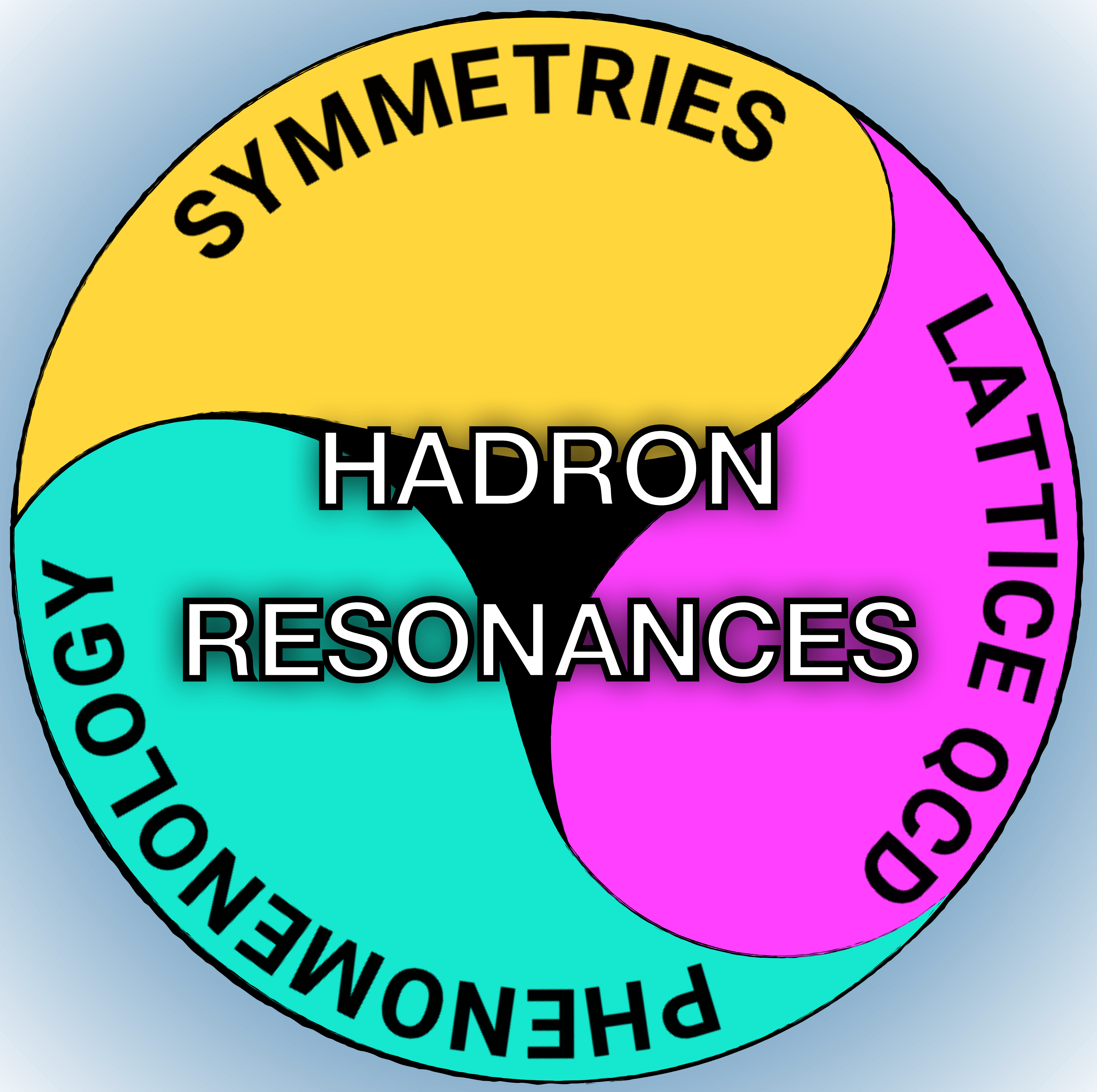


MAXIM MAI

UNIVERSITY OF BERN (MAIN)

THE GEORGE WASHINGTON UNIVERSITY

TRR110: NSFC Grant No. 12070131001, DFG Project-ID 196253076

DOE: DE-SC0016582, DE-AC05-06OR23177, DE-FG02-95ER40907

DFG: Heisenberg Programme (project number: 532635001)

NSF: PHY-2012289

Seminar@ ITP/CAS Beijing 29.09.2024

HADRON SPECTRUM

Many/mostly excited states

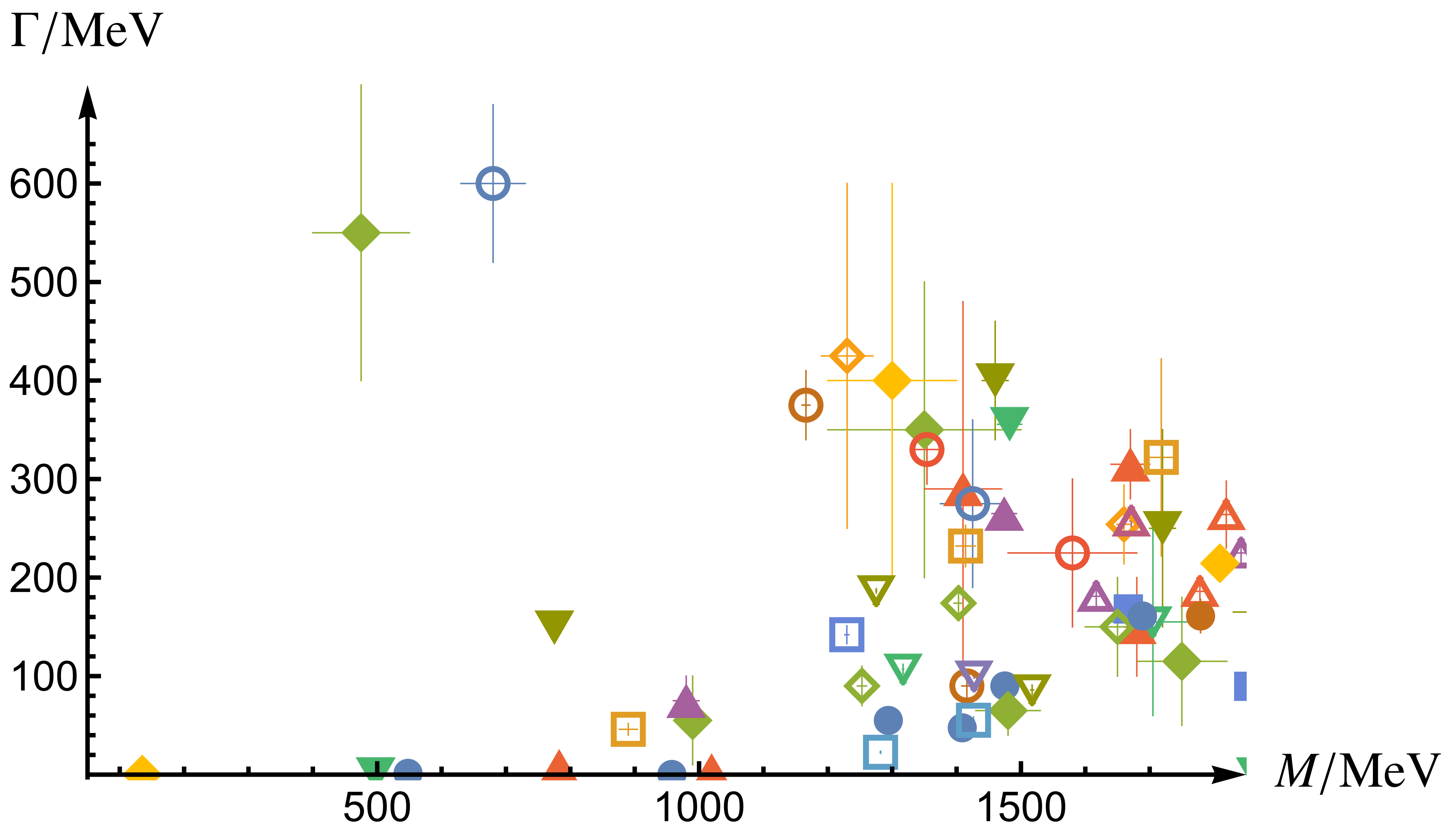
Review: MM/Meißner/Urbach *Phys.Rept.* 1001 (2023) 1-66

$\approx$ 150 mesons

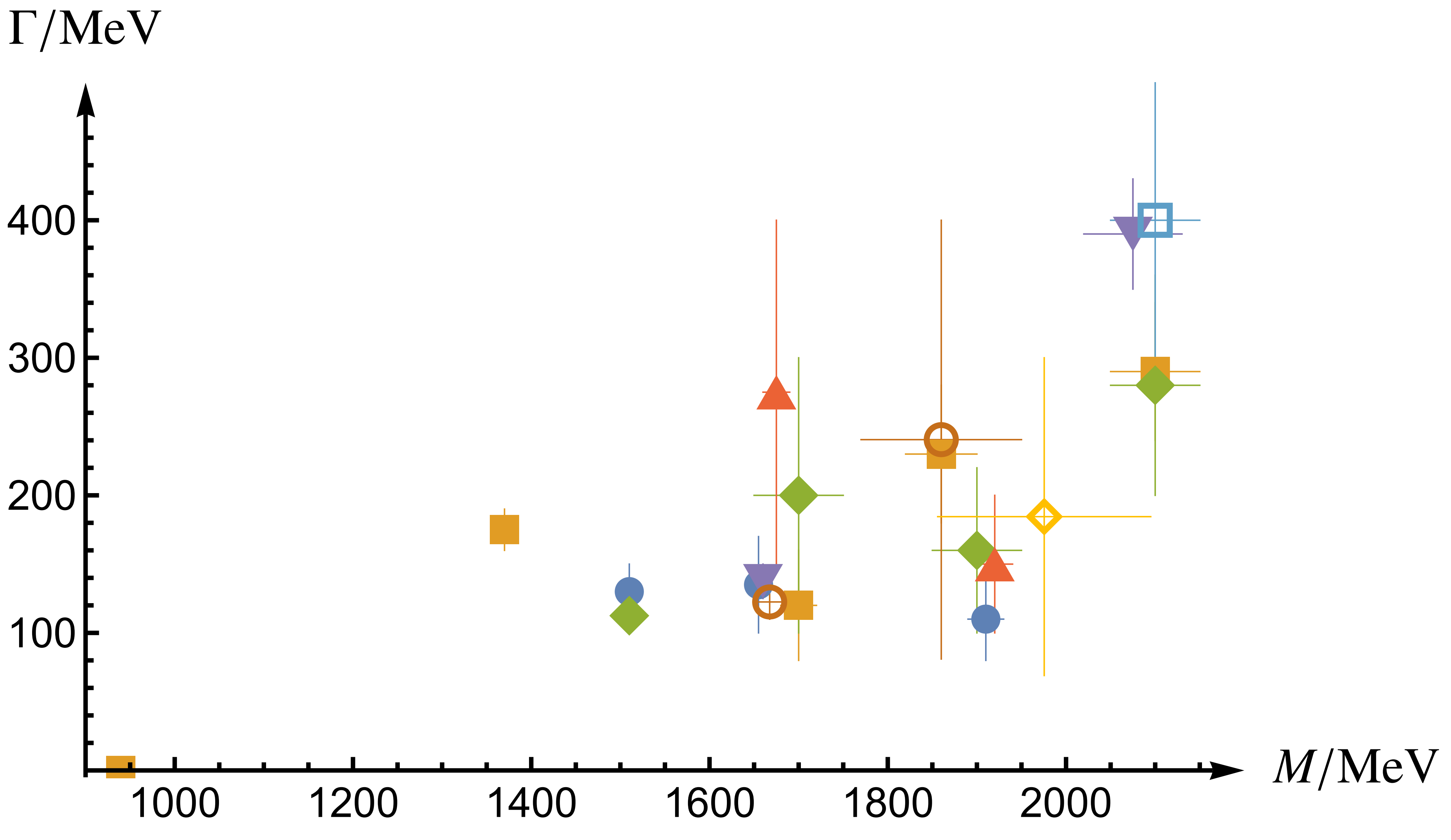
$\approx$ 50 baryons (****)

"If I could remember the names of all these particles, I would have been a botanist."

Enrico Fermi



Unflavoured mesons @ PDG 2023



N Baryons @ PDG/2023

...

HADRON SPECTRUM

Many/mostly excited states

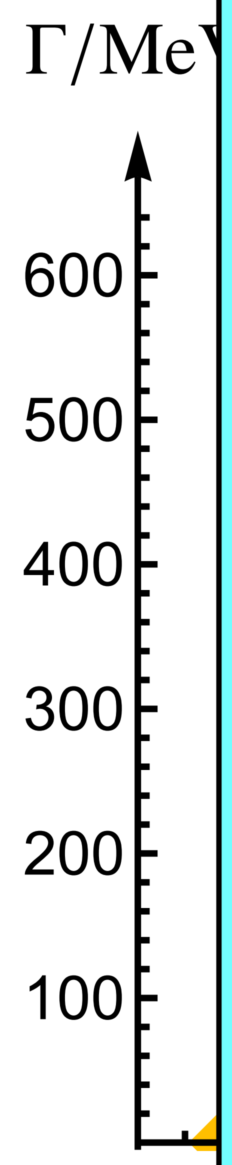
Review: MM/Meißner/Urbach *Phys.Rept.* 1001 (2023) 1-66

≈ 150 mesons

≈ 50 baryons (****)

"If I could remember the names of all these particles, I would have been a botanist."

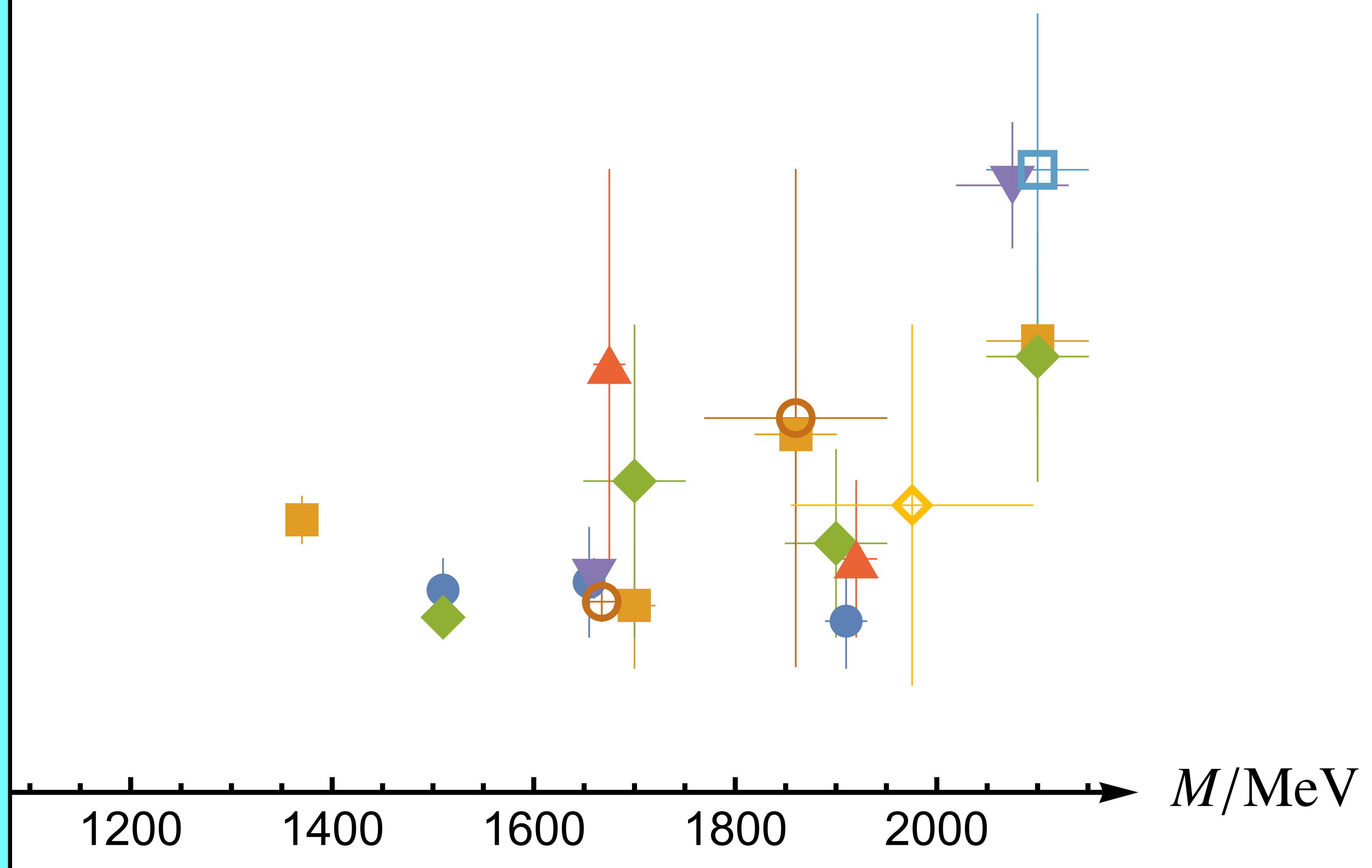
Enrico Fermi



BIRD'S PERSPECTIVE QUESTIONS

- **Are there some patterns?**
- **Minimal spectrum?**

>> More experiments
>> Cross-channel models
>> Statistics/Machine learning tools



...

N Baryons @ PDG/2023

HADRON SPECTRUM

Many/mostly excited states

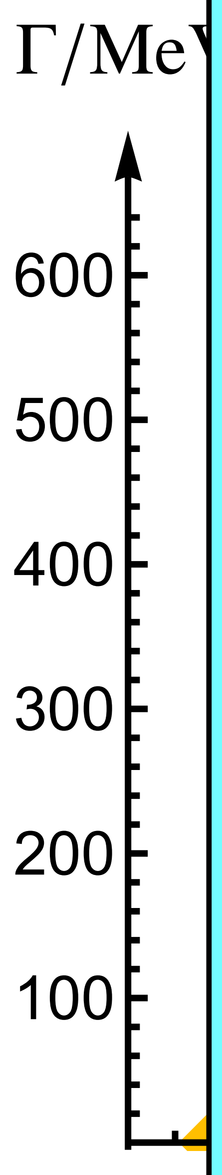
Review: MM/Meißner/Urbach *Phys.Rept.* 1001 (2023) 1-66

≈ 150 mesons

≈ 50 baryons (****)

"If I could remember the names of all these particles, I would have been a botanist."

Enrico Fermi



BIRD'S PERSPECTIVE QUESTIONS

- **Are there some patterns?**
- **Minimal spectrum?**

>> More experiments
>> Cross-channel models
>> Statistics/Machine learning tools

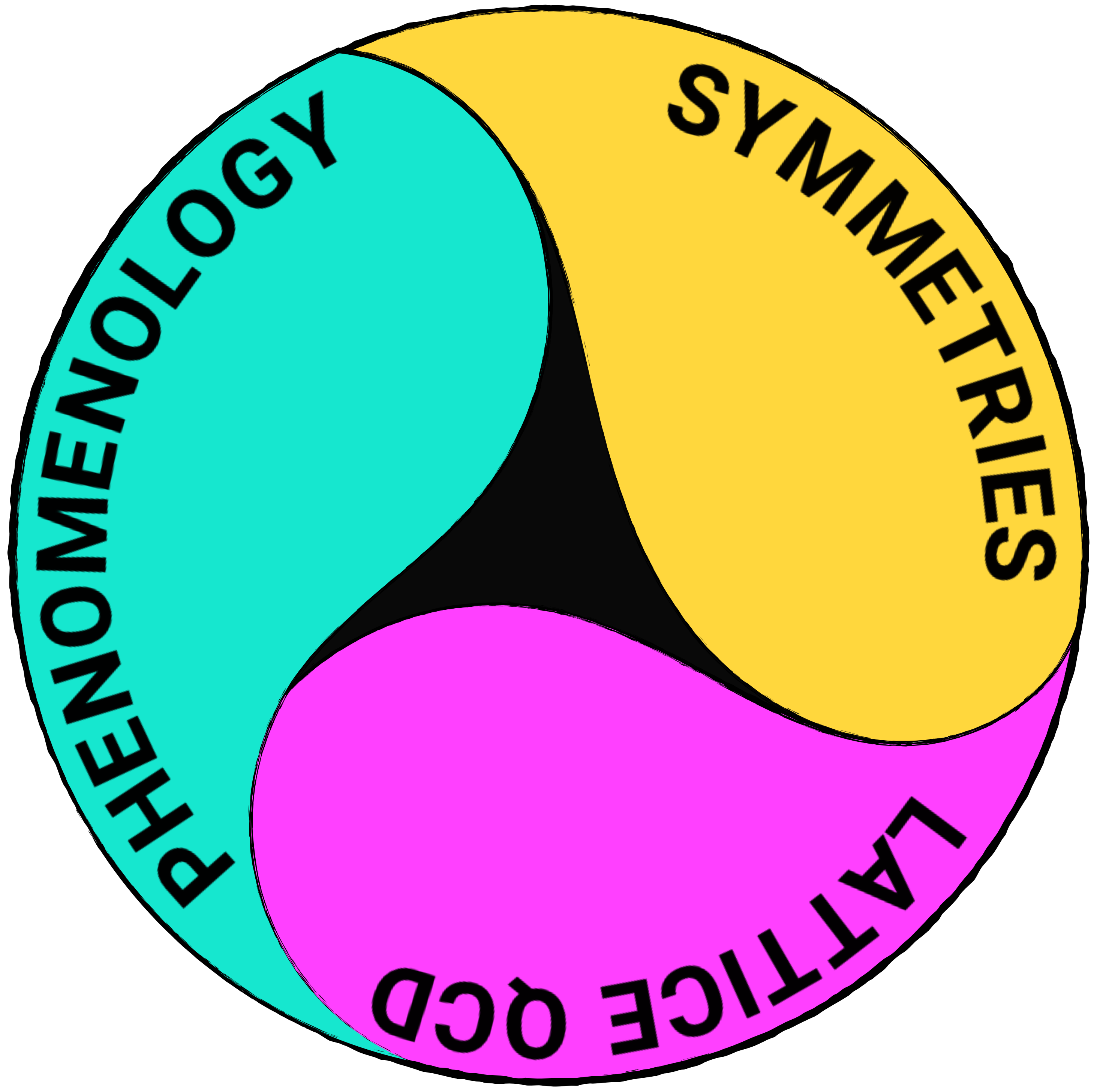
FROG'S PERSPECTIVE QUESTIONS

- **How do they form?**
- **What are their reaction mechanisms?**

>> Change reaction probe
>> Float QCD parameter
>> New interaction modes

N Baryons @ PDG/2023

UNIVERSAL
RESONANCE PARAMETER



RESONANCE PARAMETER

S-matrix theory: ***transition amplitude***

- Unitarity/Analyticity/Crossing symmetry
- Poles on unphysical Riemann Sheets

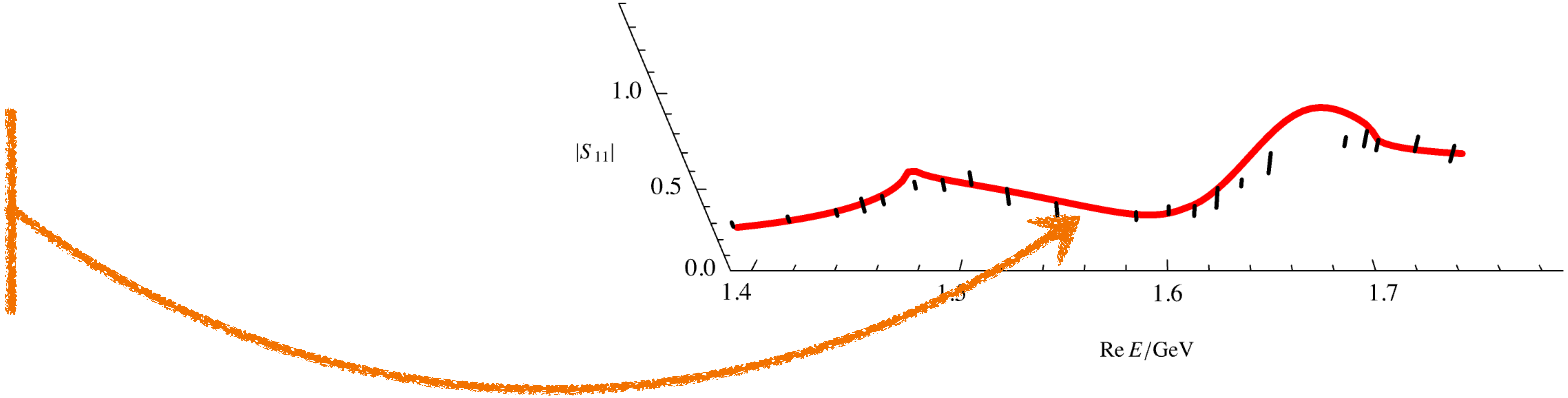
RESONANCE PARAMETER

S-matrix theory: ***transition amplitude***

- Unitarity/Analyticity/Crossing symmetry
- Poles on unphysical Riemann Sheets

Boundary ($E \in \mathbb{R}$):

- Experiment
- Lattice QCD
- CHPT



Data: SAID: Phys. Rev. C 74 (2006) 045205
Model: MM et al. Phys.Rev.D 86 (2012) 094033

Review: MM/Urbach/Meißner Phys.Rept. 1001 (2023) 1-66

RESONANCE PARAMETER

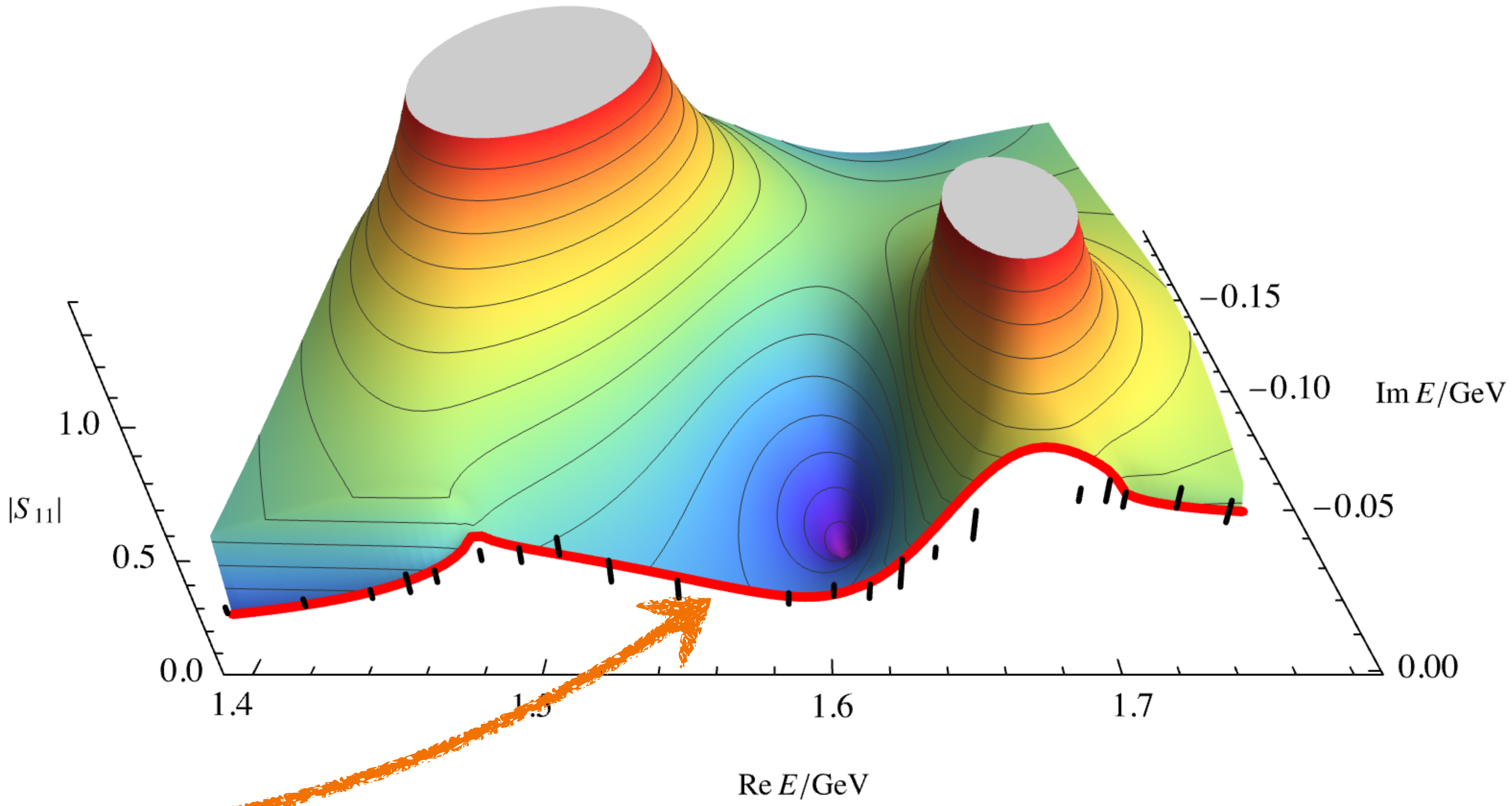
S-matrix theory: ***transition amplitude***

- Unitarity/Analyticity/Crossing symmetry
- Poles on unphysical Riemann Sheets

Boundary ($E \in \mathbb{R}$):

