中国科学院高能物理研究所

Institute of High Energy Physics, CAS

中国科学院

CHINESE ACADEMY OF SCIENCES

Lattice QCD studies on charmonium-like states around 3.9 GeV

Ying Chen

Division of Theoretical Physics,

Institute of High Energy Physics,

Chinese Academy of Sciences, China

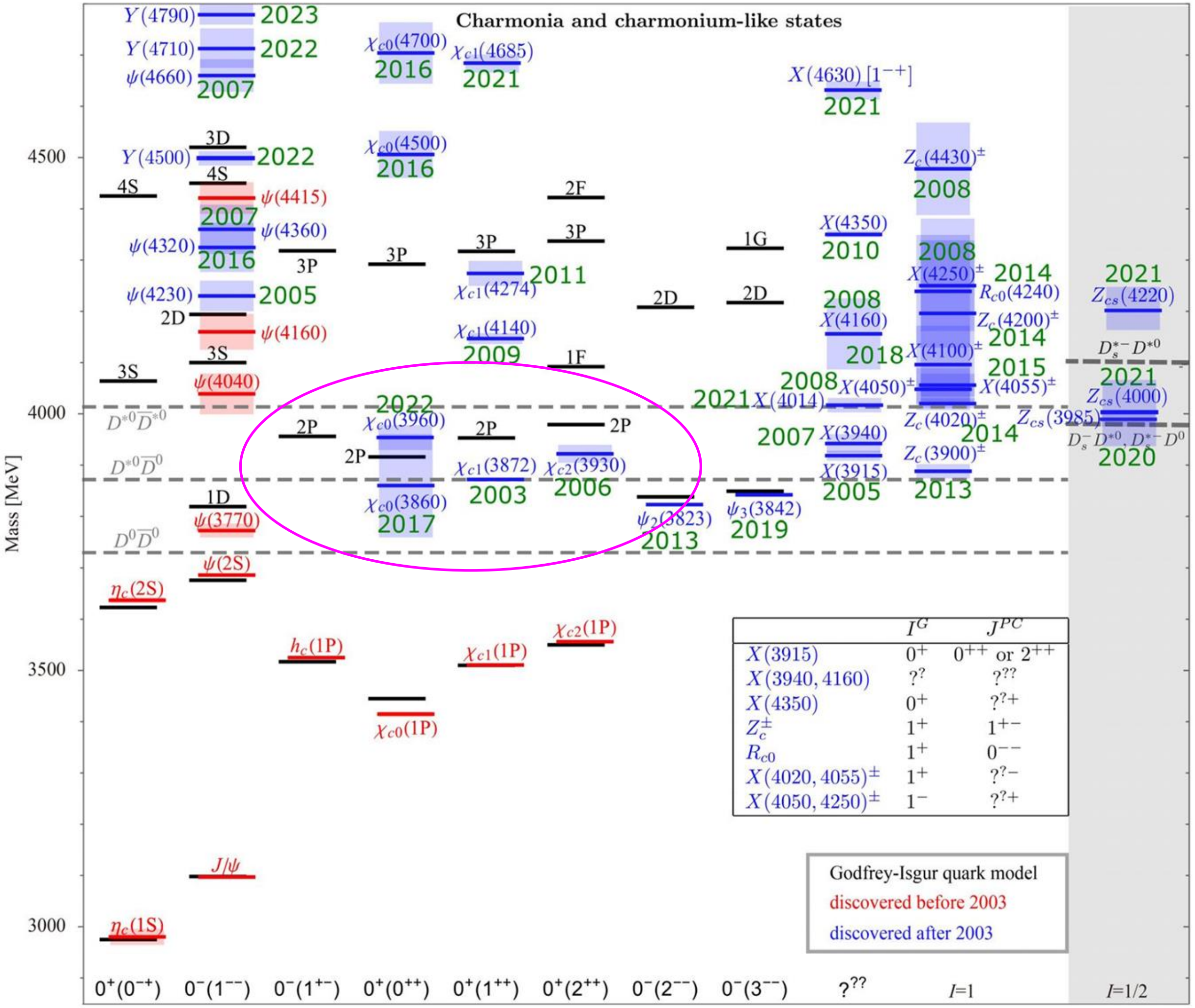
Jun 21, 2024, HAPOF no.99, Beijing

Outline

I. Introduction
II. Methodology for hadron-hadron scattering
III. $X(3872)$ and $\chi_{c1}(2P)$
IV. $(0,2)^{++}$ Charmonium-like resonance
VI. Summary and perspectives

I. Introduction

1. Charmonium(-like) states from experiments vs. Quark Model predictions



Mult.	State	Input mass (NR model)	Theor. mass NR	GI
1S	$J/\psi(1^3S_1)$	3097	3090	3098
	$\eta_c(1^1S_0)$	2979	2982	2975
2S	$\psi'(2^3S_1)$	3686	3672	3676
	$\eta_c'(2^1S_0)$	3638	3630	3623
3S	$\psi(3^3S_1)$	4040	4072	4100
	$\eta_c(3^1S_0)$		4043	4064
4S	$\psi(4^3S_1)$	4415	4406	4450
	$\eta_c(4^1S_0)$		4384	4425
1P	$\chi_2(1^3P_2)$	3556	3556	3550
	$\chi_1(1^3P_1)$	3511	3505	3510
	$\chi_0(1^3P_0)$	3415	3424	3445
	$h_c(1^1P_1)$		3516	3517
2P	$\chi_2(2^3P_2)$		3972	3979
	$\chi_1(2^3P_1)$		3925	3953
	$\chi_0(2^3P_0)$		3852	3916
	$h_c(2^1P_1)$		3934	3956
3P	$\chi_2(3^3P_2)$		4317	4337
	$\chi_1(3^3P_1)$		4271	4317
	$\chi_0(3^3P_0)$		4202	4292
	$h_c(3^1P_1)$		4279	4318

2. Experimental results of $(0,1,2)^{++}$ charmonium-like states (PDG 2024)

- X(3872) (aka $\chi_{c1}(3872)$): $I^G J^{PC} = 0^+ 1^{++}$, $m_X = 3871.64 \pm 0.06$ MeV, $\Gamma_X = 1.19 \pm 0.21$ MeV
  X(3872) decay channels:
  $D^0 \bar{D}^{0*}$ $(34(12)\%)$, $J/\psi\omega$ $(4.3(1.2)\%)$, $\gamma J/\psi$ $(0.8(3)\%)$, $\gamma\psi(2S)$ [$4.1(1.4)\%$ (PDG2022)]
  Isospin violating: $J/\psi\rho$ ($J/\psi\pi^+\pi^-, 3.5(9)\%$), $\pi^0\chi_{c1}$ $\left((3.1^{+1.5}_{-1.3})\%\right)$

- X(3860) (aka $\chi_{c0}(3860)$) $I^G J^{PC} = 0^+ 0^{++}$ (preferred), $m_X = 3860^{+26+40}_{-32-13}$MeV, $\Gamma_X = 201^{+154+88}_{-67-82}$ MeV
  X(3860) decay channels: $D^0\bar{D}^0$, D^+D^- (seen by Belle (2017), not seen by LHCb (2020))

- X(3915) (aka $\chi_{c0}(3915)$) : $I^G J^{PC} = 0^+ 0^{++}$, $m_X = 3922.1 \pm 1.8$ MeV, $\Gamma_X = 20 \pm 4$ MeV
  X(3915) decay channels: D^+D^-, $D_s^+D_s^-$, $\omega J/\psi$, $\gamma\gamma$
  $\Gamma(D^+D^-)/\Gamma(D_s^+D_s^-) = 0.29 \pm 0.09 \pm 0.10 \pm 0.08$

- $\chi_{c2}(3930)$: $I^G J^{PC} = 0^+ 2^{++}$, $m_X = 3922.5 \pm 1.0$ MeV, $\Gamma_{\chi_{c2}} = 35.2 \pm 2.2$ MeV
  X(3915) decay channels: D^+D^-, $D^0\bar{D}^0$, $\gamma\gamma$
  $\Gamma(D^+D^-)/\Gamma\left(D^0\bar{D}^0\right) = 0.74 \pm 0.43 \pm 0.16$

- X(3940) : $I^G J^{PC} =?^? ?^{??}$, $m_X = 3942^{+7}_{-6} \pm 6$ MeV, $\Gamma_X = 37^{+26}_{-15} \pm 8$ MeV
  X(3940) decay channels: $D\bar{D}^*$ ($e^+e^- \to J/\psi X$, Belle 2007, 2008)

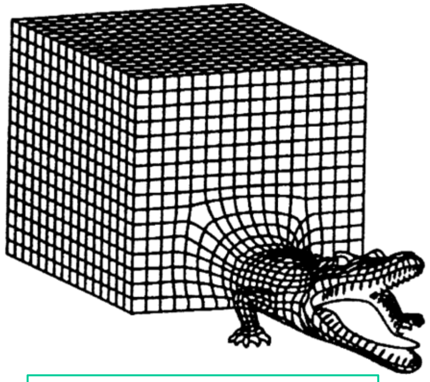
3. Formalism of Lattice QCD

- **Path integral quantization on finite Euclidean spacetime lattices**

$$Z = \int DA D\psi D\bar{\psi}\, e^{iS[A,\psi,\bar{\psi}]} \rightarrow \int DU \det M[U]\ e^{-S_g[U]}$$

$$\langle \hat{\mathcal{O}}[U,\psi,\bar{\psi}] \rangle = \frac{1}{Z}\int DU\ \det M[U]\ e^{-S_g[U]}\ \mathcal{O}[U]$$

Green's functions **Product of fields**



Spacetime discretization

QCD拉氏量

大规模数值模拟（Monte Carlo重点抽样）

产生QCD物质的状态（组态）

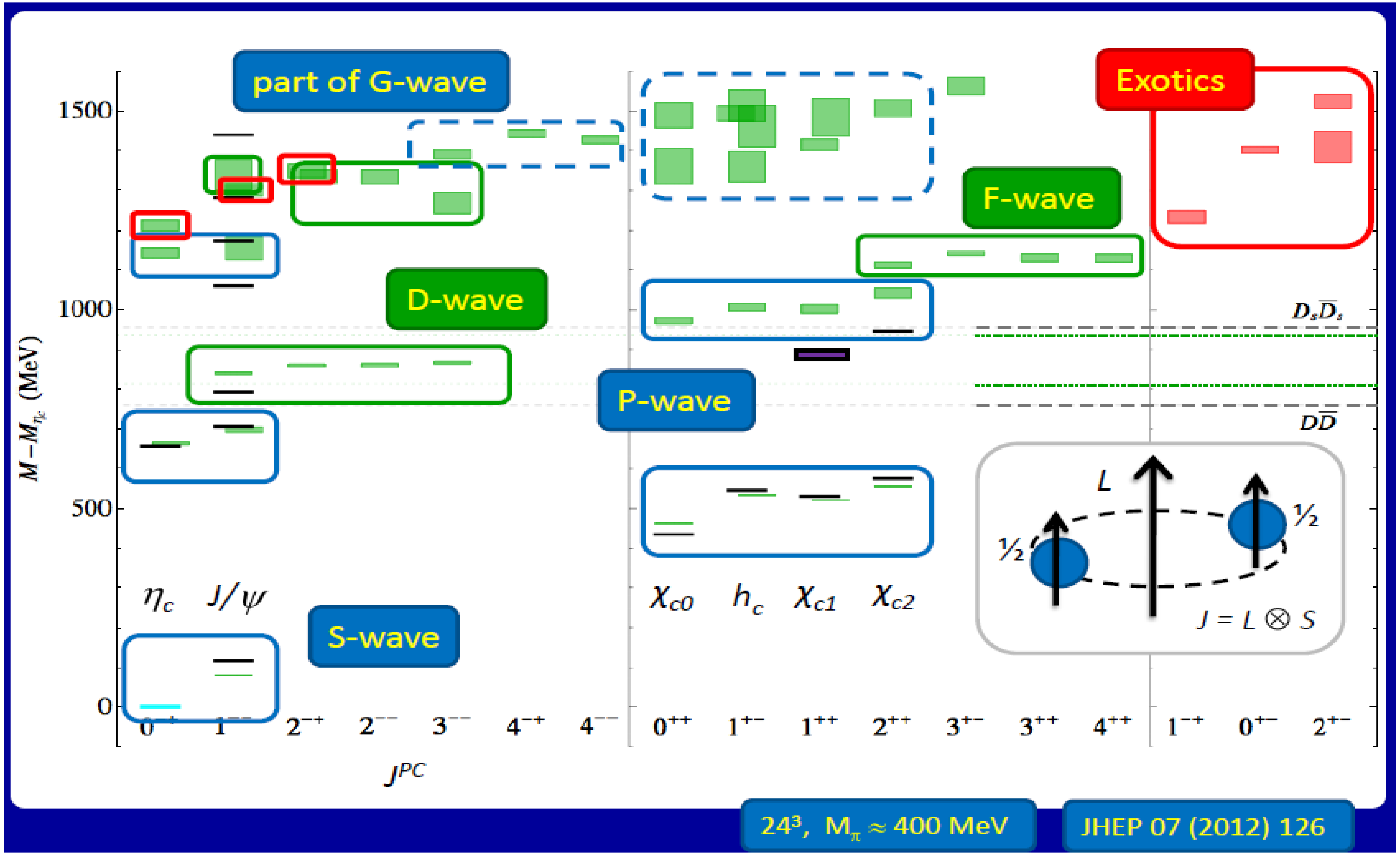
物理观测量的统计平均

数据分析，得到最终物理结果

- **Very similar to a statistical physics system**
- **Monte Carlo simulation——importance sampling according to** $\mathcal{P}[U] \propto \det M[U]\ e^{-S_g[U]}$

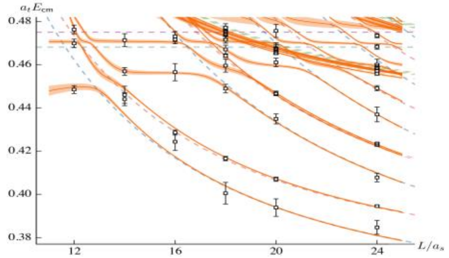
Gauge ensemble: $\{U_i(\text{spacetime}), i = 1, \dots, N\}$ $\Rightarrow$ $\langle \hat{\mathcal{O}}[U,\psi,\bar{\psi}] \rangle = \frac{1}{N}\sum_i \mathcal{O}[U_i] + O\left(\frac{1}{\sqrt{N}}\right)$

4. Spectrum of charmonium-like states from lattice QCD (only $c\bar{c}(g)$ operators)



Courtesy to S. Prelovsek

II. Lattice Methodology for Hadron-hadron Scattering



1. State-of-art Approach——Lüscher's formalism

(see R. Briceno et al., Rev. Mod. Phys. 90 (2018) 025001 for a review).

$$\det\left[F^{-1}\left(\vec{P}, E, L\right) + \mathcal{M}(E)\right] = 0$$

$E_n(L)$: Eigen-energies of lattice Hamiltonian.

- **Interpolation field operator set for a given J^{PC}**

  $\mathcal{O}_i$: $\bar{q}_1\Gamma q_2$ $[\bar{q}_1\Gamma_1 q][\bar{q}\Gamma_2 q_2]$ $[q_1^T\Gamma_1 q][\bar{q}\Gamma_2\bar{q}_2^T]$, ...
- **Correlation function matrix —— Observables**

$$C_{ij}(t)\& = \left\langle\Omega\middle|\mathcal{O}_i(t)\mathcal{O}_j^+(0)\middle|\Omega\right\rangle$$

$$= \sum_n \langle\Omega|\mathcal{O}_i|n\rangle\left\langle n\middle|\mathcal{O}_j^+\middle|\Omega\right\rangle e^{-E_n t}$$

All the energy levels $E_n(L)$ are discretized.

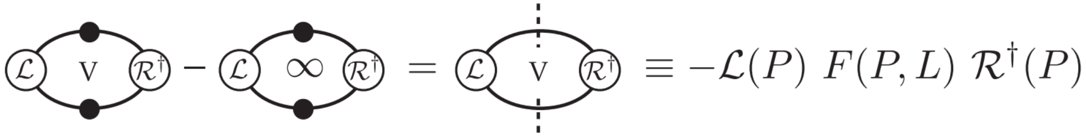
$\mathcal{M}(E)$: Scattering matrix.

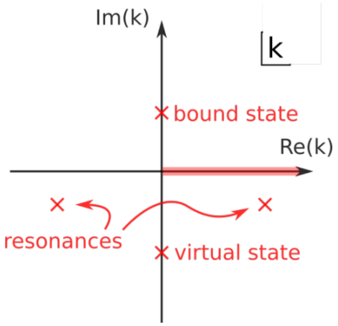
- Unitarity requires

$$\mathcal{M}_{ab}^{-1} = (\mathcal{K}^{-1})_{ab} - i\delta_{ab}\frac{2q_a^*}{E_{cm}}$$

- $\mathcal{K}$ is a real function of s for real energies above kinematic threshold.
- The pole singularities of $\mathcal{M}(s)$ in the complex s-plane correspond to bound states, virtual states, resonances, etc..

$F\left(\vec{P}, E, L\right)$: Mathematically known function matrix in the channel space (the explicit expression omitted





$$k = \frac{1}{2}\sqrt{s - 4m^2}$$

Comparison of the hadron spectra

Euclidean spacetime lattice

One particle states

Multiple particle states
with discrete relative spatial
Momentum (scattering
States in a finite volume

All the energies are
Discretized.

