag-pole spinor** and belongs to the **spin-half representation** of the Lorentz group.

What are the physics of flag-dipole spinors?

Are there higher-spin generalisations?

Flag-dipole spinors

It is possible to construct QFT with flag-dipole spinors.

CYL, *Mass dimension one fermions from flag-dipole spinors*. arXiv: 1809.04381

Cavalcanti, R. T., *Classification of singular spinor fields and other mass dimension one fermions*. Int. J. Mod. Phys. D23 (14), 1444002. arXiv: 1408.0720

There exists a simple transformation from Elko to flag-dipole spinors

$$\chi_z(\mathbf{p},\sigma) = \mathcal{Z}(z)\chi(\mathbf{p},\sigma)$$

$$\mathcal{Z}(z) = \begin{bmatrix} (z^*)^{-1} I & O \\ O & zI \end{bmatrix}, \quad |z|^2 \neq 1$$

The spin-sums for the flag-dipole spinors are given by

$$\sum_{\sigma} \xi_z(\mathbf{p},\sigma)\widetilde{\xi}_z(\mathbf{p},\sigma) = m[\mathcal{G}_z(\phi) + I]$$

$$\sum_{\sigma} \zeta_z(\mathbf{p},\sigma)\widetilde{\zeta}_z(\mathbf{p},\sigma) = m[\mathcal{G}_z(\phi) - I]$$

where $\mathcal{G}_z(\phi)$ is defined as

$$\mathcal{G}_z(\phi) = \begin{pmatrix} 0 & 0 & 0 & -ie^{-i\phi}|z|^{-2} \\ 0 & 0 & ie^{i\phi}|z|^{-2} & 0 \\ 0 & -ie^{-i\phi}|z|^{2} & 0 & 0 \\ ie^{i\phi}|z|^{2} & 0 & 0 & 0 \end{pmatrix}.$$

Their traces are

$$\mathrm{Tr}[\mathcal{G}_z(\phi_1)\mathcal{G}_z(\phi_2)\cdots\mathcal{G}_z(\phi_{2n})] = \mathrm{Tr}[\mathcal{G}(\phi_1)\mathcal{G}(\phi_2)\cdots\mathcal{G}(\phi_{2n})]$$

Different values of z

$$\mathrm{Tr}[\mathcal{G}_{z_1}(\phi_1)\mathcal{G}_{z_2}(\phi_2)] = 2\left(\frac{|z_1|^2}{|z_2|^2} + \frac{|z_2|^2}{|z_1|^2}\right)\cos(\phi_1 - \phi_2)$$

$$\mathrm{Tr}[\mathcal{G}_{z_1}(\phi_1)\cdots\mathcal{G}_{z_4}(\phi_4)] = 2\left(\frac{|z_1 z_3|^2}{|z_2 z_4|^2} + \frac{|z_2 z_4|^2}{|z_1 z_3|^2}\right)\cos(\phi_1-\phi_2+\phi_3-\phi_4)$$

Higher-spin generalisation

We can generalise the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ construct to $(j,0)\oplus(0,j)$

CYL, *Self-interacting mass-dimension one fields for any spin*, Int.J.Mod.Phys. A30 (2015) 1550048. arXiv:1210.7916

We may define a charge-conjugation operator in the $(j,0)\oplus(0,j)$ representation

$$\mathcal{C}=\begin{pmatrix} O & -i\Theta^{-1} \\ i\Theta & O \end{pmatrix}\mathcal{K},\quad \Theta\mathbf{J}\Theta^{-1}=-\mathbf{J}^{*}$$

where $\mathbf{J}$ is the rotation generator of the $(j,0)$ and $(0,j)$ representation.

Higher-spin Elko

The construct parallels the spin-half theory

$$\chi(\boldsymbol{\epsilon},\sigma) = \begin{bmatrix} \vartheta\phi^*(\boldsymbol{\epsilon},\sigma) \\ \phi(\boldsymbol{\epsilon},\sigma) \end{bmatrix}$$

The eigenvalues for integer and half-integer representations are opposite

$$\mathcal{C}\chi(\boldsymbol{\epsilon},\sigma) = \mp(-1)^{2j}\chi(\boldsymbol{\epsilon},\sigma)|_{\theta=\pm i}$$

The structure the spin-sums are particularly interesting

$$\sum_{\sigma} \chi(\mathbf{p},\sigma)\overset{\neg}{\chi}(\mathbf{p},\sigma) = m\left[\mathcal{G}(\phi) \pm I\right]$$

where

$$\mathcal{G}(\phi) = \beta(j)(-1)^{2j}\begin{bmatrix} O & g(\phi) \\ g(\phi) & O \end{bmatrix}, \quad \beta(j) = \begin{cases} -i, j = \frac{1}{2}, \frac{3}{2}, \cdots \\ 1, j = 1, 2, \cdots \end{cases}$$

$$g_{\ell m}(\phi) = (-1)^{j+\ell} e^{-2i\ell\phi}\delta_{\ell,-m}, \quad \ell = -j, \cdots j$$