- Experiment
- Lattice QCD
- CHPT

Analytic continuation to unphysical Riemann Sheet



Data: SAID: Phys. Rev. C 74 (2006) 045205
Model: MM et al. Phys.Rev.D 86 (2012) 094033

Review: MM/Urbach/Meißner Phys.Rept. 1001 (2023) 1-66

RESONANCE PARAMETER

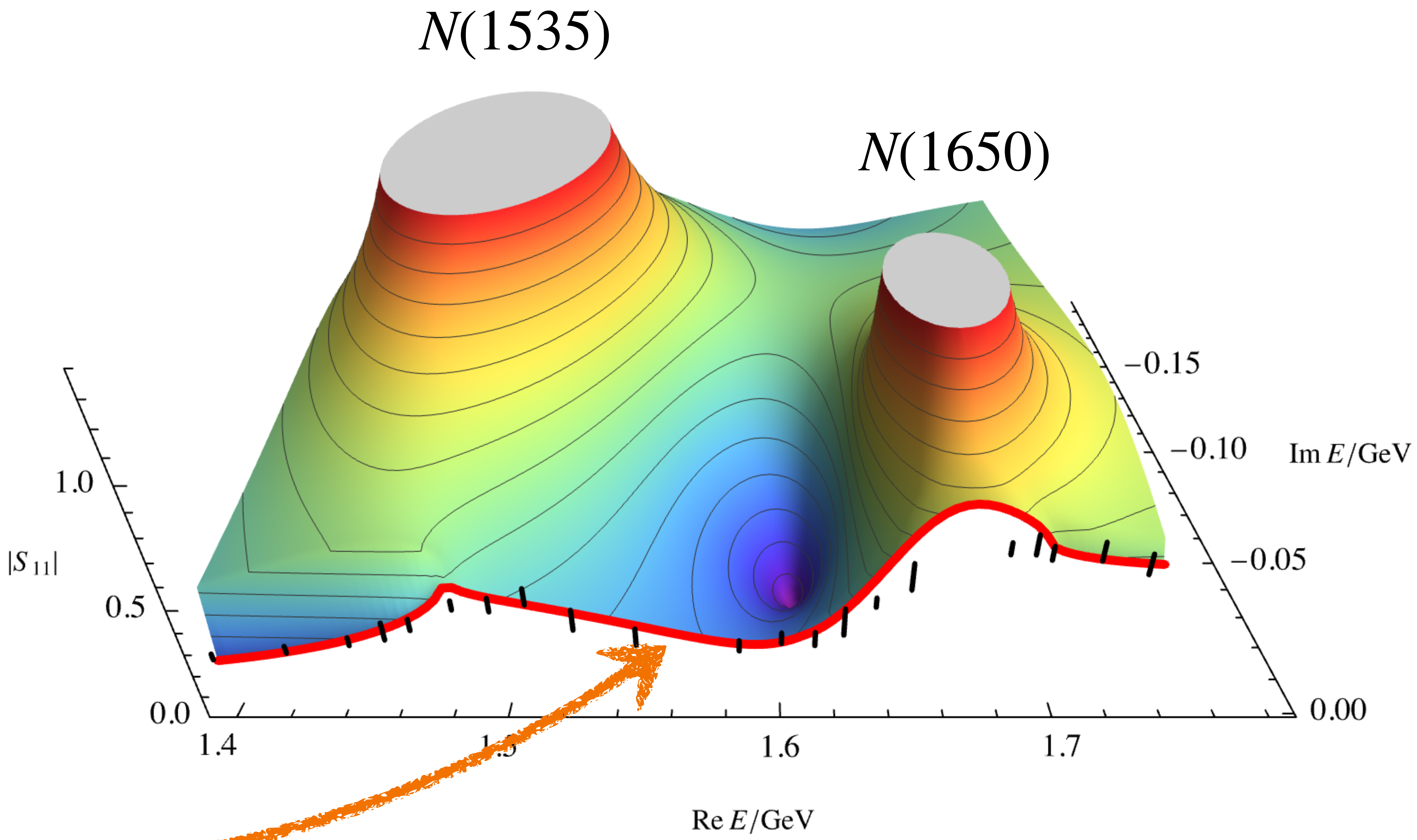
Analytic continuation to unphysical Riemann Sheet

S-matrix theory: *transition amplitude*

- Unitarity/Analyticity/Crossing symmetry
- Poles on unphysical Riemann Sheets

Boundary ($E \in \mathbb{R}$):

- Experiment
- Lattice QCD
- CHPT



Data: SAID: Phys. Rev. C 74 (2006) 045205
Model: MM et al. Phys.Rev.D 86 (2012) 094033

Review: MM/Urbach/Meißner Phys.Rept. 1001 (2023) 1-66

RESONANCE PARAMETER

S-matrix theory: ***transition amplitude***

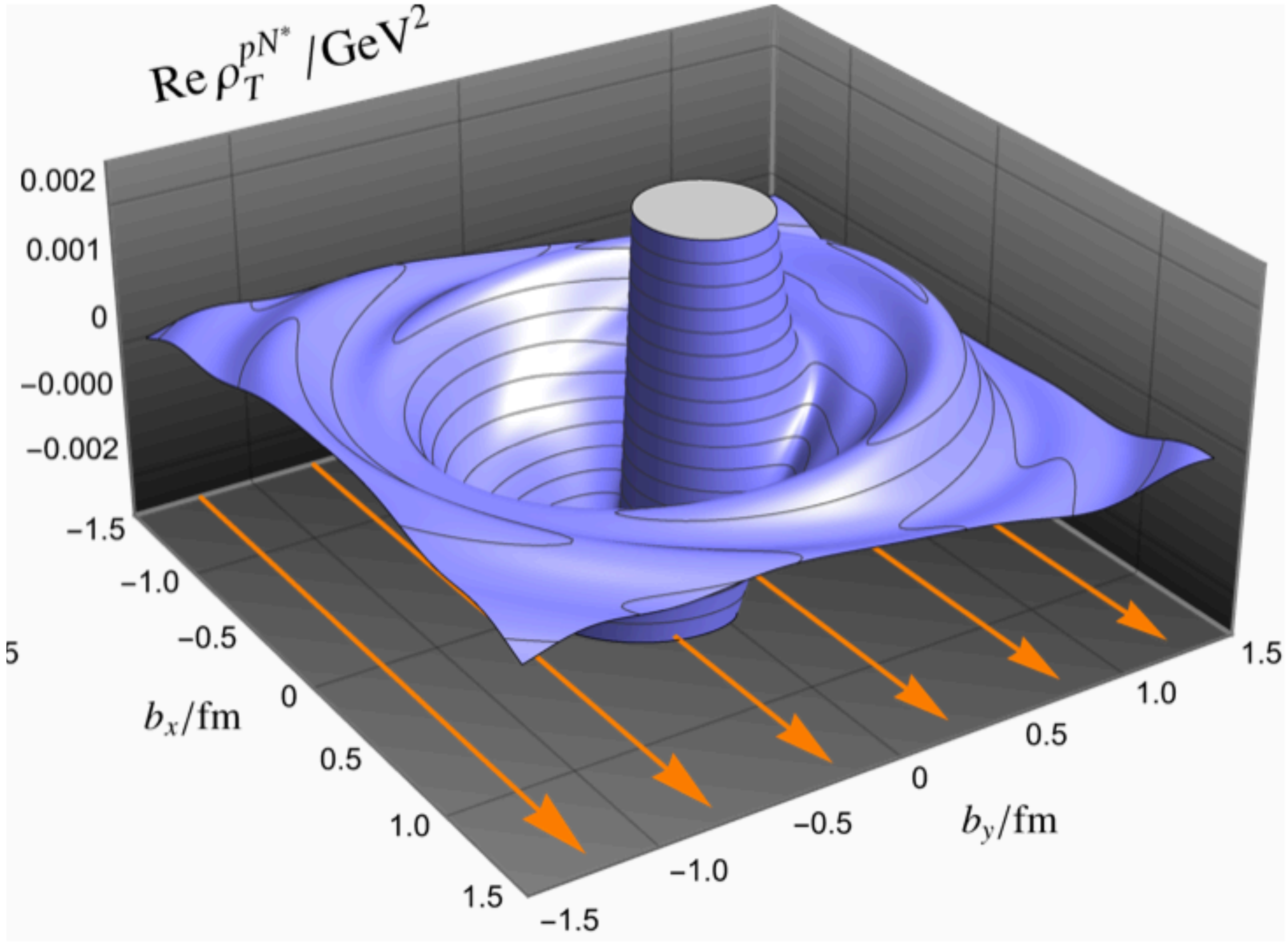
- Unitarity/Analyticity/Crossing symmetry
- Poles on unphysical Riemann Sheets

Boundary ($E \in \mathbb{R}$):

- Experiment
- Lattice QCD
- CHPT

Transition form factors of resonances

Light front charge density of Roper resonance



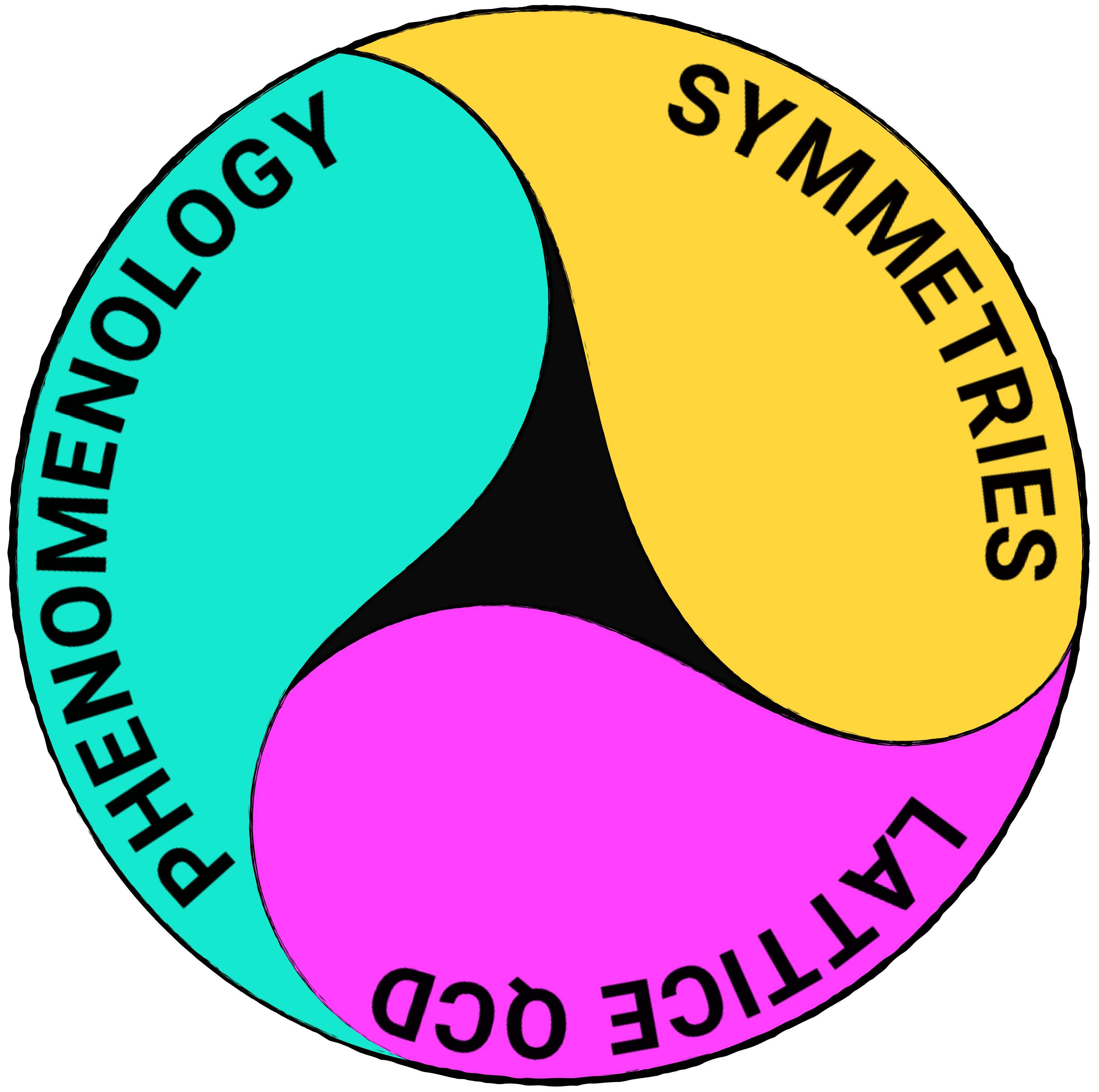
[JBW] Wang et al. in print at PRL e-Print: 2404.17444

Review: MM/Urbach/Meißner Phys.Rept. 1001 (2023) 1-66

A CURIOUS CASE OF A STRANGENESS RESONANCE

- **REVIEW: MM, *Eur.Phys.J.ST* 230 (2021) 6, 1593-1607**

- MM/Ulf-G. Meißner *Nucl.Phys.A* 900 (2013) 51 - 64
- MM/Ulf-G. Meißner *Eur.Phys.J.A* 51 (2015) 3, 30
- J.-X. Lu/L.-S. Geng/MM, M.Döring Phys.Rev.Lett. 130 (2023) 7
- F-K Guo/Y. Kamyia/MM, Ulf-G. Meißner Phys.Lett.B 846 (2023) 138264
- P.C.Bruns/A.Cieply/MM*Phys.Rev.D* 106 (2022) 7
- D. Sadasivan et al. Front.Phys. 11 (2023) 1139236
- Pittler/MM in progress



LOW-ENERGY EFT@QCD

CHPT = EFT of QCD

Weinberg (1979) Gasser, Leutwyler (1981)

- an appropriate tool for low-energy hadronic interactions
- effective degrees of freedom
- well defined power counting

Reviews:
V. Bernard and U.-G. Meißner, Ann. Rev. Nucl. Part. Sci. 57, 33 (2007), arXiv:hep-ph/0611231.119
V. Bernard, Prog. Part. Nucl. Phys. 60, 82 (2008), arXiv:0706.0312 [hep-ph]
S. Scherer, Adv. Nucl. Phys. 27, 277 (2003), arXiv:hep-ph/0210398

$$\mathcal{L} = \mathcal{L}_{\phi}^{(2)} + \mathcal{L}_{\phi}^{(4)} + \mathcal{L}_{\phi B}^{(1)} + \mathcal{L}_{\phi B}^{(2)} + \mathcal{L}_{\phi B}^{(3)}.$$

$$\begin{aligned}\mathcal{L}_{\phi B}^{(2)} = {} & b_{D/F}\langle \bar{B}[\chi_+, B]_\pm\rangle + b_0\langle \bar{B}B\rangle\langle \chi_+\rangle + b_{1/2}\langle \bar{B}[u_\mu, [u^\mu, B]_\mp]\rangle + b_3\langle \bar{B}\{u_\mu, \{u^\mu, B\}\}\rangle + b_4\langle \bar{B}B\rangle\langle u_\mu u^\mu\rangle \\ & + i\sigma^{\mu\nu}(b_{5/6}\langle \bar{B}[[u_\mu, u_\nu], B]_\mp\rangle + b_7\langle \bar{B}u_\mu\rangle\langle u_\nu B\rangle) + \frac{ib_{8/9}}{2m_0}(\langle \bar{B}\gamma^\mu[u_\mu, [u_\nu, [D^\nu, B]]_\mp]\rangle + \langle \bar{B}\gamma^\mu[D_\nu, [u^\nu, [u_\mu, B]]_\mp]\rangle) \\ & + \frac{ib_{10}}{2m_0}(\langle \bar{B}\gamma^\mu\{u_\mu, \{u_\nu, [D^\nu, B]\}\}\rangle + \langle \bar{B}\gamma^\mu[D_\nu, \{u^\nu, \{u_\mu, B\}\}]\rangle) + \frac{ib_{11}}{2m_0}(2\langle \bar{B}\gamma^\mu[D_\nu, B]\rangle\langle u_\mu u^\nu\rangle \\ & + \langle \bar{B}\gamma^\mu B\rangle\langle [D_\nu, u_\mu]u^\nu + u_\mu[D_\nu, u^\nu]\rangle), \qquad (8)\end{aligned}$$

LOW-ENERGY EFT@QCD

CHPT = EFT of QCD

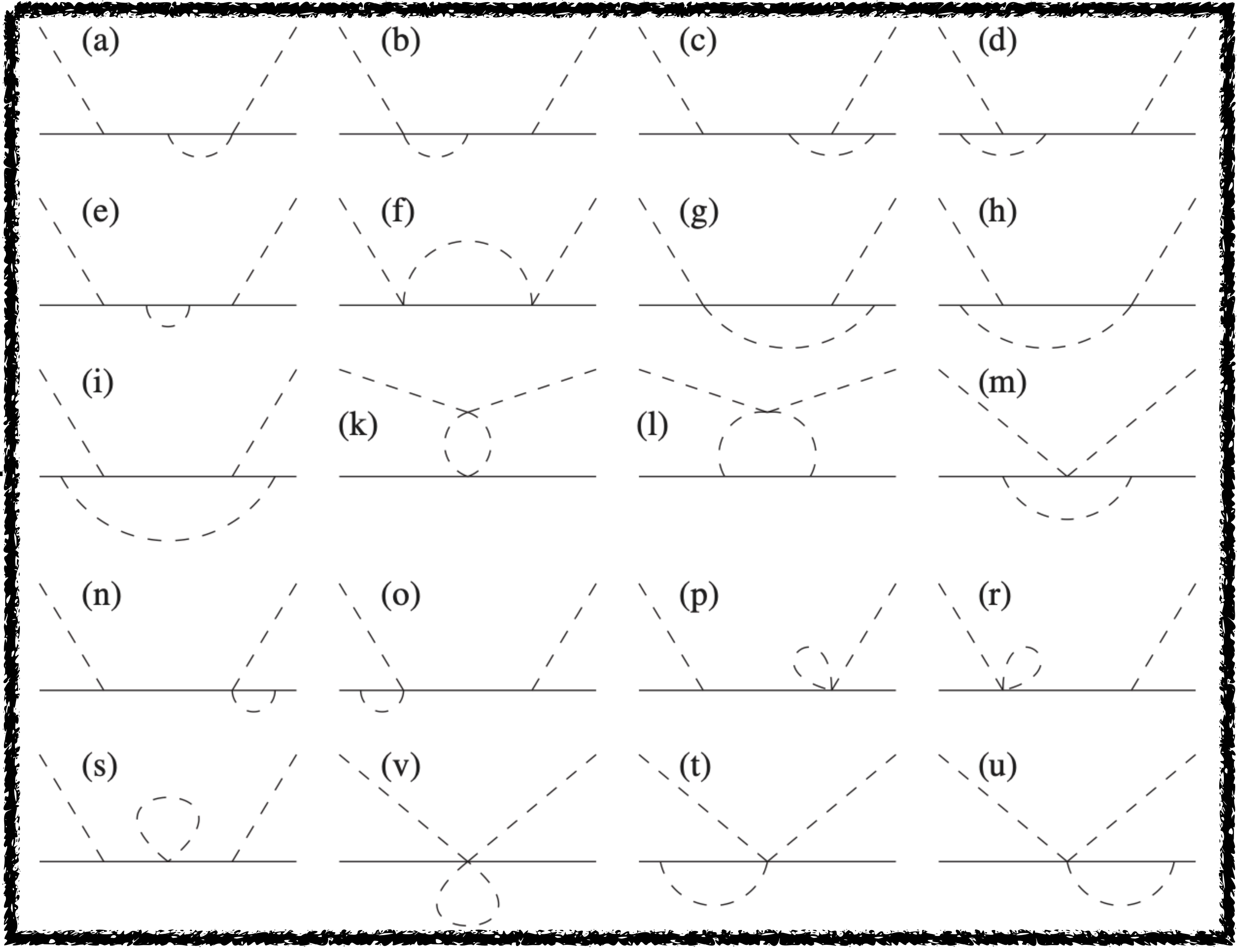
Weinberg (1979) Gasser, Leutwyler (1981)

- an appropriate tool for low-energy hadronic interactions
- effective degrees of freedom
- well defined power counting

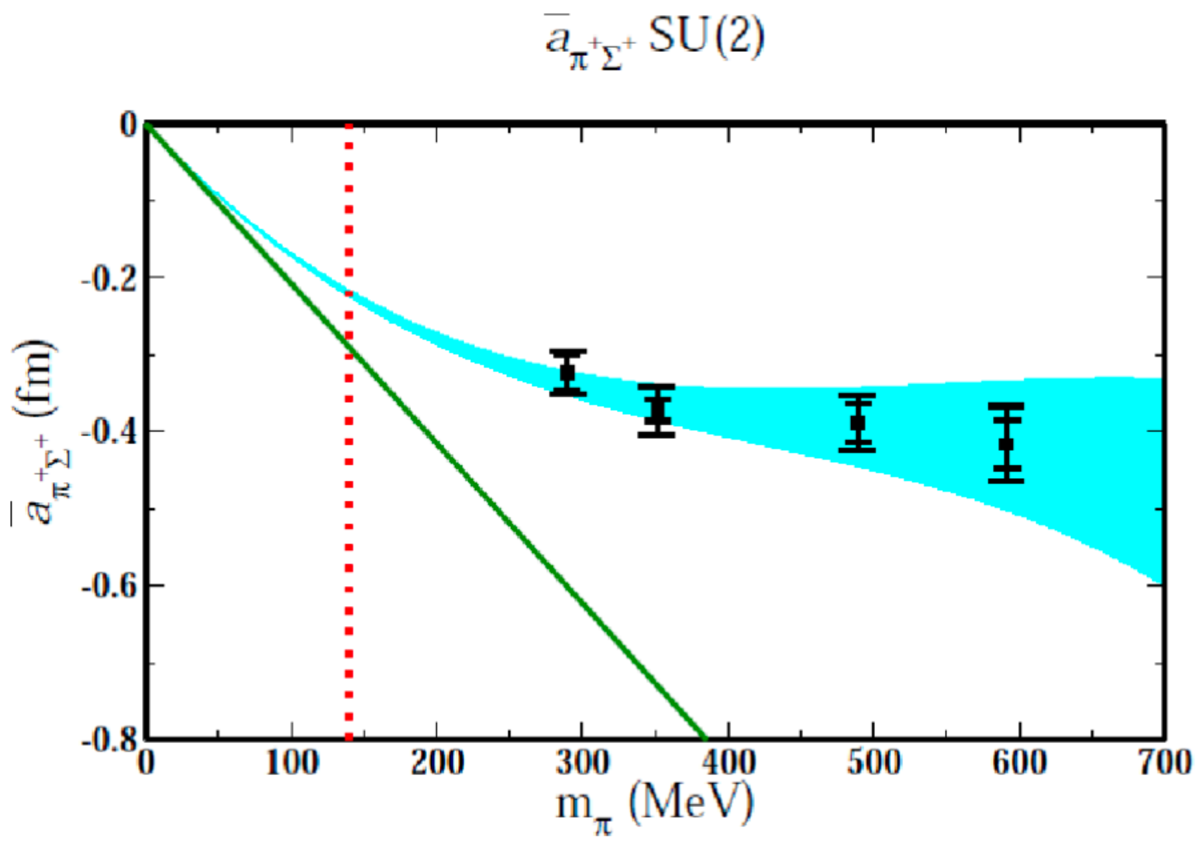
Reviews:
V. Bernard and U.-G. Meißner, Ann. Rev. Nucl. Part. Sci. 57, 33 (2007), arXiv:hep-ph/0611231.119
V. Bernard, Prog. Part. Nucl. Phys. 60, 82 (2008), arXiv:0706.0312 [hep-ph]
S. Scherer, Adv. Nucl. Phys. 27, 277 (2003), arXiv:hep-ph/0210398

$$\mathcal{L} = \mathcal{L}_{\phi}^{(2)} + \mathcal{L}_{\phi}^{(4)} + \mathcal{L}_{\phi B}^{(1)} + \mathcal{L}_{\phi B}^{(2)} + \mathcal{L}_{\phi B}^{(3)}.$$

$$\begin{aligned}\mathcal{L}_{\phi B}^{(2)} = {} & b_{D/F}\langle\bar{B}[\chi_+, B]_\pm\rangle + b_0\langle\bar{B}B\rangle\langle\chi_+\rangle + b_{1/2}\langle\bar{B}[u_\mu,[u^\mu,B]_\mp]\rangle + b_3\langle\bar{B}\{u_\mu,\{u^\mu,B\}\}\rangle + b_4\langle\bar{B}B\rangle\langle u_\mu u^\mu\rangle \\ & + i\sigma^{\mu\nu}(b_{5/6}\langle\bar{B}[[u_\mu,u_\nu],B]_\mp\rangle + b_7\langle\bar{B}u_\mu\rangle\langle u_\nu B\rangle) + \frac{ib_{8/9}}{2m_0}(\langle\bar{B}\gamma^\mu[u_\mu,[u_\nu,[D^\nu,B]]_\mp]\rangle + \langle\bar{B}\gamma^\mu[D_\nu,[u^\nu,[u_\mu,B]]_\mp]\rangle) \\ & + \frac{ib_{10}}{2m_0}(\langle\bar{B}\gamma^\mu\{u_\mu,\{u_\nu,[D^\nu,B]\}\}\rangle + \langle\bar{B}\gamma^\mu[D_\nu,\{u^\nu,\{u_\mu,B\}\}]\rangle) + \frac{ib_{11}}{2m_0}(2\langle\bar{B}\gamma^\mu[D_\nu,B]\rangle\langle u_\mu u^\nu\rangle \\ & + \langle\bar{B}\gamma^\mu B\rangle\langle[D_\nu,u_\mu]u^\nu + u_\mu[D_\nu,u^\nu]\rangle), \qquad (8)\end{aligned}$$



- Compare with experiment
- Compare with Lattice QCD ($\pi\Sigma$)



MM/P.C.Bruns/Ulf-G. Meißner/B.Kubis *Phys.Rev.D* 80 (2009) 094006
Torok/Beane/Detmold/Luu/… *Phys.Rev.D* 81 (2010) 074506

LOW-ENERGY EFT@QCD

CHPT = EFT of QCD Weinberg (1979) Gasser, Leutwyler (1981)

- For hadron (strangeness) resonances it fails (perturbatively)!

  Review: MM, Eur.Phys.J.ST 230 (2021) 6, 1593-1607

  - Kaon mass is large → convergence
  - Relevant thresholds are widely separated → convergence
  - Resonance just below $\bar{K}N$ threshold → non-perturbative effect

LOW-ENERGY EFT@QCD

CHPT = EFT of QCD

Weinberg (1979) Gasser, Leutwyler (1981)

- For hadron (strangeness) resonances it fails (perturbatively)!

  Review: MM, Eur.Phys.J.ST 230 (2021) 6, 1593-1607

  - Kaon mass is large → convergence
  - Relevant thresholds are widely separated → convergence
  - Resonance just below $\bar{K}N$ threshold → non-perturbative effect

$$a^{I=0}_{\bar{K}N} = \Big((+0.53)_{\rm LO} + (+0.97)_{\rm NLO} + (-0.40 + 0.22i)_{\rm NNLO} + \ldots\Big)\ {\rm fm}\,,$$

$$a^{I=1}_{\bar{K}N} = \Big((+0.20)_{\rm LO} + (+0.22)_{\rm NLO} + (-0.26 + 0.18i)_{\rm NNLO} + \ldots\Big)\ {\rm fm}\,.$$

LOW-ENERGY EFT@QCD

CHPT = EFT of QCD Weinberg (1979) Gasser, Leutwyler (1981)

- For hadron (strangeness) resonances it fails (perturbatively)!