Minkowski continuum spacetime

Stable particles

Bound states of hadrons

Resonances

Continuum scattering states

Luescher's Relation:

$$E_n = (m_1^2 + p^2)^{1/2} + (m_2^2 + p^2)^{1/2}$$

$$\tan\delta(p) = \frac{\sqrt{\pi}\ p\ L}{2\ \mathcal{Z}_{00}\left(1; (\frac{pL}{2\pi})^2\right)}$$

Resonances

$$T(p) = \frac{-\sqrt{s}\,\Gamma(p)}{s - m_R^2 + i\sqrt{s}\,\Gamma(p)} = \frac{1}{\cot\delta(p) - i}$$

$$\Gamma(p) = g^2\,\frac{p^{2l+1}}{s}\,, \quad \frac{p^{2l+1}}{\sqrt{s}}\cot\delta(p) = \frac{1}{g^2}(m_R^2 - s)$$

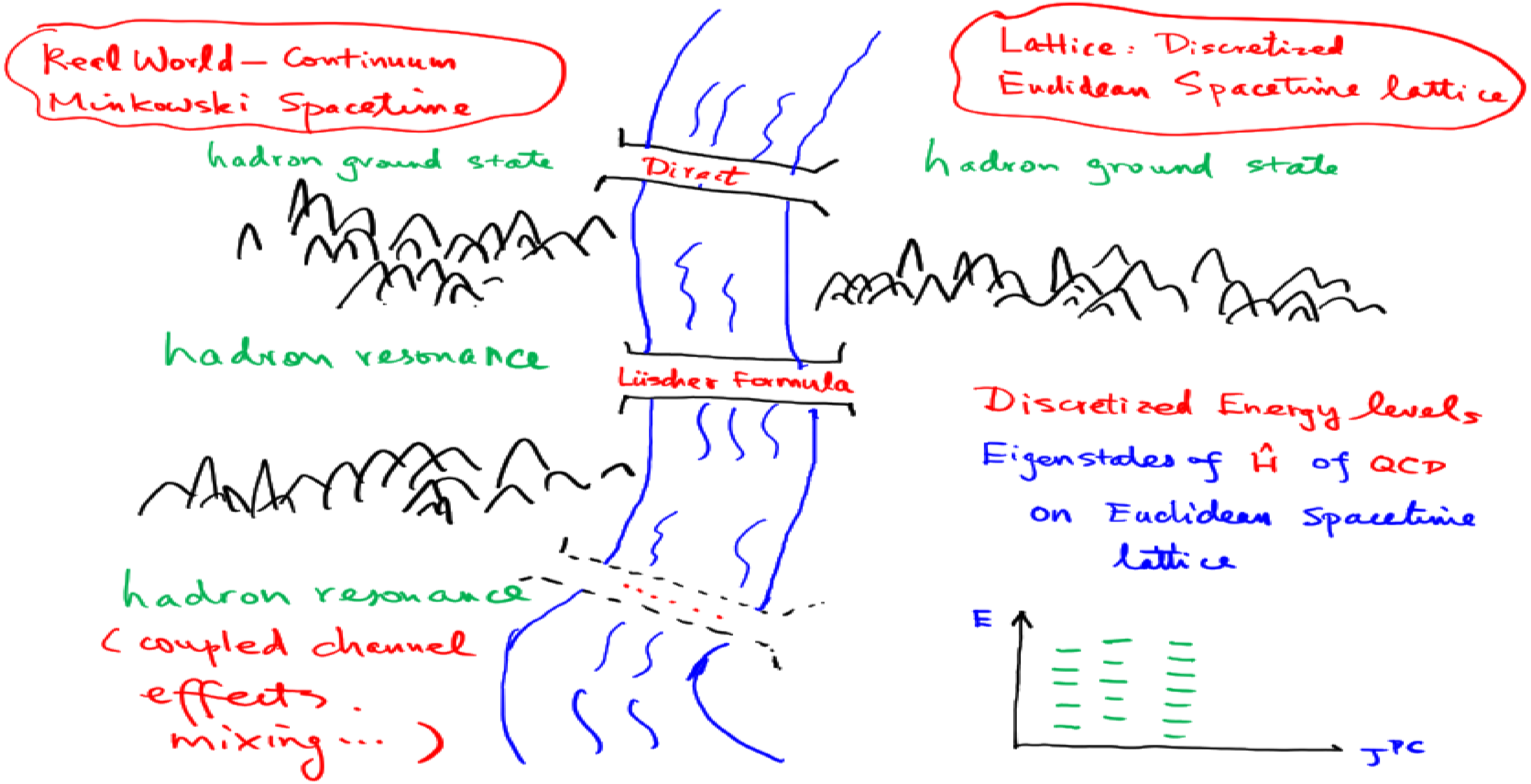
Bound states

$$p\cot(\delta_0(p)) = \frac{1}{a_0} + \frac{1}{2}r_0\,p^2\,, \quad -|p_B| = \frac{1}{a_0} - \frac{1}{2}r_0\,|p_B|^2$$

$$T = \frac{}{\cot(\delta_l(p_B)) - i} = \infty$$

$$m_B = E_{H_1}(p_B) + E_{H_2}(p_B)\,,\ p_B = i|p_B|$$

2. Present status of lattice QCD study on hadron spectroscopy



J^{PC}	hadron-hadron channels below **4.1 GeV**
0^{-+}	$\eta_c f_0$, $\chi_{c0}\eta$ $\{^1S_0\}$; $\psi\omega$, $D\bar{D}^*$, $D^*\bar{D}^*$ $\{^3P_0\}$; $\chi_{c2}\eta\{^5D_0\}$;
0^{++}	$\eta_c\eta$, $D\bar{D}$, $\eta_c\eta'$, $D_s\bar{D}_s$, $\psi\omega$, $D^*\bar{D}^*$, $\psi\phi$ $\{^1S_0\}$; $\chi_{c1}\eta$ $\{^3P_0\}$; $\psi\omega$, $D^*\bar{D}^*$, $\psi\phi$ $\{^5D_0\}$;
1^{-+}	$\eta_c\eta$, $\eta_c\eta'$, $\psi\omega$, $\psi\phi$ $\{^1P_1\}$; $\psi\omega$, $D\bar{D}^*$, $D^*\bar{D}^*$, $\psi\phi$ $\{^3P_1\}$; $\psi\omega$, $\psi\phi$ $\{^5P_1\}$; $\chi_{c1}\eta$ $\{^3S_1\}$; $\chi_{c1}\eta$ $\{^3D_1\}$; $\chi_{c2}\eta$ $\{^5D_1\}$;
1^{++}	$\psi\omega$, $D\bar{D}^*$ $\{^3S_1\}$; $\psi\omega$, $D\bar{D}^*$ $\{^3D_1\}$; $\psi\omega$, $D^*\bar{D}^*$ $\{^5D_1\}$; $\chi_{c0}\eta$, $\eta_c f_0$ $\{^1P_1\}$; $\chi_{c1}\eta$ $\{^3P_1\}$; $\chi_{c2}\eta$ $\{^5P_1\}$;
2^{-+}	$\psi\omega$, $D\bar{D}^*$, $D_s\bar{D}_s^*$, $\psi\phi$ $\{^3P_2\}$; $\chi_{c2}\eta$ $\{^5S_2\}$; $\chi_{c1}\eta$ $\{^3D_2\}$; $\chi_{c2}\eta$ $\{^5D_2\}$; $\eta_c f_0$ $\{^1D_2\}$
2^{++}	$\psi\omega$, $D^*\bar{D}^*$, $\psi\phi$, $D_s^*\bar{D}_s^*$ $\{^5S_2\}$; $\eta_c\eta$, $D\bar{D}$, $\eta_c\eta'$, $D_s\bar{D}_s$, $\psi\omega$, $D^*\bar{D}^*$, $\psi\phi$ $\{^1D_2\}$;
	$\psi\omega$, $D\bar{D}^*$, $D_s\bar{D}_s^*$, $D^*\bar{D}^*$, $\psi\phi$ $\{^3D_2\}$; $\chi_{c1}\eta$ $\{^3P_2\}$; $\chi_{c2}\eta$ $\{^5P_2\}$;
3^{-+}	$\psi\omega$, $\psi\phi$ $\{^5P_3\}$; $\chi_{c1}\eta$ $\{^3D_3\}$; $\chi_{c2}\eta$ $\{^5D_3\}$; $\eta_c\eta$, $\eta_c\eta'$, $\psi\omega$, $\psi\phi$ $\{^1F_3\}$; $\psi\omega$, $D\bar{D}^*$, $D^*\bar{D}^*$, $\psi\phi$ $\{^3F_3\}$; $\psi\omega$, $\psi\phi$ $\{^5F_3\}$;
3^{++}	$D\bar{D}^*$, $D_s\bar{D}_s^*$, $\psi\omega$, $\psi\phi$ $\{^3D_3\}$; $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $\psi\omega$, $\psi\phi$ $\{^5D_3\}$;
	$\eta_c f_0$ $\{^1F_3\}$; $\chi_{c2}\eta$ $\{^5P_3\}$; $\chi_{c0}\eta$ $\{^1F_3\}$; $\chi_{c1}\eta$ $\{^3F_3\}$; $\chi_{c2}\eta$ $\{^5F_3\}$;
4^{-+}	$\psi\omega$, $D\bar{D}^*$, $D^*\bar{D}^*$, $D_s\bar{D}_s^*$ $\{^3F_4\}$;
4^{++}	$\psi\omega$, $D^*\bar{D}^*$, $\psi\phi$, $D_s^*\bar{D}_s^*$ $\{^5D_4\}$; $\chi_{c1}\eta\{^3F_4\}$; $\chi_{c2}\eta\{^5F_4\}$;

III. $X(3872)$ and $\chi_{c1}(2P)$

H. Li, C. Shi, YC et al, arXiv: 2402.14541(hep-lat)

1. Our lattice setup

- Anisotropic lattice ($\xi = \frac{a_s}{a_t} \approx 5$): heavy particles ($J/\psi$) involved, compromise of the resolution in the time direction and the computational expenses.

- Lattice actions: Tadpole improved Symanzik's gauge action (C. Morningstar, PRD60(1999)034509)
  Tadpole impoved Clover action for $N_f = 2$ degenerate u, d sea quarks
  and also the valence charm quark.

ens.	m_π (MeV)	a_t^{-1}	$N_{\rm cfg}$	$N_V^{(l)}$	$N_V^{(c)}$	$m_{\chi_{c1}}$ (MeV)	m_D (MeV)	m_{D^*} (MeV)	$E_{D\bar{D}^*}^{q=0}$ (MeV)	$E_D^{q=1}$ (MeV)	$E_{D^*}^{q=1}$ (MeV)	$E_{D\bar{D}^*}^{q=1}$ (MeV)
M245	250(3)	7.276	401	70	120	3489(3)	1873(1)	1985(2)	3858(3)	1958(2)	2064(4)	4022(5)
M305	307(2)	7.187	401	70	120	3496(2)	1881(1)	1990(2)	3871(3)	1962(2)	2070(2)	4032(4)
M360	362(1)	7.187	401	70	120	3502(2)	1884(1)	2003(2)	3888(2)	1970(1)	2084(3)	4054(4)
M415	417(1)	7.219	401	70	120	3509(2)	1896(1)	2017(1)	3913(2)	1978(1)	2094(2)	4072(3)

H. Li et al, arXiv: 2402.14541(hep-lat)

- Calculation of disconnected diagrams: distillation method (M. Peardon et al. (HSC), PRD80(2009)054506).

2. Lattice operators

- Flavor structure of $D\bar{D}^*$ with quantum numbers $I^G J^{PC} = 0^+1^{++}$

$$|D\bar{D}^*\rangle^{Q=0}_{I=0} = \frac{1}{2}\left(|D^+\bar{D}^{*-}\rangle + |D^0\bar{D}^{*0}\rangle - |\bar{D}^0 D^{*0}\rangle - |D^- D^{*+}\rangle\right)$$

- Two-particle operators with a relative momentum $\vec{q}$

$$\mathcal{O}^q_{AB} = \frac{1}{N_q}\sum_{R\in O}\mathcal{O}_A(R\circ\mathbf{q})\mathcal{O}_B(-R\circ\mathbf{q})$$

- Several spatially extended $c\bar{c}$ operators with the same quantum numbers

$$\mathcal{O}^r_{c\bar{c}}(t) = \frac{1}{N_r}\sum_{|\mathbf{y}-\mathbf{x}|=r}\bar{c}(\mathbf{x},t)\gamma_5\gamma_i K_U(\mathbf{x},\mathbf{y};t)c(\mathbf{y},t) \qquad K_U(\mathbf{x},\mathbf{y};t) = \mathcal{P}e^{ig\int_{\mathbf{y}}^{\mathbf{x}}\mathbf{A}\cdot dr}$$

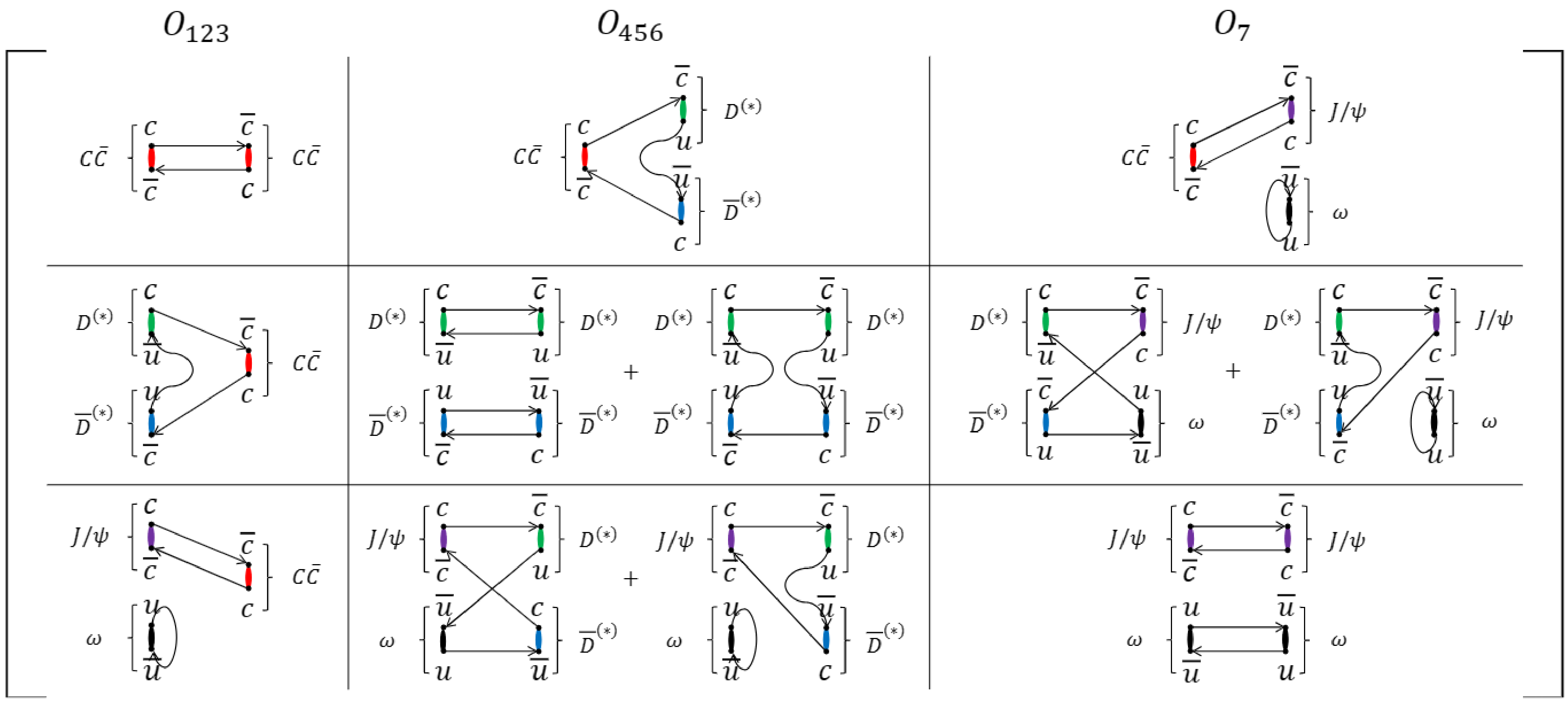
- Our operator set for charmonium-like systems with $I^G J^{PC} = 0^+1^{++}$

$$\boxed{\mathcal{S} = \{\mathcal{O}_\alpha|\alpha = 1,\cdots,7\} = \{\mathcal{O}^{r=0}_{c\bar{c}}, \mathcal{O}^{r=1}_{c\bar{c}}, \mathcal{O}^{r=2}_{c\bar{c}}, \mathcal{O}^{\mathbf{q}=0,\gamma_5}_{D\bar{D}}, \mathcal{O}^{q=0,\gamma_4\gamma_5}_{D\bar{D}}, \mathcal{O}^{q=1}_{D\bar{D}}, \mathcal{O}^{q=0}_{J/\psi\omega}\}}$$

- Finite volume energy levels can be obtained by solving the generalized eigenvalue problem

$$C_{ij}(t_1)v^{(n)}_j(t_1,t_0) = \lambda^{(n)}(t_1,t_0)C_{ij}(t_0)v^{(n)}_j(t_1,t_0)$$

3. Schematic quark diagrams for the correlation matrix $C_{ij}(t)$ after Wick's contraction



3. Effective energy plateaus from the GEVP analysis

a) $\boldsymbol{c\bar{c} - D\bar{D}^* - J/\psi\omega}$ operators.

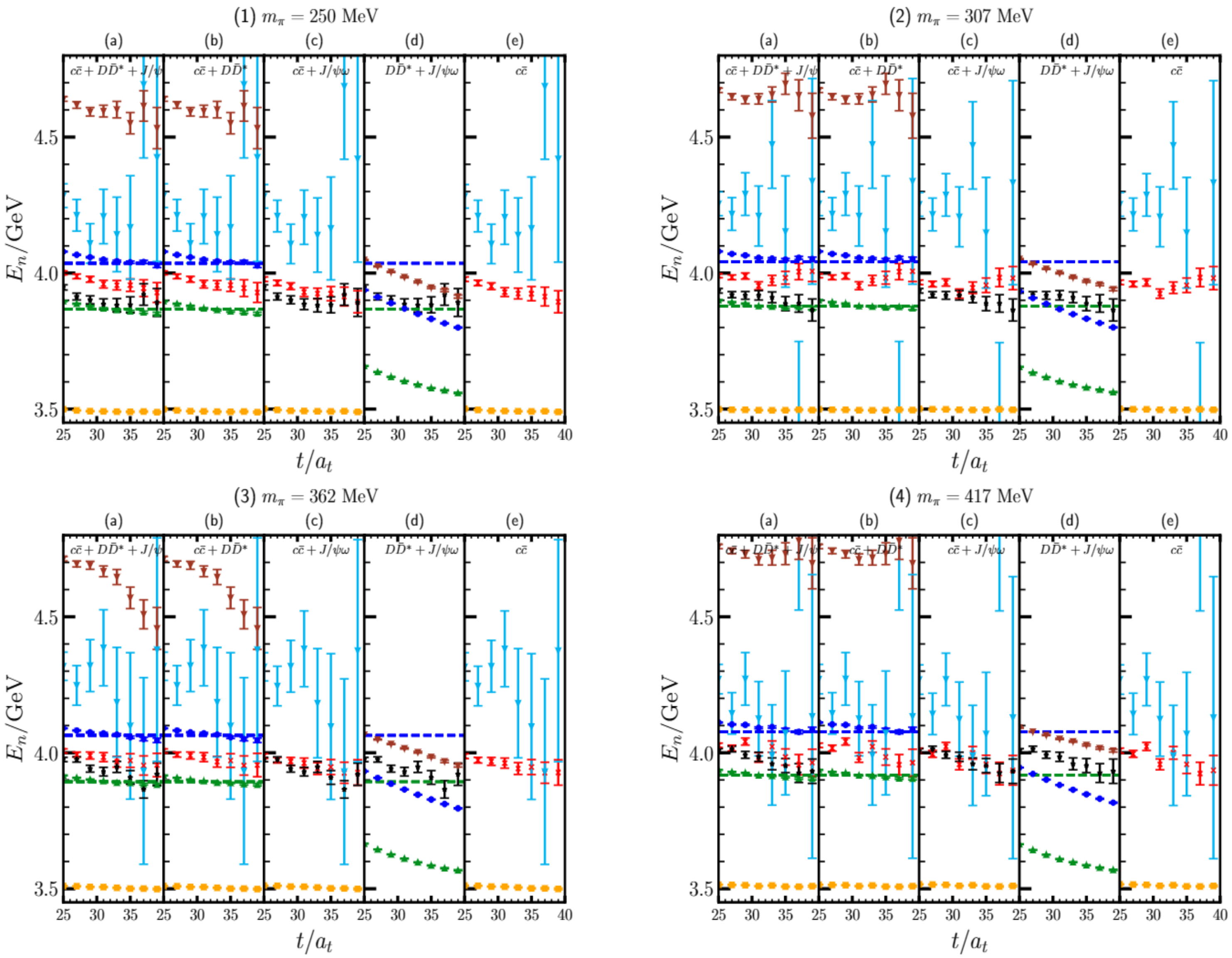
b) $\boldsymbol{c\bar{c} - D\bar{D}^*}$: black points (correspond to $J/\psi\omega$ state) disappear.

