Estimate the rotation speed.

$$M_{halo} = 4\pi \int_0^{R_{halo}} dr\, r^2 \rho$$

$$= \frac{4\pi}{3} \rho R_{halo}^3 = 10^{12} M_\odot$$

$$\underline{M_\odot = 2\times10^{30}\,\mathrm{kg}} = 2\times10^{33}\,\mathrm{g} = 2\times10^{33} \times 5.6\times10^{23}\,\mathrm{GeV}$$

$$R_{halo} \sim \sqrt[3]{\frac{M_{halo}}{\pi \rho}} \sim \sqrt[3]{\frac{2\times6\times10^{56}\,\mathrm{GeV}\times10^{12}}{\pi\times 0.3\,\mathrm{GeV/cm^3}}}$$

$$\sim \left(10^{69}\right)^{1/3} \mathrm{cm} \sim 10^{23}\,\mathrm{cm}$$

$$\mathrm{kpc} = 3\times10^{21}\,\mathrm{cm}$$

$$\underline{G = 4.3\times10^{-3}}\,\mathrm{pc}\,M_\odot^{-1} \cdot (\mathrm{km/s})^2$$

$$\langle v \rangle = \sqrt{\frac{G M_{halo}}{R_{halo}}} \sim \sqrt{\frac{4.3\times10^{-6}\,\mathrm{kpc}\times10^{12}}{100\,\mathrm{kpc}}}$$

$$\sim 200\,\mathrm{km/s}$$

Calculate the density profile

$\rho \propto e^{\Psi/\sigma^2}$ $\qquad \Psi/\sigma^2 = \ln\rho + c$

$\Psi = \sigma^2 \ln\rho + c$

$\nabla^2 \Psi = -4\pi G\rho$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) = -4\pi G\rho$

$\rho = A \cdot r^n \qquad \Psi = n\sigma^2 \ln r + \cancel{c}$

$r^2 \frac{\partial \Psi}{\partial r} = r^2 \cdot n\sigma^2 \cdot \frac{1}{r} = n\sigma^2 \cdot r$

$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) = n\sigma^2 \cdot \frac{1}{r^2} = -4\pi G A \cdot r^n$

$n = -2 \qquad -2\sigma^2 = -4\pi G A$

$A = \frac{\sigma^2}{2\pi G}$

$\rho = \frac{\sigma^2}{2\pi G} \cdot \frac{1}{r^2}$

$$M = \int_0^R \rho \, d^3 r = 4\pi \int_0^R \frac{\sigma^2}{2\pi G r^2} \cdot r^2 \, dr$$

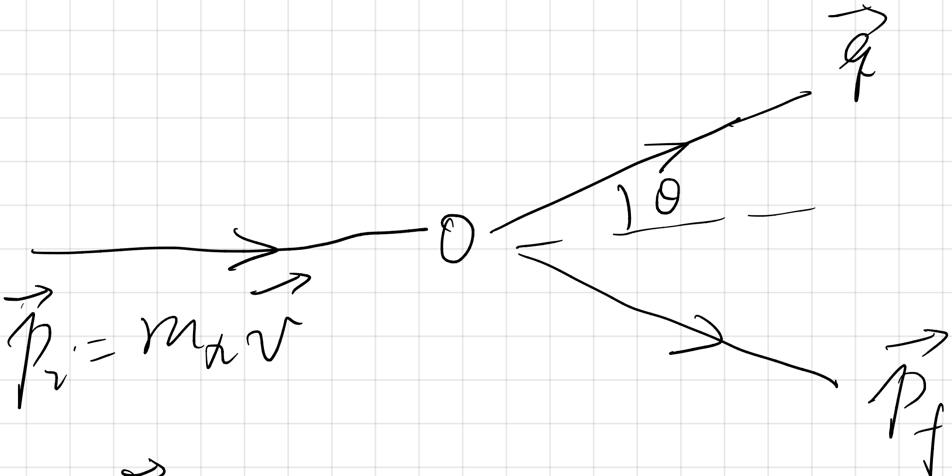
$$= \frac{2\sigma^2 \cdot R}{G}$$

$$v_c = \sqrt{\frac{GM}{R}} = \sqrt{2}\,\sigma = v_0$$

$$\sigma = \frac{1}{\sqrt{2}} v_0$$

$$f(v) \propto e^{-v^2/2\sigma^2} = e^{-v^2/v_0^2}$$

Kinematics.