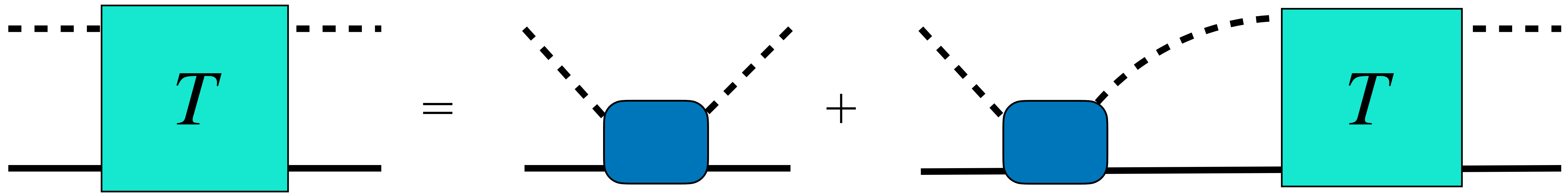
  Review: MM, Eur.Phys.J.ST 230 (2021) 6, 1593-1607

  - Kaon mass is large → convergence
  - Relevant thresholds are widely separated → convergence
  - Resonance just below $\bar{K}N$ threshold → non-perturbative effect

$$a^{I=0}_{\bar{K}N} = \Big((+0.53)_{\rm LO} + (+0.97)_{\rm NLO} + (-0.40 + 0.22i)_{\rm NNLO} + ...\Big)\ {\rm fm}\,,$$
$$a^{I=1}_{\bar{K}N} = \Big((+0.20)_{\rm LO} + (+0.22)_{\rm NLO} + (-0.26 + 0.18i)_{\rm NNLO} + ...\Big)\ {\rm fm}\,.$$

- Extension to higher energies, unitarity restoration — Chiral Unitary Approach (**UCHPT**)

  Weise/Kaiser/Meißner/Lutz/Oset/Oller/Ramos/Hyodo/Borasoy/MM/Bruns/…

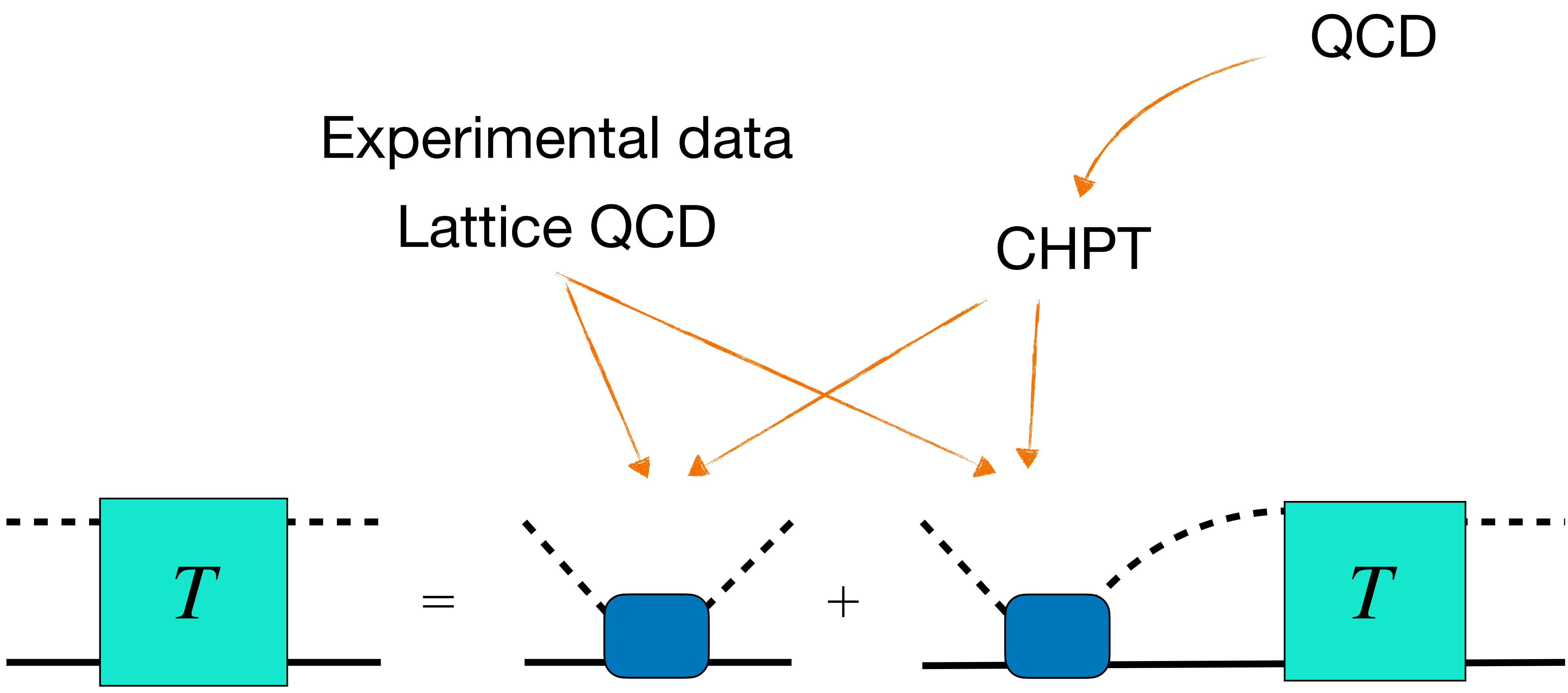


CHIRAL UNITARY APPROACH

GOOD

- Non-perturbative scheme
- Record complex pole-positions
- Often works:
  $N(1535), N(1650), \Lambda(1405), \Lambda(1380), \ldots$

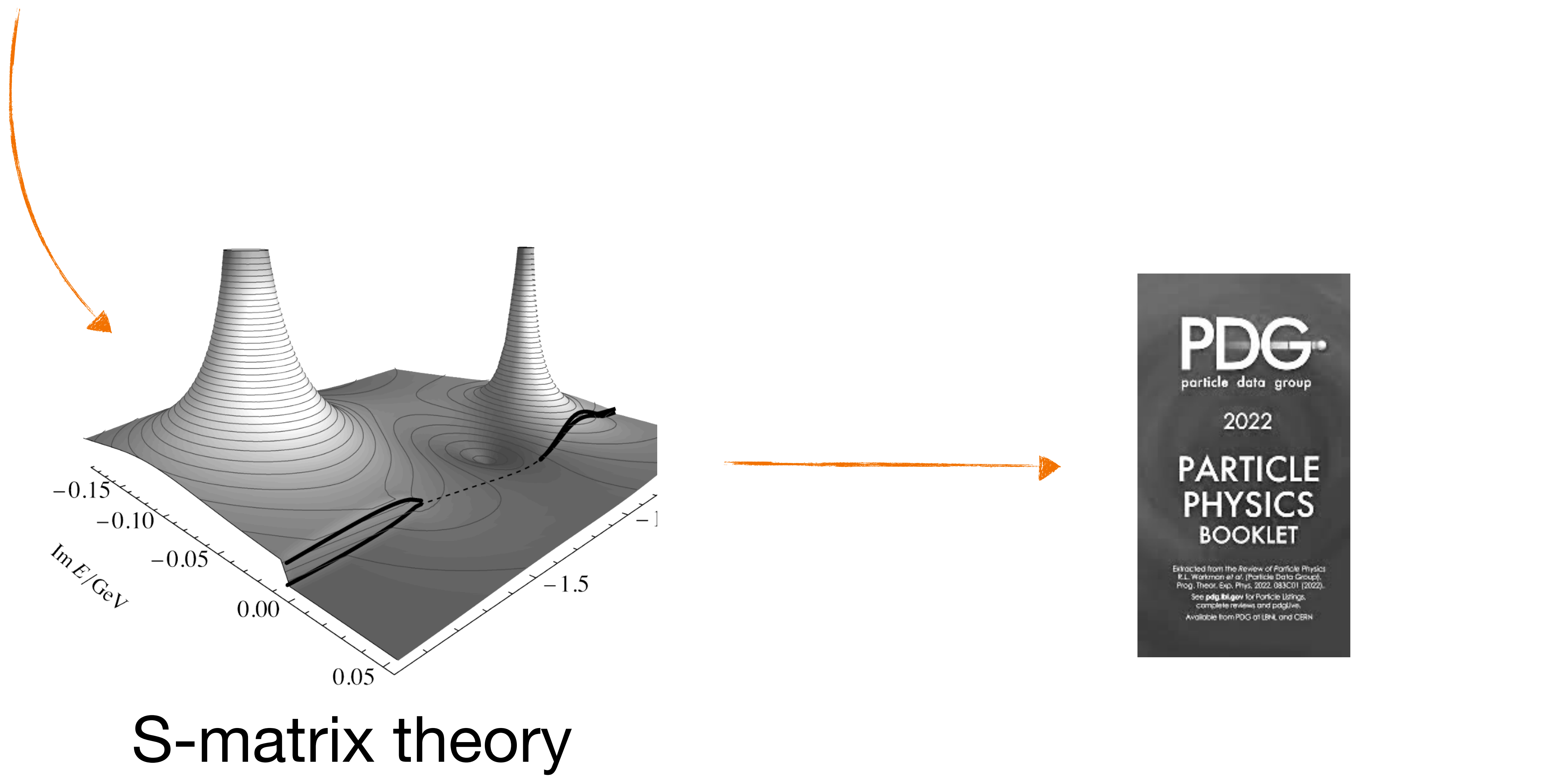
Kaiser/Siegel/Weise Phys.Lett.B 362 (1995)
Lutz/Soyeur Nucl.Phys.A 773 (2006);
MM et al. Phys.Lett.B 697 (2011); …



BAD

- Renormalisation/Crossing symmetry/Power counting only perturbatively
- Choice of potential MM et al. Phys.Lett.B 697 (2011); …

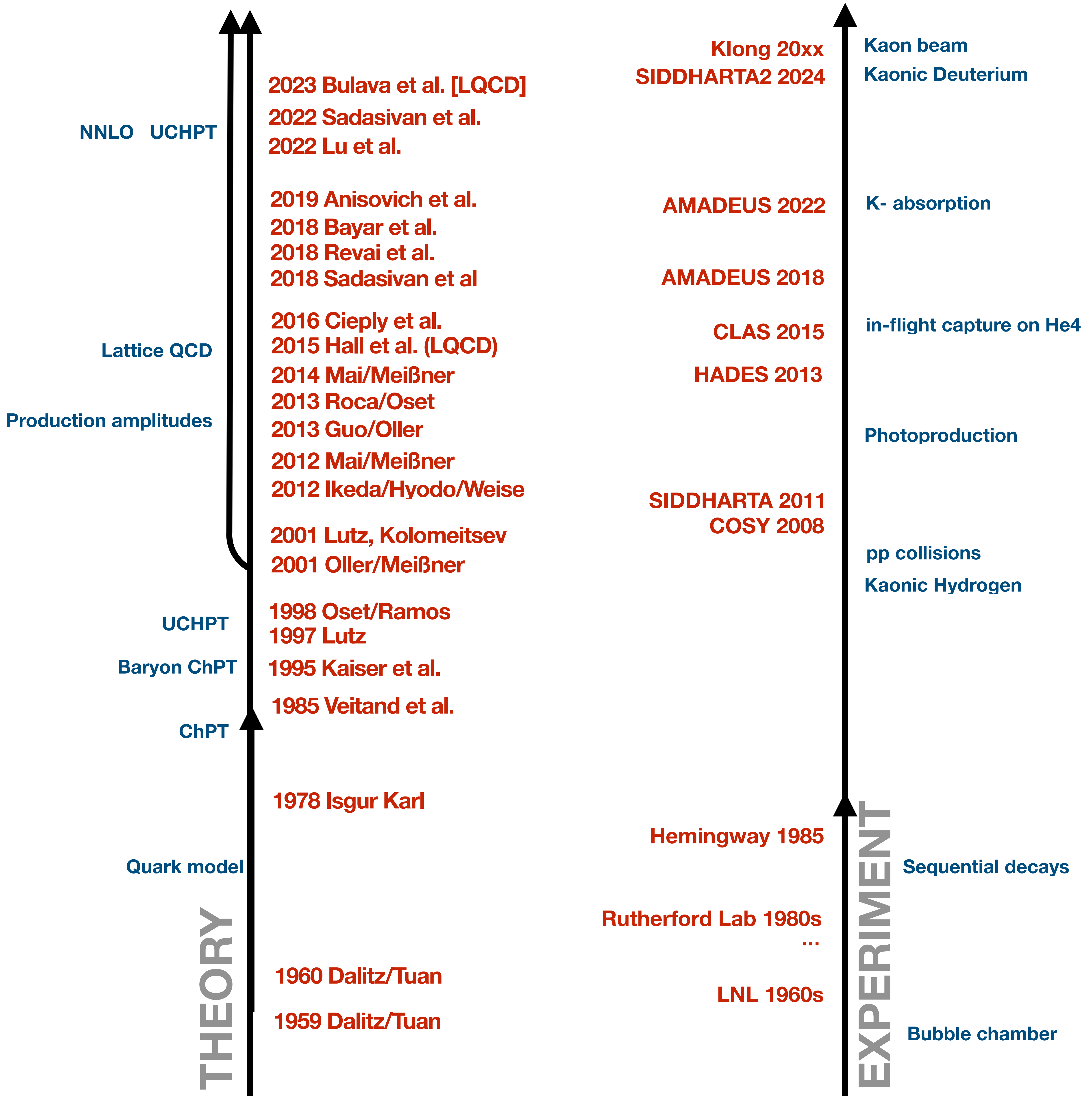
$$V(\not{q}_2, \not{q}_1; p) = A_{WT}(\not{q}_1 + \not{q}_2) + Born(s) + Born(u)$$
$$+A_{14}(q_1 \cdot q_2) + A_{57}[\not{q}_1, \not{q}_2] + A_M + A_{811}\left(\not{q}_2(q_1 \cdot p) + \not{q}_1(q_2 \cdot p)\right)$$



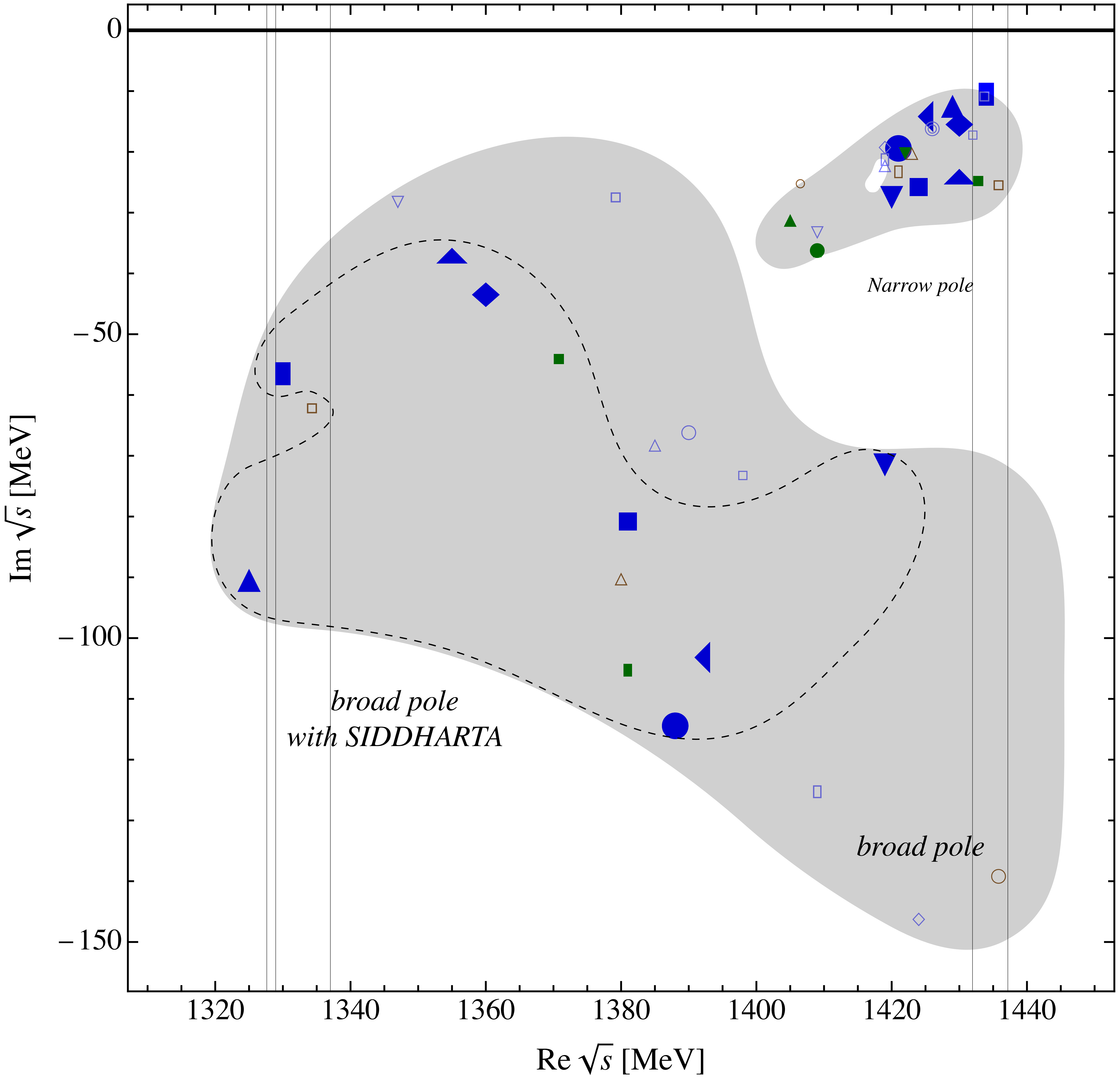
THE ENIGMA OF THE $\Lambda(1405)$

Long history of experimental and theoretical efforts

- Sub-($\bar{K}N$)-threshold $\Lambda(1405)$ resonance
- second state $\Lambda(1380)$ predicted from UCHPT
- no direct experimental verification
- confirmed by many critical tests & LQCD



THE ENIGMA OF THE $\Lambda(1405)$



CURRENT STATUS

UCHPT w. SIDDHARTA
- Ikeda:2011pi
- Guo:2012vv
- Mai:2014xna(2)
- Mai:2014xna(4)
- Feijoo:2018den
- Sadasivan:2018jig
- Lu:2022hwm
- Sadasivan:2022srs

UCHPT w/o SIDDHARTA
- Oller:2000fj
- Oset:2001cn
- Jido:2003cb
- Borasoy:2005ie
- Roca:2013av(1)
- Roca:2013av(2)
- Morimatsu:2019wvk(B)
- Morimatsu:2019wvk(C)

DCC
- Haidenbauer:2010ch
- Zhang:2013sva
- Fernandez−Ramirez:2015tfa
- Anisovich:2020lec(1)
- Anisovich:2020lec(2)

PM
- Cieply:2011nq
- Shevchenko:2011ce
- Shevchenko:2011ce
- Hassanvand:2012dn
- Revai:2019ipq

@ Maxim Mai 08/2024

UNPHYSICAL QUARK MASSES

CHPT encodes quark mass dependence

- SU(3) limit provides a simpler resonance structure

Jido et al. Nucl.Phys.A 725 (2003); Garcia-Recio/Lutz/Nieves Phys.Lett.B 582 (2004) 49-54;

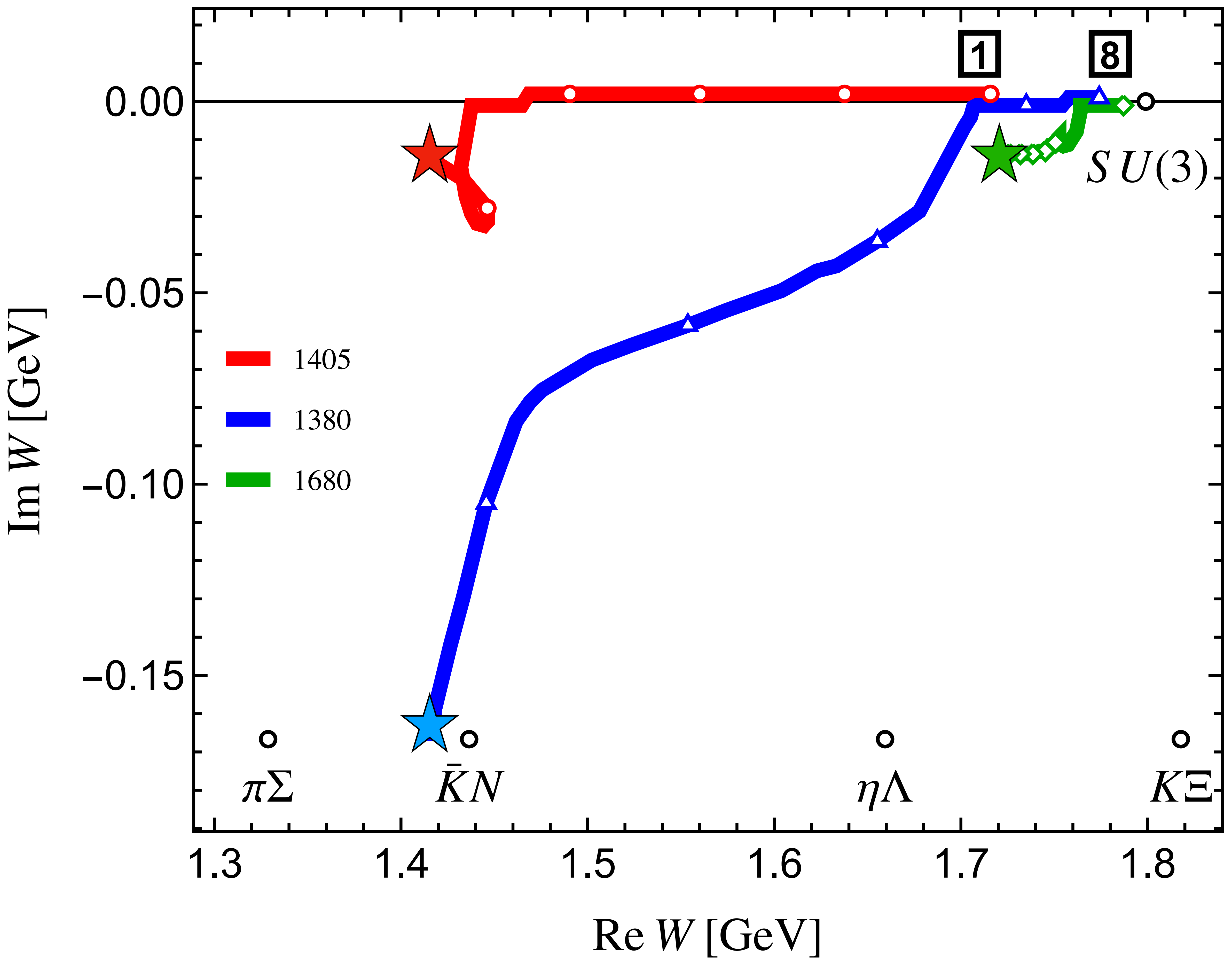
→ 1 singlet + 2 octet poles

→ LO/NLO “tracks” differ

Guo/Kamyia/MM/Meißner Phys.Lett.B 846 (2023)

- Resonance ↭ virtual bound state ↭ bound state

(?) Lattice QCD



UNPHYSICAL QUARK MASSES

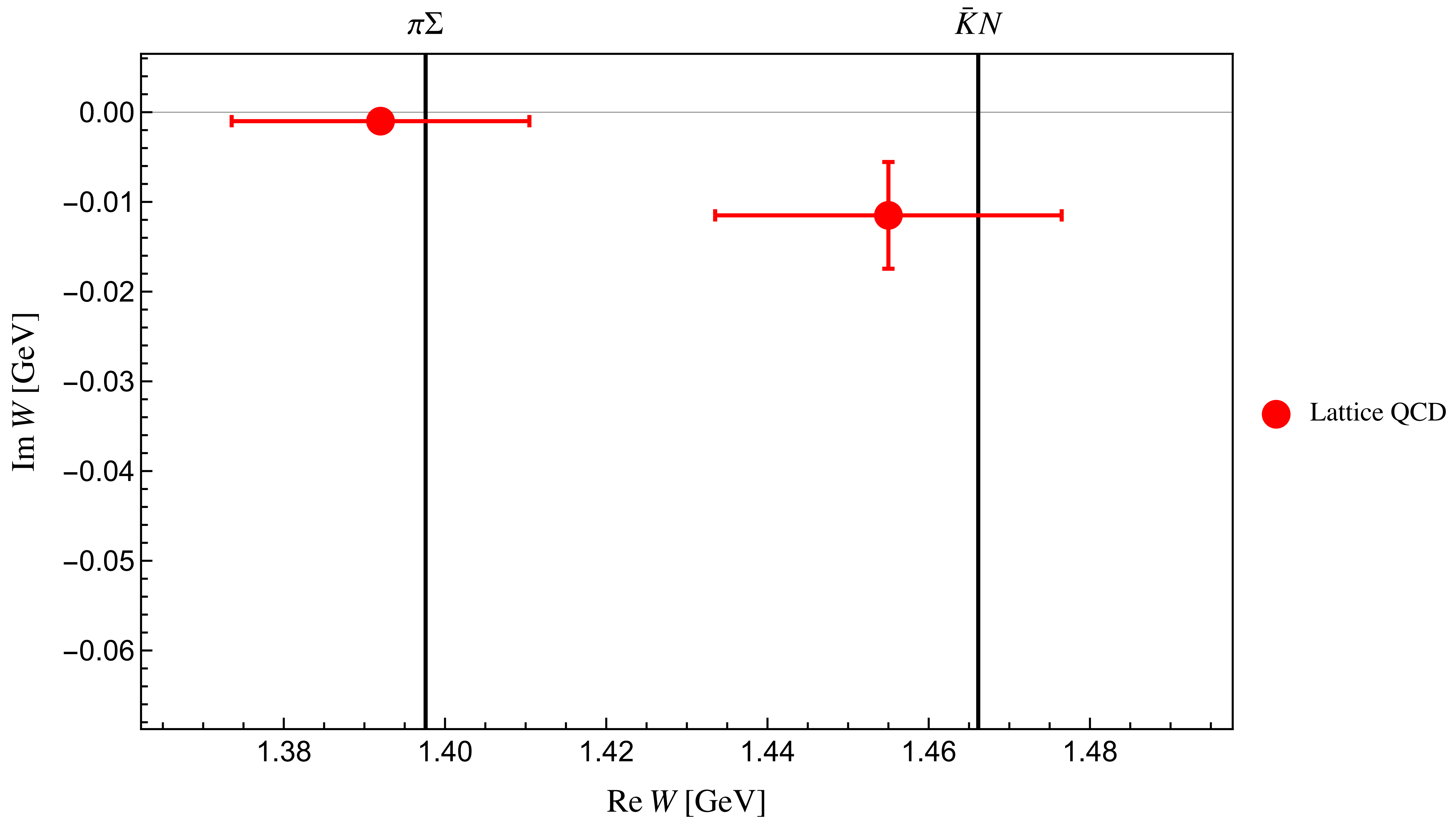
CHPT encodes quark mass dependence

- Available Lattice spectrum

[BaSc] Bulava et al. Phys.Rev.Lett. 132 (2024) 5; 2307.13471

$M_\pi \approx 200\,\mathrm{MeV}\;\; M_K \approx 487\,\mathrm{MeV}$

$M_\pi L = 4.181(16) \quad a = 0.0633(4)(6)\,\mathrm{fm}$



UNPHYSICAL QUARK MASSES

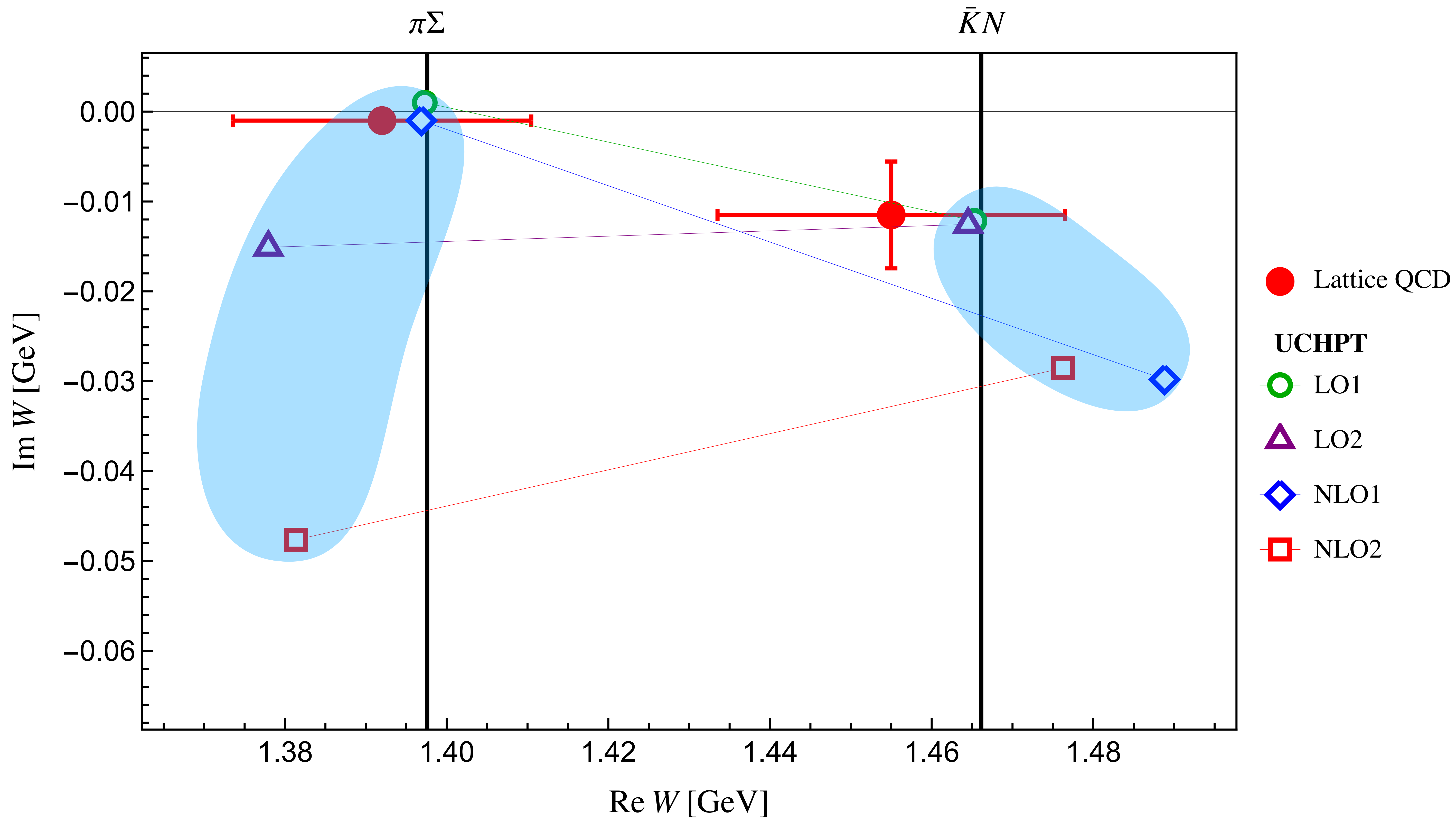
CHPT encodes quark mass dependence

- Available Lattice spectrum
  [BaSc] Bulava et al. Phys.Rev.Lett. 132 (2024) 5; 2307.13471