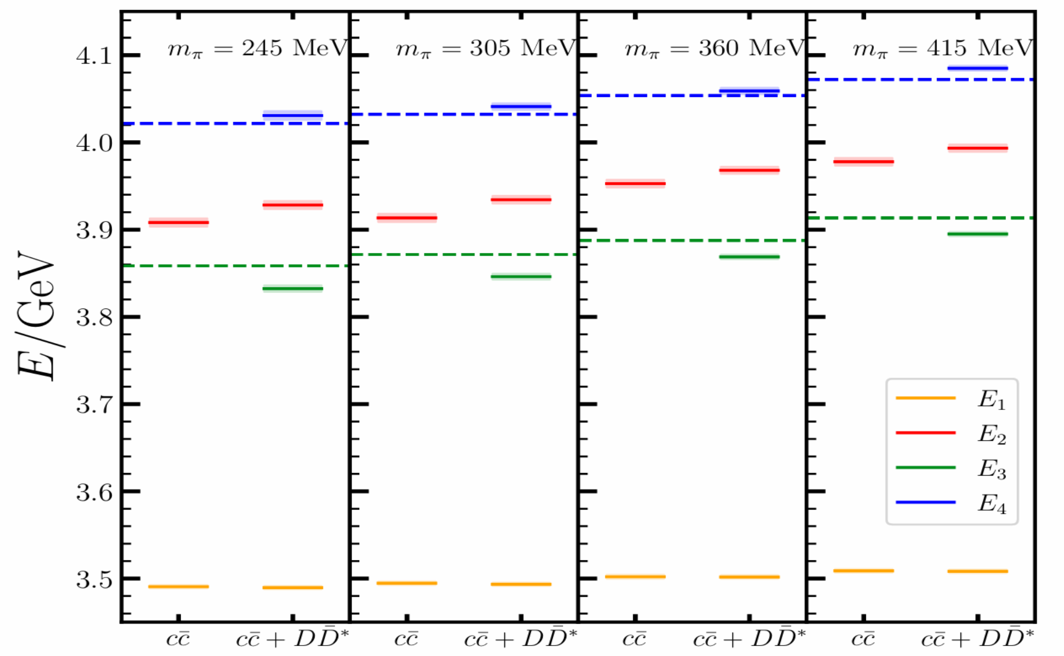
c) $\boldsymbol{c\bar{c} - J/\psi\omega}$: Energy levels close to non-interacting $D\bar{D}^*$ energies disappear.

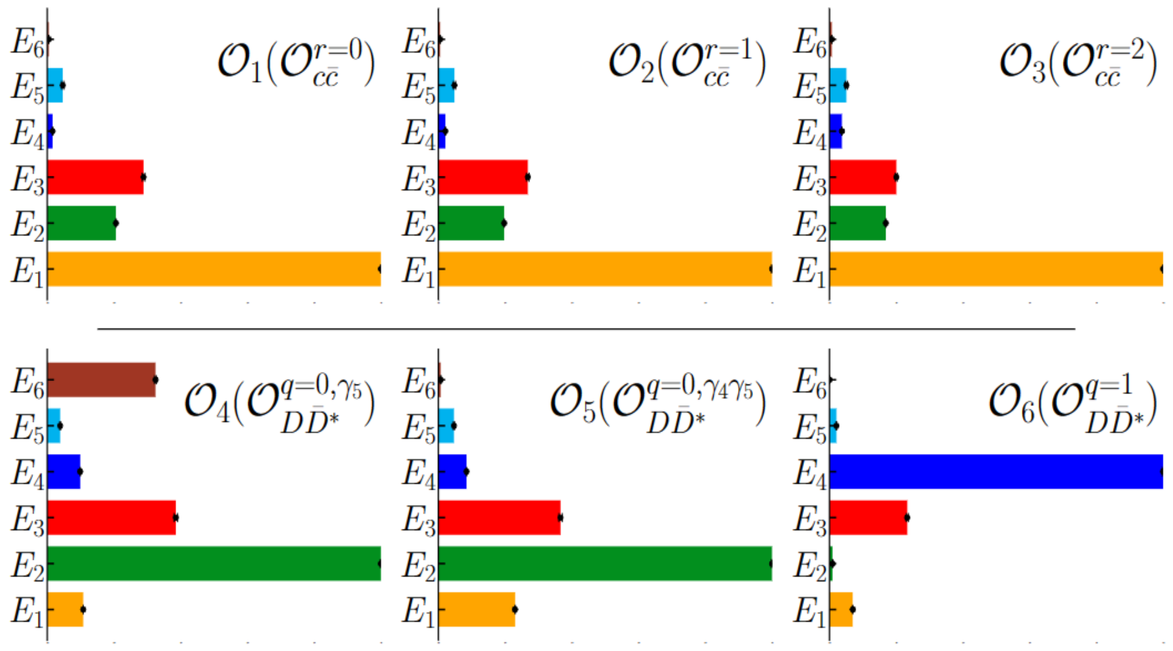
d) $\boldsymbol{D\bar{D}^* - J/\psi\omega}$: Energy levels close to non-interacting $D\bar{D}^*$ and $J/\psi\omega$ states.

e) $\boldsymbol{c\bar{c}}$: Energy levels close to χ_{c1} states.

f) In summary: In all the cases, $J/\psi\omega$ energy has no sizable changes w/o $c\bar{c}$ and $D\bar{D}^*$ operators. So $J/\psi\omega$ almost decouples from other states and is neglected from the discussion in this work.







Identify the energy levels:

- $\boldsymbol{E_1}$: around 3.5 GeV, should be χ_{c1}.
- $\boldsymbol{E_2}$: close but below the $D\bar{D}^*$ threshold.
- $\boldsymbol{E_3}$: far from and in middle of the non-interacting $D\bar{D}^*$ energies $E_{D\bar{D}^*}^{q=0}$ and $E_{D\bar{D}^*}^{q=1}$.
- $\boldsymbol{E_4}$: close but above $E_{D\bar{D}^*}^{q=1}$.

Operator couplings:

- $\boldsymbol{E_1}$: coupled most by $c\bar{c}$ operators.
- $\boldsymbol{E_2}$: coupled most by $\mathcal{O}_{D\bar{D}^*}^{q=0}$ and substantially by $c\bar{c}$ operators.
- $\boldsymbol{E_3}$: coupled substantially by $\mathcal{O}_{D\bar{D}^*}^{q=0}$, $\mathcal{O}_{D\bar{D}^*}^{q=1}$ and $c\bar{c}$ operators.
- $\boldsymbol{E_4}$: coupled most by $\mathcal{O}_{D\bar{D}^*}^{q=1}$ and a little by $\mathcal{O}_{D\bar{D}^*}^{q=0}$ and $c\bar{c}$ operators.

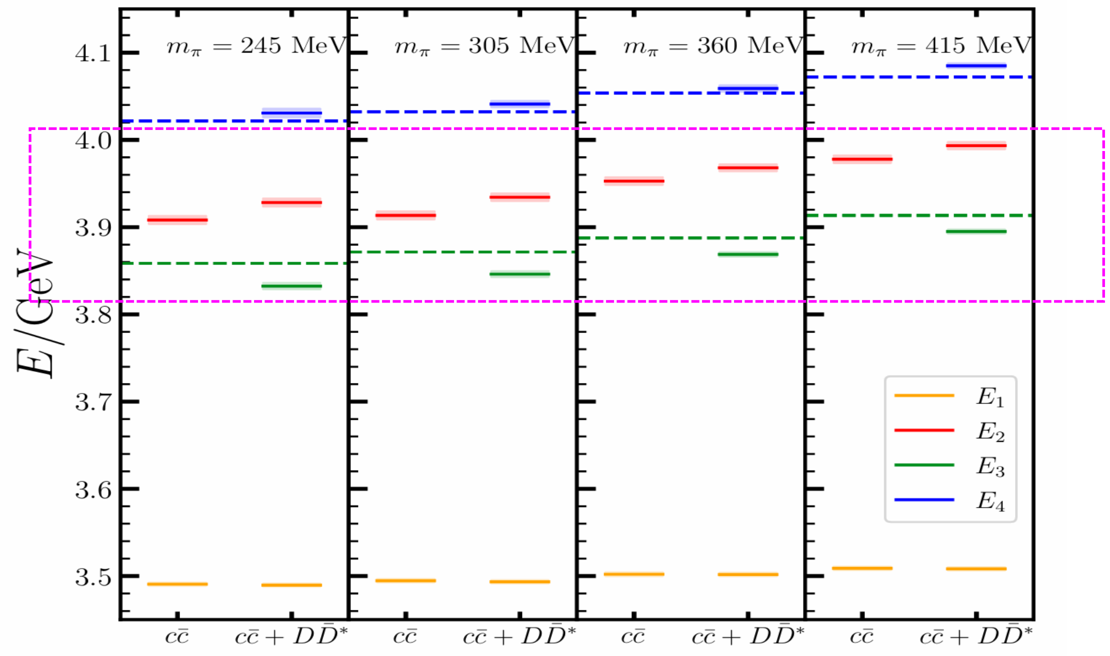
4. A bound state below $D\bar{D}^*$ threshold?

- Physical implication of E_2 and E_3 :
  (S. Prelovsek, PRL111(2013)192001)

Leuscher formula for S-wave scattering

$$p\cot\delta_0(p) = \frac{2}{\sqrt{\pi}\,L}\,\mathcal{Z}_{00}(1;q^2), \qquad q^2 \equiv \left(\frac{L}{2\pi}\right)^2 p^2$$

$$E_n(p_n) = \sqrt{m_D^2 + p_n^2} + \sqrt{m_{D^*}^2 + p_n^2}$$



Effective range expansion (ERE):

$$p\cot\delta_0(p) = \frac{1}{a_0} + \frac{1}{2}r_0 p^2$$

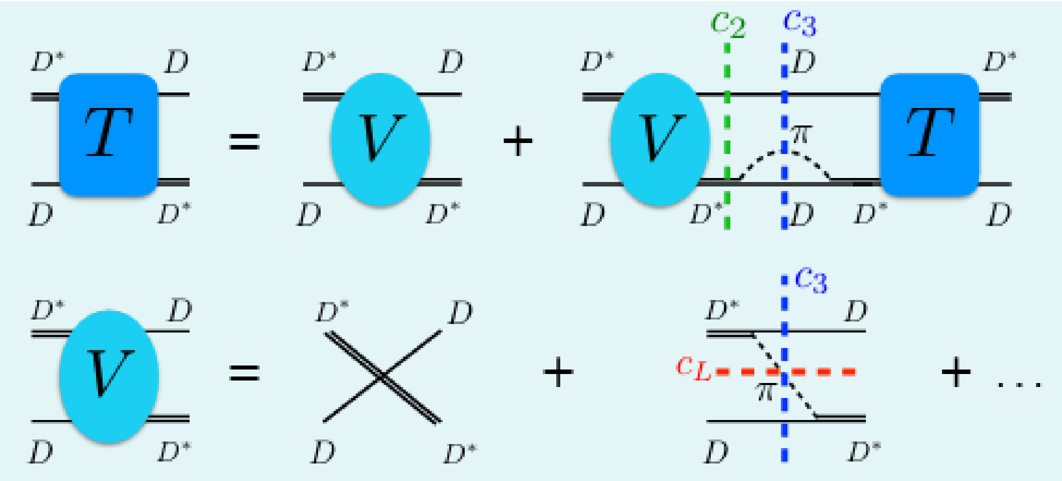
Pole singularity of the scattering amplitude (in the infinite volume):
$$\mathcal{T} \propto \frac{1}{p\cot\delta_0(p) - ip}$$

- Solving ERE with E_2 and E_3, we can obtain the parameters (a_0, r_0)
- Using the derived (a_0, r_0) as the approximation in the $V \to \infty$ limit, then the pole equation gives the banding energy $E_B = E_{D\bar{D}^*}(p_B) - (m_D + m_{D^*})$, where p_B satisfies $p_B\cot\delta_0(p_B) - i\,p_B = 0$.

	m_π(MeV)	250(3)	307(2)	362(1)	417(1)
E_4	Δ_4(MeV) $= E_4 - E^{q=1}_{D\bar{D}^*}$	9.1(1.3)	8.9(1.2)	5.3(1.3)	12.8(1.3)
	p^2(GeV^2)	0.339(8)	0.335(6)	0.340(6)	0.342(4)
	$p\cot\delta_0(p)$(GeV)	-2.02(66)	-2.35(65)	-2.76(89)	-1.79(28)
	δ_0	$(163.9^{+4.0}_{-7.3})^\circ$	$(166.1^{+3.0}_{-5.1})^\circ$	$(168.1^{+2.9}_{-5.5})^\circ$	$(161.9^{+7.4}_{-3.2})^\circ$
E_3	Δ_3(MeV)$= E_3 - E^{q=0}_{D\bar{D}^*}$	70(3)	63(3)	80(3)	80(3)
	p^2(GeV^2)	0.135(5)	0.122(6)	0.158(6)	0.158(6)
	$p\cot\delta_0(p)$(GeV)	-0.054(19)	-0.097(19)	0.012(22)	0.026(24)
	δ_0	$(98.4^{+3.3}_{-3.6})^\circ$	$(105.4^{+3.0}_{-3.1})^\circ$	$(88.2^{+3.3}_{-3.3})^\circ$	$(86.2^{+4.0}_{-4.1})^\circ$
E_2	Δ_2(MeV)$= E_2 - E^{q=0}_{D\bar{D}^*}$	-26.1(9)	-25.4(11)	-19.0(7)	-18.6(8)
	p^2(GeV^2)	-0.050(2)	-0.049(2)	-0.037(1)	-0.036(1)
	$p\cot\delta_0(p)$(GeV)	-0.154(9)(*)	-0.146(10)(*)	-0.063(11)(*)	-0.066(13)
	$(p^{1\pi}_{\rm lhc})^2$(GeV^2)	-0.0135(4)	-0.0210(4)	-0.0292(3)	-0.0400(3)
	a_0 (fm)	-1.55(10)(*)	-1.50(12)(*)	-4.03(91)(*)	$\mathbf{-4.0(1.0)}$
	r_0 (fm)	0.211(30)(*)	0.113(34)(*)	0.153(34)(*)	$\mathbf{0.187(38)}$
	E_B (MeV)	$-9.7^{+2.1}_{-2.2}$ (*)	$-9.7^{+1.9}_{-2.0}$ (*)	$-1.3^{+0.6}_{-0.8}$ (*)	$\mathbf{-1.3^{+0.8}_{-1.0}}$

- $\boldsymbol{E_2}$: The lattice energy is lower than the $D\bar{D}^*$ threshold by 20 MeV or even more.
- $\boldsymbol{a_0}$: Large negative, implies a bound state.
- $\boldsymbol{r_0}$: Small positive, implies the compositeness $X \sim 1$ up to a $\mathcal{O}(p^2)$ correction
- (Y. Li et al., PRD105(2022)L071502).
- The bound state is predominantly a $D\bar{D}^*$ molecule.
- Maybe suffering from the Left Hand Cut (lhc) issue.

(M.-L. Du et al., PRL131(2023)131901, L. Meng et al., arXiv:2312.01930 [hep-lat])



M.-L. Du et al., PRL131(2023)131901

FIG. 1. The cut structure in the DD^* system: (i) the blue dotted vertical lines (c_3) indicate the three-body right-hand cuts, (ii) the green dotted vertical line (c_2) shows the two-body cut, and (iii) the red dotted horizontal line (c_L) is for the left-hand cut. T and V denote the amplitude and interaction potential, respectively.

