The On-Shell Construction of Effective Field Theory

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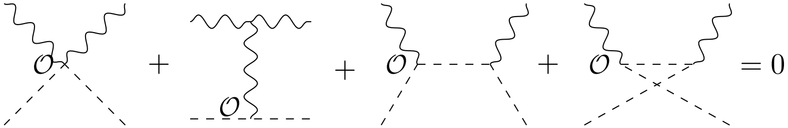
Why consider on-shell construction?

What is the dim-6 operator contribution to the $W^+\pi \to W^+\pi$ amplitude (massless limit $v = 0$)?

$$\mathcal{O}_{hw} = (D^\mu H^\dagger)\tau^i(D^\mu H)W^i_{\mu\nu}$$
$$\mathcal{O}_w = (iH^\dagger\tau^i \overleftrightarrow{D}^\mu H)D^\nu W^i_{\mu\nu}$$



A One-line Proof:
There is NO dim-6 on-shell amplitude basis $\mathcal{M}^{(6)}(V^+V^-SS)$.

Massless Helicity Spinors I

Rep of Lorentz Vector: $p_\mu \in \left(\frac{1}{2}, \frac{1}{2}\right), \quad A_\mu \in (1,0) \oplus (0,1)$.

- When $p^2 = m^2 = 0$, momentum can be decomposed into "helicity spinors":

$$p_{i\mu} = [i|_{\dot\alpha}(\sigma_\mu)^{\alpha\dot\alpha}|i\rangle_\alpha$$

  with LG transformation $|i\rangle \to e^{-i\phi/2}|i\rangle, |i] \to e^{i\phi/2}|i]$.

- But the gauge boson does not decompose in the same way:

$$F_{i\mu\nu} = \langle i|^\alpha(\sigma_{\mu\nu})_\alpha{}^\beta|i\rangle_\beta + [i|_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}|i]^{\dot\beta}$$

- An on-shell amplitude of massless particle scattering should only carry appropriate Little Group weights:

$$\mathcal{A} \sim C^{\alpha\ldots}_{\dot\alpha\ldots} \prod_{i,h_i<0} (|i\rangle_\alpha)^{2|h_i|} \prod_{j,h_j>0} (|j]^{\dot\alpha})^{2|h_j|}$$

- 3-pt on-shell amplitude has special kinematics:

$$\text{either} \quad \langle ij \rangle = 0 \quad \text{or} \quad [ij] = 0$$

As a result, all minimal gauge couplings have "spurious poles"

$$\mathcal{A}(V^+SS) = g\frac{[12][31]}{[23]},$$

$$\mathcal{A}(V^+\psi^+\psi^-) = g\frac{[12]^2}{[23]},$$

$$\mathcal{A}(V^+V^+V^-) = g\frac{[12]^3}{[23][31]}.$$

- A famous example: MHV amplitudes in Yang-Mills theory

$$\mathcal{A}_{\text{color}-\text{ordered}} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n-1, n \rangle \langle n, 1 \rangle}.$$

- Amplitude defined as function of *complex* momentum has the property of analyticity except for poles and branch cuts.
- Amplitude factorizes on poles and branch cuts due to unitarity where on-shell particles could be generated in the process.

Make complex momentum shift $\Delta p \propto z$:

$$\oint_{z=0} \frac{\mathcal{A}(z)}{z} + \sum_I \frac{1}{z_I} \oint_{z=z_I} \mathcal{A}(z) = \oint_{z=\infty} \frac{\mathcal{A}(z)}{z}$$

$$\Rightarrow \mathcal{A}_{\text{phy}} + \sum_I \frac{1}{s_I} \left(\mathcal{A}_L(z_I) \cdot \mathcal{A}_R(z_I) \right) = \mathcal{B}$$

- BCFW (2-line) shift $[i,j\rangle$: $|i] \to |i] + z|j], \ |j\rangle \to |j\rangle - z|i\rangle$.
- All-line shift: $|i\rangle \to |i\rangle + c_i z|\zeta\rangle$ or $|i] \to |i] + c_i z|\zeta]$.

On-Shell Constructibility

The amplitudes with $\mathcal{B} = 0$ are constructible iff

$$\lim_{z\to\infty} \mathcal{A}(z) \sim z^{\gamma} \to 0.$$

- All renormalizable massless theories are at least 5-line constructible. An additional $U(1)$ gauge group makes them 3-line constructible.
- Non-renormalizable massless theories are m-line constructible for at most $(m-1)$-valent interactions (seed amplitudes) without derivatives.
- The criterion for all-line constructibility is

$$\gamma = d - n - |\sum_i h_i| < 0$$

- When $p^2 = m^2 \neq 0$, Little Group is now $SO(3)$ and we need an additional LG index $I = \pm$ to represent spins:

$$\text{Spin-0}: \quad p_{i\mu} = \epsilon_{IJ}[i|^I_{\dot\alpha}(\sigma_\mu)^{\alpha\dot\alpha}|i\rangle^J_\alpha$$
$$\text{Spin-1}: \quad \epsilon_{i\mu} = \frac{1}{M}[i|^{\{I}_{\dot\alpha}(\sigma_\mu)^{\alpha\dot\alpha}|i\rangle^{J\}}_\alpha$$