  $M_\pi \approx 200\,\mathrm{MeV}\ \ M_K \approx 487\,\mathrm{MeV}$

  $M_\pi L = 4.181(16)\quad a = 0.0633(4)(6)\,\mathrm{fm}$

- Compare to prediction of UHPT
  Guo/Kamyia/MM/Meißner Phys.Lett.B 846 (2023)



UNPHYSICAL QUARK MASSES

CHPT encodes quark mass dependence

- Available Lattice spectrum
  [BaSc] Bulava et al. Phys.Rev.Lett. 132 (2024) 5; 2307.13471

  $M_\pi \approx 200\,\mathrm{MeV}\ \ M_K \approx 487\,\mathrm{MeV}$

  $M_\pi L = 4.181(16) \quad a = 0.0633(4)(6)\,\mathrm{fm}$

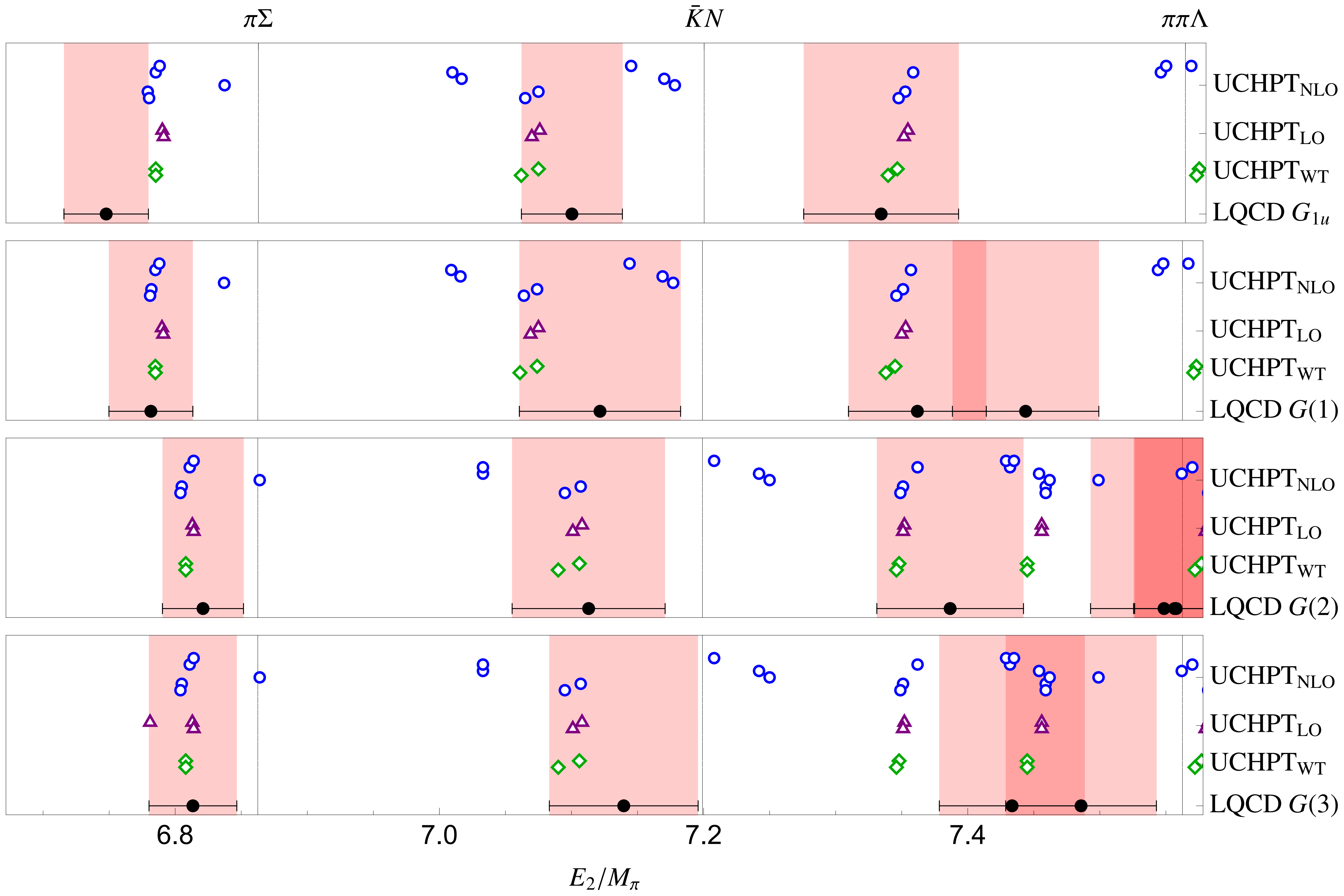
- Compare to prediction of UHPT
  Guo/Kamyia/MM/Meißner Phys.Lett.B 846 (2023)

- Unified analysis LQCD+UCHPT+EXPERIMENT

  … mostly ok, but not always

  … ongoing work

preliminary



UNPHYSICAL QUARK MASSES

CHPT encodes quark mass dependence

- Available Lattice spectrum

  [BaSc] Bulava et al. Phys.Rev.Lett. 132 (2024) 5; 2307.13471

  $M_\pi \approx 200\,\text{MeV} \;\; M_K \approx 487\,\text{MeV}$

  $M_\pi L = 4.181(16) \quad a = 0.0633(4)(6)\,\text{fm}$

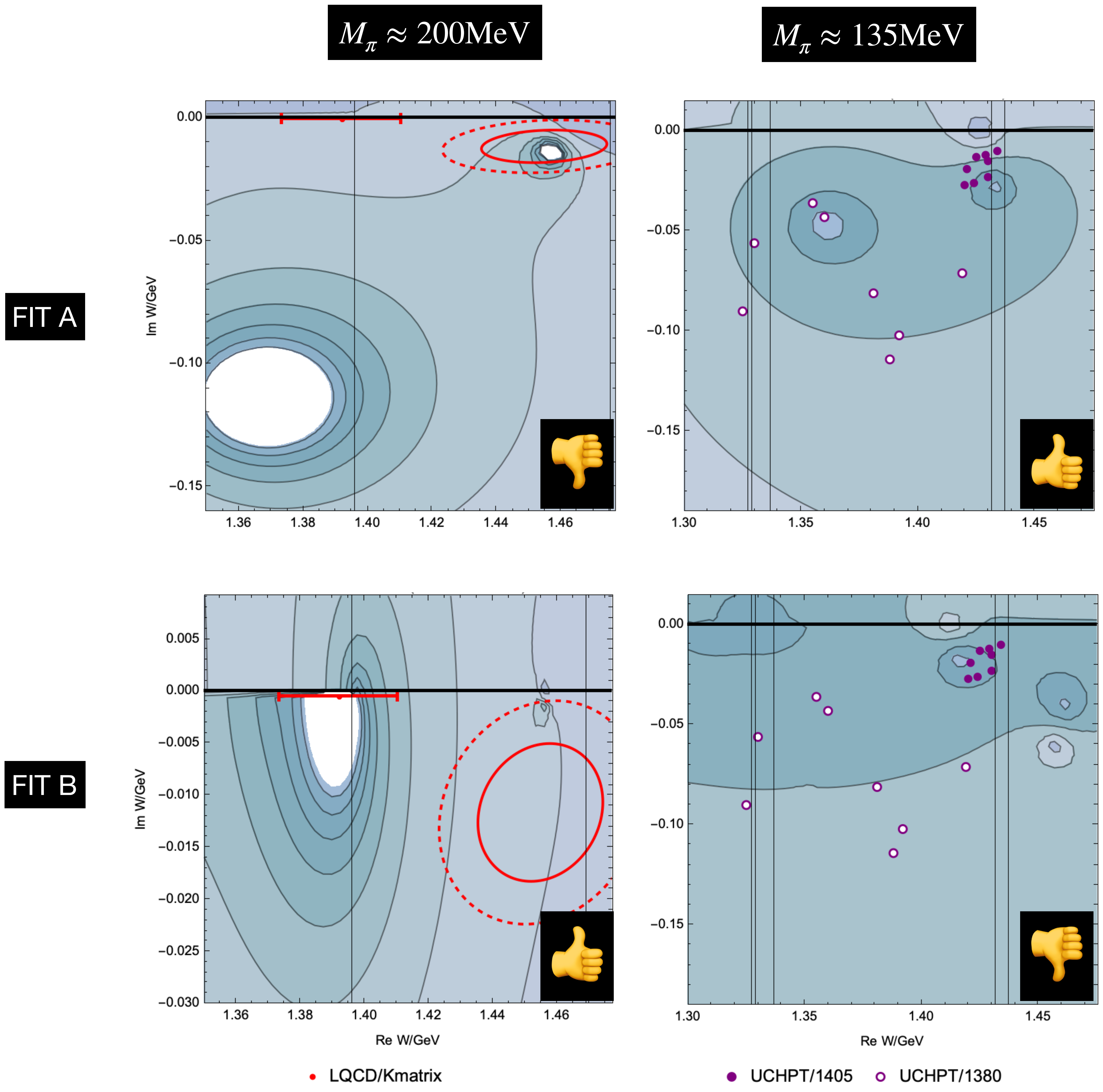
- Compare to prediction of UHPT

  Guo/Kamyia/MM/Meißner Phys.Lett.B 846 (2023)

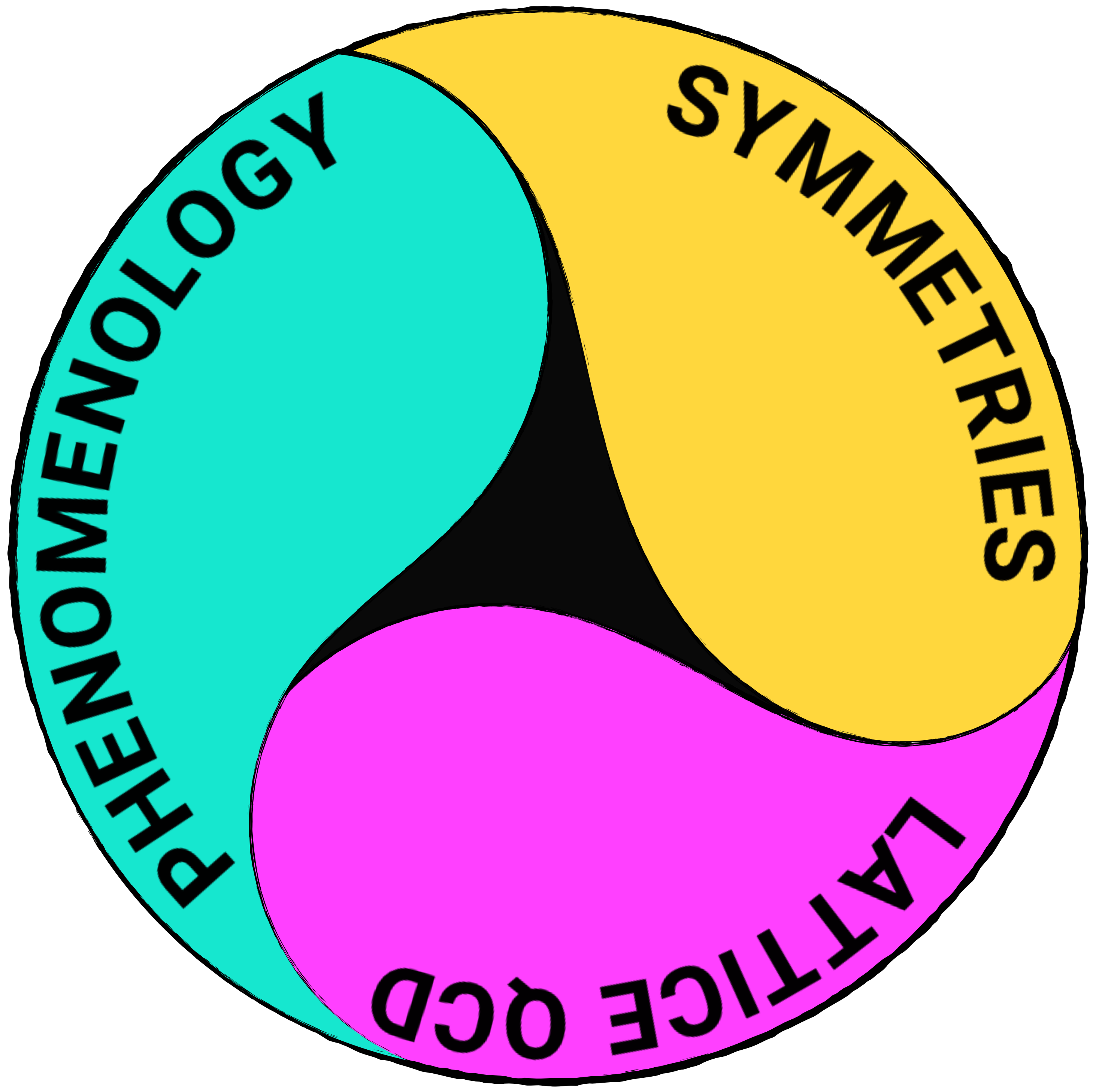
- Unified analysis LQCD+UCHPT+EXPERIMENT

  … mostly ok, but not always

  … ongoing work



THREE-HADRON STATES

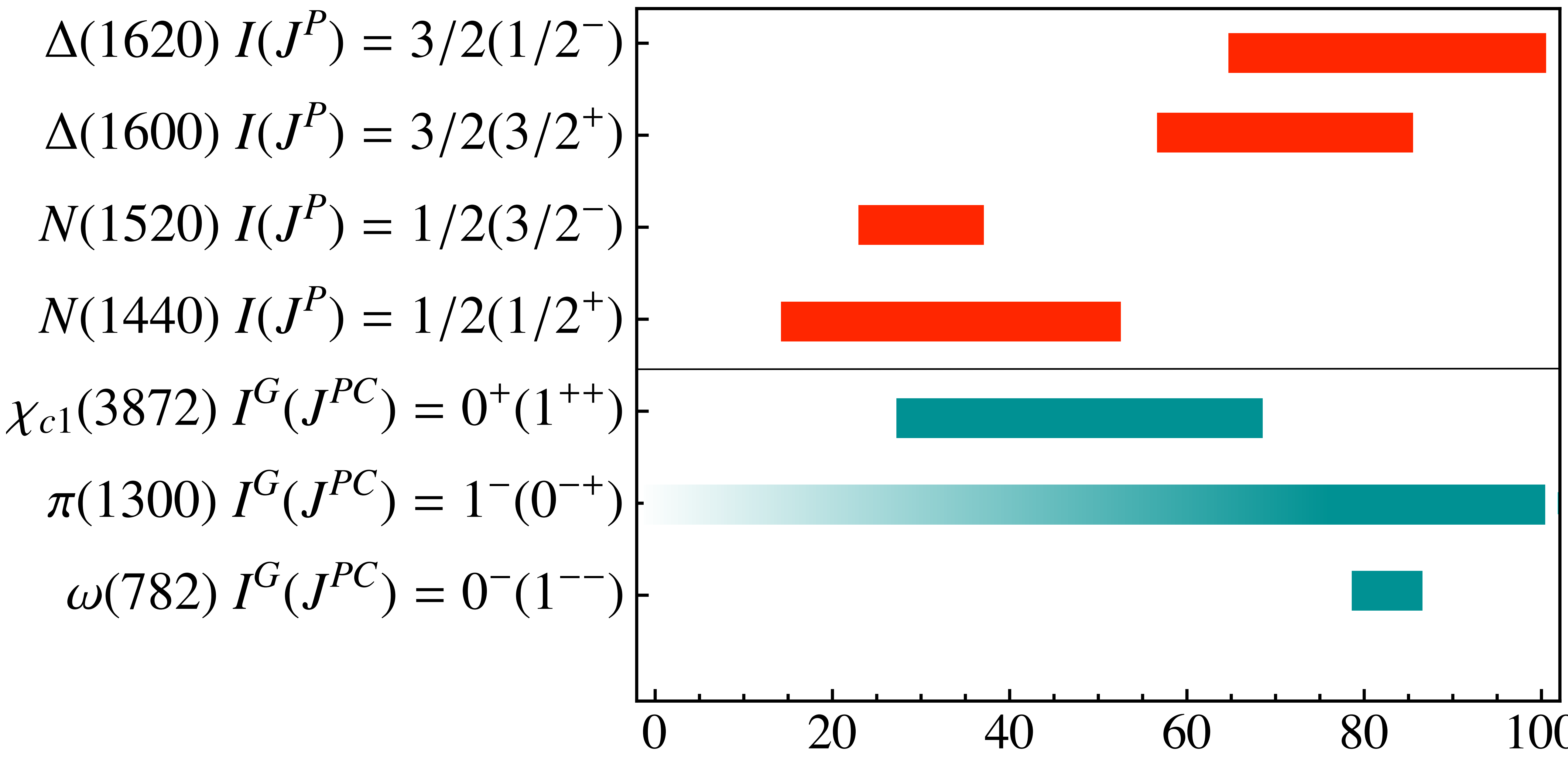


3-HADRON STATES

Most known states have large 3-body content

Review: MM/Meißner/Urbach *Phys.Rept.* 1001 (2023) 1-66

- $\omega(782) \to \pi\pi\pi$
- $a_1(1260) \to \pi\pi\pi$
- $N(1440) \to \pi\pi N$
- $X(3872) \to DD\pi$



Relative decay ratio to 3-body states [%] @PDG

Beyond Standard Model searches (τ-EDM/…)

Exotic states of matter

GlueX@JLAB; COMPASS@CERN;

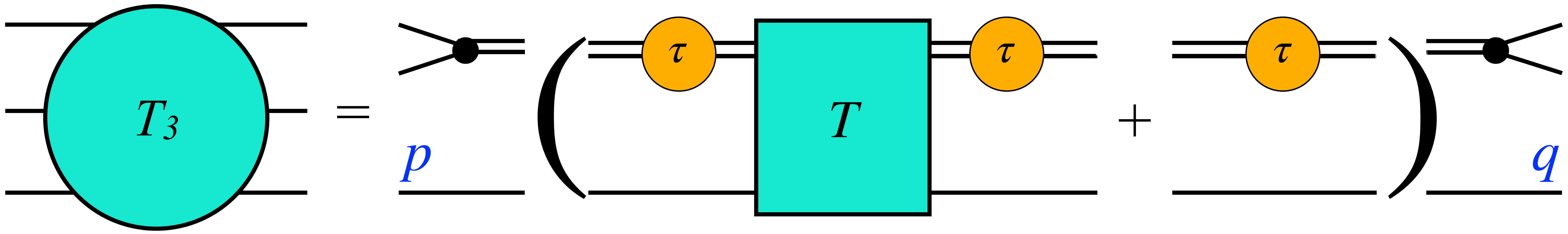
Triangle Singularities, long-range forces …

Review: FK Guo Prog.Part.Nucl.Phys. 112 (2020) 103757

TRANSITION AMPLITUDE

- 3 asymptotic states (scalar particles of equal mass m)
- Connectedness structure of matrix elements

Olive/Eden/… “The Analytic S-matrix”



isobar-parametrization of two-body amplitude by

Bedaque, Griesshammer (1999)

- “*isobars*” with right-hand-singularities $\tau(\sigma_p = s + m^2 - 2\sqrt{p^2+m^2}\sqrt{s})$
- in general a tower of “isobars” for ***L=0,1,2,...***
- coupling to asymptotic states: cut-free vertex function $v_{a\to bc}(p_b, p_c)$

Connected part: isobar-spectator interaction → $T(q,p;s)$

R. Aaron, R. D. Amado, and J. E. Young, Phys.Rev. 174, 2022 (1968)
>> Stable isobars assumed <<

THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int\prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}\,(2\pi)\delta^+(k_\ell^2 - m^2)\right]\,(2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\,\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int \prod_{\ell=1}^{3}\left[\frac{d^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\,\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

Exact identification

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle_{(1a)}$$
$$= i\int \prod_{\ell=1}^{3}\left[\frac{d^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)$$
$$\times \frac{1}{3!}\sum_{n,i} v(q_{\bar{n}}, q_{\bar{\bar{n}}})S^\dagger(\sigma(q_n))v(k_{\bar{i}}, k_{\bar{\bar{i}}})\,2E_{\mathbf{q}_n}(2\pi)^3\delta^3(\mathbf{q}_n - \mathbf{k}_i)$$
$$\times \frac{1}{3!}\sum_{j,m} v(k_{\bar{j}}, k_{\bar{\bar{j}}})S(\sigma(k_j))\langle k_j|T|p_m\rangle S(\sigma(p_m))v(p_{\bar{m}}, p_{\bar{\bar{m}}}) \times \delta_{ij}$$
$$= i\frac{1}{3!}\sum_{n,m,i}\frac{1}{3!}\int \prod_{\ell=1}^{3}\left[\frac{d^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)$$
$$\times v(q_{\bar{n}}, q_{\bar{\bar{n}}})S^\dagger(\sigma(q_n))v(k_{\bar{i}}, k_{\bar{\bar{i}}})\,2E_{\mathbf{q}_n}(2\pi)^3\delta^3(\mathbf{q}_n - \mathbf{k}_i) \times v(k_{\bar{i}}, k_{\bar{\bar{i}}})S(\sigma(k_i))[...]S(\sigma(p_m))v(p_{\bar{m}}, p_{\bar{\bar{m}}})$$
$$= i\frac{1}{3!}\sum_{n,m,i}\frac{1}{3!}\int \prod_{\ell\neq i}\left[\frac{d^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell\neq i}k_\ell - q_n\right)$$
$$\times v(q_{\bar{n}}, q_{\bar{\bar{n}}})S^\dagger(\sigma(q_n))v(k_{\bar{i}}, k_{\bar{\bar{i}}}) \times v(k_{\bar{i}}, k_{\bar{\bar{i}}})S(\sigma(q_n))\langle q_n|T|p_m\rangle S(\sigma(p_m))v(p_{\bar{m}}, p_{\bar{\bar{m}}})$$
$$= i\frac{1}{3!}\sum v(q_{\bar{n}}, q_{\bar{\bar{n}}})S^\dagger(\sigma(q_n))S(\sigma(q_n))\langle q_n|T|p_m\rangle S(\sigma(p_m))v(p_{\bar{m}}, p_{\bar{\bar{m}}})$$

let us unpack it …

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle_{(4a)}$$
$$= i\int \prod_{\ell=1}^{3}\left[\frac{d^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)$$
$$\times \frac{1}{3!}\sum v(q_{\bar{n}}, [...])S^\dagger(\sigma(q_n))\langle q_n|T^\dagger|k_i\rangle S^\dagger(\sigma(k_i))v(k_{\bar{i}}, k_{\bar{\bar{i}}})$$
$$[...]\sum_{j,m} v(k_{\bar{j}}, k_{\bar{\bar{j}}})S(\sigma(k_j))\langle k_j|T|p_m\rangle S(\sigma(p_m))v(p_{\bar{m}}, p_{\bar{\bar{m}}}) \times \delta_{ij}$$
$$= i\int \prod_{\ell=1}^{3}\left[\frac{d^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)$$
$$\times \frac{1}{3!}\sum_{n,m} v(q_{\bar{n}}, q_{\bar{\bar{n}}})S^\dagger(\sigma(q_n))S(\sigma(p_m))v(p_{\bar{m}}, p_{\bar{\bar{m}}}) \times \frac{1}{3!}\sum_{i}\langle q_n|T^\dagger|k_i\rangle S^\dagger(\sigma(k_i))v(k_{\bar{i}}, k_{\bar{\bar{i}}})^2 S(\sigma(k_i))\langle k_i|T|p_m\rangle$$
$$= i\frac{1}{3!}\sum_{n,m} v(q_{\bar{n}}, q_{\bar{\bar{n}}})S^\dagger(\sigma(q_n))S(\sigma(p_m))v(p_{\bar{m}}, p_{\bar{\bar{m}}})$$
$$\times \int \prod_{\ell\neq 1}\left[\frac{d^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell\neq 1}k_\ell - k_1\right)v(k_2, k_3)^2$$

THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3} k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\,\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

all possible 3-body configurations

THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}\,(2\pi)\delta^+(k_\ell^2 - m^2)\right]\,(2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3} k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\,\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

all possible 3-body configurations

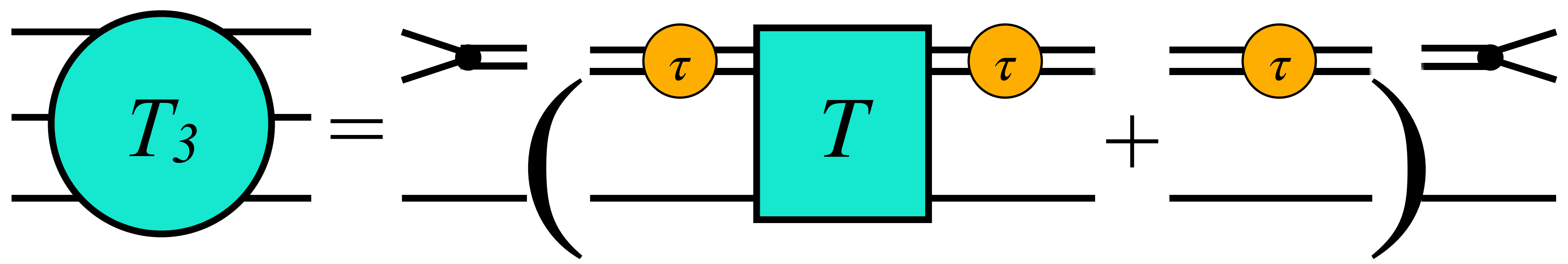
for higher energies also 4,5,… particle states are needed

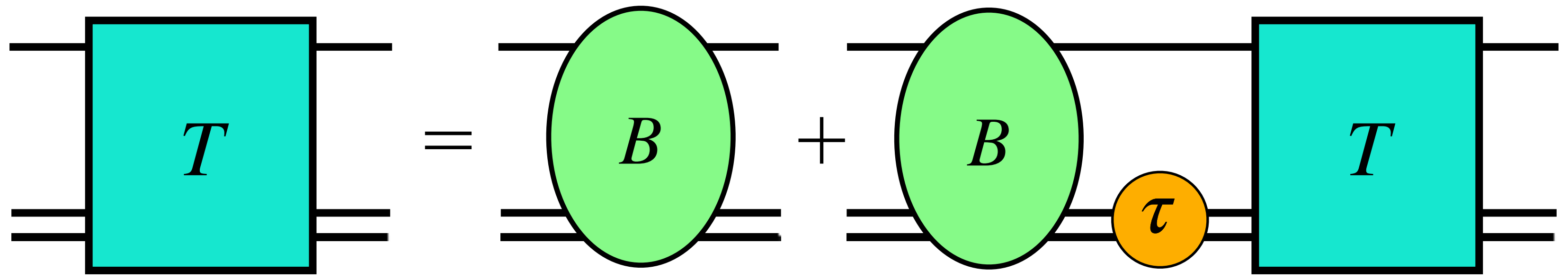
THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int\prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

Bethe-Salpeter integral equation (general building blocks)



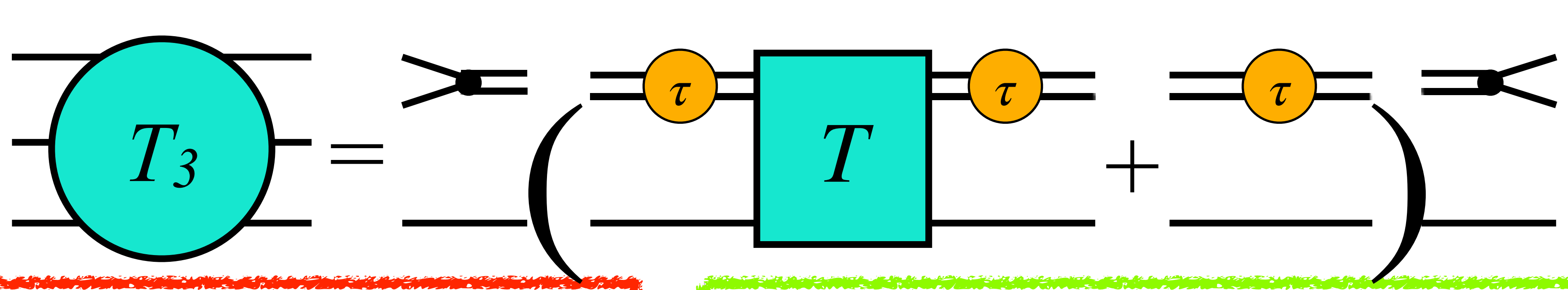


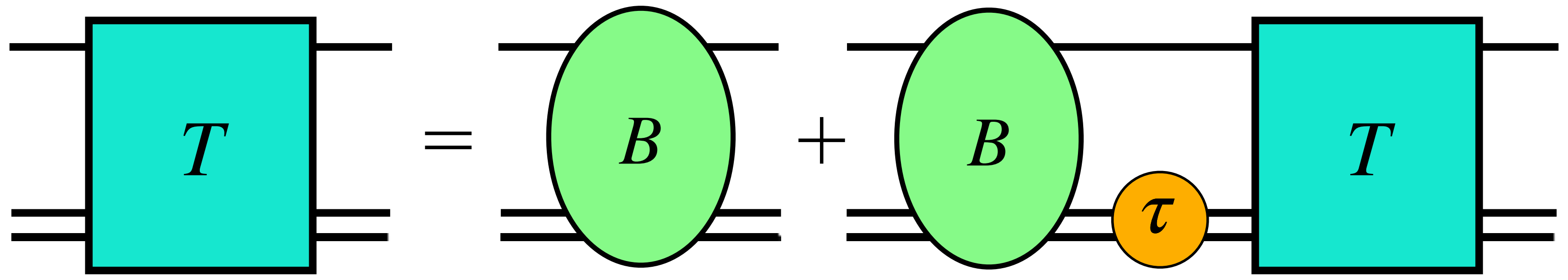
THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int\prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3}k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\,\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