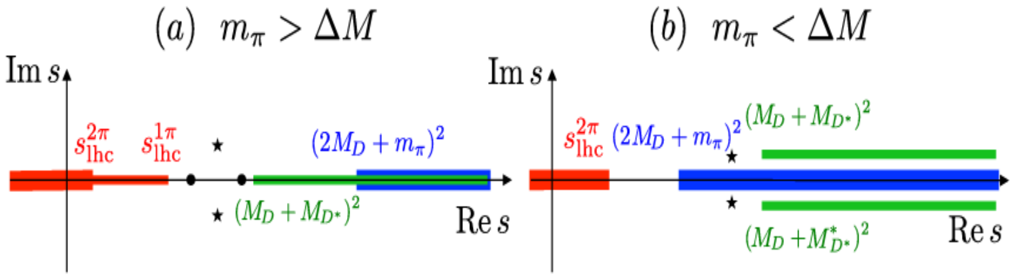
$$G_\pi^{-1}(E, \boldsymbol{k}', \boldsymbol{k}) = \Delta M + \frac{p^2}{2\mu} - \frac{k^2 + k'^2}{2M_D} - \omega_\pi(q^2)$$

One Pion Exchange (OPE)

New singularities emerge in the on-shell partial-wave amplitudes at imaginary values of $k^2 = k'^2 = p^2 < 0$:

$$\left(p_{lhc}^{1\pi}\right)^2 \approx \frac{1}{4}\left[(\Delta M)^2 - m_\pi^2\right]$$

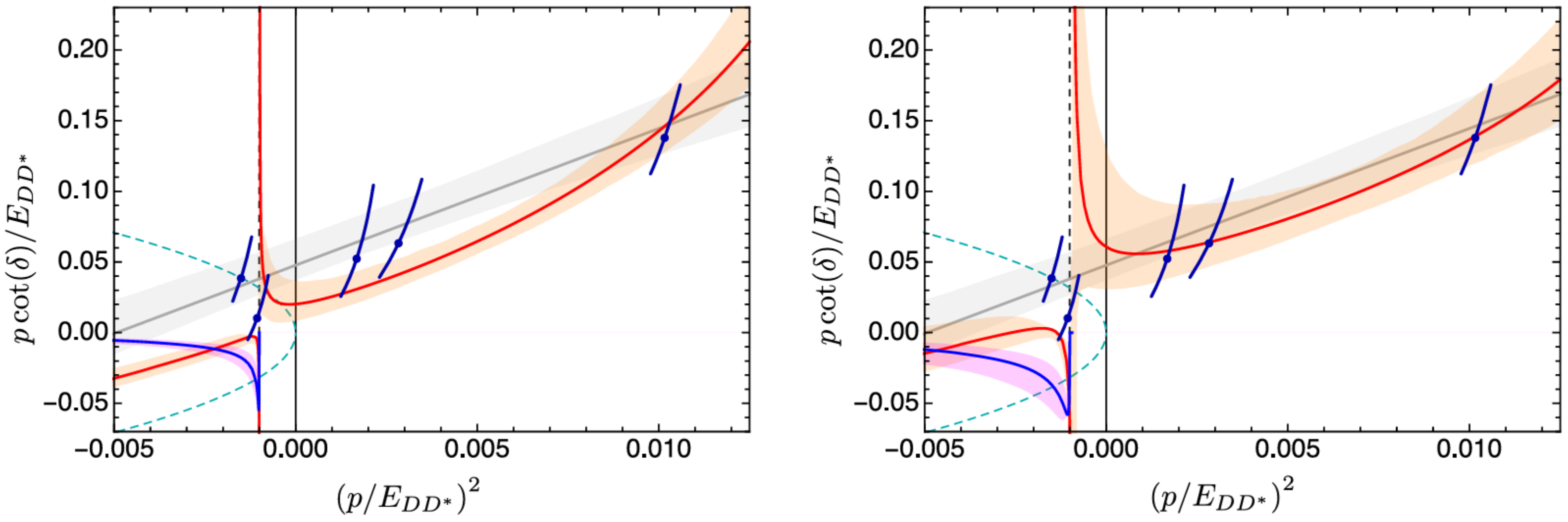
A lhc introduces nonanalyticity to $p \cot \delta$ and sets the upper bound on the convergence radius of ERE ($p \cot \delta$ acquires an imaginary part for $p^2 < \left(p_{lhc}^{1\pi}\right)^2$).



Stars: resonance poles.
Dots: virtual states poles

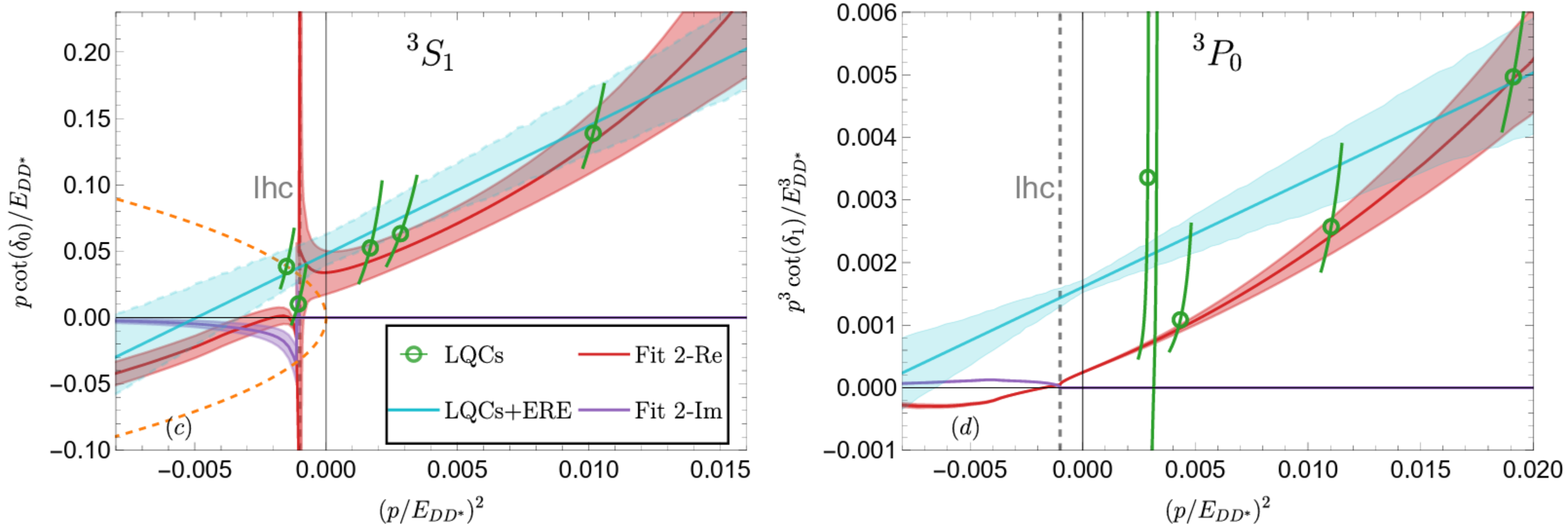
M.-L. Du et al., PRL131(2023)131901

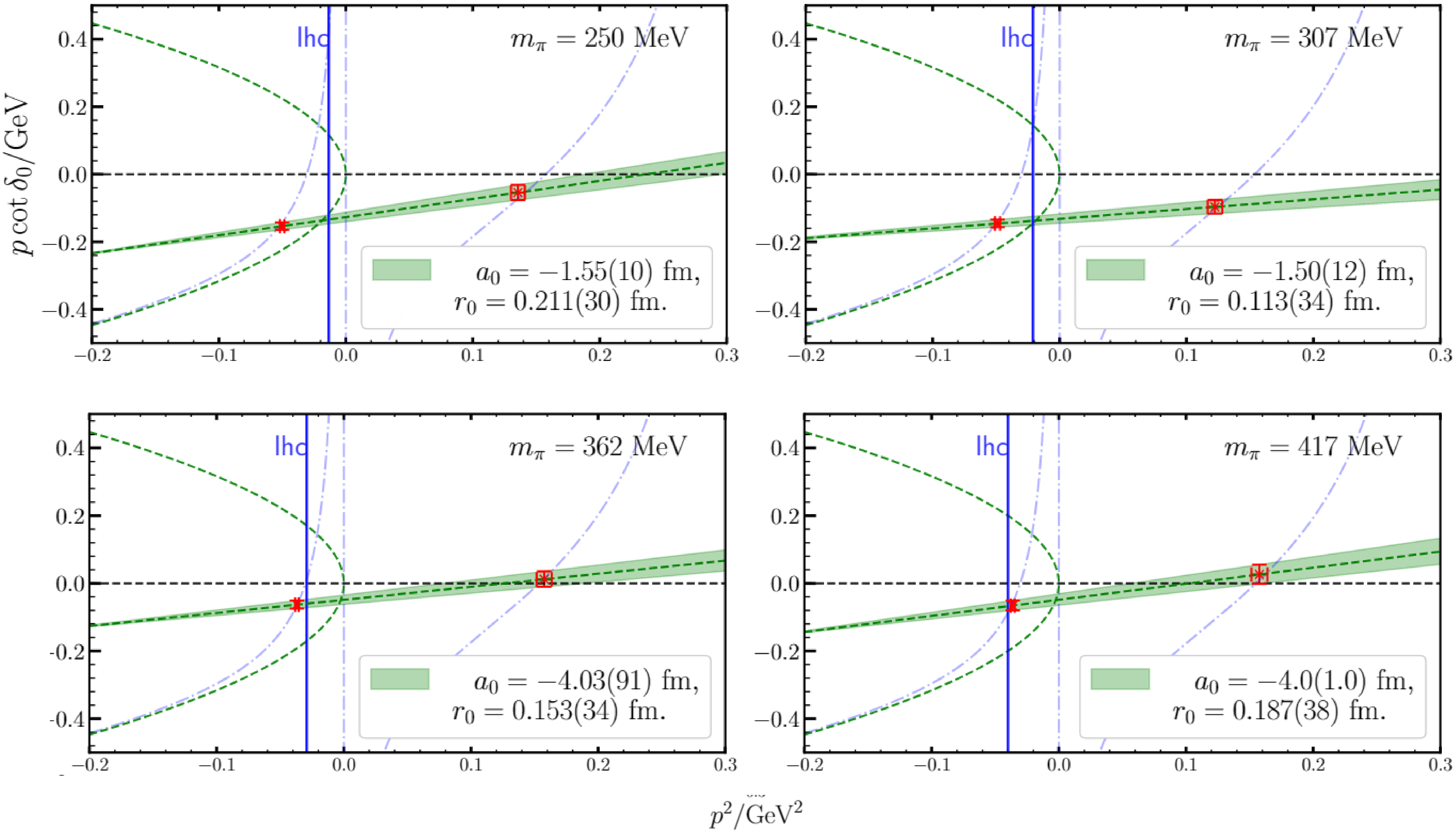
Case studies on $T_{cc}^{+}(3872)$ relevant DD^{*} scattering. The data points are from lattice QCD calculation (M. Padmanath et al. PRL129(2022)032002)

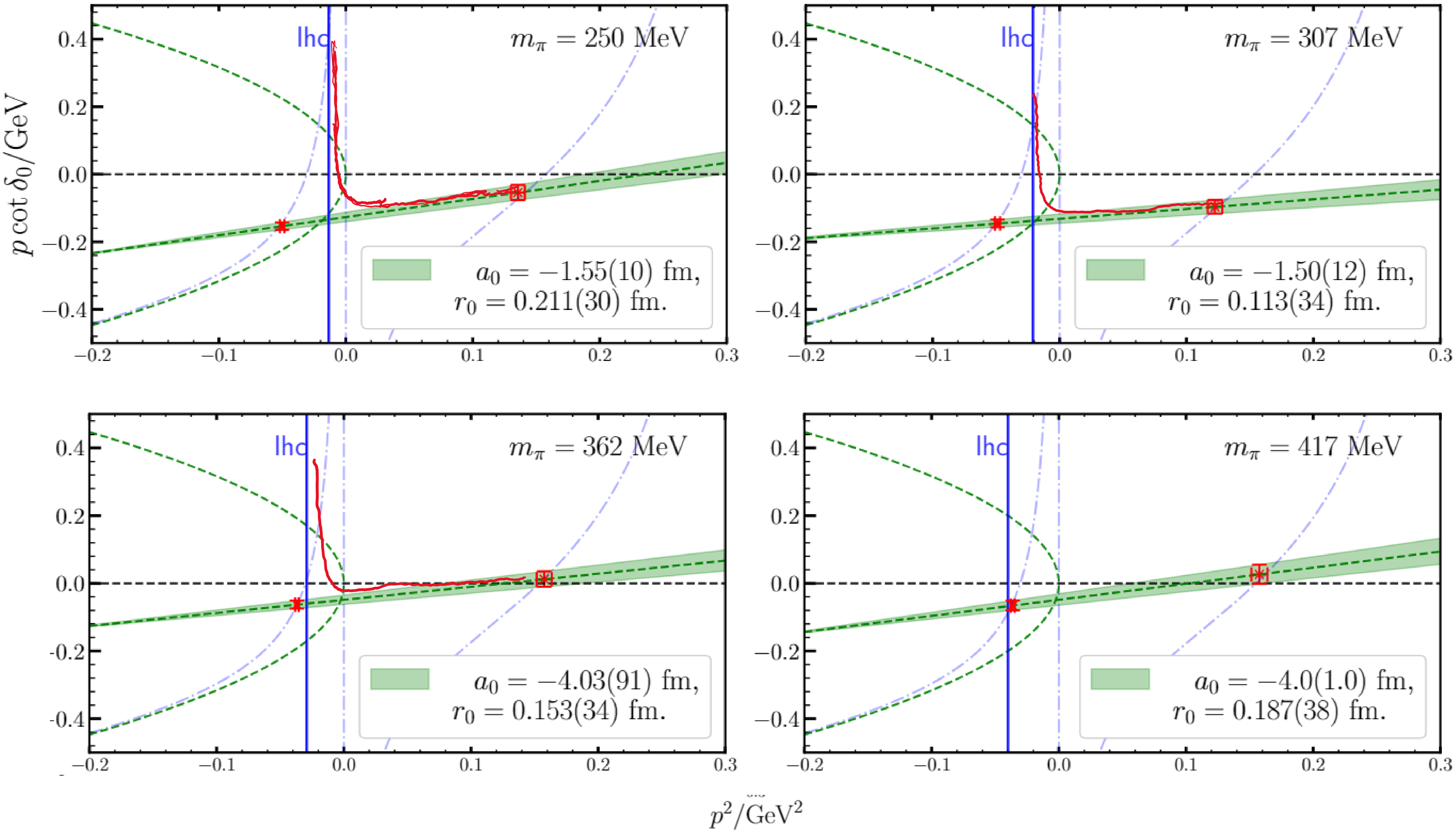


L. Meng et al., arXiv:2312.01930 [hep-lat]

Similar to the discussion above. The difference is that, the lattice finite volume energy levels are used to fix the parameters in the EFT involved. Then prediction of the EFT (red curves) are compared with ERE with out OPE.




$m_\pi = 250$ MeV
$a_0 = -1.55(10)$ fm, $r_0 = 0.211(30)$ fm.
$m_\pi = 307$ MeV
$a_0 = -1.50(12)$ fm, $r_0 = 0.113(34)$ fm.
$m_\pi = 362$ MeV
$a_0 = -4.03(91)$ fm, $r_0 = 0.153(34)$ fm.
$m_\pi = 417$ MeV
$a_0 = -4.0(1.0)$ fm, $r_0 = 0.187(38)$ fm.
lhc
$p \cot \delta_0/\mathrm{GeV}$
p^2/GeV^2



lhc
$m_\pi = 250$ MeV
$a_0 = -1.55(10)$ fm,
$r_0 = 0.211(30)$ fm.
lhc
$m_\pi = 307$ MeV
$a_0 = -1.50(12)$ fm,
$r_0 = 0.113(34)$ fm.
lhc
$m_\pi = 362$ MeV
$a_0 = -4.03(91)$ fm,
$r_0 = 0.153(34)$ fm.
lhc
$m_\pi = 417$ MeV
$a_0 = -4.0(1.0)$ fm,
$r_0 = 0.187(38)$ fm.
$p \cot \delta_0/\mathrm{GeV}$
p^2/GeV^2

To summarize on the bound state:

- $\boldsymbol{m_\pi = 417}$ **MeV** : Free from the OPE lhc issue, a bound state exists in the $V \to \infty$ limit

$$E_B = -1.3^{+0.8}_{-1.0}\ MeV, \qquad X \approx 1 + \mathcal{O}(p^2)$$

  This pole may correspond to $X(3872)$, which is mainly a $D\bar{D}^*$ molecule.

- $\boldsymbol{m_\pi < 360}$ **MeV** : OPE lhc may have the effects on the existence and the pole position of a bound state.

- If the OPE lhc effects are similar to the case of $T_{cc}^+(3872)$ relevant scattering in that, ERE can give a ballpark description of the p^2 behavior of $p\cot\delta_0$, the singularity induced by OPE lhc permit the existence of a bound state, and result in a smaller binding energy.

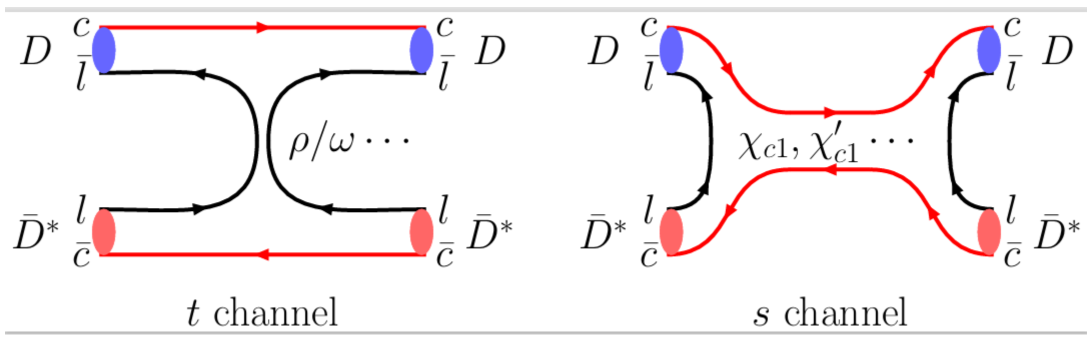
- However, this conclusion may be debatable!