- General amplitude with massive external state

$$\mathcal{A}^{\{I_1\ldots I_{2j}\}_i} = \mathcal{A}^{\{\alpha_1\ldots\alpha_{2j}\}}|i\rangle^{I_1}_{\alpha_1}\ldots|i\rangle^{I_{2j}}_{\alpha_{2j}}.$$

- Factorization on massive channels:

$$\mathcal{A} \to \frac{\mathcal{A}_L^{\{I_1\ldots I_{2j}\}}(\epsilon_{IJ})^{2j}\mathcal{A}_R^{\{J_1\ldots J_{2j}\}}}{P^2 - M^2}$$

Using $|i\rangle^I_\alpha \epsilon_{IJ}\langle i|^J_\beta = -m\epsilon_{\alpha\beta}$, the numerator becomes

$$\mathcal{A}_L^{\{\alpha_1\ldots\alpha_{2j}\}}(-m\epsilon_{\alpha\beta})^{2j}\mathcal{A}_R^{\{\beta_1\ldots\beta_{2j}\}}.$$

- The $2j$ power of $\epsilon_{\alpha\beta}$ eventually generate $n_\times = 2j$ spinor products "across the channel"! In other words

$$j = n_\times / 2$$

is the total angular momentum of the channel.

- In case of "mixed pole" *i.e.* the residues contain pole in other channel, there are spinor products in the denominators

$$n_\times = (\# \text{ in numerator}) - (\# \text{ in denominator})$$

- Symmetrization is important

$$[13][24] - [14][23] = [12][34] \text{ (Schouten Identity)}$$

A general effective field theory is described by Wilson coefficients

$$\mathcal{L} = \sum_i c_i \mathcal{O}_i$$

where $\mathcal{O}_i$ are independent gauge invariant operators.

How to choose the basis?

- Integration By Part (IBP)

$$\mathcal{O}_i \sim \mathcal{O}_i + d\Omega$$

- Equation Of Motion (EOM)

$$\mathcal{O}_i \sim \mathcal{O}_i + E \quad \text{if } \langle E \rangle = 0$$

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- Equation Of Motion (EOM) $\Leftrightarrow$ (on-shell condition)

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Only on-shell information is important!

Goal: Using Local On-Shell Amplitudes to describe EFT.

Operator Basis $\Leftrightarrow$ Amplitude Basis

Amplitudes in the language of helicity spinors are manifestly gauge invariant and satisfy on-shell conditions.

- What is the dim-6 contribution to the $W^+\pi \to W^+\pi$ process, *i.e.* $\mathcal{A}(V^+SV^-S)$?

$$\mathcal{A}(V^+SV^-S) = [1|p_2 - p_4|3\rangle^2 \left(\frac{g^2}{su} + K(s,u) \right)$$

Amplitude dimension: $d = n + N_p = 4 - [g]$.

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$$\mathcal{A}(V^+SV^-S) = \mathcal{M}^{(8)}(V^+SV^-S)\left(\frac{g^2}{su} + K(s,u)\right)$$

No dim-6 amplitude basis can be written.

- Dimension 6 operator $\mathcal{O}_{HW} = H^\dagger H W_{\mu\nu}W^{\mu\nu}$ only contributes to helicity changing process.

$$\mathcal{M}^{(6)}(V^+SV^+S) = [13]^2$$

All dim-6 amplitudes:

$h=3$	$\mathcal{M}^{(6)}(V^+V^+V^+) = [12][23][31]$
$h=2$	$\mathcal{M}^{(6)}(V^+SV^+S) = [13]^2$ $\mathcal{M}^{(6)}(V^+\psi^+\psi^+S) = [12][13]$ $\mathcal{M}^{(6)}(\psi^+\psi^+\psi^+\psi^+) = [13][24] \pm [14][23]$
$h=0$	$\mathcal{M}^{(6)}(SSSS) = t \pm u$ $\mathcal{M}^{(6)}(\psi^+\psi^-SS) = [1\|(p_2-p_4)\|3\rangle$ $\mathcal{M}^{(6)}(\psi^+\psi^+\psi^-\psi^-) = [12]\langle 34\rangle$

$+$Group Factor $+$ Spin Statistics $\Leftrightarrow$ Warsaw Basis!