Bethe-Salpeter integral equation (general building blocks)





THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3} k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\,\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

Exact identification

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle =$$
$$i\int \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3} k_\ell\right)\langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\,\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

Exact identification

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

these are the ones of
R. Aaron, R. D. Amado, and J. E. Young
Phys.Rev. 174, 2022 (1968)

DISCONTINUITY RELATION

all 8 equations are solved simultaneously if

$$\text{Disc}\, B = iv(P-p-q,q)(2\pi)\delta^+((P-q-p)^2-m^2)v(P-p-q,p)$$

$$\text{Disc}\,\frac{1}{\tau(\sigma(k))} = \frac{i}{64\pi^2 K_{\text{cm}}}\int \mathrm{d}^3\bar{\mathbf{K}}\,\frac{\delta\left(|\bar{\mathbf{K}}|-K_{\text{cm}}\right)}{\sqrt{(\bar{\mathbf{K}})^2+m^2}}v^2$$

DISCONTINUITY RELATION

all 8 equations are solved simultaneously if

$$\text{Disc}\, B = iv(P-p-q,q)(2\pi)\delta^+((P-q-p)^2-m^2)v(P-p-q,p)$$

$$\Downarrow$$

$$\langle q\,|\,B(s)\,|\,p\rangle = -\frac{v(Q,q)v(Q,p)}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$

$$\text{Disc}\,\frac{1}{\tau(\sigma(k))} = \frac{i}{64\pi^2 K_{\text{cm}}}\int \mathrm{d}^3\bar{\mathbf{K}}\,\frac{\delta\left(|\bar{\mathbf{K}}|-K_{\text{cm}}\right)}{\sqrt{(\bar{\mathbf{K}})^2+m^2}}v^2$$

$$\Downarrow$$

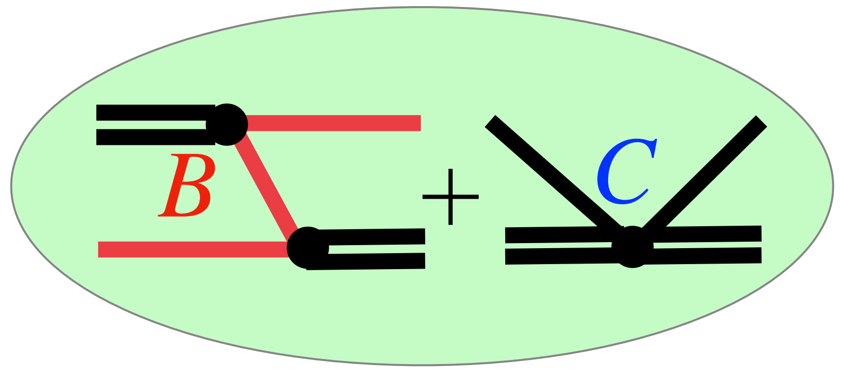
$$\frac{1}{\tau(\sigma(k))} = A + B\,\sigma(k) + \frac{\sigma(k)^2}{\pi}\int_{4m^2}^{\infty} d\sigma'\,\frac{\text{Im}\,\tau^{-1}(\sigma')}{\sigma'^2(\sigma'-\sigma(k)-i\epsilon)}$$

DISCONTINUITY RELATION

all 8 equations are solved simultaneously if

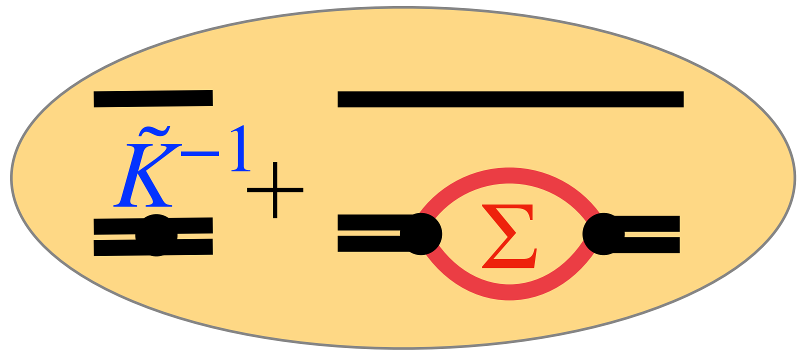
$$\mathrm{Disc}\, B = iv(P-p-q,q)(2\pi)\delta^+((P-q-p)^2-m^2)v(P-p-q,p)$$

$$\langle q|B(s)|p\rangle = -\frac{v(Q,q)v(Q,p)}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$



$$\mathrm{Disc}\,\frac{1}{\tau(\sigma(k))} = \frac{i}{64\pi^2 K_{\rm cm}}\int \mathrm{d}^3\bar{\mathbf{K}}\,\frac{\delta\left(|\bar{\mathbf{K}}|-K_{\rm cm}\right)}{\sqrt{(\bar{\mathbf{K}})^2+m^2}}v^2$$

$$\frac{1}{\tau(\sigma(k))} = A + B\,\sigma(k) + \frac{\sigma(k)^2}{\pi}\int_{4m^2}^{\infty} d\sigma'\,\frac{\mathrm{Im}\,\tau^{-1}(\sigma')}{\sigma'^2(\sigma'-\sigma(k)-i\epsilon)}$$



DISCONTINUITY RELATION

all 8 equations are solved simultaneously if

$$\mathrm{Disc}\, B = iv(P-p-q,q)(2\pi)\delta^+((P-q-p)^2-m^2)v(P-p-q,p)$$

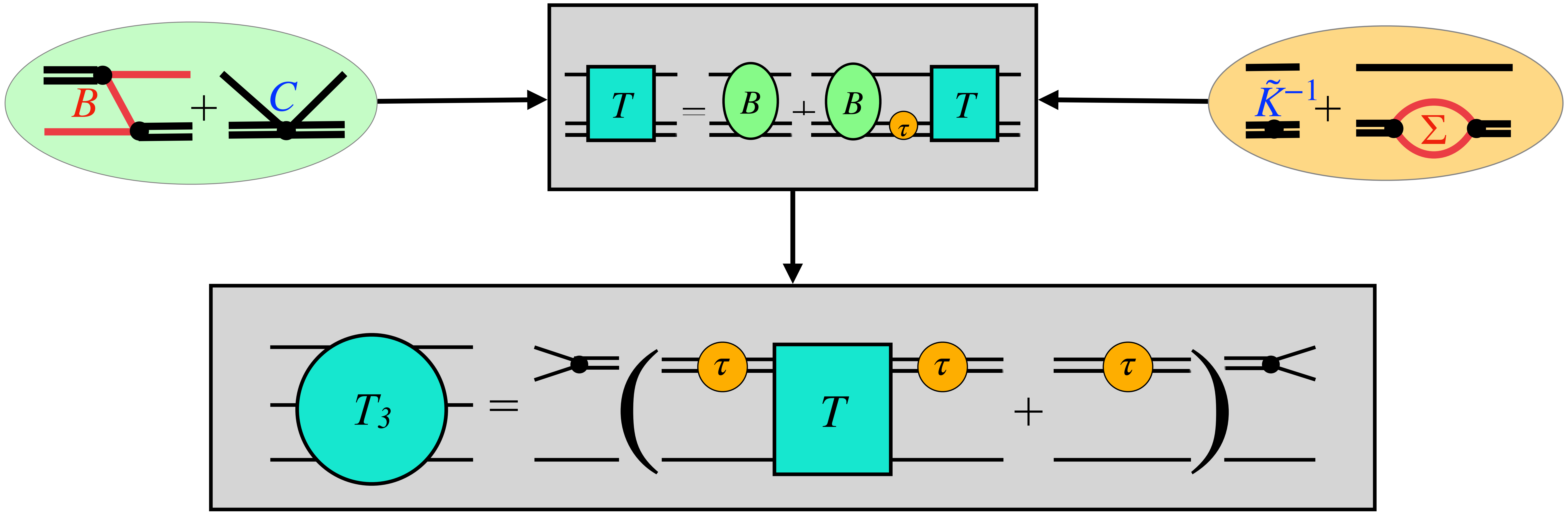
⇓

$$\langle q\,|\,B(s)\,|\,p\rangle = -\frac{v(Q,q)v(Q,p)}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$

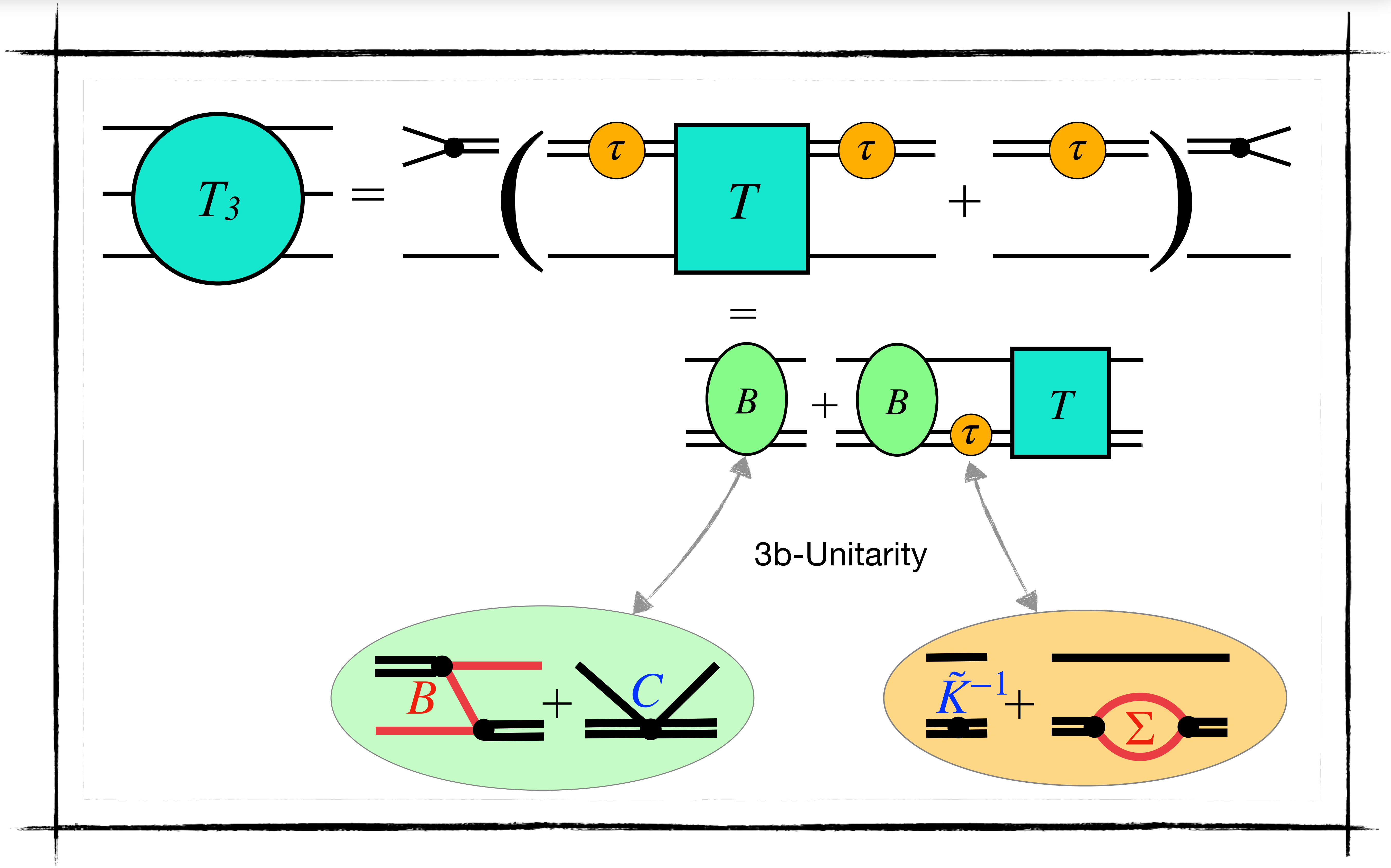
$$\mathrm{Disc}\,\frac{1}{\tau(\sigma(k))} = \frac{i}{64\pi^2 K_{\mathrm{cm}}}\int \mathrm{d}^3\bar{\mathbf{K}}\,\frac{\delta\left(|\bar{\mathbf{K}}|-K_{\mathrm{cm}}\right)}{\sqrt{(\bar{\mathbf{K}})^2+m^2}}v^2$$

⇓

$$\frac{1}{\tau(\sigma(k))} = A + B\,\sigma(k) + \frac{\sigma(k)^2}{\pi}\int_{4m^2}^{\infty} d\sigma'\,\frac{\mathrm{Im}\,\tau^{-1}(\sigma')}{\sigma'^2(\sigma'-\sigma(k)-i\epsilon)}$$



SUMMARY



"Infinite Volume Unitarity" — IVU formalism

- Three-body scattering amplitude
  MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)
  Related: Hansen/Sharpe(2014), Wunderlich et al. JHEP 08 (2019); Jackura et al. Eur.Phys.J.C 79 (2019);
- Express 3-body through 2+1 system
- 3b-Unitarity:
  - On-shell configs (B, Σ) — fixed
  - Off-shell parts $(C, \tilde{K})$ — input
- Generalised to all channels + strangeness

  Feng+ 2407.08721 [nucl-th]

IVU

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

SOLUTION TECHNIQUES

- 3b scattering equation necessarily: Integral equation

IVU

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

MM/Hu/Döring/Pilloni/Szczepaniak
Eur.Phys.J.A 53 (2017)

Singularities:

- One-particle exchange diagram
- Two-body self energy

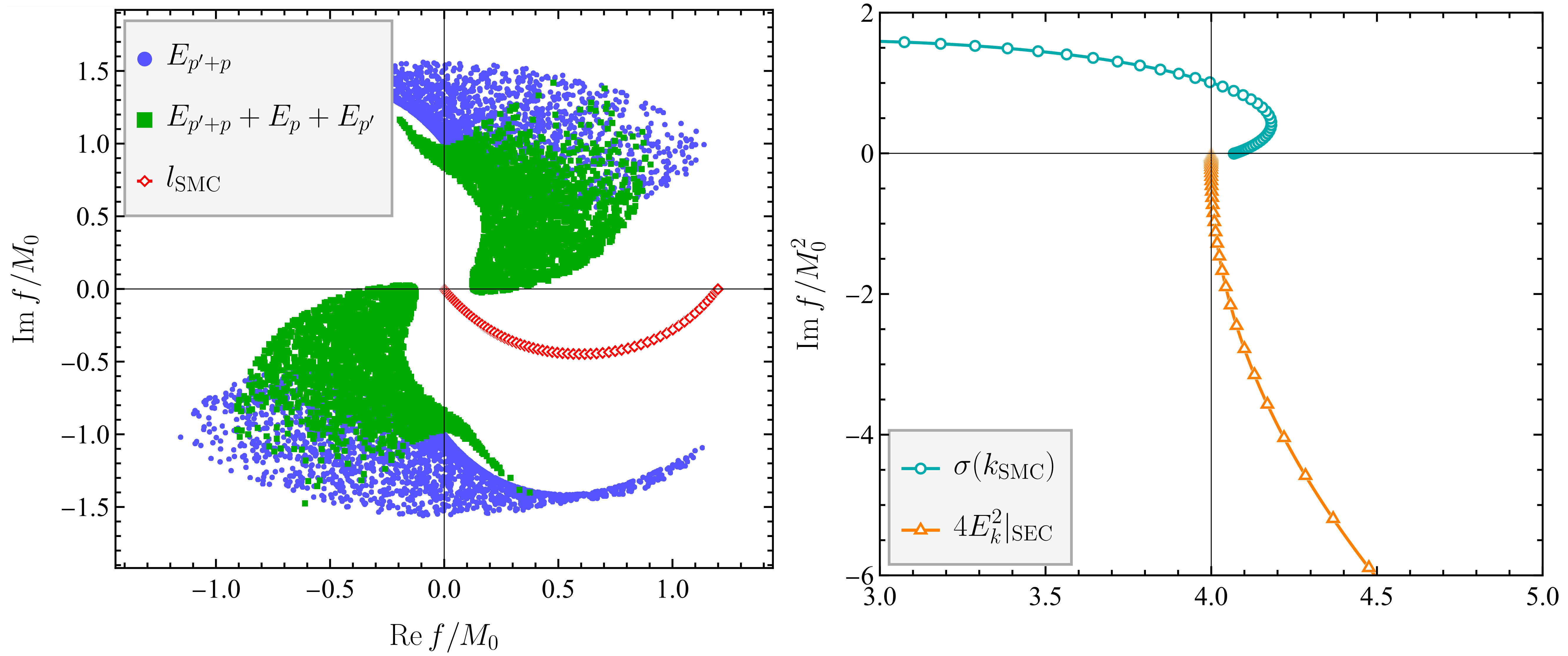
Avoid singularities:

- Complex contour deformation $|\ell| \in \mathbb{C}$
- Analytic continuation to the real axis

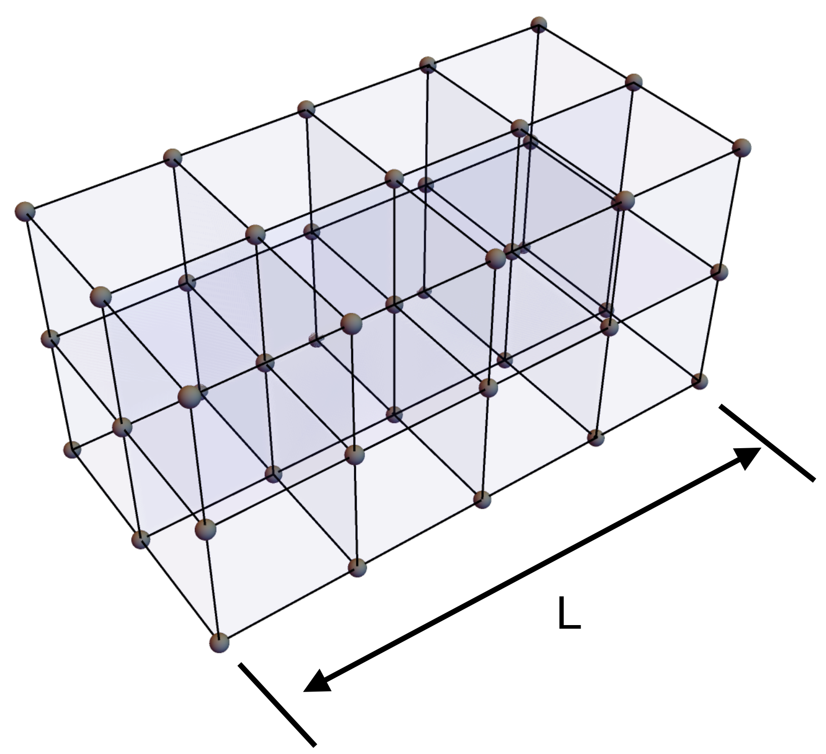
J. H. Hetherington and L. H. Schick, Phys. Rev. 137, B935–B948 (1965).
R. T. Cahill and I. H. Sloan, Nucl. Phys. A 165, 161–179 (1971)
Erich W. Schmid and Horst Ziegelmann,
S. K. Adhikari and R. D. Amado, Phys. Rev. D 9, 1467–1475 (1974)

- Application to Triangle singularities

Ajay S. Sakthivasan+ 2407.17969 [hep-ph]



FINITE-VOLUME SPECTRUM

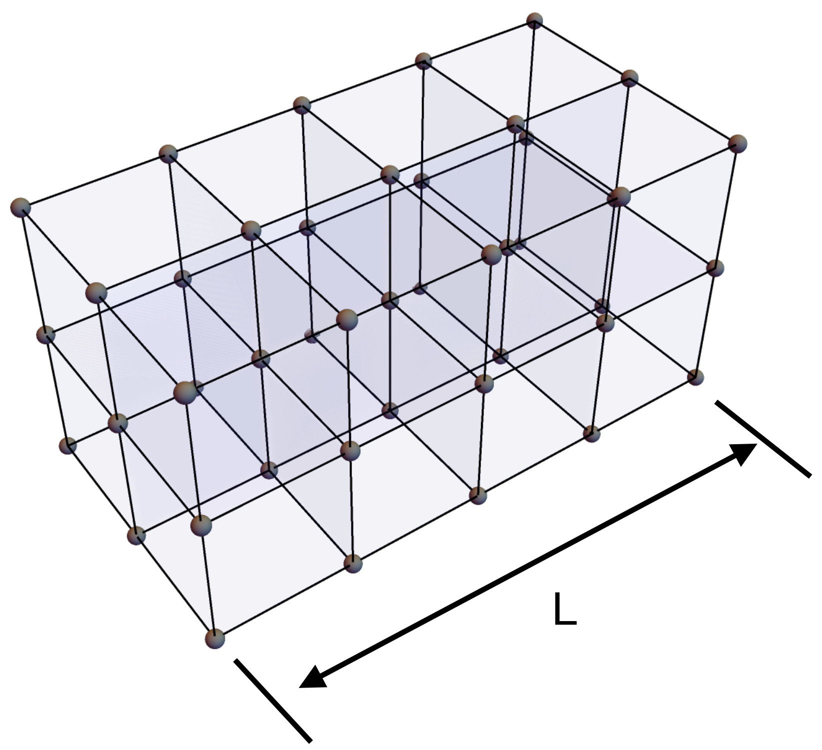


Lattice QCD — QCD Green's functions:

- Euclidean space-time
- unphysical pion mass
- **finite-volume**

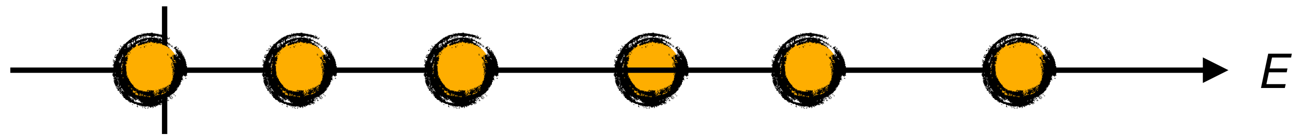
Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, …
Reviews: Briceno/Dudek/Young (2017) Rev.Mod.Phys. 90 (2018); Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Doring/Rusetsky Eur.Phys.J.ST 230 (2021);

FINITE-VOLUME SPECTRUM

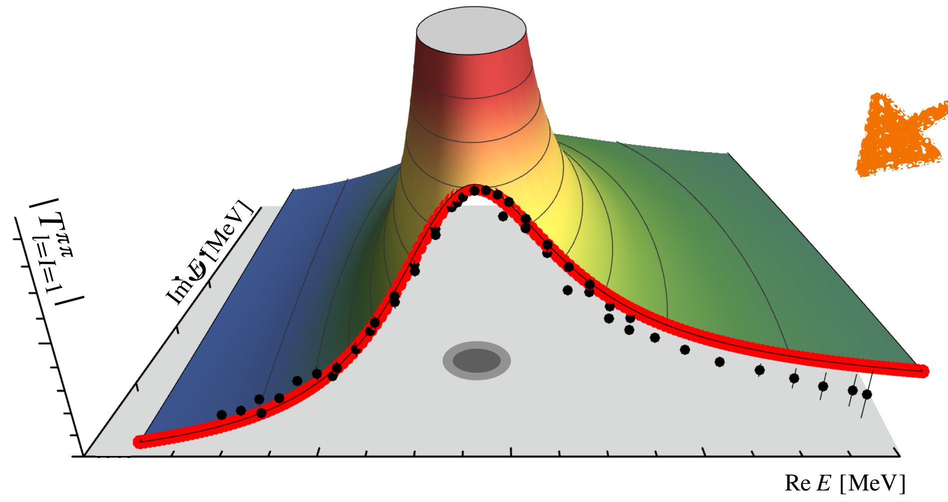


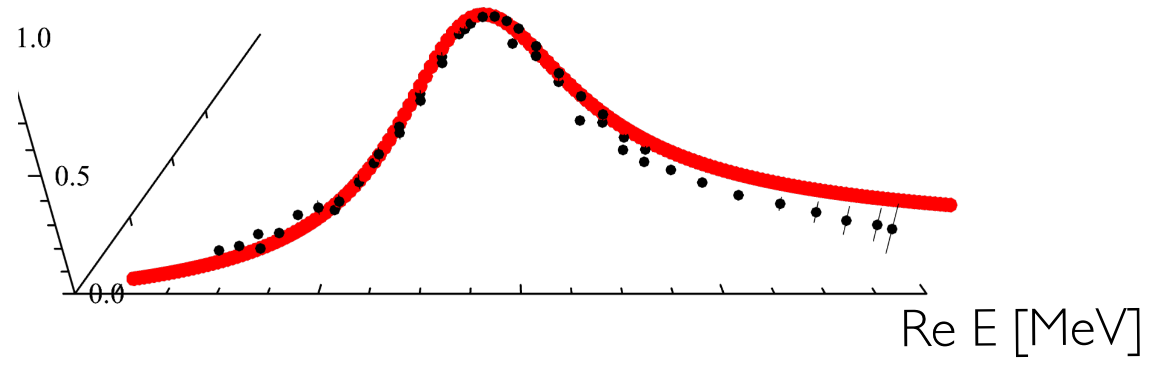
Lattice QCD — QCD Green's functions:

- Euclidean space-time
- unphysical pion mass
- **finite-volume**



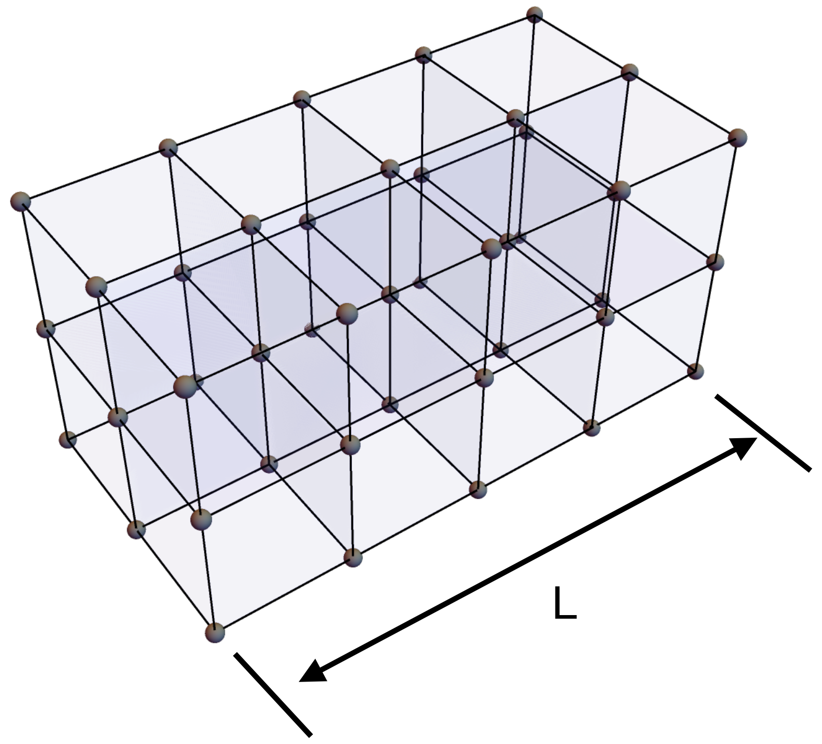
Quantization condition





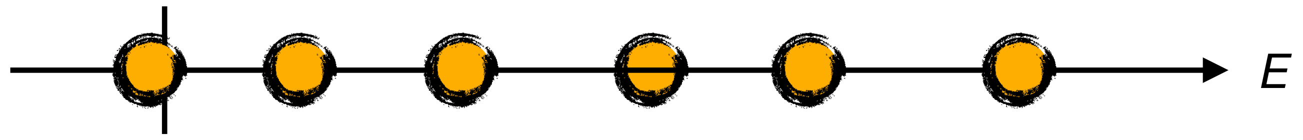
Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, …
Reviews: Briceno/Dudek/Young (2017) Rev.Mod.Phys. 90 (2018); Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Doring/Rusetsky Eur.Phys.J.ST 230 (2021);

FINITE-VOLUME SPECTRUM



Lattice QCD — QCD Green's functions:

- Euclidean space-time
- unphysical pion mass
- **finite-volume**



Quantization condition

- on-shell states "feel" the box-size
- off-shell configurations decay exponentially $\sim e^{-ML}$
- unitarity separates those
- "Finite Volume Unitarity" 3-body quantization condition