5. Question: Where is the expected $\chi_{c1}(2P)$

a) Non-relativistic quark model expects $\chi_{c1}(2P)$ with a mass around 3.95 GeV.
b) $X(3872)$ is likely a $D\bar{D}^*$ molecule.
c) There should be a state that has a large component of $\chi_{c1}(2P)$.
d) It might appear as a resonance.
e) The dynamics for the $D\bar{D}^*$ scattering in 0^+1^{++} channel

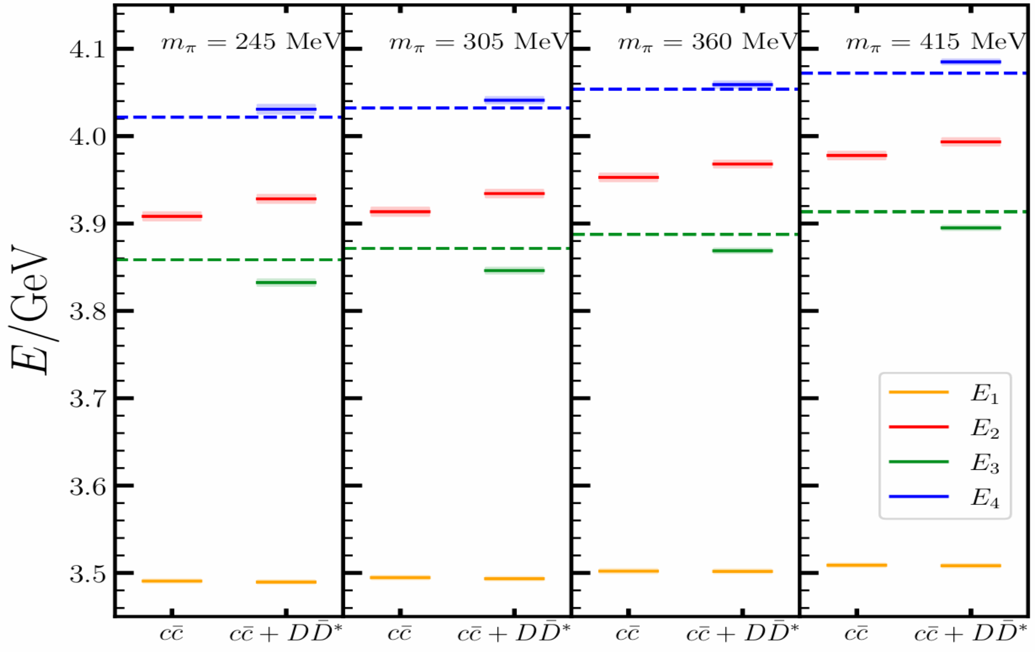
Multiplet	State	Expt.	Input (NR)	Theor. NR	Theor. GI
2P	$\chi_2(2^3P_2)$			3972	3979
	$\chi_1(2^3P_1)$			3925	3953
	$\chi_0(2^3P_0)$			3852	3916
	$h_c(2^1P_1)$			3934	3956

T. Barnes et al.,PRD72(2005)054026



Meson exchange

Charmonium mediation

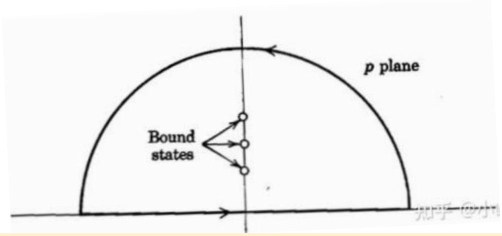


6. Problematic interpretation of the E_3 level in S. Prelovsek, PRL111(2013)192001

- **Levinson' theorem**:

$$\delta_l(0^+) - \delta_l(\infty) = n_l \pi, \qquad \delta_l(\infty) = 0$$

where n_l is the number of bound states in a NR scattering against a spherical potential.

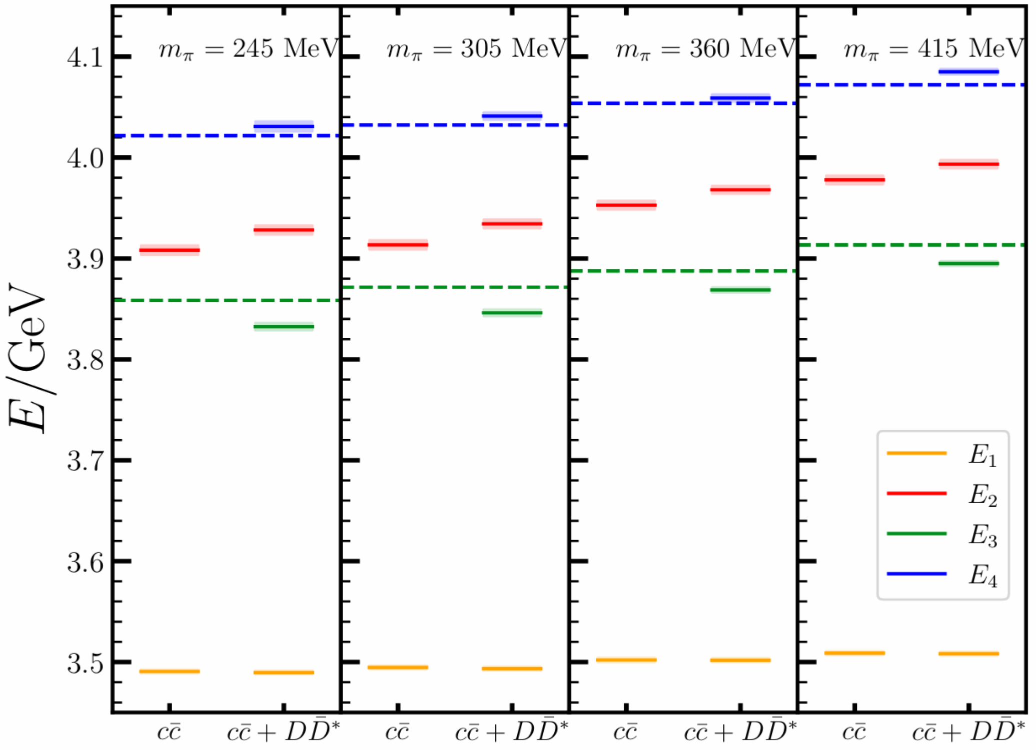


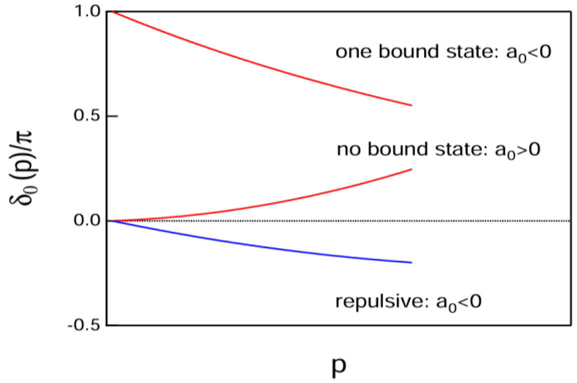
- The authors in PRL111(2013)192001 claims:

✓ If a bound state is formed below the $D\bar{D}^*$ threshold, The next energy level (E_3) correspond to a $D\bar{D}^*$ scattering state with the relative momentum $\vec{p} = 0$.

✓ This state is pushed up due to the negative scattering length.

✓ They did not pay enough attention to the E_4 energy level.





S. Sasaki et al., PRD74(2006)114507

7. Our interpretation of the E_3 and E_4 levels

- **Generalized Levinson' theorem:**

$$\delta_l(0^+) - \delta_l(p_{max}) = (n_l - n_b)\pi,$$

n_l: the number of bound states,

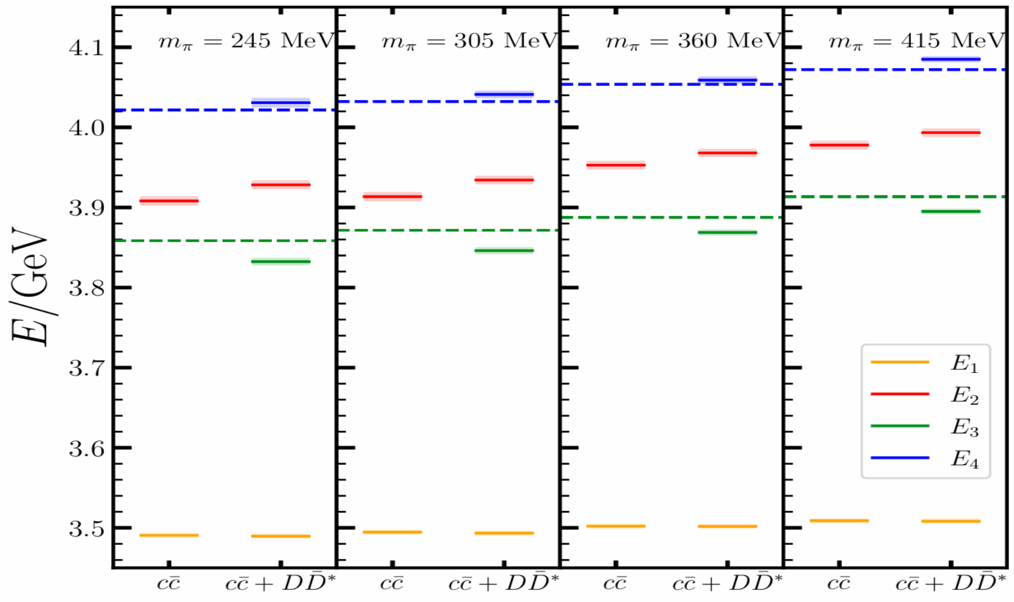
n_b: the number of bare states below the energy corresponding to p_{max}.

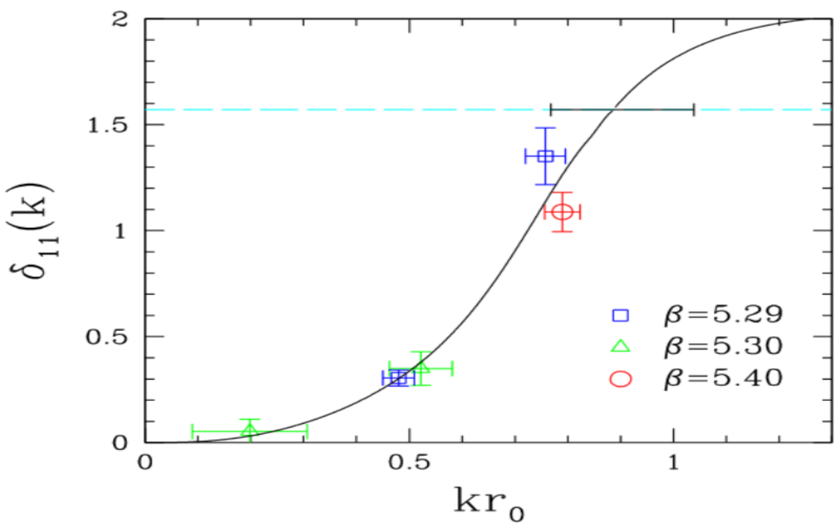
(F. Vidal et al., PRC45(1992)418,
Y. Li et al., PRD105(2022)116024)

- Take the $\pi\pi$ scattering relevant to ρ for instance,

$$n_1 = 0, \quad n_b = 1 \text{ when } E_{\pi\pi}(p) > m_\rho.$$

The evolution of $\delta_{l=1}(p)$:





- For the case of this study:

$$\boldsymbol{n_0 = 1, \quad n_b = 1} \text{ when } \boldsymbol{E_{\pi\pi}(p) > m_{\chi_{c1}(2P)}}.$$

The evolution of $\delta_0(p)$:

✓ Starts with $\delta_0(0^+) = \pi$;
✓ Goes down rapidly when p increases;
✓ Goes up when $E_{D\bar{D}^*}(p)$ approaching $m_{\chi_{c1}(2P)}$;
✓ Goes up to $\delta_0(p) \approx \pi$ when passing $m_{\chi_{c1}(2P)}$.

	m_π(MeV)	250(3)	307(2)	362(1)	417(1)
E_4	Δ_4(MeV) $= E_4 - E^{q=1}_{D\bar{D}^*}$	9.1(1.3)	8.9(1.2)	5.3(1.3)	12.8(1.3)
	p^2(GeV^2)	0.339(8)	0.335(6)	0.340(6)	0.342(4)
	$p\cot\delta_0(p)$(GeV)	-2.02(66)	-2.35(65)	-2.76(89)	-1.79(28)
	δ_0	$(163.9^{+4.0}_{-7.3})^\circ$	$(166.1^{+3.0}_{-5.1})^\circ$	$(168.1^{+2.9}_{-5.5})^\circ$	$(161.9^{+7.4}_{-3.2})^\circ$
E_3	Δ_3(MeV)$= E_3 - E^{q=0}_{D\bar{D}^*}$	70(3)	63(3)	80(3)	80(3)
	p^2(GeV^2)	0.135(5)	0.122(6)	0.158(6)	0.158(6)
	$p\cot\delta_0(p)$(GeV)	-0.054(19)	-0.097(19)	0.012(22)	0.026(24)
	δ_0	$(98.4^{+3.3}_{-3.6})^\circ$	$(105.4^{+3.0}_{-3.1})^\circ$	$(88.2^{+3.3}_{-3.3})^\circ$	$(86.2^{+4.0}_{-4.1})^\circ$
E_2	Δ_2(MeV)$= E_2 - E^{q=0}_{D\bar{D}^*}$	-26.1(9)	-25.4(11)	-19.0(7)	-18.6(8)
	p^2(GeV^2)	-0.050(2)	-0.049(2)	-0.037(1)	-0.036(1)
	$p\cot\delta_0(p)$(GeV)	-0.154(9)(*)	-0.146(10)(*)	-0.063(11)(*)	-0.066(13)
	$(p^{1\pi}_{\rm lhc})^2$(GeV^2)	-0.0135(4)	-0.0210(4)	-0.0292(3)	-0.0400(3)
	a_0 (fm)	-1.55(10)(*)	-1.50(12)(*)	-4.03(91)(*)	$\mathbf{-4.0(1.0)}$
	r_0 (fm)	0.211(30)(*)	0.113(34)(*)	0.153(34)(*)	$\mathbf{0.187(38)}$
	E_B (MeV)	$-9.7^{+2.1}_{-2.2}$ (*)	$-9.7^{+1.9}_{-2.0}$ (*)	$-1.3^{+0.6}_{-0.8}$ (*)	$\mathbf{-1.3^{+0.8}_{-1.0}}$

- $\boldsymbol{E_3}$: Gives a scattering phase around $\delta(E_3) \sim 90°$;
- $\boldsymbol{E_4}$: Gives a scattering phase close to $\delta(E_4) \sim 180°$.
- Exactly as the expectation of the generalized Levinson's theorem.
- Hint at the existence of a resonance.

8. The possible existence of a resonance below 4.0 GeV

- **Breit-Wigner ansatz** for a resonance :

$$\mathcal{T} \approx \frac{1}{\cot\delta_0 - i} \sim \frac{1}{(m_R - E) - \frac{i\Gamma_R}{2}},$$

- Resonance parameters derived through

$$\delta_0(E) = \arctan\left(\frac{\Gamma_R}{2(m_R - E)}\right)$$

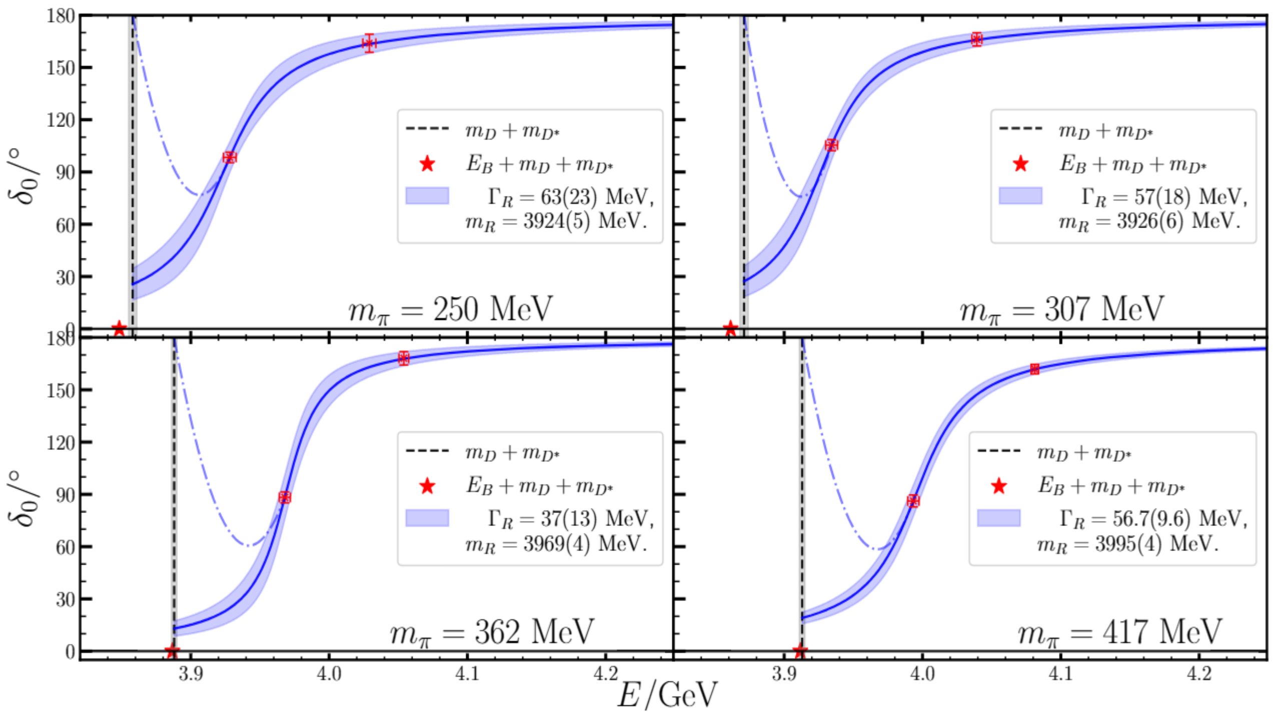
by using E_3 and E_4.

- Caution: The parameters (m_R, Γ_R) may change, since they are derived from only two energy levels.