$$\frac{\vec p_i^{\,2}}{2m_\chi} = \frac{\vec q^{\,2}}{2m_N} + \frac{\vec p_f^{\,2}}{2m_\chi}.$$

$$\vec p_f = \vec p_i - \vec q$$

$$\frac{\cancel{\vec p_i^{\,2}}}{2m_\chi} = \frac{\vec q^{\,2}}{2m_N} + \frac{(\vec p_i - \vec q)^2}{2m_\chi}$$

$$= \frac{\vec q^{\,2}}{2m_N} + \frac{\cancel{\vec p_i^{\,2}} + \vec q^{\,2}}{2m_\chi} + \frac{p_i q \cos\theta}{m_\chi}$$

$$q\left(\frac{1}{m_N} + \frac{1}{m_\chi}\right) = -2\frac{1}{m_\chi}\cos\theta\, p_i$$

$$q \cdot \frac{m_N + m_\chi}{m_N m_\chi} = -2v\cos\theta$$

$$q = -2\mu_{\chi N} v \cos\theta$$

$$E_R \sim \frac{2\mu_{\chi N}^2 v^2}{2m_N} = \frac{\mu_{\chi N}^2 v^2}{m_N} \sim m_N v^2$$

Electron recoil: $v_e \sim \alpha >> v_\chi$

$$\omega = \frac{1}{2} m_\chi v^2 - \frac{(m_\chi \vec{v} - \vec{q})^2}{2m_\chi}$$

$$= \frac{1}{2} m_\chi v^2 - \frac{m_\chi^2 v^2 + q^2 - 2m_\chi \vec{v}\cdot\vec{q}}{2m_\chi}$$

$$= \vec{q}\cdot\vec{v} - \frac{q^2}{2m_\chi} = qv\cos\theta - \frac{q^2}{2m_\chi}$$

$$\omega \leq qv - \frac{q^2}{2m_\chi}$$

$$\frac{\partial \omega}{\partial q} = 0 \quad q = m_\chi v$$

$$\omega = m_\chi v^2 - \frac{m_\chi^2 v^2}{2m_\chi} = \frac{1}{2} m_\chi v^2$$

isotropic gamma background

$$\frac{dN}{dE} = \int dt \, \frac{dN_\gamma}{dE} \frac{n^2(z)}{2} \langle \sigma v_{rel} \rangle dV_z$$

$$\frac{dN}{dE dV_0} = \int_\infty^0 \frac{dt}{dz} dz \cdot \frac{dN_\gamma(z)}{dE} \frac{n^2(z)}{2} \langle \sigma v_{rel} \rangle \frac{dV_z}{dV_0}$$

$\leftarrow$ physical

$\uparrow$ comoving

$$\frac{dV_z}{dV_0} = \frac{a^3}{a_0^3} = \frac{1}{(1+z)^3} \qquad a = \frac{1}{1+z}$$

$$E_z = (1+z)E \qquad \frac{dE}{dE_z} = \frac{1}{1+z}$$

$$H(z) = \frac{\dot{a}}{a} = \frac{da/dt}{a} = -\frac{1}{(1+z)^2} \frac{1}{a} \frac{dz}{dt}$$

$$= -a \frac{dz}{dt}$$

$$\Rightarrow \frac{dt}{dz} = -\frac{a}{H} \qquad n(z) = \frac{\rho(z)}{m_\chi}$$