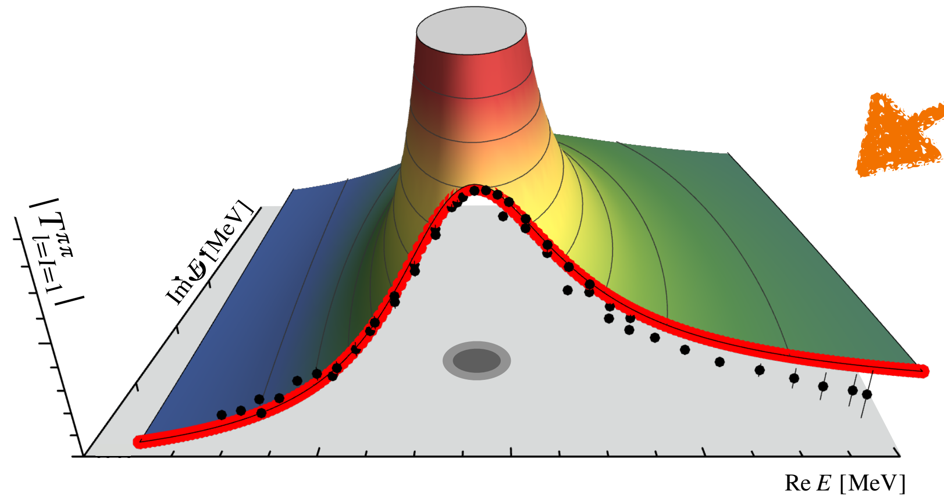
FVU

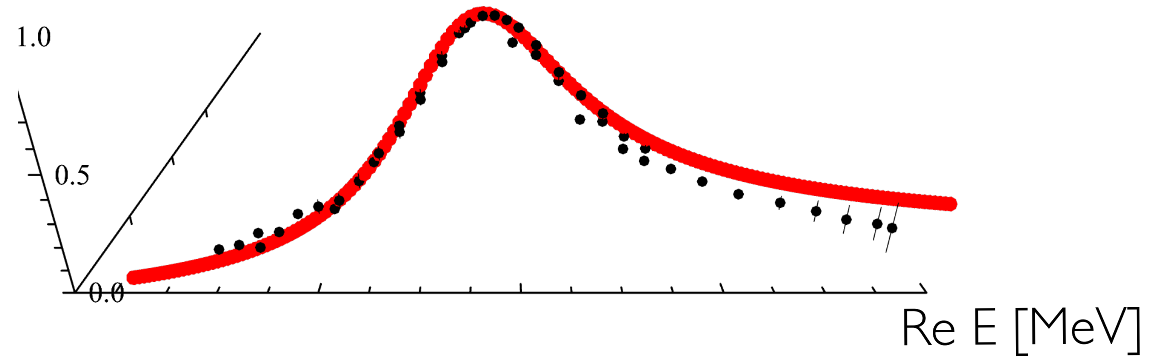
$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}^{-1}-\Sigma^L\right)-B-C\,\right]^{\Lambda}\equiv 0$$

MM/Döring
Eur.Phys.J.A 53 (2017) 12, 240

Soon as a python package: H.Yan+MM

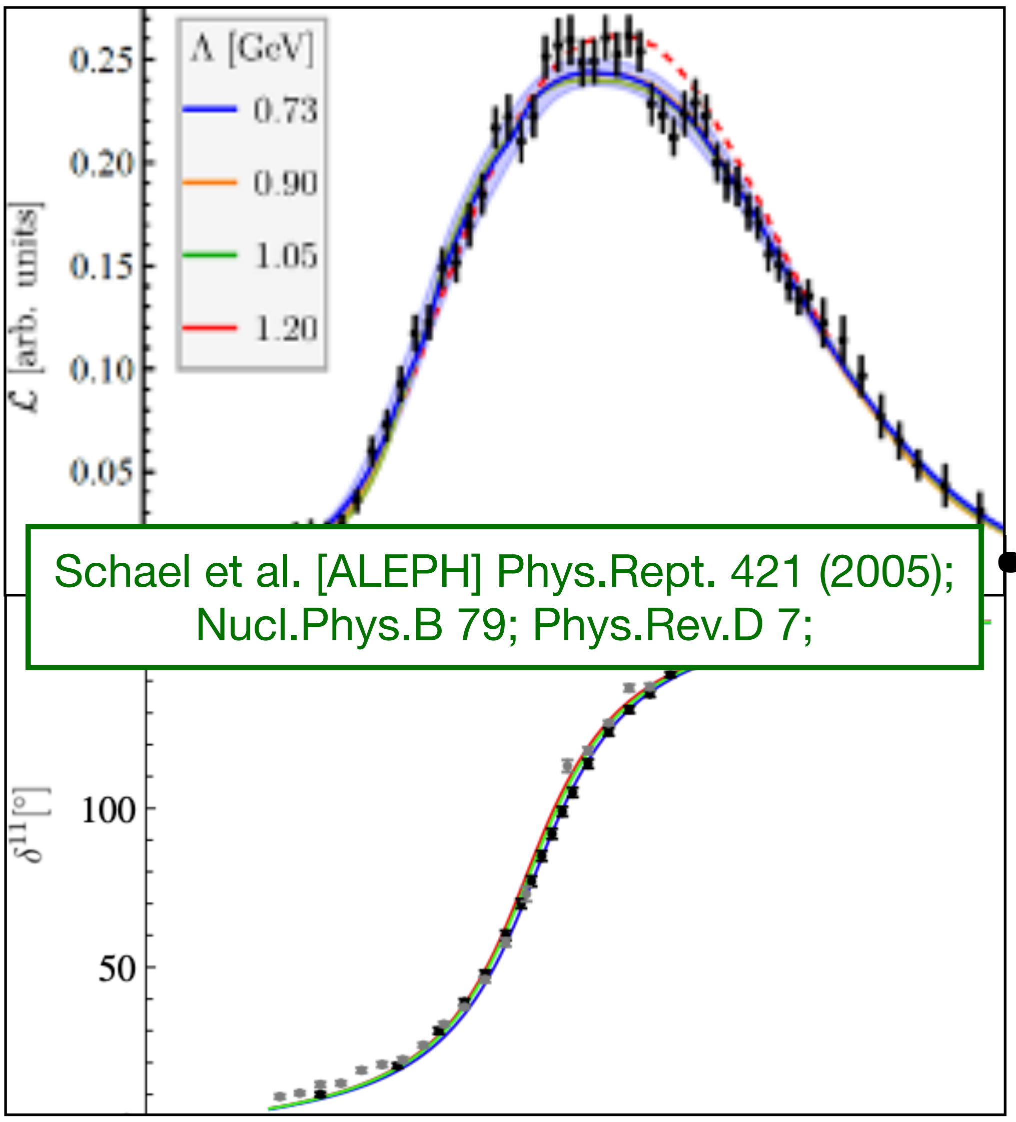
- Alternatives:
  - RFT(Hansen/Sharpe 2014)
  - NREFT(Rusetsky/Hammer/Pang 2017)





Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, …
Reviews: Briceno/Dudek/Young (2017) Rev.Mod.Phys. 90 (2018); Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Doring/Rusetsky Eur.Phys.J.ST 230 (2021);

AXIAL-VECTOR MESON



Schael et al. [ALEPH] Phys.Rept. 421 (2005); Nucl.Phys.B 79; Phys.Rev.D 7;

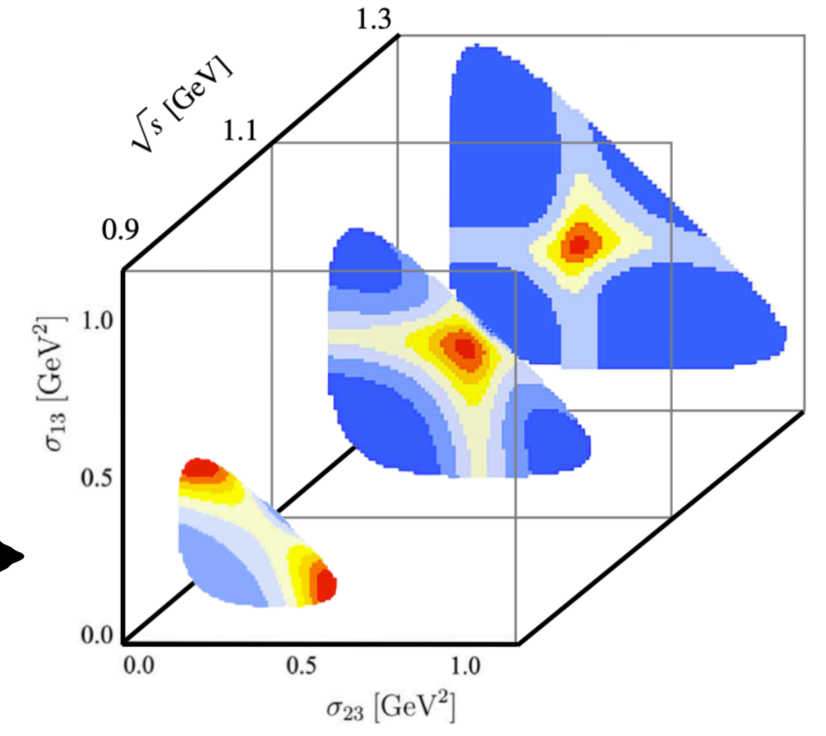
INPUT EXPERIMENT

IVU

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

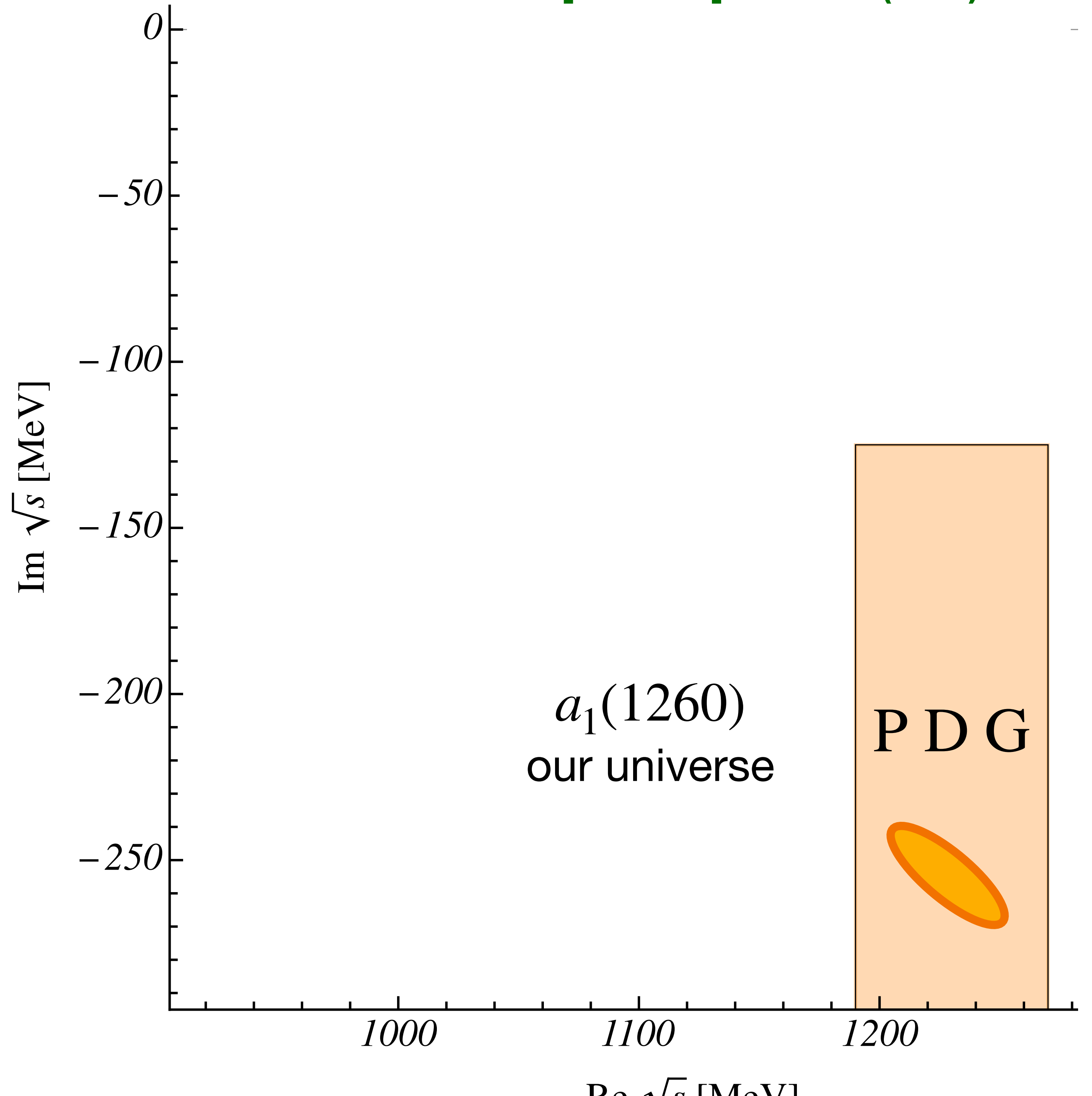
OUTPUT

Sadasivan/MM/… Phys.Rev.D 101 (2020);



OUTPUT

MM/Culver/… [GWQCD] PRL 127 (2021)



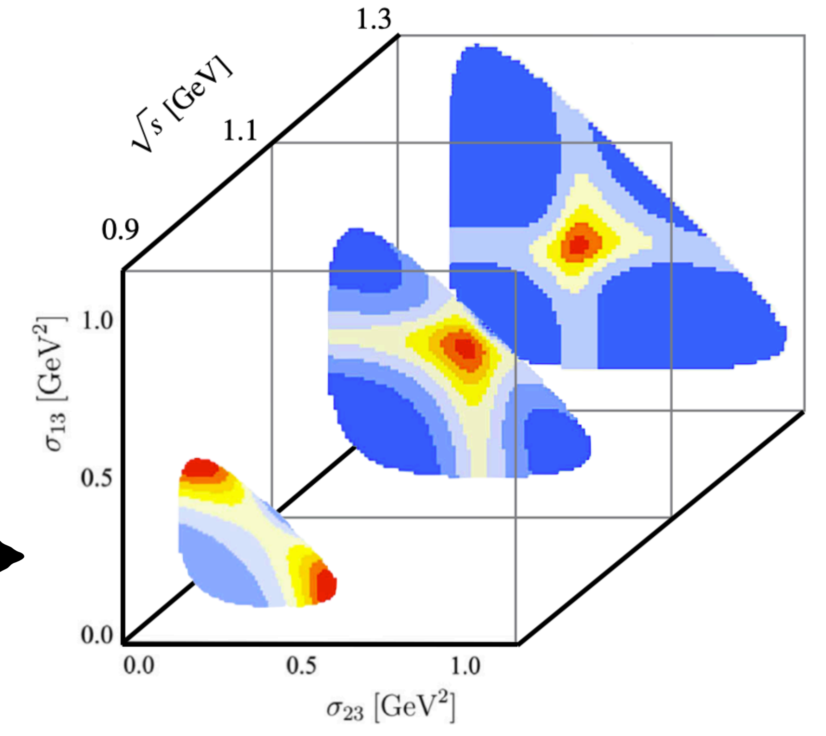
AXIAL-VECTOR MESON

IVU

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

OUTPUT

Sadasivan/MM/... Phys.Rev.D 101 (2020);

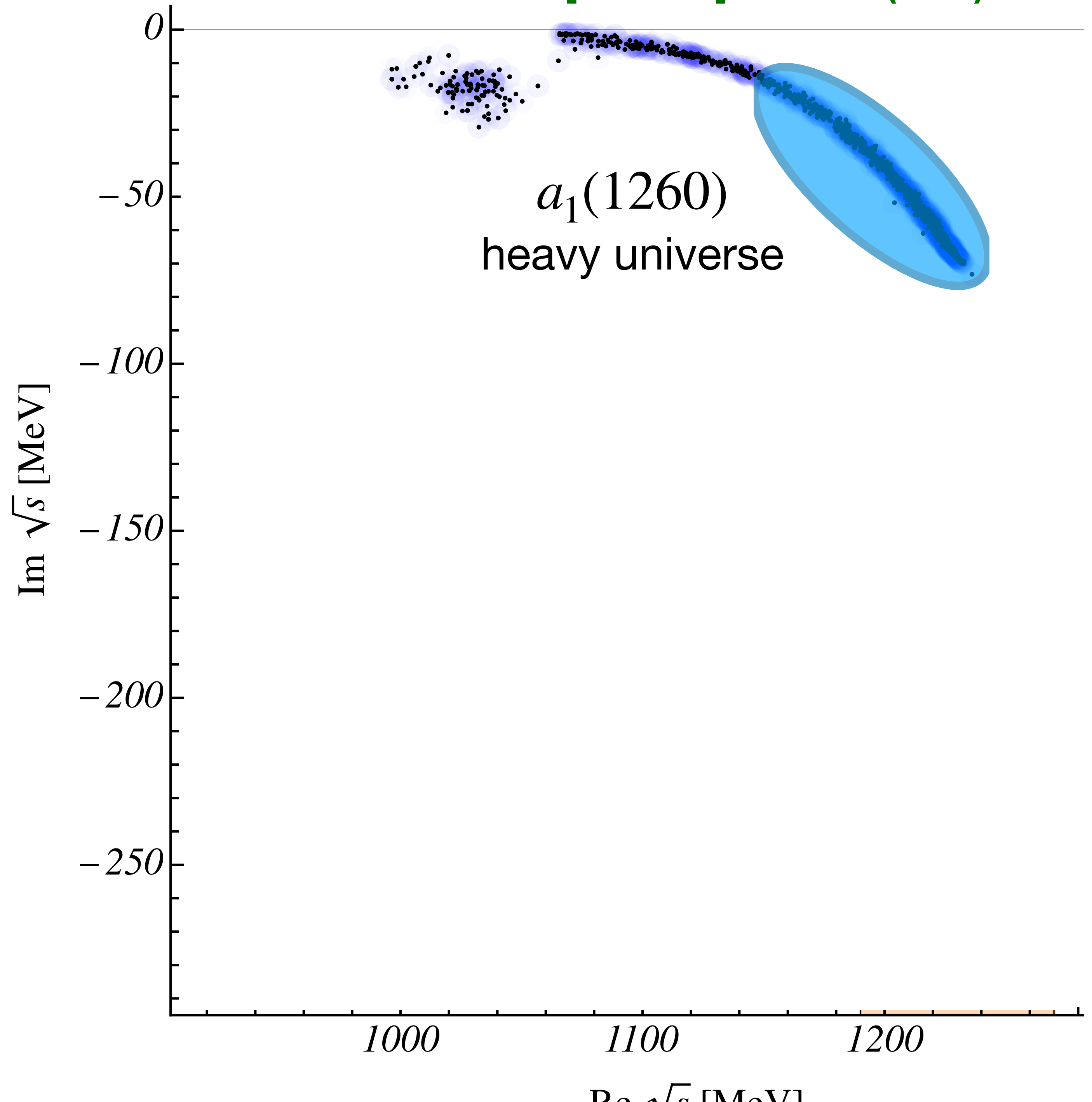


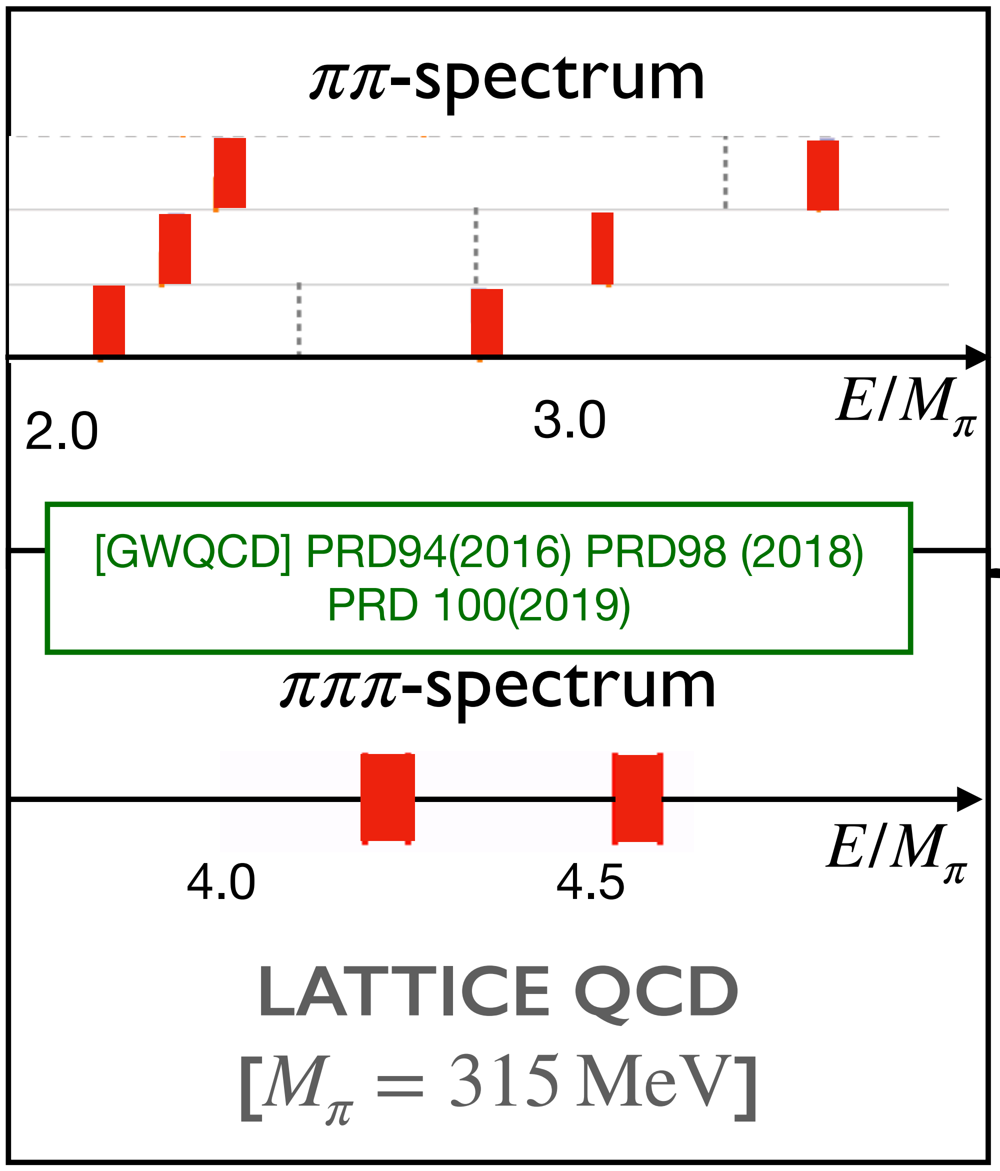
$C, \tilde{K}$

volume independent

OUTPUT

MM/Culver/... [GWQCD] PRL 127 (2021)



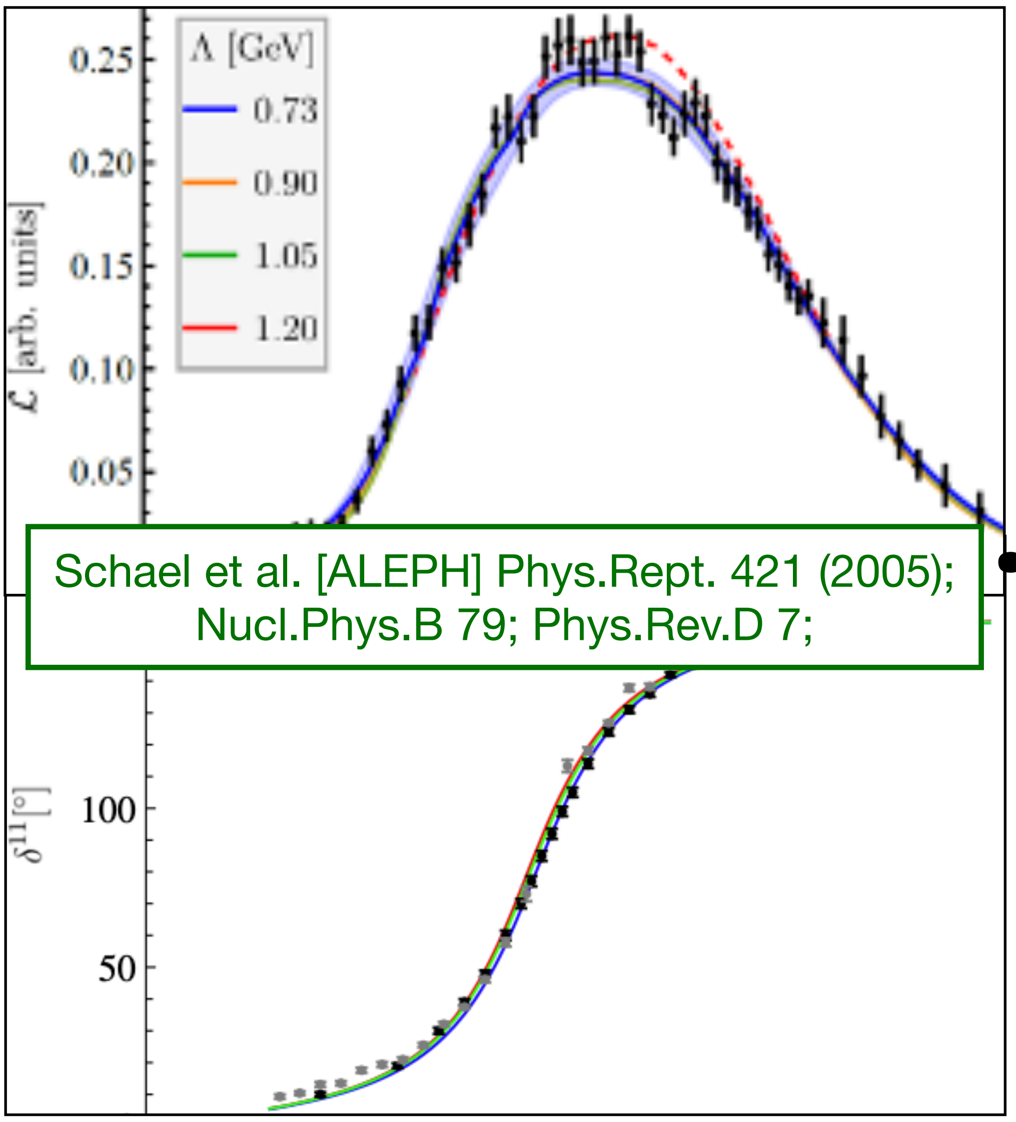


INPUT
Lattice QCD

FVU

$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^\Lambda \equiv 0$$

AXIAL-VECTOR MESON



INPUT EXPERIMENT

IVU

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

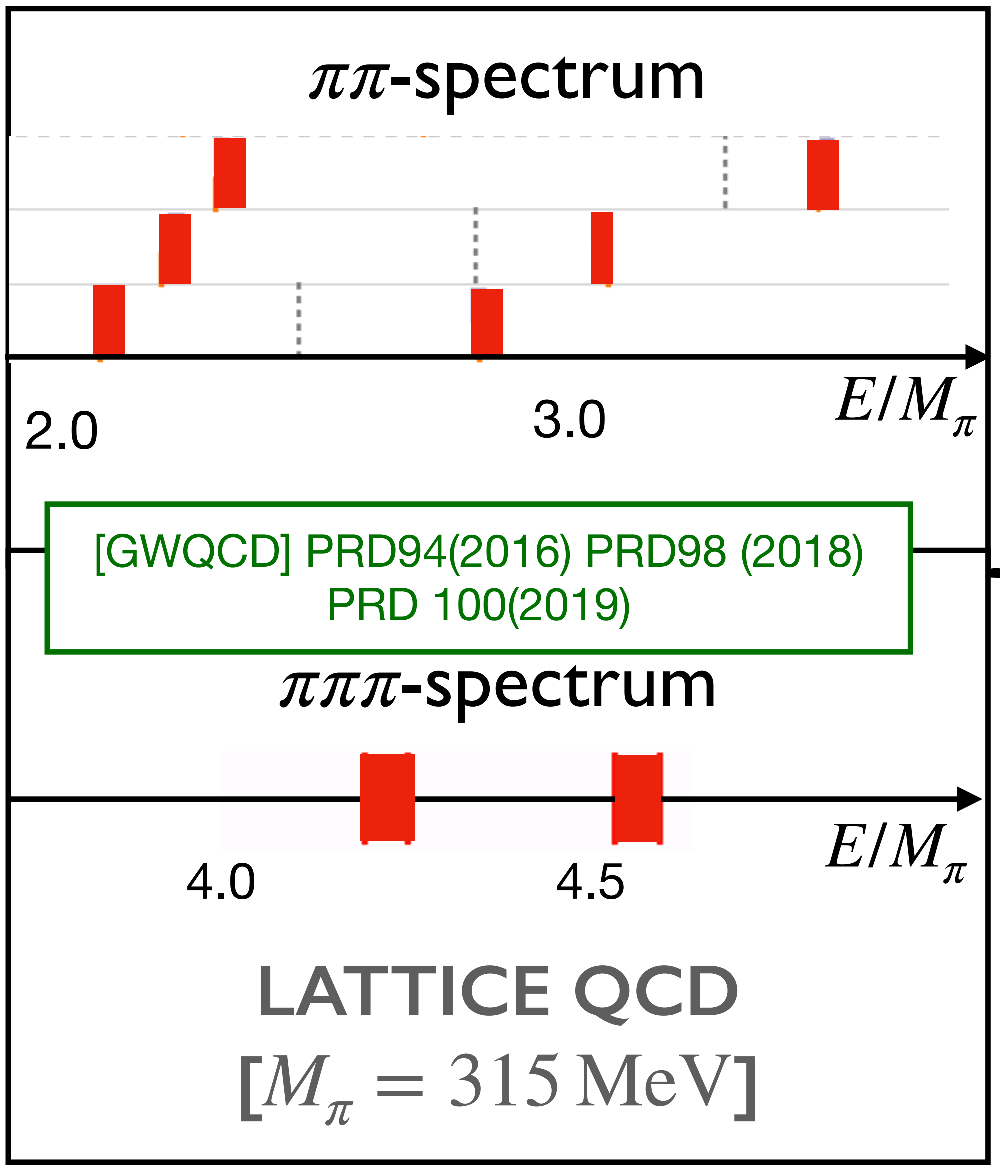
$C, \tilde{K}$

volume independent

FVU

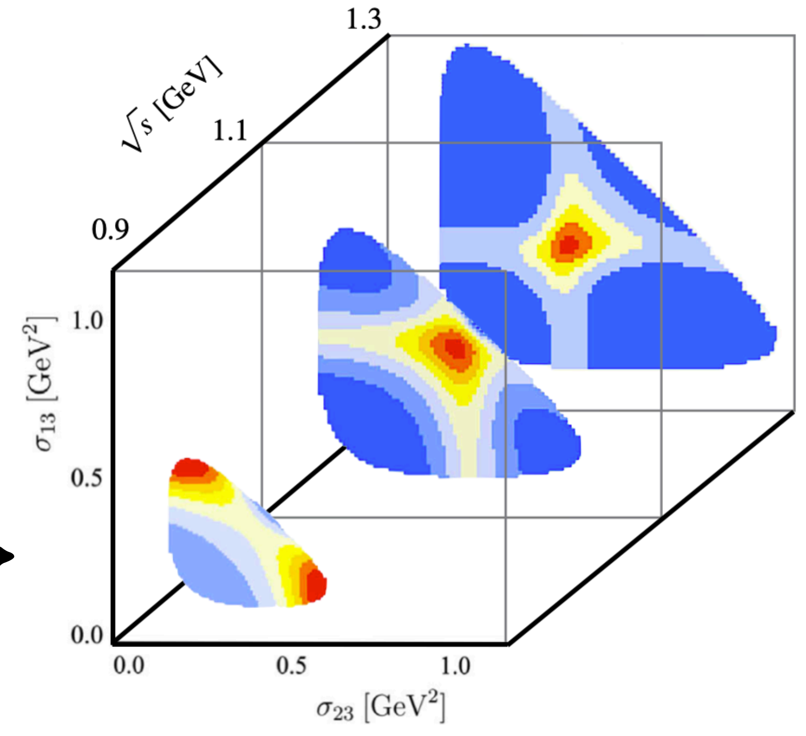
$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Lambda} \equiv 0$$