Operator-Amplitude Correspondence:

$$|i] \to \bar{\psi}^{\dot{\alpha}},\, |i\rangle \to \psi_\alpha,\, |i][i| \to (F_i^{\mu\nu}\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}},\, |i\rangle[i| \to \sigma^\mu_{\alpha\dot{\alpha}} D_\mu \Psi_i$$

- Example: $\mathcal{M}^{(6)}(V^+\psi^+\psi^+S) = [12][13] \;\to\; F^{\mu\nu}(\bar{\psi}\bar{\sigma}_{\mu\nu}\bar{\psi})\phi.$

On-Shell Non-Constructible?

How about the gauge field in covariant derivatives $\mathcal{O}^\mu D_\mu \Psi$?

$$\mathcal{A}(\mathcal{O}\Psi\gamma) = \langle \mathcal{O}|\mathcal{O}^\mu(-igA_\mu)\Psi|\Psi\gamma\rangle + \langle \mathcal{O}|(\mathcal{O}^\mu\partial_\mu \Psi)(J^\mu_\Psi A_\mu)|\Psi\gamma\rangle$$

The gauge invariant amplitude involves pole of the charged particle, whose residue factorize due to unitarity

$$\mathcal{A}(\mathcal{O}\Psi\gamma) \to \langle \mathcal{O}|\mathcal{O}^\mu\partial_\mu\Psi|\Psi\rangle \times \langle\Psi|J^\mu_\Psi A_\mu|\Psi\gamma\rangle.$$

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Does recursion relation reconstruct $\mathcal{A}(\mathcal{O}\Psi\gamma)$ from amplitude basis?

$$\gamma = d - n - |h| = N_p - |h| \geq 0$$

The answer is NO without additional information: symmetry ...

Single Particle Resonance: Reproducing CDE

The real problem of recursion relation:

Full Amplitude = Unitarity modulo Res=0 contributions

$\mathcal{B} = 0$ is often not a physical way to fix the Res=0 contributions.

- This ambiguity corresponds to some operator identities

$$D^2(\bar{\psi}\psi) = 2D_\mu\bar{\psi}D^\mu\psi + \bar{\psi}\not{D}^2\psi - gF^{\mu\nu}\bar{\psi}\sigma_{\mu\nu}\psi.$$

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If high-dim seed amplitude is generated through single particle resonance (SPR), the spin of the heavy particle fixes the angular momentum of certain channel, providing additional constraints on the full amplitude:

Full Amplitude $\stackrel{?}{=}$ Unitarity + Angular Momentum

- Starting with seed amplitude $\mathcal{M}^{(8)}(\psi^+\psi^+\psi^+\psi^+) = c_0[12][34]s_{12}$, compute the relevant contribution to $\mathcal{A}(\psi^+\psi^+\psi^+\psi^+\gamma^+)$.

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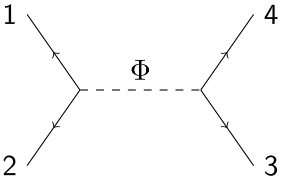

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- Consider angular momentum of $[12][34]s$ ($j = n_\times/2$): $j_s = 0$.

- No amplitude basis $\mathcal{M}^{(8)}(\psi^+\psi^+\psi^+\psi^+\gamma^+)$ has $j_{12} = 0$ and $j_{34} = 0$, thus fixing the Res=0 contribution, and the full amplitude is determined.

$$
\begin{aligned}
&\mathcal{A}(\psi^+\psi^+\psi^+\psi^+\gamma^+) \\
&= c_0 g\left(\frac{[14]s_{12}s_{34}}{\langle 25\rangle\langle 35\rangle} + \frac{[12][45]s_{34}}{\langle 35\rangle} + \frac{[15][34]s_{12}}{\langle 25\rangle}\right) \\
&\qquad + c_1[12][35][45] + c_2[13][25][45] + c_3[14][25][35]
\end{aligned}
$$

This is a result consistent with that based on effective operator $c_0(\bar{\psi}\psi)D^2(\bar{\psi}\psi)$, generated in CDE.

Conclusion & Outlook

- On-shell amplitude provides a better way to examine helicity amplitudes from effective operators than effective operators.
- We proposed a correspondence between operator basis and amplitude basis. Counting amplitude basis is a more intuitive alternative of counting operator basis.
- On-shell constructibility of EFT is essential for the validity of amplitude basis, but traditional recursion relation does not work in general EFT.
- Under SPR ansatz, angular momentum is an additional criterion for on-shell construction. It produces result consistent with CDE.

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- Under SPR ansatz, angular momentum is an additional criterion for on-shell construction. It produces result consistent with CDE.
- Mass correction to amplitude basis, gauge symmetry breaking.
- Positivity bound for multi-particle scattering.

Thank You

Fine

- Schouten Identity

$$[12][34] + [13][42] + [14][23] = 0$$