$$= -\frac{1}{H(z)(1+z)} \qquad = \frac{\rho_0 (1+z)^3}{m_\chi}$$

$$\frac{dN}{dE dV_0} = \int_\infty^0 dz \cdot \frac{-1}{H(z)(1+z)} \cdot \frac{dN_\gamma}{dE_z}(1+z) \frac{1}{2m_\chi^2} \rho_0^2 (1+z)^6$$

$$\langle \sigma v_{rel} \rangle \cdot \frac{1}{(1+z)^3}$$

$$\frac{dN}{dE\,dV_0} = \int_0^\infty dz\, \frac{(1+z)^3}{H(z)} \frac{dN_\gamma}{dE_z} \rho_0^2 \cdot \frac{\langle \sigma v_{rel} \rangle}{2 m_\chi^2}$$

$\downarrow$ $dA\,dt$

$$\frac{d\Phi}{dE} = \int_0^\infty dz\, \frac{1}{H(z)(1+z)^3} \frac{dN_\gamma}{dE_z} \rho^2_{(z)} \frac{\langle \sigma v_{rel} \rangle}{2 m_\chi^2}$$

Axion conversion:

$$\left[-i\frac{d}{dr}+\frac{1}{2k}\begin{pmatrix} m_a^2-\xi\omega_p^2 & \Delta_B \\ \Delta_B & 0\end{pmatrix}\right]\begin{pmatrix}\tilde{A}_\parallel \\ \tilde{a}\end{pmatrix}=0$$

$$\xi=\frac{\sin\tilde{\theta}}{1-\frac{\omega_p^2}{\omega^2}\cos^2\tilde{\theta}} \quad \Delta_B=Bg_{a\gamma}m_a\frac{\xi}{\sin\tilde{\theta}}$$

drop all ~

$$i\frac{d}{dr}\begin{pmatrix}A_\parallel \\ a\end{pmatrix}=\frac{1}{2k}\begin{pmatrix} m_a^2-\xi\omega_p^2 & \Delta_B \\ \Delta_B & 0\end{pmatrix}\begin{pmatrix}A_\parallel \\ a\end{pmatrix}$$

$$i\frac{d}{dr}\vec{A}=H\vec{A} \quad H=H_0+H_1$$

$$=\frac{1}{2k}\begin{pmatrix} m_a^2-\xi\omega_p^2 & 0 \\ 0 & 0\end{pmatrix}+\frac{1}{2k}\begin{pmatrix} 0 & \Delta_B \\ \Delta_B & 0\end{pmatrix}$$

0_{th} order, $i\frac{d}{dr}\vec{A}=\frac{1}{2k}\begin{pmatrix} m_a^2-\xi\omega_p^2 & 0 \\ 0 & 0\end{pmatrix}\vec{A}$

$$\vec{A}^0(r)=e^{-\frac{i}{2k}\int H_0 dr}\vec{A}(0)$$

$$=U\vec{A}(0)$$

$$U=\begin{pmatrix} e^{-\frac{i}{2k}\int(m_a^2-\xi\omega_p)^2 dr} & 0 \\ 0 & 1\end{pmatrix}$$

$$\vec{A}^{(0)}_{int}(r)=U^\dagger\vec{A}^0(r)=U^\dagger U\vec{A}(0)=\vec{A}(0)$$

$$H_{int} = U^{+} H_1 U$$

$$= \begin{pmatrix} b^{+} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \Delta_B \\ \Delta_B & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2k}$$

$$= \begin{pmatrix} 0 & b^{+}\Delta_B \\ \Delta_B & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2k}$$

$$= \begin{pmatrix} 0 & b^{+}\Delta_B \\ \Delta_B b & 0 \end{pmatrix} \frac{1}{2k} \qquad i\frac{d}{dr}\vec{A}_{int} = \begin{pmatrix} 0 & b^{+}\Delta_B \\ \Delta_B b & 0 \end{pmatrix} \begin{pmatrix} A^{(1)}_{int} \\ a^{(1)}_{int} \end{pmatrix}$$