For example, when $j = 1$

$$g(\phi) = \begin{pmatrix} 0 & 0 & e^{-2i\phi} \\ 0 & 1 & 0 \\ e^{2i\phi} & 0 & 0 \end{pmatrix}$$

It is interesting to compare the higher-spin Elko and Dirac spin-sums

$$\sum_{\sigma} \psi(\mathbf{p},\sigma)\overline{\psi}(\mathbf{p},\sigma) = \gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}} \pm m^{2j} I$$

$$\sum_{\sigma} \chi(\mathbf{p},\sigma)\overset{\neg}{\chi}(\mathbf{p},\sigma) = m\left[\mathcal{G}(\phi) \pm I\right]$$

$$\sum_{\sigma} \chi(\mathbf{p},\sigma)\overline{\chi}(\mathbf{p},\sigma) = \frac{1}{2m^{2j-1}}\left[I \pm \mathcal{G}(\phi)\right]\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}}$$

Introducing the operator

$$\begin{aligned}\Xi(\mathbf{p}) &= \frac{1}{m}\sum_{\sigma}\left[\xi(\mathbf{p},\sigma)\bar{\xi}(\mathbf{p},\sigma)-\zeta(\mathbf{p},\sigma)\bar{\zeta}(\mathbf{p},\sigma)\right]\\ &= \frac{1}{m^{2j}}\mathcal{G}(\phi)\gamma^{\mu_1\cdots\mu_{2j}}p_{\mu_1}\cdots p_{\mu_{2j}}\end{aligned}$$

One obtains

$$\gamma^{\mu_1\cdots\mu_{2j}}p_{\mu_1}\cdots p_{\mu_{2j}} = m^{2j}\mathcal{G}(\phi)\Xi(\mathbf{p})$$

It would be interesting to rewrite the higher-spin Elko in the **vector** and **tensor** representations.

Conclusions and future prospects

Mass dimension one fermions are physically distinct from the Dirac fermions. They are potential **self-interacting dark matter candidates**.

The theory is Lorentz-violating with a **preferred axis**. The τ-deformed theory is Lorentz-invariant. Does the deformation make sense?

Is there a symmetry for the Lorentz-violating theory? What are the **Lorentz-violating effects** (effective action, loop corrections etc.)?

Classification of higher-spin representations. What are their physical implications?

Study the phenomenologies of mass dimension one fermions. There are important **consistency conditions**.

CYL. *Consistent interactions for mass dimension one fermions.* **17th October 2019**.

Thank You!

Local quantum field theory

Let $\Psi(x)$ be a quantum field operator

$$\Psi(x) = \kappa\Psi^+(x) + \lambda\Psi^-(x)$$

$$\Psi^+(x) = \int \frac{d^3p}{(2\pi)^3} \sum_\sigma u(x, \mathbf{p}, \sigma) a(\mathbf{p}, \sigma)$$

$$\Psi^-(x) = \int \frac{d^3p}{(2\pi)^3} \sum_\sigma v(x, \mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)$$

Local quantum field theory

$$U(\Lambda,\alpha)\Psi^{\pm}_{\ell}(x)U^{-1}(\Lambda,\alpha)=\sum_{\bar{\ell}}D_{\ell\bar{\ell}}(\Lambda^{-1})\Psi^{\pm}_{\bar{\ell}}(\Lambda x+\alpha)$$

$$[\Psi(x),\Psi(y)]_{\pm}=[\Psi(x),\Psi^{\dagger}(y)]_{\pm}=0$$

Translation invariance

$$\Psi_\ell^+(x) = \int \frac{d^3p}{(2\pi)^3} \sum_\sigma e^{-ip\cdot x} u_\ell(\mathbf{p},\sigma) a(\mathbf{p},\sigma)$$

$$\Psi_\ell^-(x) = \int \frac{d^3p}{(2\pi)^3} \sum_\sigma e^{ip\cdot x} v_\ell(\mathbf{p},\sigma) b^\dagger(\mathbf{p},\sigma)$$

Boosts

$$u_\ell(\mathbf{p},\sigma) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(L(p))u_{\bar{\ell}}(0,\sigma)$$

$$v_\ell(\mathbf{p},\sigma) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(L(p))v_{\bar{\ell}}(0,\sigma)$$

Rotation invariance

Constraints on $u(\mathbf{0}, \sigma)$ and $v(\mathbf{0}, \sigma)$:

$$\sum_{\bar{\sigma}} u_{\bar{\ell}}(\mathbf{0}, \bar{\sigma}) \mathbf{J}_{\bar{\sigma}\sigma} = \sum_{\ell} \mathcal{J}_{\bar{\ell}\ell} u_{\ell}(\mathbf{0}, \sigma)$$

$$\sum_{\bar{\sigma}} v_{\bar{\ell}}(\mathbf{0}, \bar{\sigma}) \mathbf{J}^*_{\bar{\sigma}\sigma} = -\sum_{\ell} \mathcal{J}_{\bar{\ell}\ell} v_{\ell}(\mathbf{0}, \sigma)$$

Choose the generators to be

$$\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}, \quad \mathcal{J} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & O \\ O & \boldsymbol{\sigma} \end{pmatrix}$$

Dirac spinors are solutions to the above constraints. Elko does not satisfy these constraints.