INPUT Lattice QCD



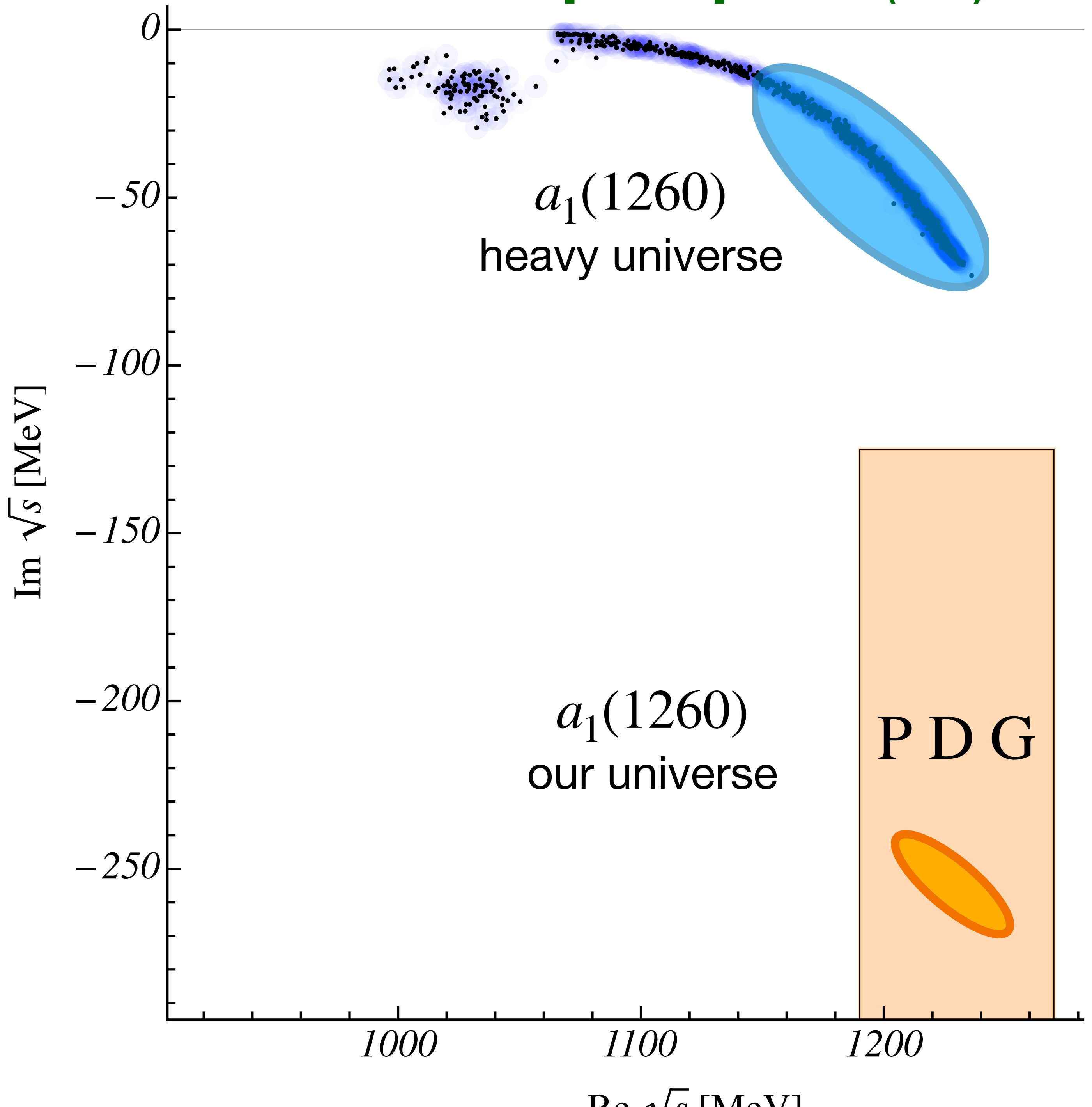
OUTPUT

Sadasivan/MM/... Phys.Rev.D 101 (2020);



OUTPUT

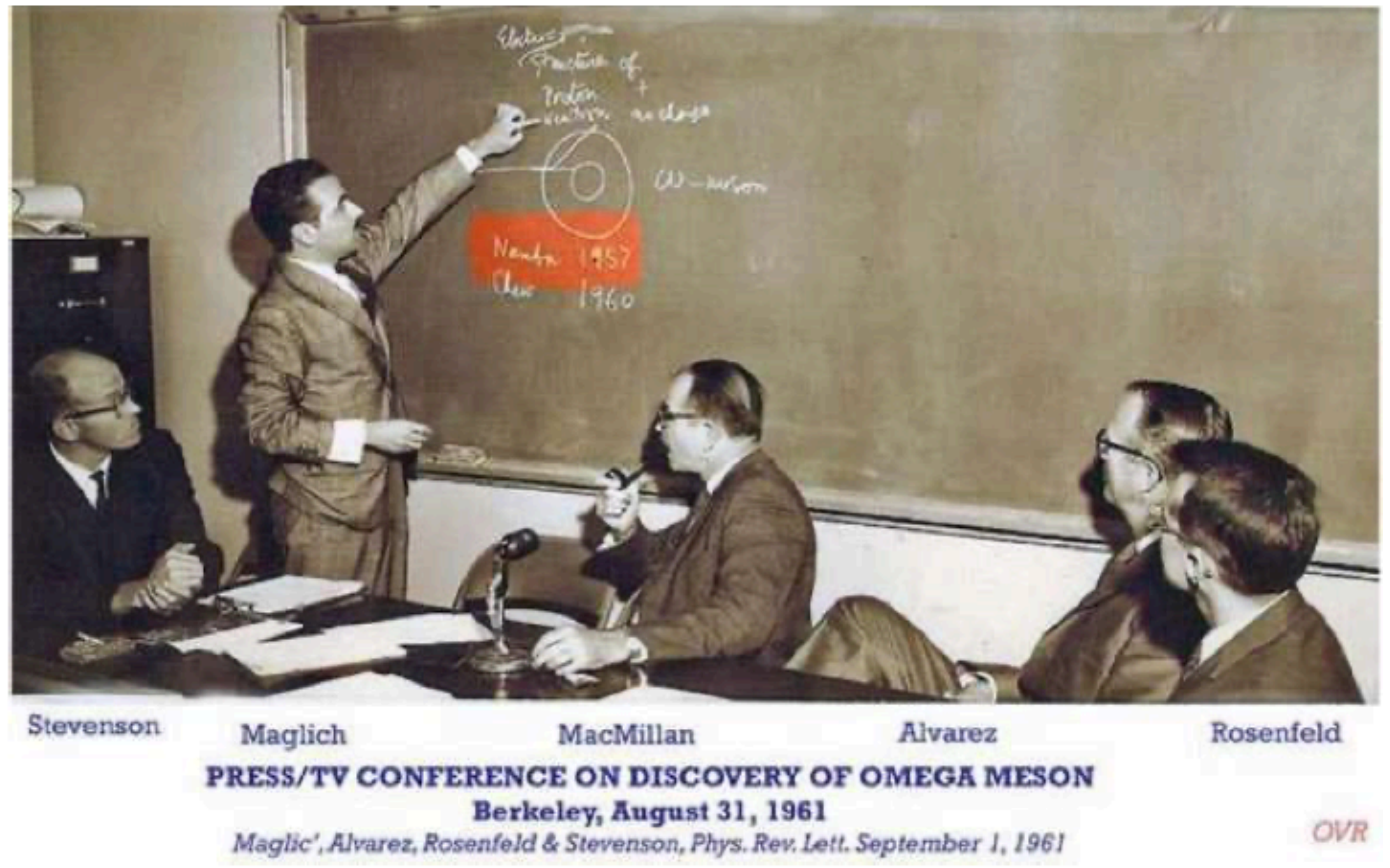
MM/Culver/... [GWQCD] PRL 127 (2021)



VECTOR MESON

$\omega(782)$

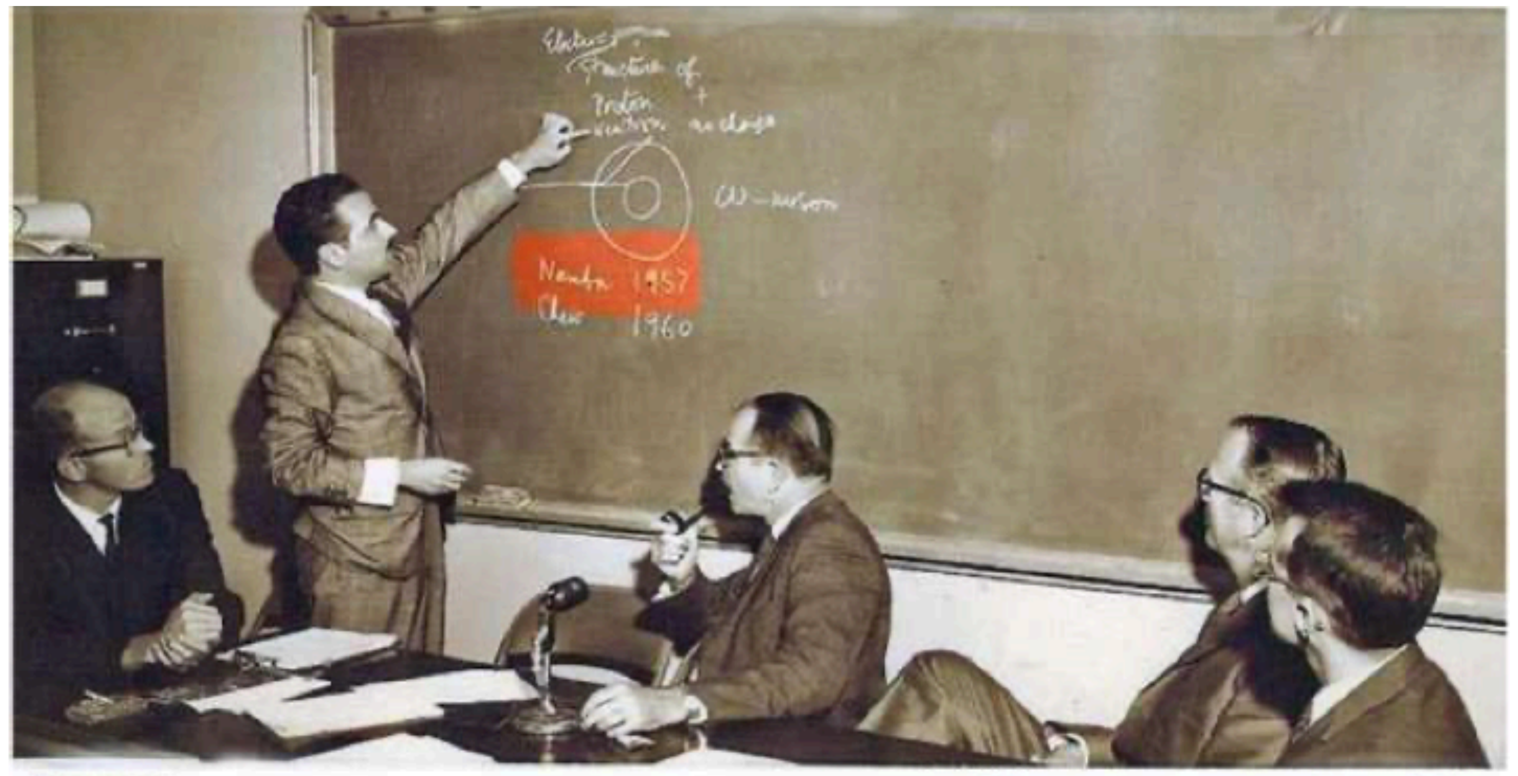
- lightest hadron decaying into three particles
  Maglich/Alvarez/Rosenfeld/Stevenson *Phys.Rev.Lett.* 7 (1961) 178-182
- dominates the isoscalar response within the VMD picture of the photon-nucleon interactions
  Sakurai (1960); Erkelenz (1974); Brown and Jackson (1976);
  Barkov et al., 1985; Connell et al. (1997); Bazavov et al. (2021)
- generates the observed repulsion at < 1 fm in the one-boson-exchange picture of the N– N interaction



VECTOR MESON

$\omega(782)$

- lightest hadron decaying into three particles
  Maglich/Alvarez/Rosenfeld/Stevenson *Phys.Rev.Lett.* 7 (1961) 178-182
- dominates the isoscalar response within the VMD picture of the photon-nucleon interactions
  Sakurai (1960); Erkelenz (1974); Brown and Jackson (1976);
  Barkov et al., 1985; Connell et al. (1997); Bazavov et al. (2021)
- generates the observed repulsion at < 1 fm in the one-boson-exchange picture of the N– N interaction



Stevenson Maglich MacMillan Alvarez Rosenfeld
PRESS/TV CONFERENCE ON DISCOVERY OF OMEGA MESON
Berkeley, August 31, 1961
Maglic', Alvarez, Rosenfeld & Stevenson, Phys. Rev. Lett. September 1, 1961
OVR

What can we learn from Lattice QCD?

Haobo Yan (燕浩波)/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

- two/three-body force
- pion-mass dependence
- KSFR/Universality relations/… in EFT
  Gell-Mann/Sharp/Wagner/Fujiwara/Kawarabayashi… **Review**: Meißner *Phys.Rept.* 161 (1988) 213

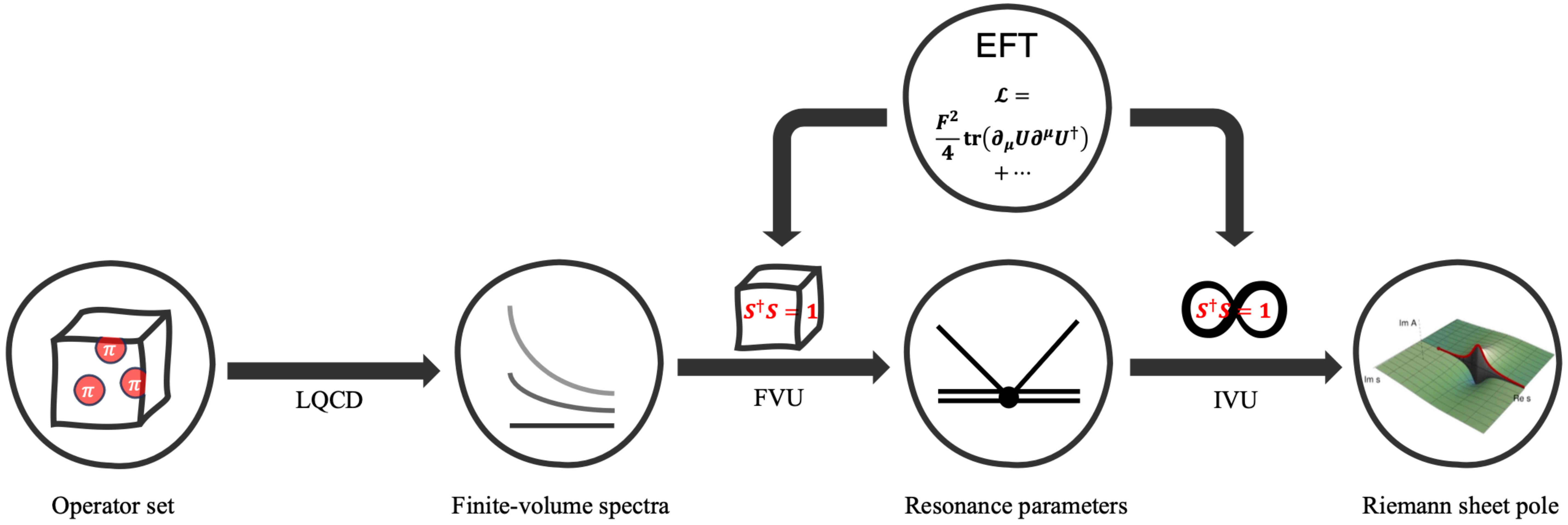
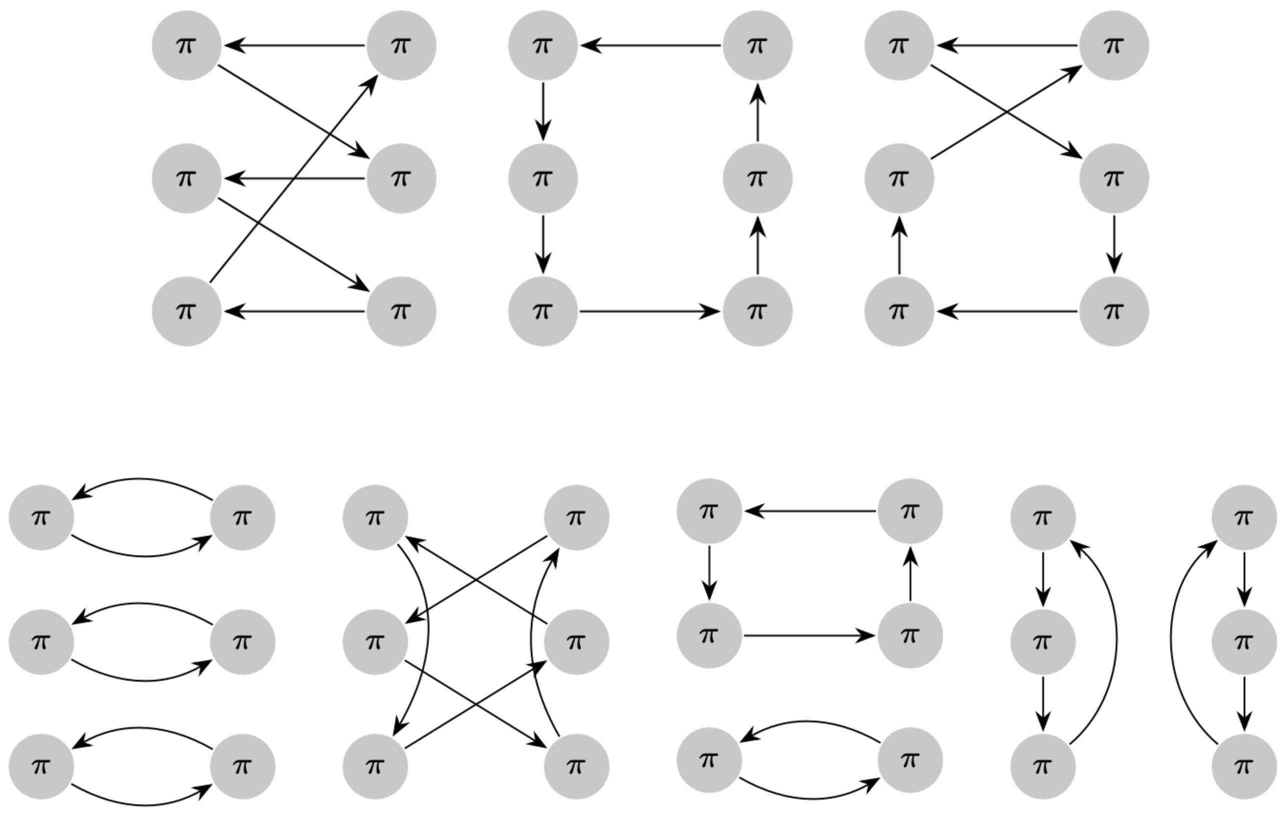


Fig by Haobo Yan (燕浩波)

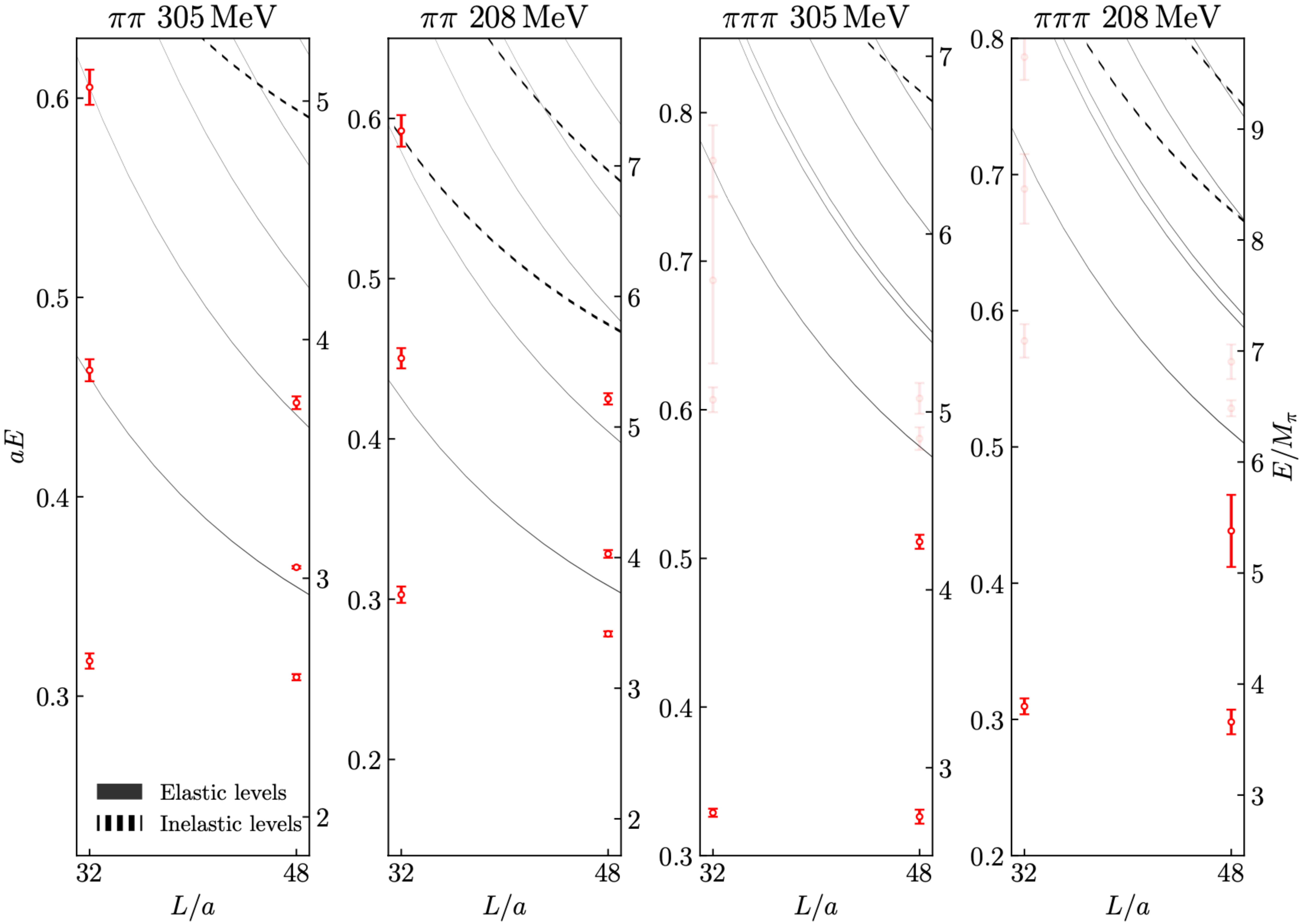
VECTOR MESON

Lattice QCD setup

- Nf = 2 + 1 Clover fermions
  CLQCD, 2024
- 2/3 particle operators
  OpTion package @HaoboYan
- 2 pion masses ($\approx 210, 305$ MeV)
- 2 volumes ($L^3 = 32^3, 48^3$)



Finite-volume spectrum = Energy eigenvalues



H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

VECTOR MESON

Mapping to infinite volume

- 3-body quantization condition

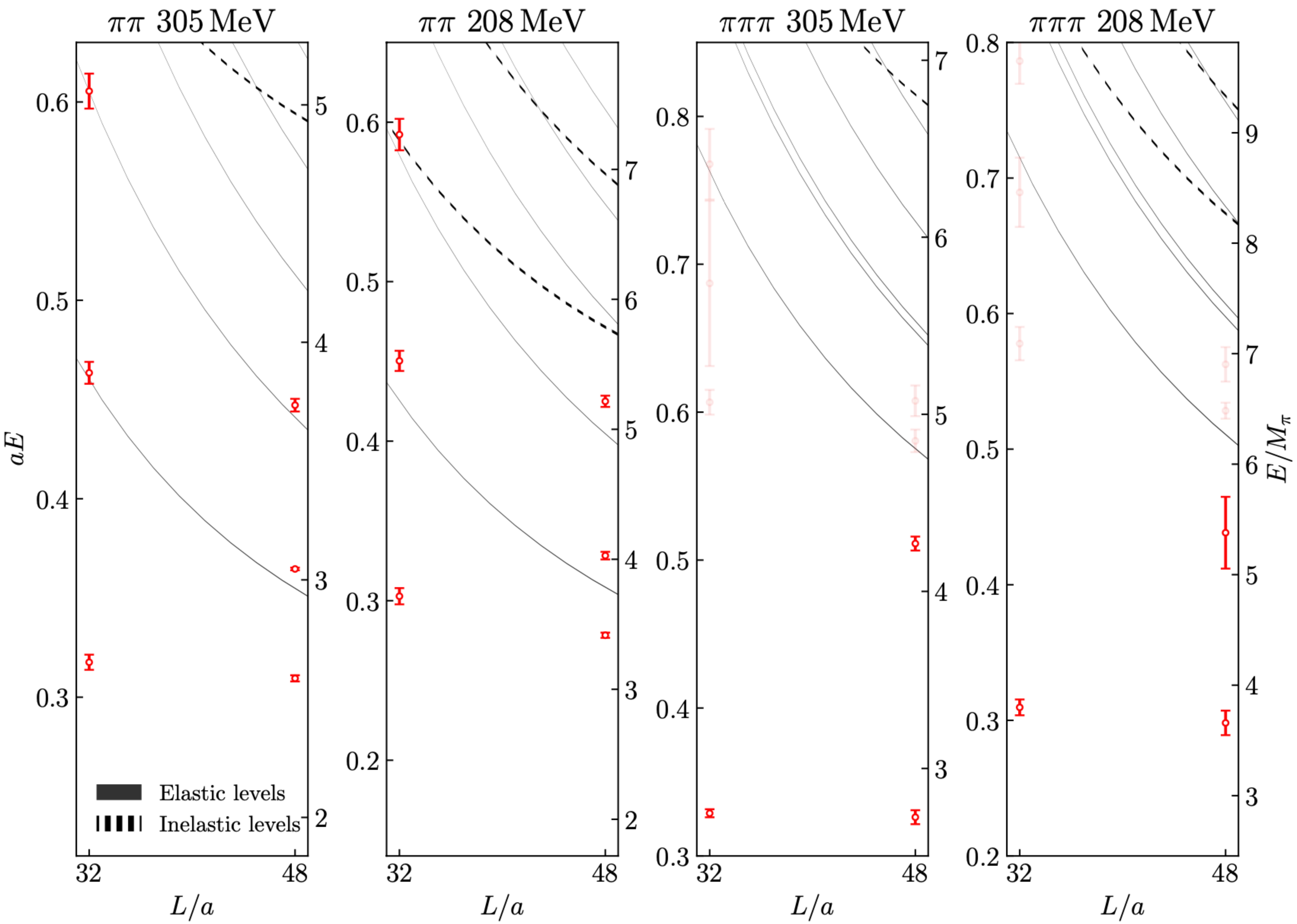
FVU

$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}^{-1}-\Sigma^L\right)-B-C\,\right]^{\Lambda}\equiv 0$$