$$i\frac{d}{dr} A^{(1)}_{int} = b^{+} \frac{\Delta_B}{2k} a^{(0)}_{int}(r) = b^{+} \frac{\Delta_B}{2k} a^{(0)}_{int}(r)$$

$$A^{(1)}_{int} = -i\int b^{+} \frac{\Delta_B}{2k} a^{(0)}_{int}\, dr = -i\int b^{+} \frac{\Delta_B}{2k} a(0)\, dr$$

$$= -i\int e^{\int \frac{i}{2k}(m_a^2 - \xi\omega_p^2)dr'} \frac{\Delta_B}{2k}\, dr\, a(0)$$

$$\frac{\Delta_B}{2k} = \frac{B g_{a\gamma\gamma} m_a \xi}{2 m_a v_c \sin\theta} = \frac{g_{a\gamma\gamma} B \xi}{2 v_c \sin\theta} \qquad G\frac{M m_a}{r_c} = \frac{1}{2} m_a v_c^2$$

$$v_c = \sqrt{\frac{2GM}{r_c}}$$

$$A^{(1)}_{\parallel} = U A^{(1)}_{int} = -i\, b \int_0^r \frac{g_{a\gamma\gamma} B(r')\xi(r')}{2 v_c \sin\theta} e^{\int_0^{r'} \frac{i}{2k}(m_a^2 - \xi\omega_p^2)dr''} dr' \cdot a(0)$$

$$P_{a\gamma} = \left| \int_0^r \frac{g_{a\gamma\gamma} B(r')\xi(r')}{2 v_c \sin\theta} e^{i\int_0^{r'} \frac{1}{2k}(m_a^2 - \xi\omega_p^2)\,dr''} dr' \right|^2$$

$$f(r') = \int_0^{r'} \frac{1}{2k}(m_a^2 - \xi\omega_p^2)\,dr''$$

$$\frac{df(r')}{dr'} = \frac{1}{2k}(m_a^2 - \xi\omega_p^2) = 0 \qquad m_a^2 = \xi\omega_p^2$$

resonant condition

$$\frac{d^2 f(r')}{dr'^2} = \frac{1}{2k}\frac{d\omega_p^2}{dr} \qquad \theta = \frac{\pi}{2} \quad \xi = 1$$

$$f(r') = \underbrace{\int_0^{r_c} \frac{1}{2k}(m_a^2 - \xi\omega_p^2)\,dr''}_{\text{const}} - \frac{1}{2}\frac{1}{2k}\frac{d\omega_p^2}{dr}(r' - r_c)^2$$

$$= C - \frac{\pi}{2}\cdot\frac{d\omega_p^2/dr}{2 m_a v_c \pi}(r' - r_c)^2 \qquad n_e \propto \frac{1}{r^3} \propto B$$

$$\omega_p = \sqrt{\frac{4\pi\alpha n_e}{m_e}} \qquad d\omega_p^2/dr = -3\omega_p^2 \cdot \frac{1}{r} = -3\frac{m_a^2}{r_c}$$

$$f(r') = C + \frac{\pi}{2}\cdot\frac{3 m_a^2}{2 m_a v_c \pi r_c}(r' - r_c)^2 = C + \frac{\pi}{2}\frac{1}{L^2}(r' - r_c)^2$$

$$L = \sqrt{2\pi r_c v_c / 3 m_a}$$

$$P_{a\gamma} = \left| \int_0^r \frac{g_{a\gamma\gamma} B}{2 v_c} e^{-i\frac{\pi}{2}\frac{\omega_p \omega_p'}{m_a v_c \pi}(r' - r_c)^2} dr' \right|^2$$

$$= \frac{g_{a\gamma\gamma}^2 B^2}{4 v_c^2}\left(\sqrt{2} L\right)^2 = \frac{1}{2 v_c^2} g_{a\gamma\gamma}^2 B^2 L^2$$