- Only one experimental observation $X(3940)$:

$$m_X = 3942(9)\ \text{MeV}$$
$$\Gamma_X = 37^{+27}_{-17}\ \text{MeV}$$

(Belle, PRL98(2007)082001; PRL100(2008)202001)



m_π(MeV)	250(3)	307(2)	362(1)	417(1)
m_R(MeV)	3924(5)	3926(6)	3969(4)	3995(4)
Γ_R(MeV)	63(23)	57(18)	37(13)	57(10)

9. A joint analysis on E_2, E_3, E_4 (ignoring the lhc issue tentatively)

- **Effective range expansion** (**H.A. Bethe, PRD 76 (1949) 38**):

$$p\cot\delta_0(p) = \frac{1}{a_0} + \frac{1}{2}r_0 p^2 + c_2 p^4,$$

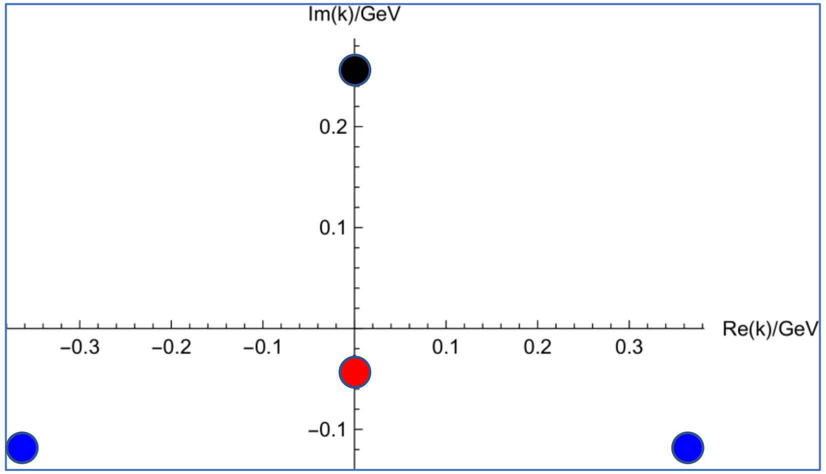
- Poles of the amplitude

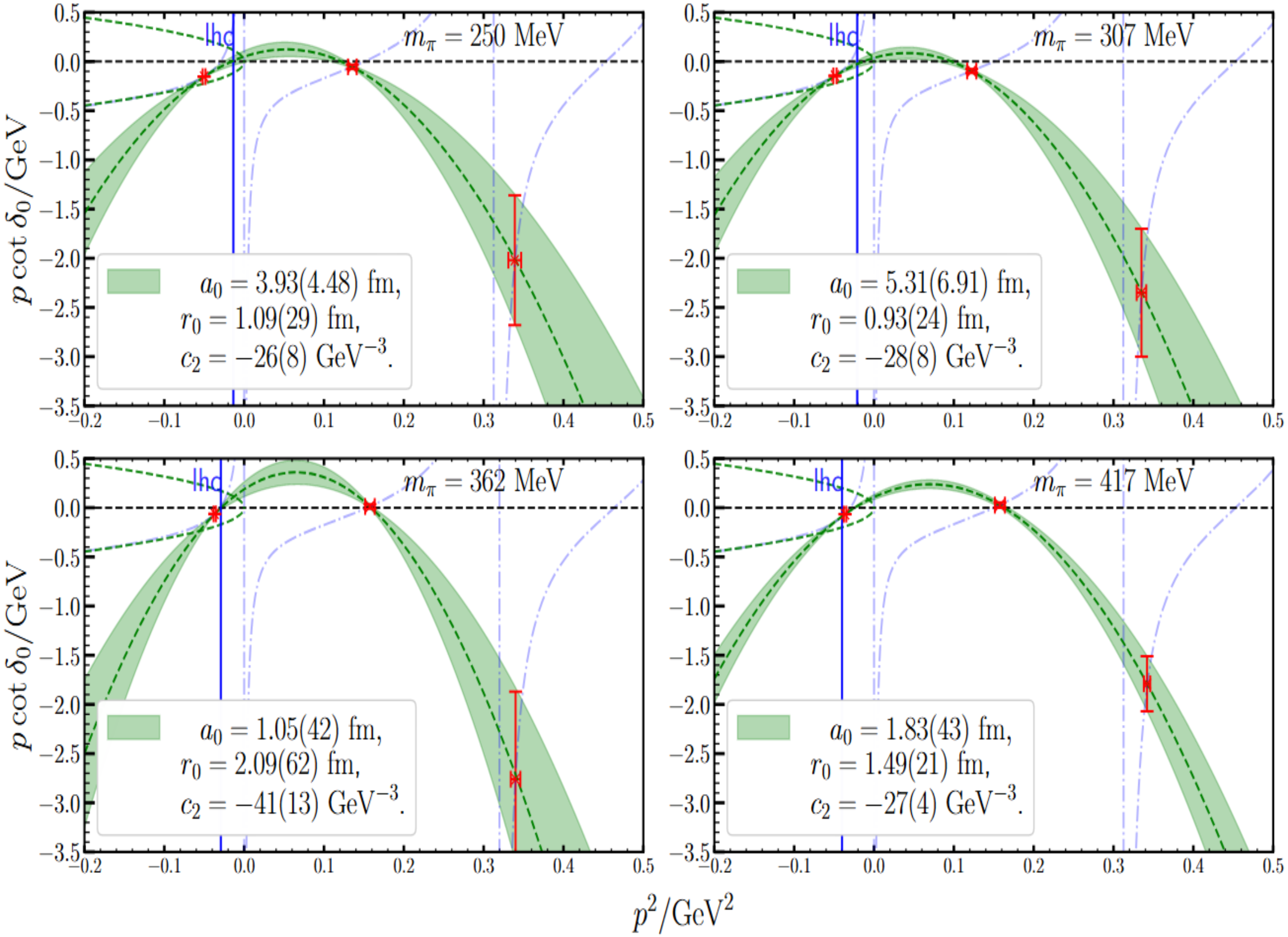
$$\mathcal{T} \propto \frac{1}{p\cot\delta_0 - ip}$$

- A "virtual state" below the $D\bar{D}^*$ threshold

$$a_0 > 0$$

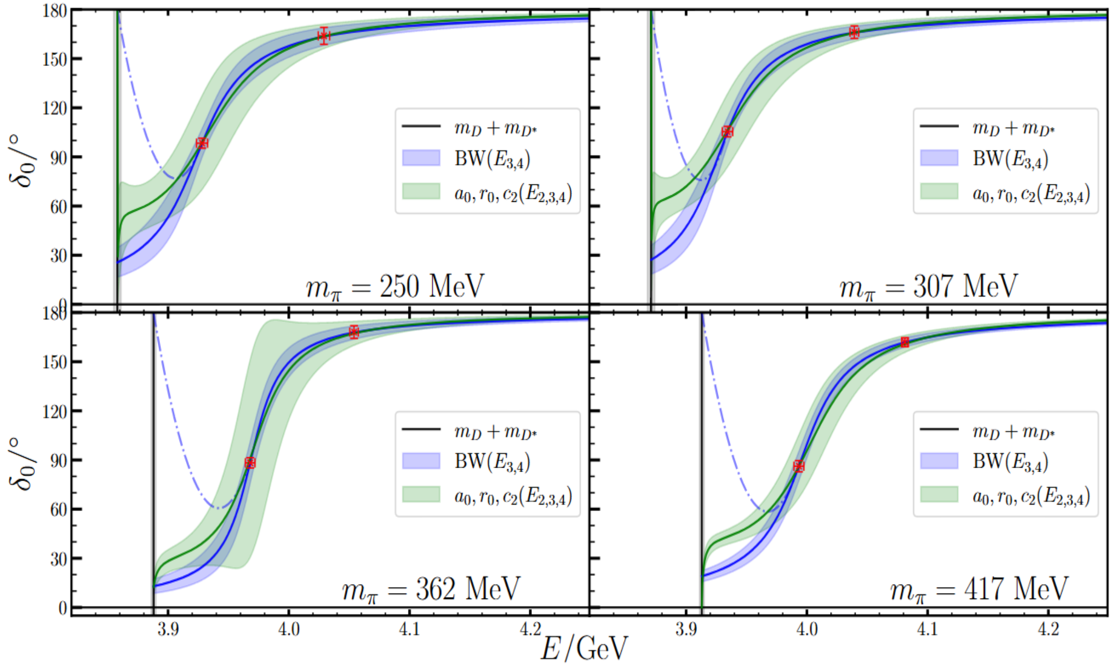
- A resonance pole:





$$p\cot\delta_0(p) = \frac{1}{a_0} + \frac{1}{2}r_0 p^2 + c_2 p^4$$

$(E_{2,3,4})$	a_0 (fm)	$3.9(4.5)(*)$	$5.9(6.9)(*)$	$1.05(42)(*)$	$\mathbf{1.83(43)}$
	r_0 (fm)	$1.09(29)(*)$	$0.93(24)(*)$	$2.09(62)(*)$	$\mathbf{1.49(21)}$
	c_2 (GeV^{-3})	$-26(8)(*)$	$-28(8)(*)$	$-41(13)(*)$	$\mathbf{-27(4)}$
	p_B (MeV)	$-i45^{+45}_{-53}(*)$	$-i35^{+35}_{-48}(*)$	$-i\ 114(25)(*)$	$\mathbf{-i\ 81(16)}$
	E_B (MeV)	$-1.1^{+1.1}_{-2.4}(*)$	$-0.6^{+0.6}_{-1.7}(*)$	$-6.7(2.9)(*)$	$\mathbf{-3.3(1.3)}$



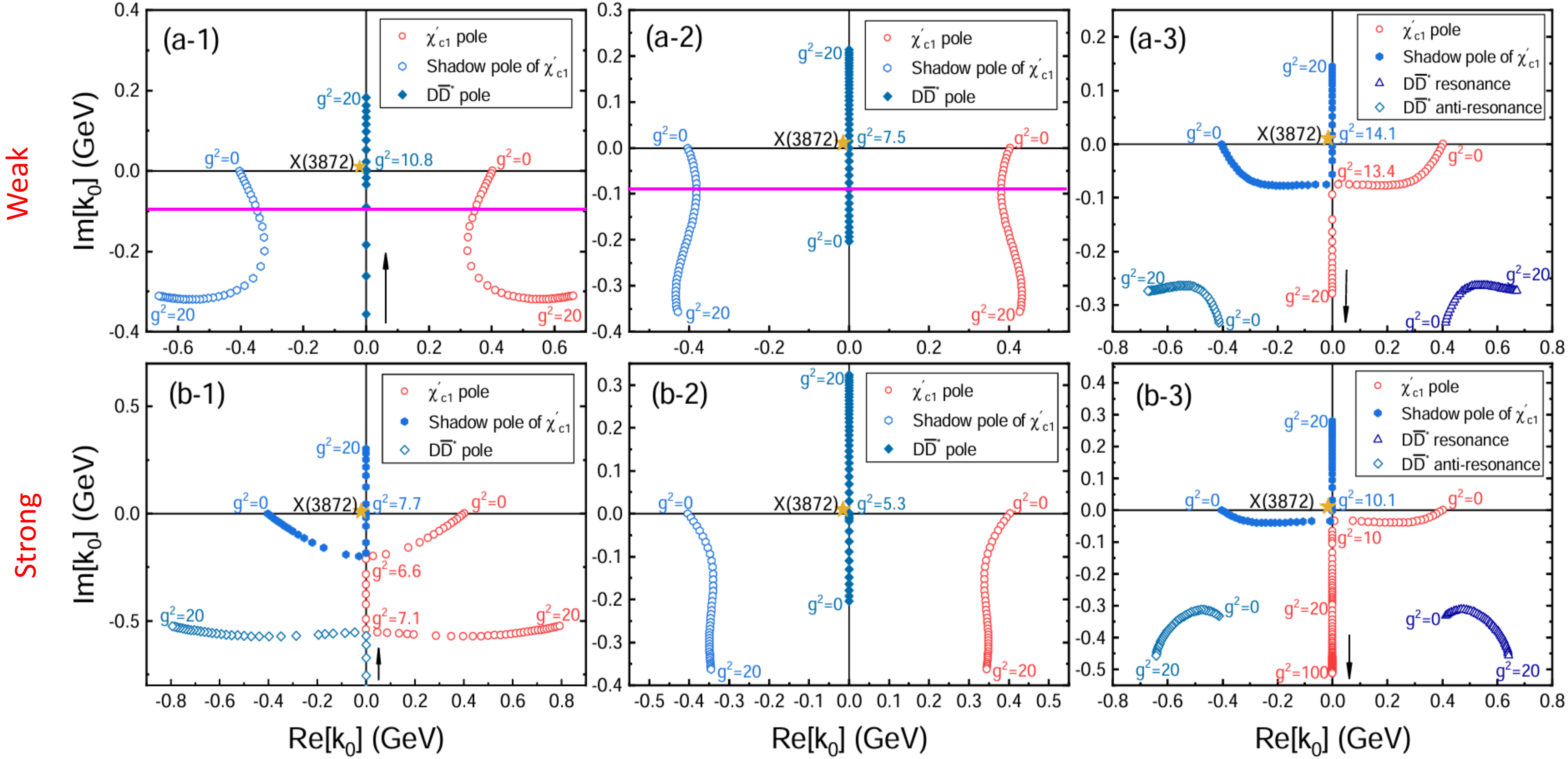
m_π(MeV)	250(3)	307(2)	362(1)	417(1)
BW fit from $E_{3,4}$				
m_R(MeV)	3924(5)	3926(6)	3969(4)	3995(4)
Γ_R(MeV)	63(23)	57(18)	37(13)	57(10)
Pole from $E_{2,3,4}$				
$\mathrm{Re}\,p_R$(MeV)	391(24)	364(20)	410(13)	427(13)
$\mathrm{Im}\,p_R$(MeV)	$-i108(34)$	$-i114(32)$	$-i61(20)$	$-i87(14)$
m_R(MeV)	3931(8)	3933(6)	3971(6)	4002(6)
Γ_R(MeV)	86(32)	85(27)	50(18)	75(13)

$$\delta_l(0^+) - \delta_l(p_{max}) = (n_l - n_b)\pi$$

Resonance pole :

$$E = m_R - i\frac{\Gamma}{2}$$

Intrinsic attractive $D\bar{D}^*$ Intrinsic respulsive $D\bar{D}^*$

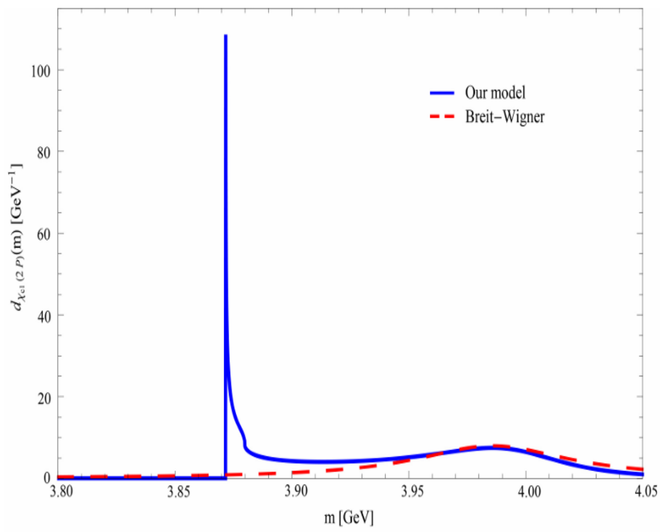


J.-Z. Wang et al., arXiv:2404.16575[hep-ph]

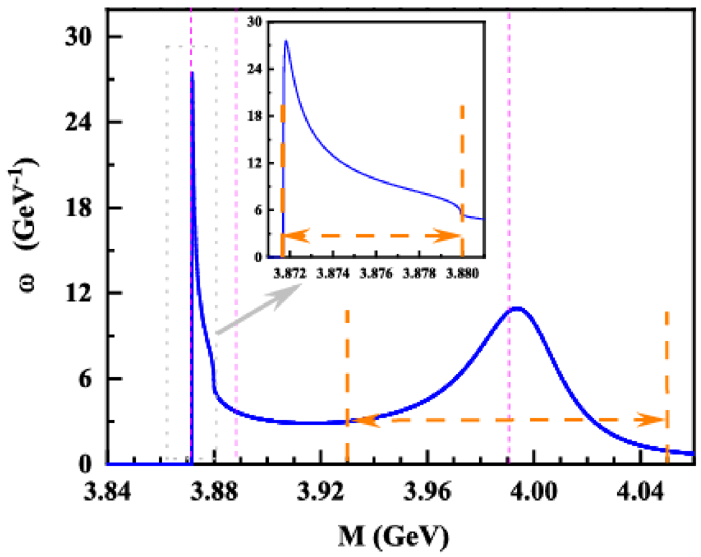
10. Related phenomenological studies

- Considering the mixing between $D\bar{D}^*$ and $\chi_{c1}(2P)$.

m_R (MeV)	3910-3925	3995	3990	3958
Γ_R (MeV)	$5-70$	72	~ 60	~ 17
Ref.	E. Cincioglu et al., EPJC76(2016)576	F. Giacosa et al., IJMPA34(2019)1950173	Q. Deng et al., 2312.10296	G.J. Wang et al., 2306.12406



E. Cincioglu et al., EPJC76(2016)576



Q. Deng et al., 2312.10296

Spectral function:

$$\omega(M) = \frac{1}{2\pi} \frac{\Gamma}{(M - m_R)^2 + \Gamma^2/4}$$

IV. $((0,2)^{++}$ charmonium-like resonance around 3.9 GeV

HSC Collab., PRD 109 (2024) 114503 (arXiv:2309.14071 [hep-lat]);
PRL 132 (2024) 241901 (arXiv:2309.14070 [hep-lat]).

1. Lattice setup——$N_f = 2+1$ QCD

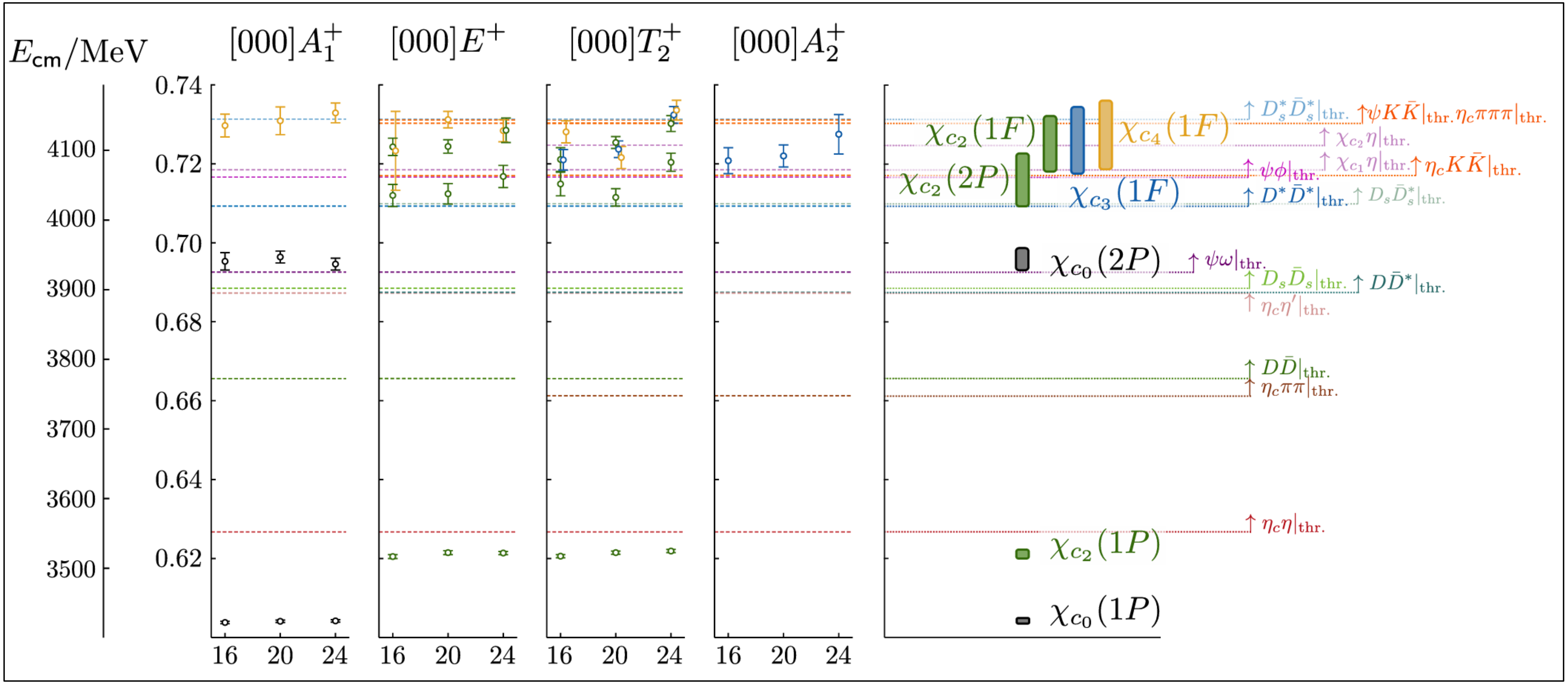
HSC has established a very sophisticated numerical framework to study the hadron-hadron scatterings based on **Lüscher's formalism**