MM/Döring
Eur.Phys.J.A 53 (2017) 12, 240

- Volume-independent 2-,3-body force $C,\tilde{K}$

Finite-volume spectrum = Energy eigenvalues



H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

VECTOR MESON

Mapping to infinite volume

- 3-body quantization condition

FVU

$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}^{-1}-\Sigma^L\right)-B-C\,\right]^{\Lambda}\equiv 0$$

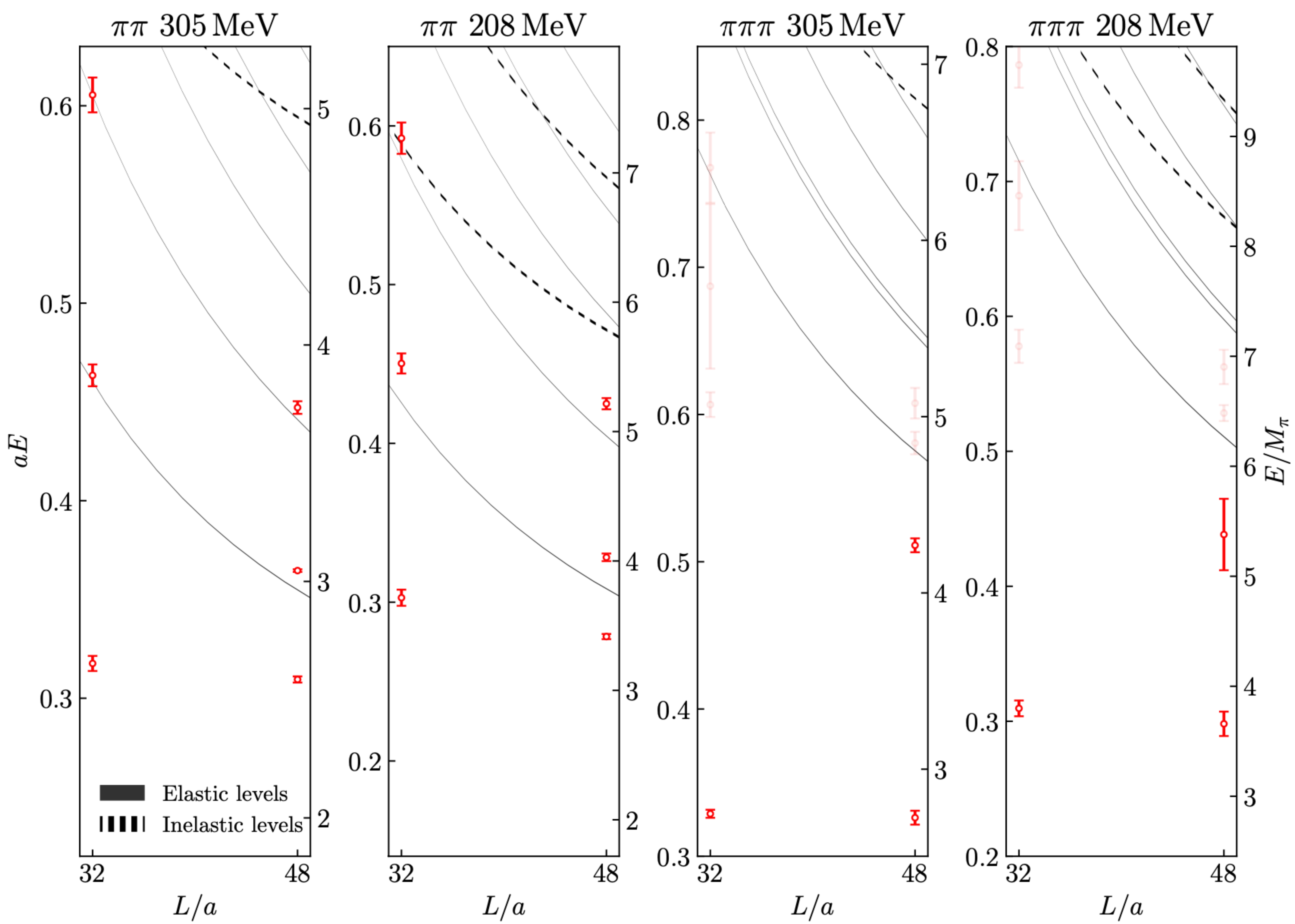
MM/Döring
Eur.Phys.J.A 53 (2017) 12, 240

- Volume-independent 2-,3-body force
  $C, \tilde{K}$
- **Geric form**

$$\tilde{K}(s)^{-1}=\delta_{\vec{p}'\vec{p}}\delta_{\lambda'\lambda}(a_0+a_1\sigma_{\vec{p}}(s))$$

$$c_{11}^{GEN}(c_0,m_\omega,c_1)=\frac{c_0}{s-m_\omega^2}+c_1$$

Finite-volume spectrum = Energy eigenvalues



H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

VECTOR MESON

Mapping to infinite volume

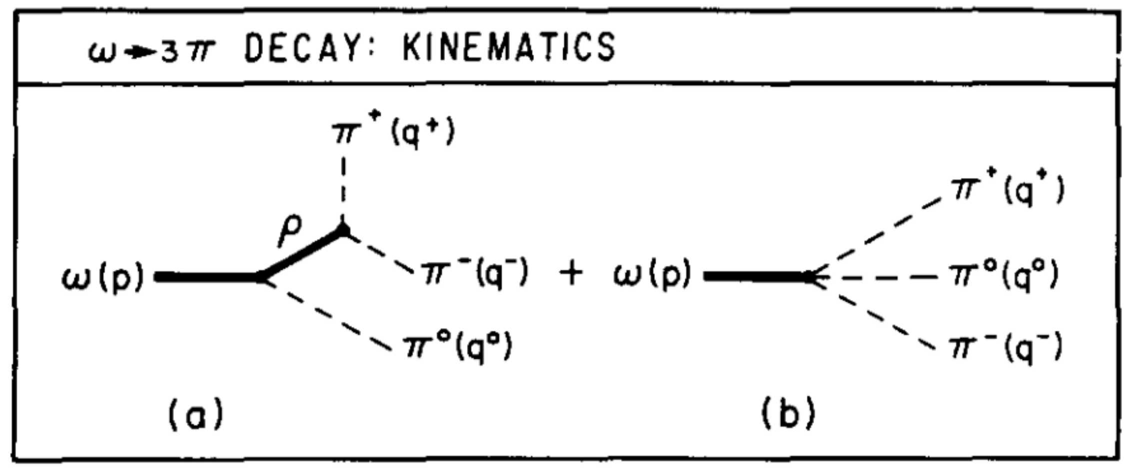
- 3-body quantization condition

FVU

$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Lambda} \equiv 0$$

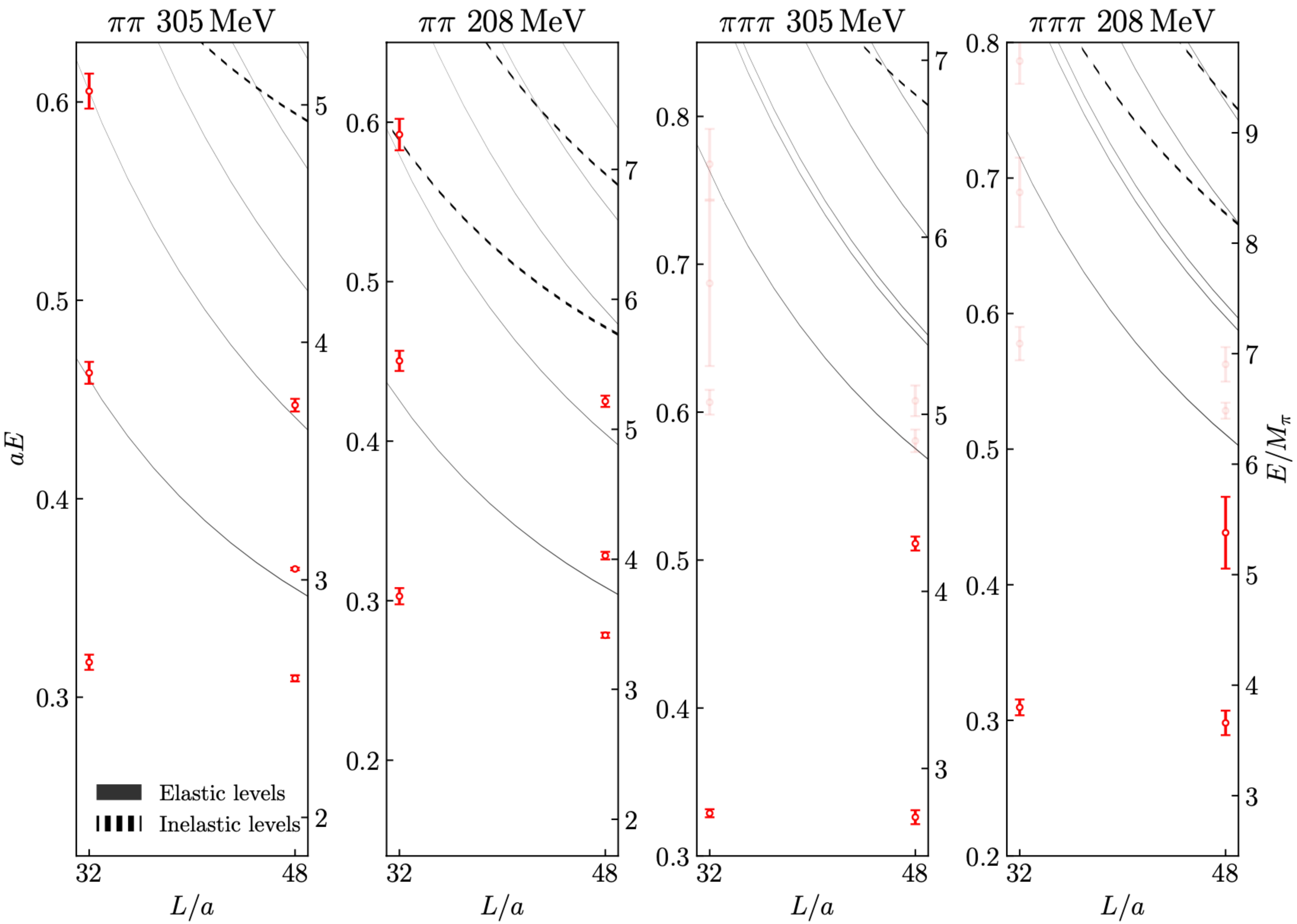
MM/Döring
Eur.Phys.J.A 53 (2017) 12, 240

- Volume-independent 2-,3-body force $C, \tilde{K}$
- saturated by meson s-channel interaction — **EFT form**



Gell-Mann/Sharp/Wagner/Fujiwara/Kawarabayashi…
Review: Meißner *Phys.Rept.* 161 (1988) 213

Finite-volume spectrum = Energy eigenvalues



H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

VECTOR MESON

Mapping to infinite volume

- 3-body quantization condition

FVU

$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}^{-1}-\Sigma^L\right)-B-C\,\right]^{\Lambda}\equiv 0$$

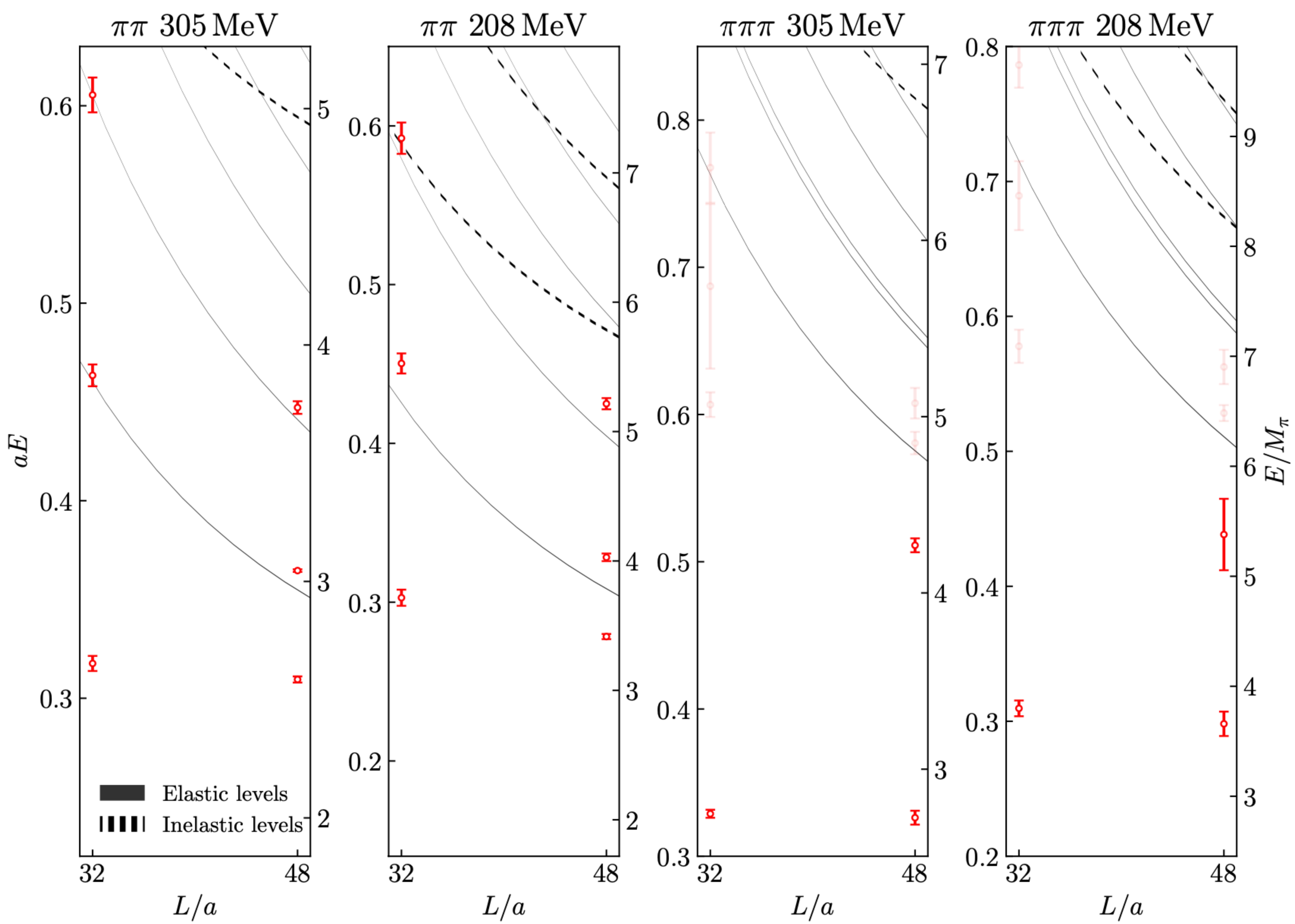
MM/Döring
Eur.Phys.J.A 53 (2017) 12, 240

- Volume-independent 2-,3-body force $C, \tilde{K}$
- saturated by meson s-channel interaction — **EFT form**

$$\left[\tilde{K}^{-1}\right]_{\mathbf{p}'\lambda',\mathbf{p}\lambda}=\delta_{\lambda'\lambda}\delta_{\mathbf{p}'\mathbf{p}}\frac{\sigma_p-M_\rho^2}{2g^2}\,,$$

$$\tilde{c}_{11}=\frac{6s(M_\rho^2-\sigma_q+6g^2f_\pi^2)(M_\rho^2-\sigma_p+6g^2f_\pi^2)}{64g^2\pi^3f_\pi^6(s-M_\omega^2)}\,,$$

Finite-volume spectrum = Energy eigenvalues



H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

VECTOR MESON

Mapping to infinite volume

- 3-body quantization condition

FVU

$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}^{-1}-\Sigma^L\right)-B-C\,\right]^{\Lambda}\equiv 0$$

MM/Döring
Eur.Phys.J.A 53 (2017) 12, 240

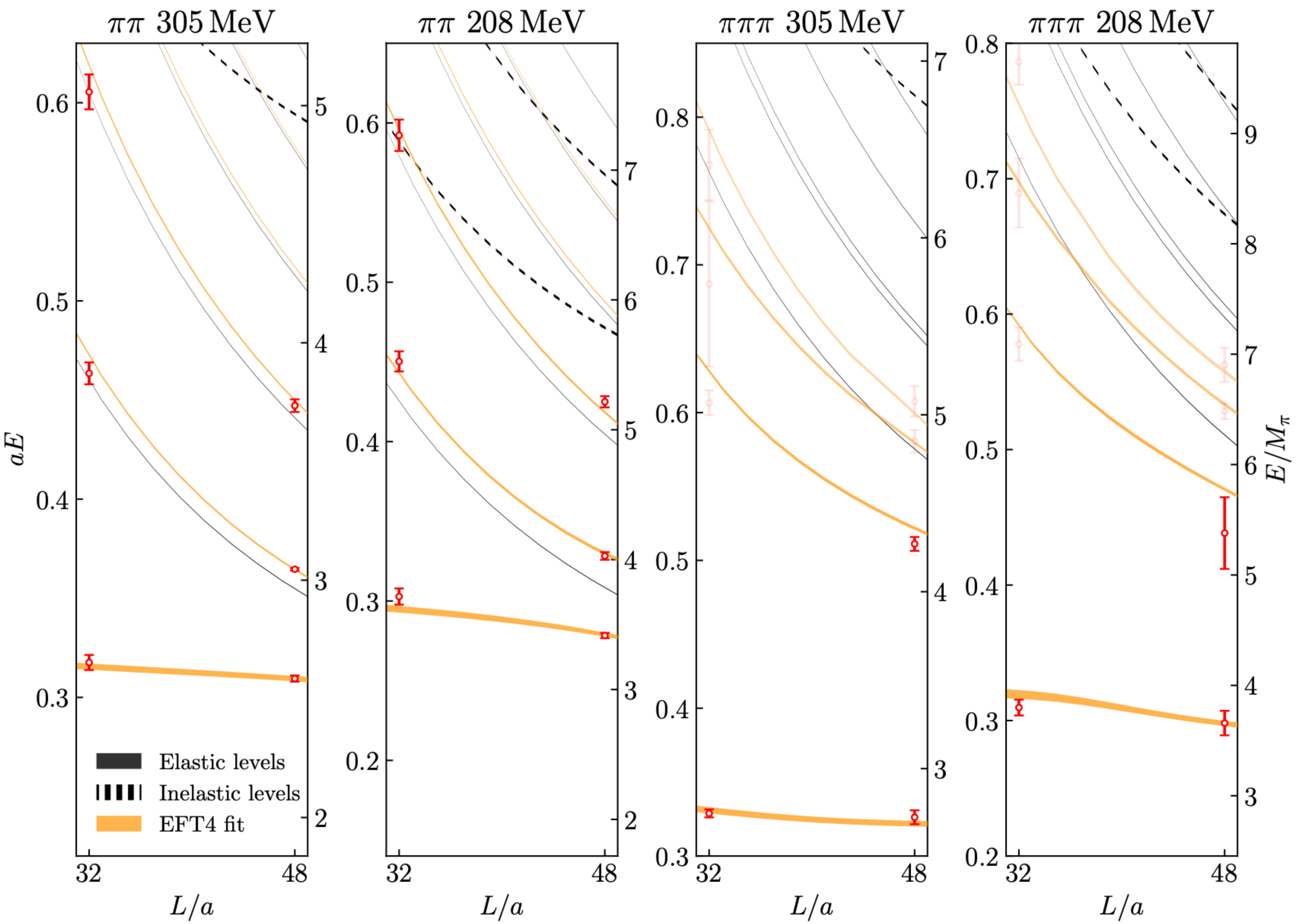
- Volume-independent 2-,3-body force
$C, \tilde{K}$

- saturated by meson s-channel interaction — **EFT form**

$$\left[\tilde{K}^{-1}\right]_{\boldsymbol{p}'\lambda',\boldsymbol{p}\lambda}=\delta_{\lambda'\lambda}\delta_{\boldsymbol{p}'\boldsymbol{p}}\frac{\sigma_p-M_\rho^2}{2g^2}\,,$$

$$\tilde{c}_{11}=\frac{6s(M_\rho^2-\sigma_q+6g^2f_\pi^2)(M_\rho^2-\sigma_p+6g^2f_\pi^2)}{64g^2\pi^3f_\pi^6(s-M_\omega^2)}\,,$$

Finite-volume spectrum = Energy eigenvalues

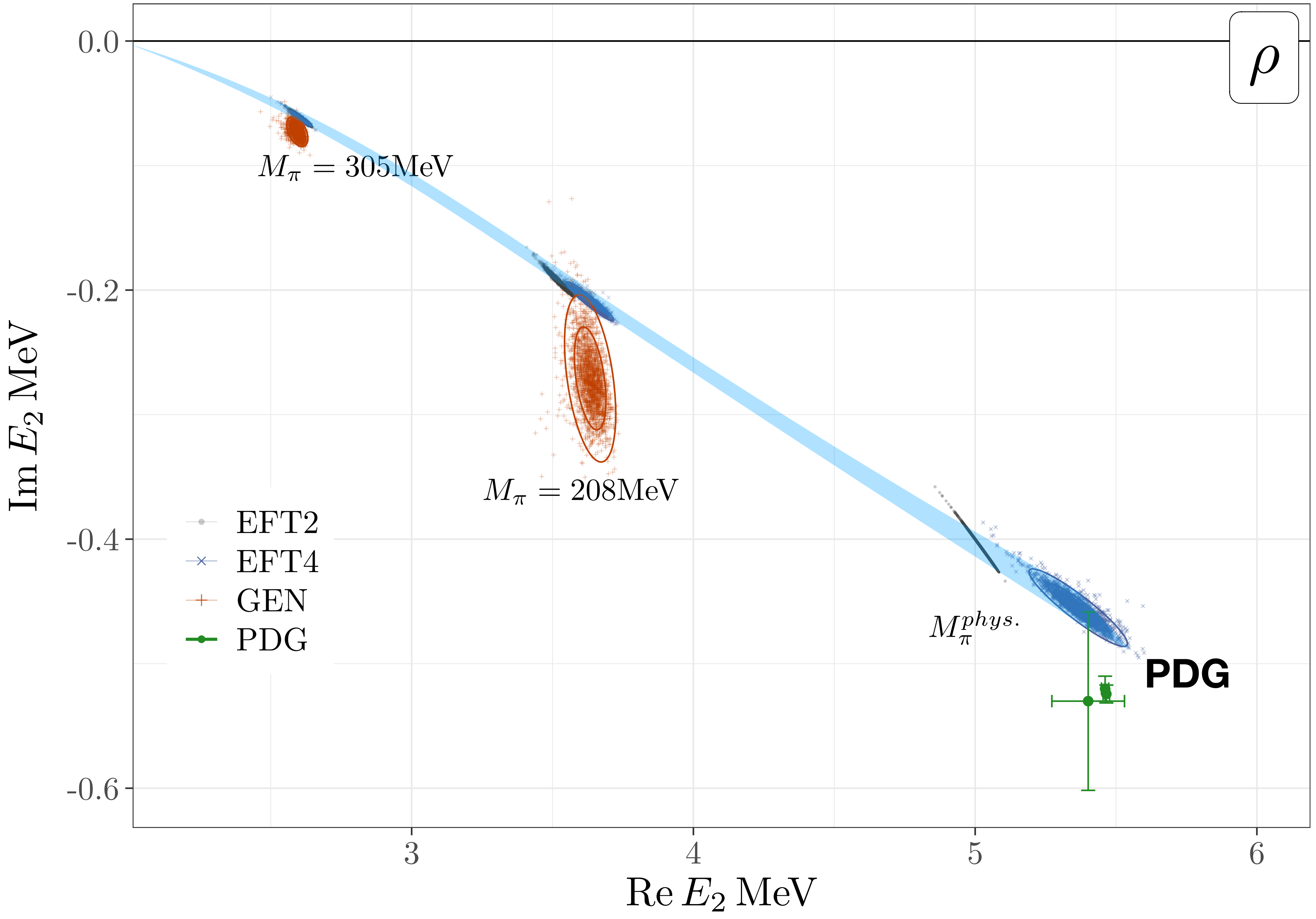


H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

VECTOR MESON

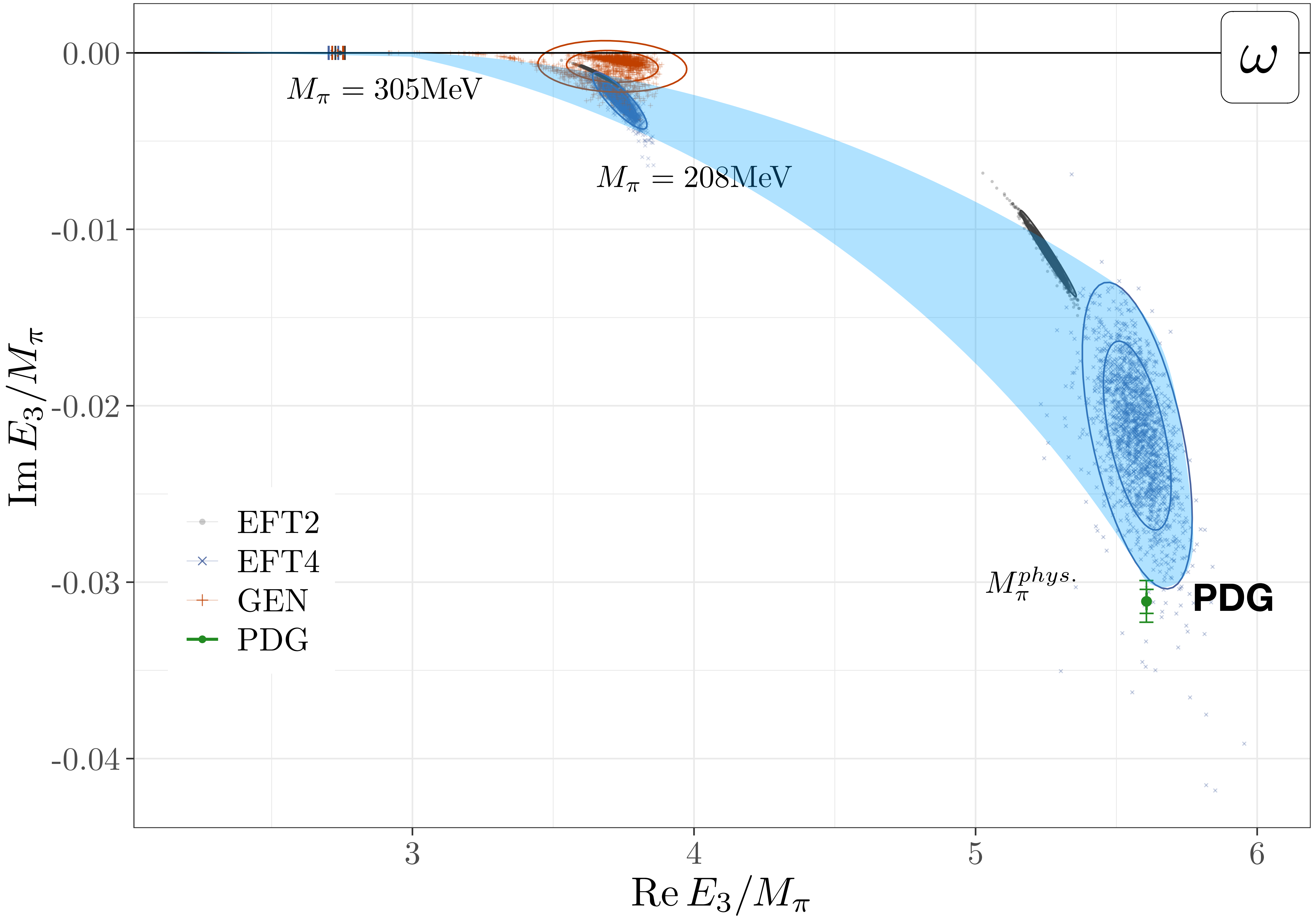
Final step

- Volume-independent 2-,3-body force $C, \tilde{K}$
- solve intergral equation
- analytically continue to the II Riemann Sheet



Result

- GEN/EFT2/EFT4 ansatzes consistent
- $\omega(782)$ becomes abound state at ~300 MeV
- at the physical point very close to the EXP value



H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

SUMMARY

Synergetic approach to hadron spectrum

- **Lattice QCD:** ab-initio QCD calculations
- **EFTs:** quark-mass dependence, symmetries
- **(S-matrix/..):** 3-body Quantization condition

2-body systems/3-body systems

- $f_0(500), \rho(770), \ldots$ well established quark-mass dependence
- $N(1535), N(1650), \Lambda(1405), \Lambda(1380)$
- pilot results on $3\pi(I=2)$, $\omega(782)$, $a_1(1260)$
- chiral trajectories

Outlook — it is just the beginning!

- $DD\pi$ — $N(1440)$ — …spin-exotics? — $a_1(1420)$
- systematic/statistics improvement
- EFT tests — Universality of $\omega \to 3\pi, \rho \to 2\pi$ coupling? — KSFR relation? …
- Cutoff treatment — Gradient flow?