L/a_s	T/a_t	$N_{\rm cfg}$	$N_{\rm vec}$	$N_{\rm tsrcs}$	$L/{\rm fm}$	$m_\pi L$
16	128	478	64	8–16	1.9	3.8
20	256	288	128	4–8	2.4	4.8
24	128	553	160	2–4	2.9	5.7

$$\det\left[F^{-1}\left(\vec{P},E,L\right)+\mathcal{M}(E)\right]=0,\qquad \mathcal{M}^{-1}_{ab}=\left(\mathcal{K}^{-1}\right)_{ab}-i\delta_{ab}\frac{2q^*_a}{E_{cm}}$$

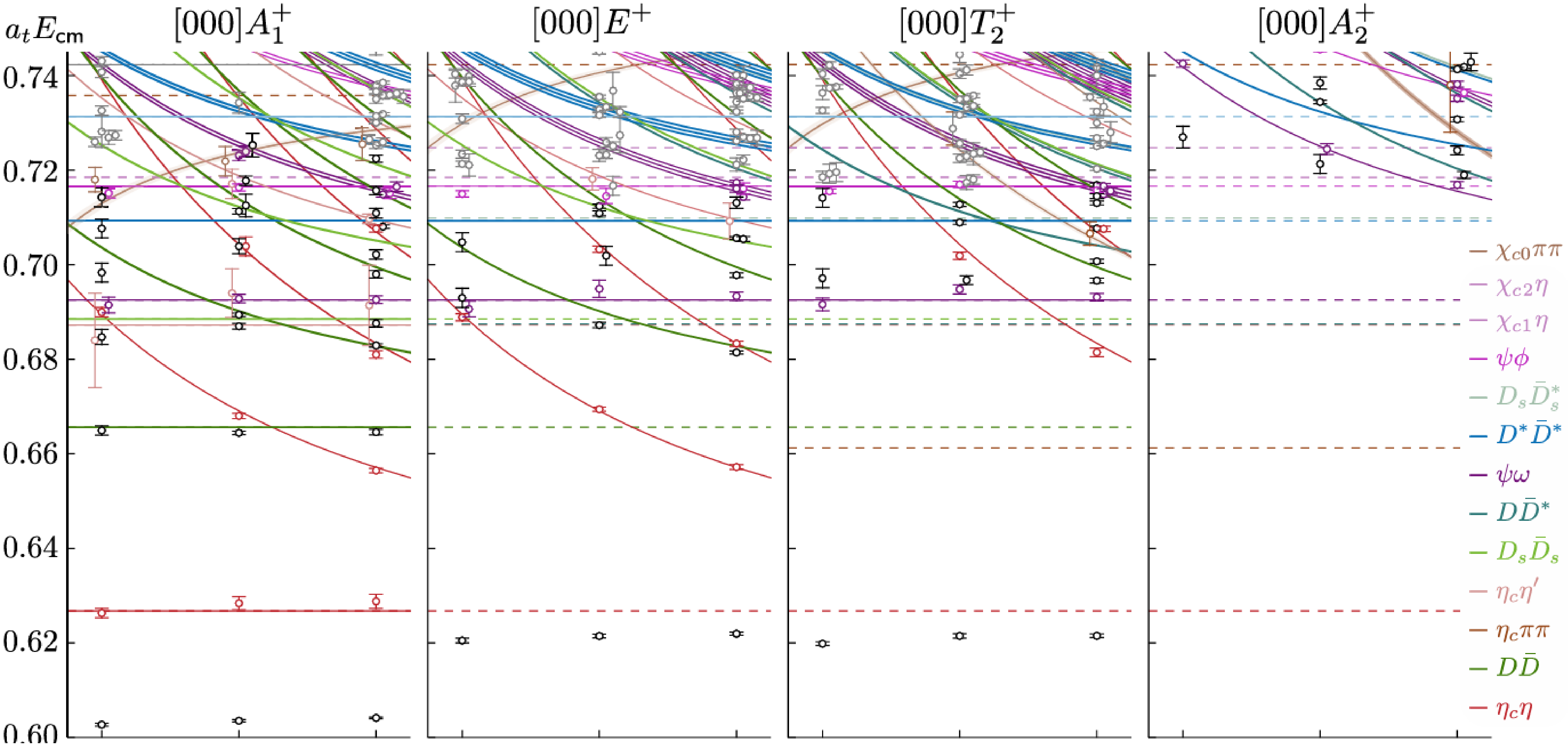
Meson mass / MeV	π	K	η	D	D_s	D^*	η_c	J/ψ	χ_{c0}	χ_{c2}
this calc.	391	550	587	1886	1951	2010	2965	3044	3423(3)	3519(2)
expt.	140	494	548	1865	1969	2007	*2984*	3097	*3415*	3556

- **Focus on 0^{++} and 2^{++} states around 3.9 GeV**

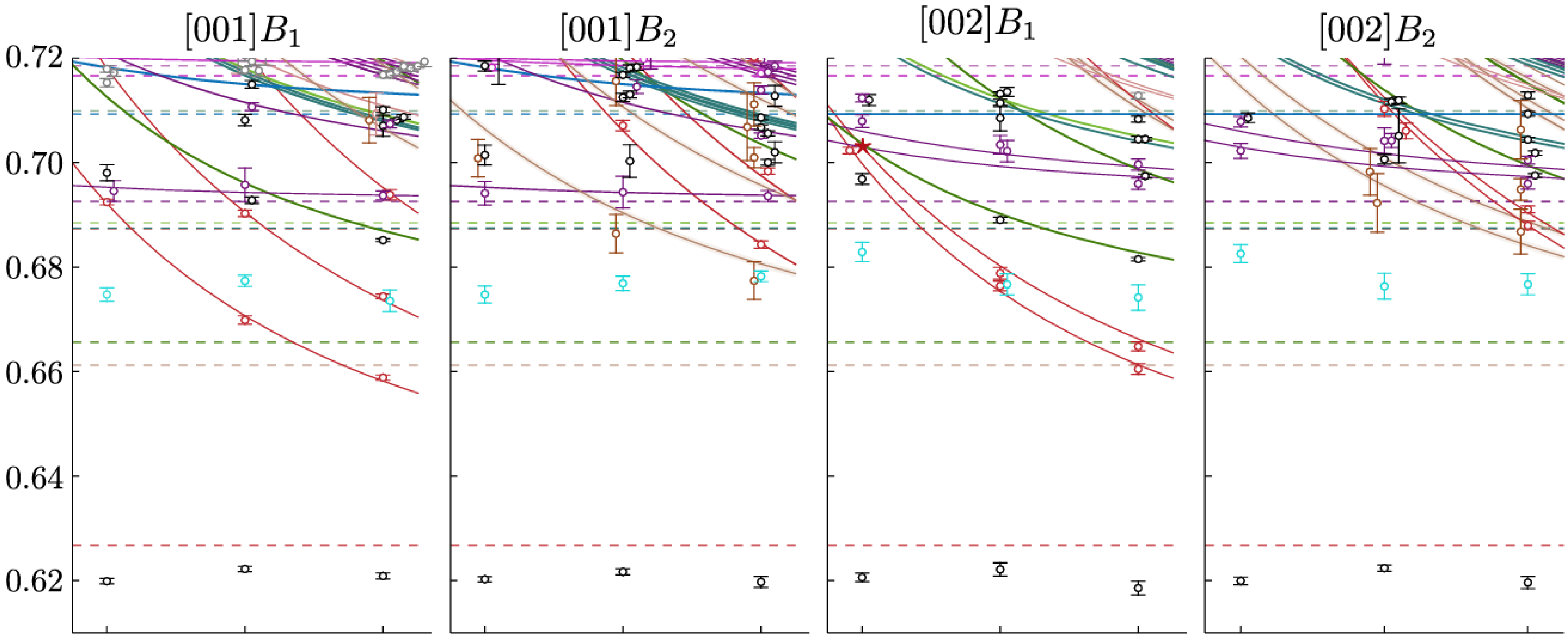


The energy levels obtained from only the $c\bar{c}$-like operators.

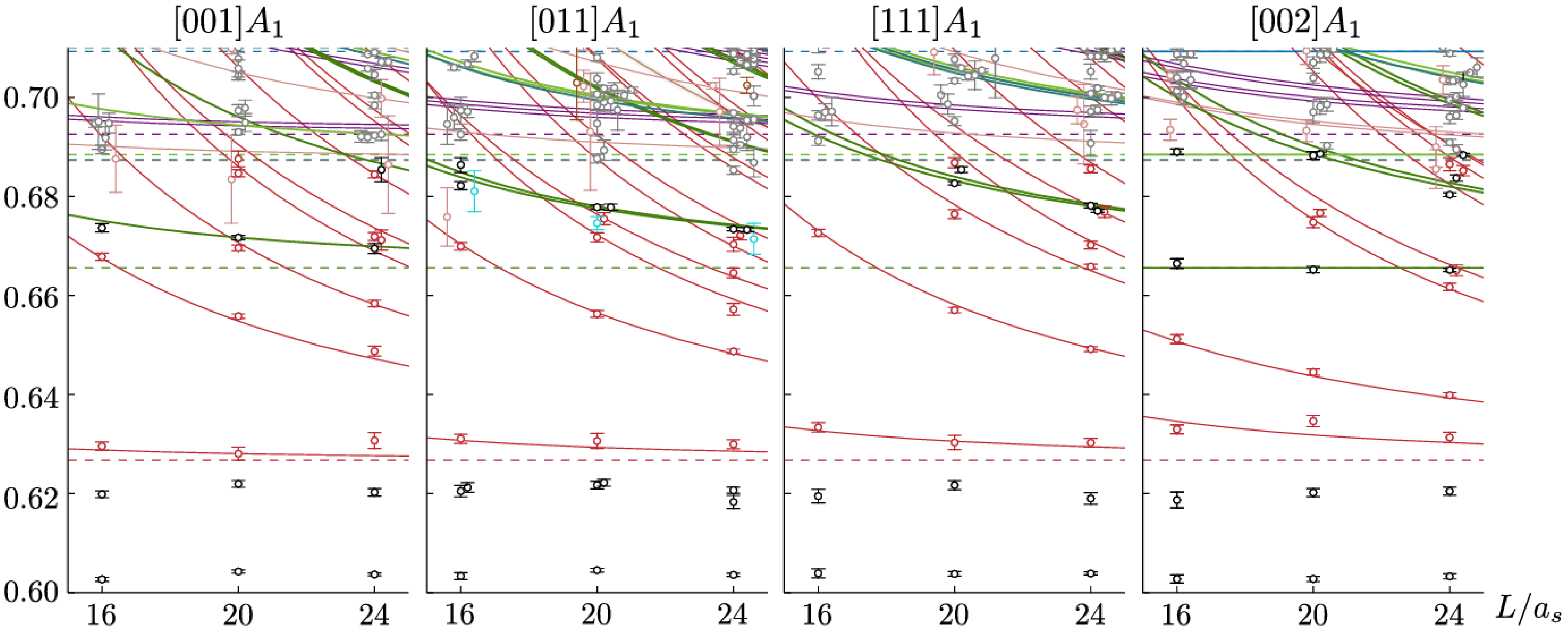
Finite volume energy levels in the rest frame of the system



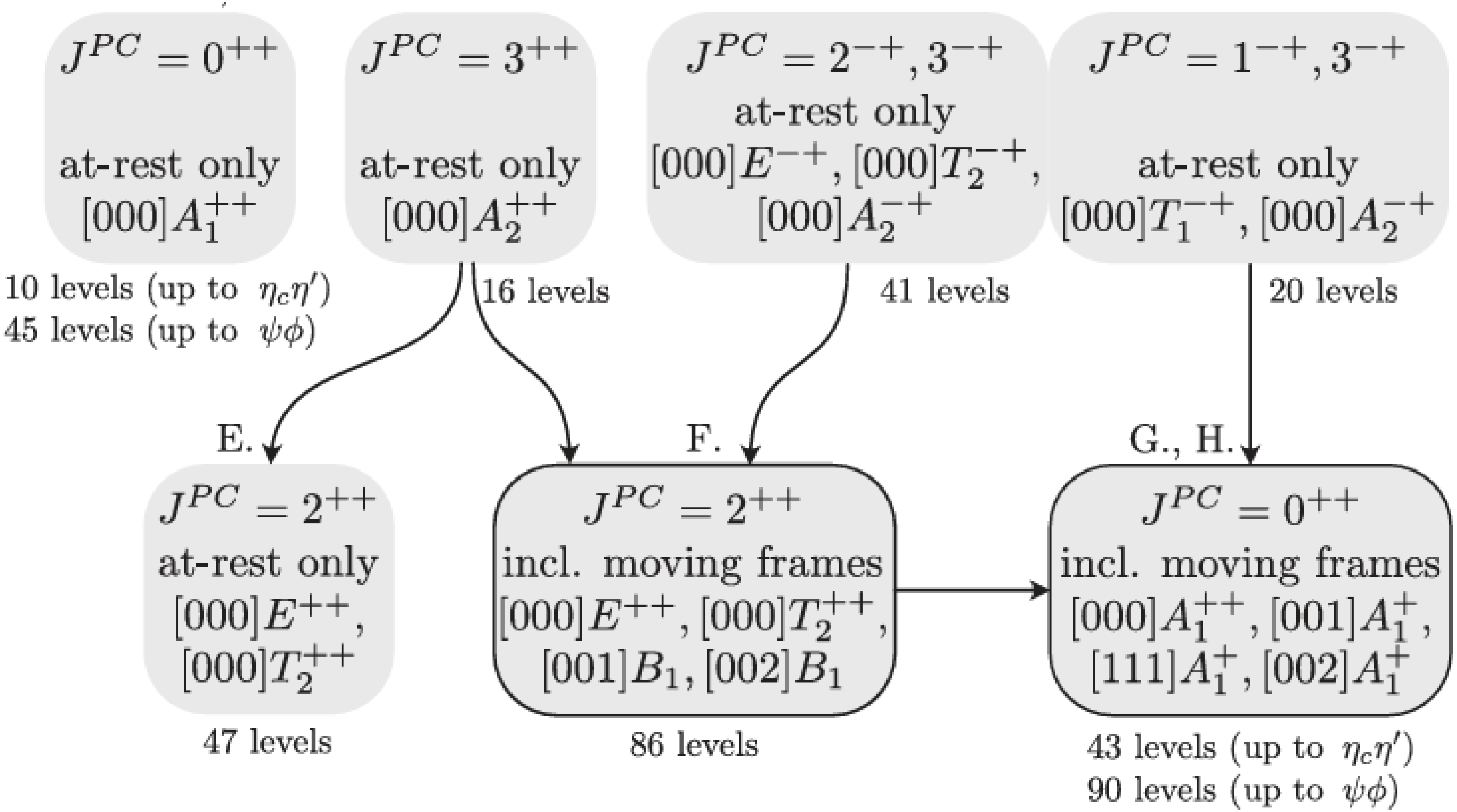
Finite volume energy levels in moving frames of the system



Finite volume energy levels in moving frames of the system



Finite volume energy levels vs. the J^{PC} systems in the continuum



- **Parameterizations of coupled-channel partial-wave t −matrices respect the unitarity.**

$$K_{ij} = \sum_p \frac{g_i^{(p)} g_j^{(p)}}{m_p^2 - s} + \sum_a \gamma_{ij}^{(a)} s^a, \qquad [t^{-1}]_{ij} = (2k_i)^{-l_i} [K^{-1}]_{ij} (2k_j)^{-l_j} + I_{ij}$$

The matrix I_{ij} is diagonal and has imaginary part $\mathrm{Im}\, I_{ij} = -\rho_i = -\frac{2k_i}{\sqrt{s}}$

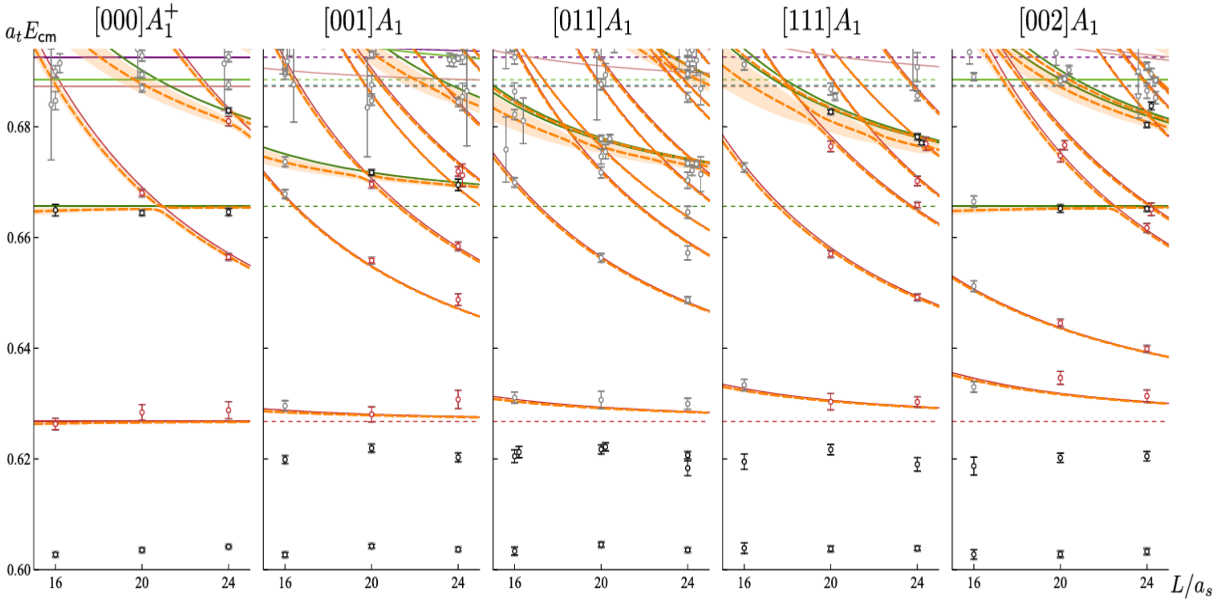
2. 0^{++} channel

- **Channels considered:**
  $\eta_c\eta, D\bar{D}, D_s\bar{D}_s, \eta_c\eta', \psi\omega, D^*\bar{D}^*, \psi\phi$

- **Both rest and moving frames are considered**

- **K-matrix parameterization and results**

$$K_{ij} = \frac{g_i g_j}{m^2 - s} + \gamma_{ij}$$



- **Parameters of the K matrix fitted from 90 energy levels up to $\psi\phi$ threshold.**

Channel couplings g_{AB} in the K-matrix

$$
\begin{aligned}
a_t m &= (0.7065 \pm 0.0015 \pm 0.0004) \\
a_t g_{D\bar{D}\{^1S_0\}} &= (0.1174 \pm 0.0226 \pm 0.0039) \\
a_t g_{D_s\bar{D}_s\{^1S_0\}} &= (0.189 \pm 0.046 \pm 0.026) \\
a_t g_{\psi\omega\{^1S_0\}} &= (-0.127 \pm 0.069 \pm 0.230) \\
a_t g_{D^*\bar{D}^*\{^1S_0\}} &= (0.330 \pm 0.095 \pm 0.023) \\
\gamma_{\eta_c\eta\{^1S_0\}\to\eta_c\eta\{^1S_0\}} &= (0.144 \pm 0.097 \pm 0.038) \\
\gamma_{D\bar{D}\{^1S_0\}\to D_s\bar{D}_s\{^1S_0\}} &= (-0.974 \pm 0.301 \pm 0.027) \\
\gamma_{\eta_c\eta'\{^1S_0\}\to\eta_c\eta'\{^1S_0\}} &= (2.55 \pm 1.03 \pm 0.73) \\
\gamma_{\psi\phi\{^1S_0\}\to\psi\phi\{^1S_0\}} &= (1.36 \pm 0.90 \pm 0.26) \\
\gamma_{\psi\omega\{^5D_4\}\to\psi\omega\{^5D_4\}} &= (162 \pm 254 \pm 43)\cdot a_t^8 \\
\chi^2/N_{\mathrm{d.o.f.}} &= \frac{91.0}{90-10-16} = 1.42,
\end{aligned}
$$

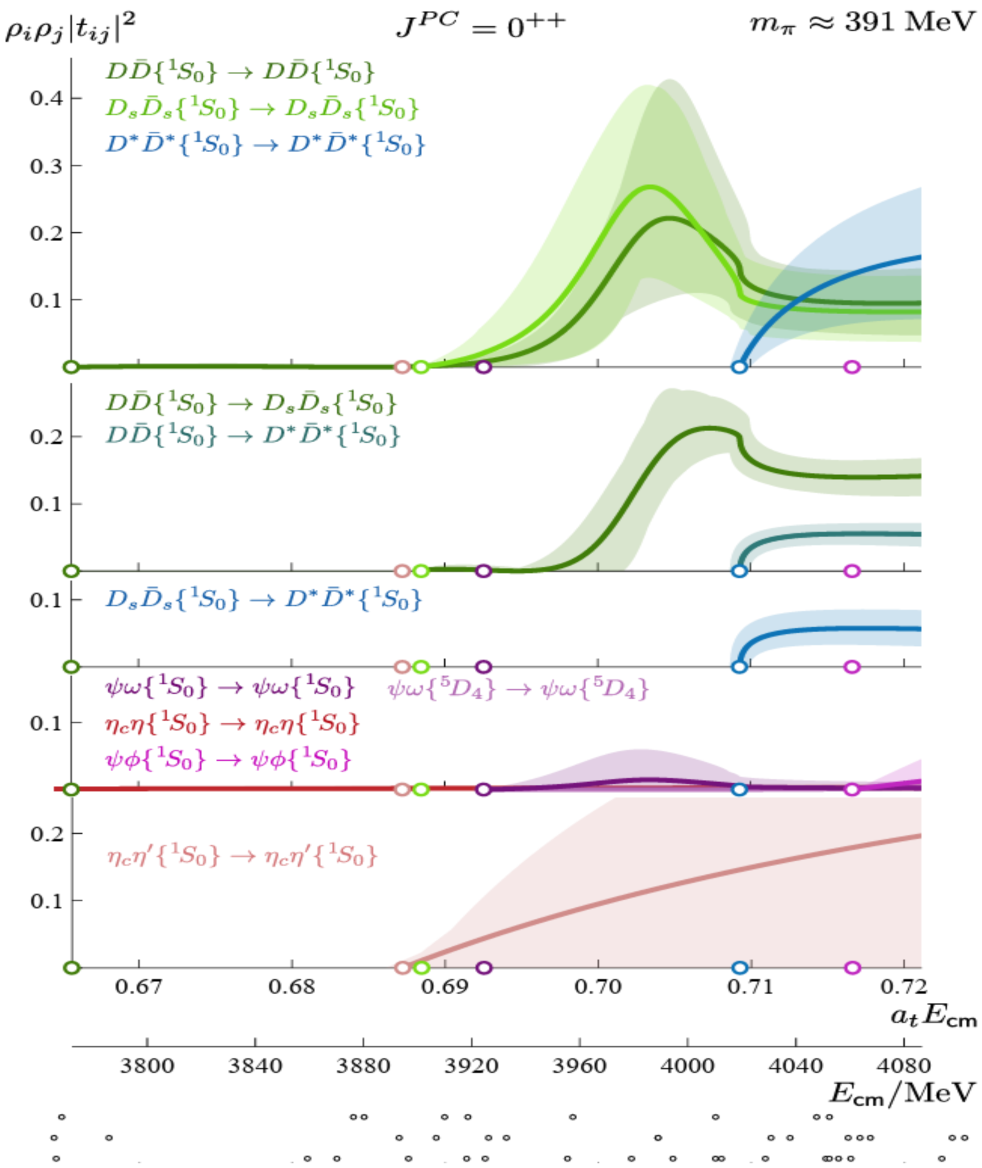
Pole couplings c_{AB} in the T-matrix

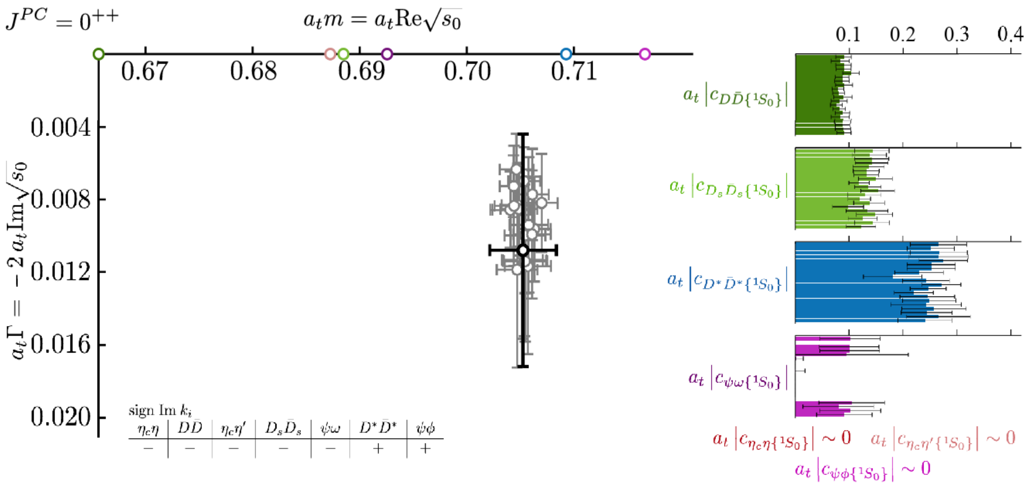
$$t_{ij}(s \approx s_0) \sim \frac{c_i c_j}{s_0 - s}$$

$$
\begin{aligned}
a_t|c_{\eta_c\eta\{^1S_0\}}| &\approx 0 \\
a_t|c_{D\bar{D}\{^1S_0\}}| &= 0.093(28) \\
a_t|c_{\eta_c\eta'\{^1S_0\}}| &\approx 0 \\
a_t|c_{D_s\bar{D}_s\{^1S_0\}}| &= 0.128(56) \\
a_t|c_{\psi\omega\{^1S_0\}}| &= 0.083(83) \\
\boxed{a_t|c_{D^*\bar{D}^*\{^1S_0\}}| = 0.227(97)} \\
a_t|c_{\psi\phi\{^1S_0\}}| &\approx 0,
\end{aligned}
$$

No significant couplings to closed charm channels.

- **There is only one pole between $\psi\omega$ and $D^*\bar{D}^*$ threshold.**





- **Pole position:**

$$\sqrt{s_0} = \left[(3995 \pm 14) - \frac{i}{2}(67 \pm 38)\right] \text{MeV}$$

- **Partial decay widths:** $\mathcal{B}_i = \frac{\Gamma_i}{\Gamma} = |c_i|^2 \frac{\rho_i(Re\, s_0)}{\sqrt{Re\, s_0}}$,

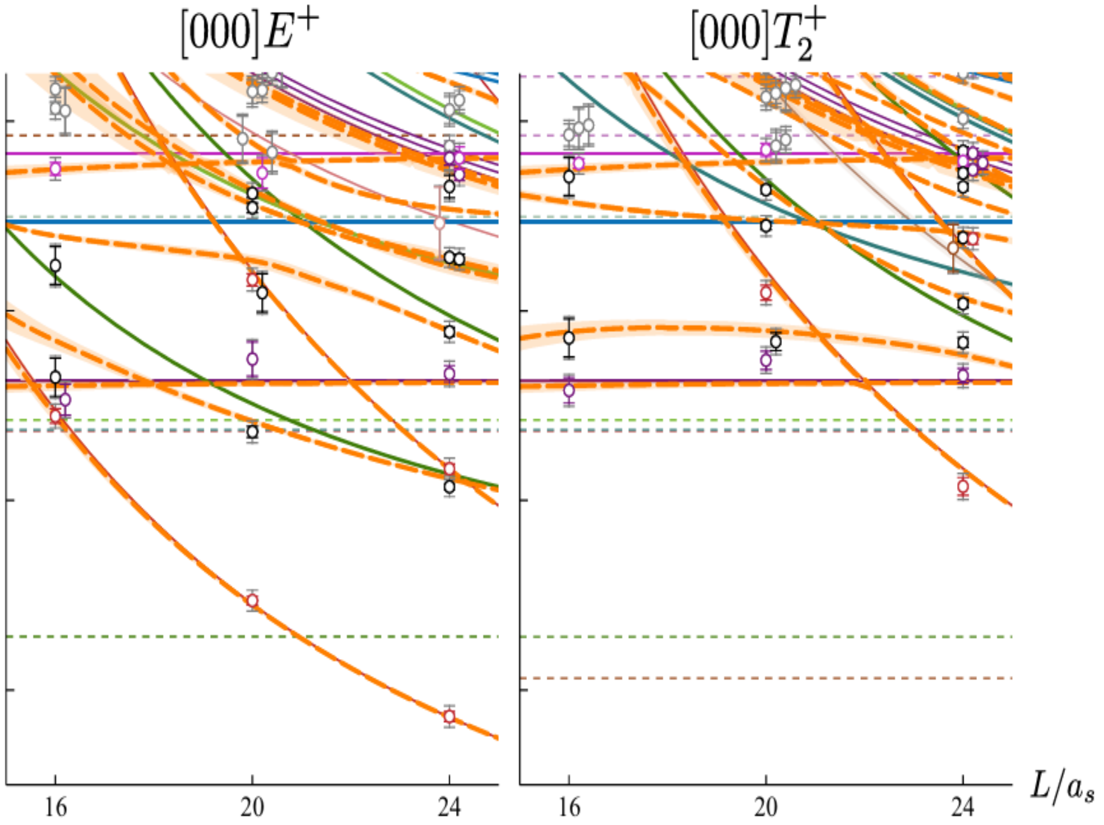
$$\Gamma_{D\bar{D}} = 23(13)\ \text{MeV},$$
$$\Gamma_{D_s\bar{D}_s} = 28(26)\ \text{MeV},$$
$$\Gamma_{\psi\omega} = 9^{+18}_{-9}\ \text{MeV}$$

- **Rapid turn-on of $D^*\bar{D}^*$ at threshold due to the large coupling $c_{D^*\bar{D}^*}$**

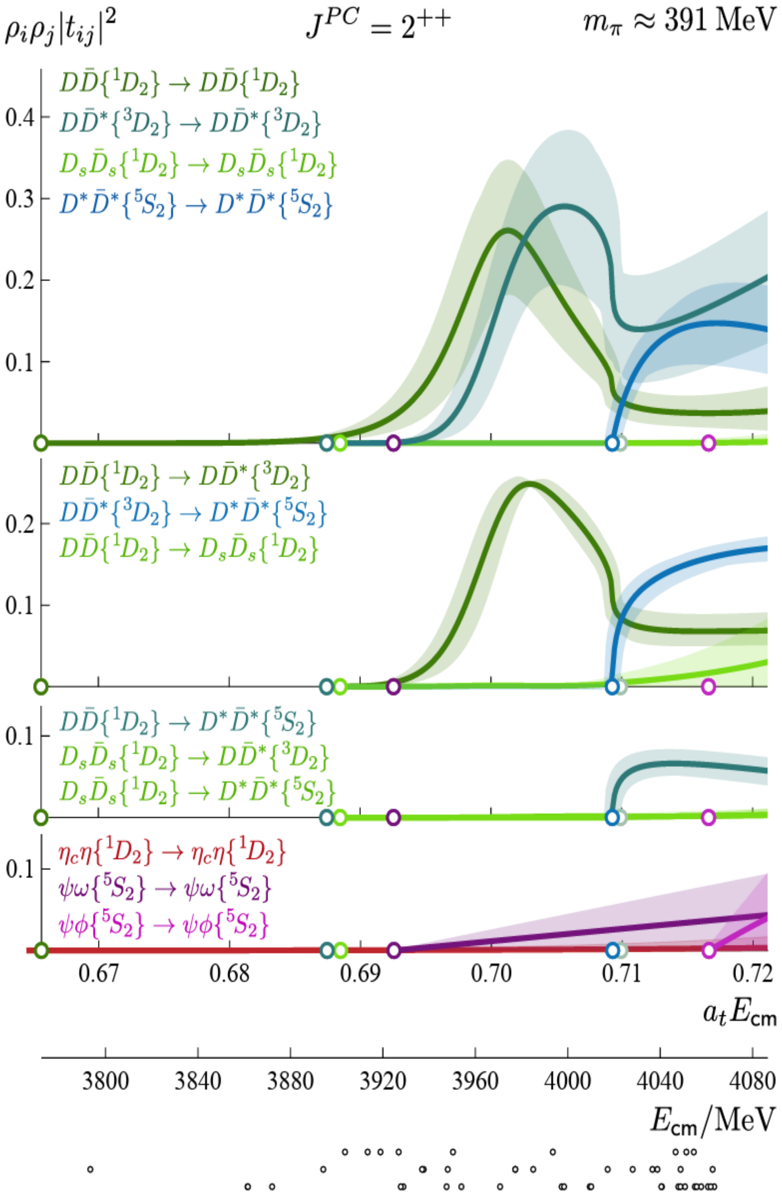
3. 2^{++} channel

- **Channels considered:** $\eta_c\eta, D\bar{D}, D\bar{D}^*, D_s\bar{D}_s, \psi\omega, D^*\bar{D}^*, \psi\phi$
- **K-matrix parameterization and results:** $K_{ij} = \frac{g_i g_j}{m^2 - s} + \sum_a s^a \gamma_{ij}^a$

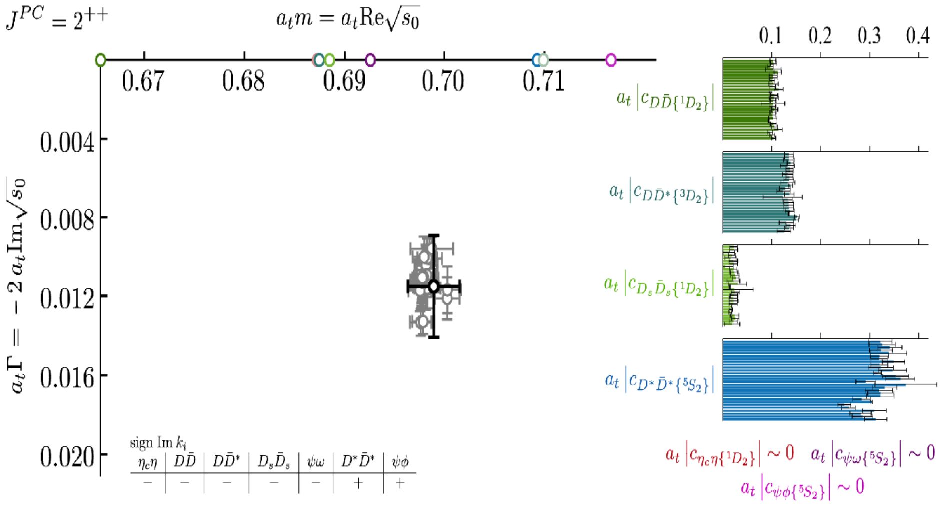
$$
\begin{aligned}
a_t m &= (0.7025 \pm 0.0012 \pm 0.0007) \\
g_{D\bar{D}^*\{^3D_2\}} &= (-37.9 \pm 5.0 \pm 3.94) \cdot a_t \\
g_{D_s\bar{D}_s\{^1D_2\}} &= (-3.3 \pm 4.3 \pm 2.5) \cdot a_t \\
g_{D^*\bar{D}^*\{^1S_2\}} &= (1.58 \pm 0.15 \pm 0.22) \cdot a_t^{-1} \\
\gamma_{\eta_c\eta\{^1D_2\}\to\eta_c\eta\{^1D_2\}} &= (16.3 \pm 23.1 \pm 7.5) \cdot a_t^4 \\
\gamma_{D\bar{D}\{^1D_2\}\to D_s\bar{D}_s\{^1D_2\}} &= (-81 \pm 129 \pm 100) \cdot a_t^4 \\
\gamma_{\psi\omega\{^5S_2\}\to\psi\omega\{^5S_2\}} &= (0.55 \pm 0.72 \pm 0.81) \\
\gamma_{\psi\phi\{^5S_2\}\to\psi\phi\{^5S_2\}} &= (2.19 \pm 0.77 \pm 0.11) \\
g_{D\bar{D}\{^1D_2\}} &= 10 \cdot a_t\,(\text{fixed}) \\
\chi^2/N_{\text{d.o.f.}} &= \tfrac{62.8}{86-8-23} = 1.14,
\end{aligned}
$$



Cross sections



Pole position and couplings c_{AB} in the T-matrix



$a_t c_{\eta_c\eta\{^1D_2\}} \approx 0$

$a_t c_{D\bar{D}\{^1D_2\}} = 0.103(25)$

$a_t c_{D\bar{D}^*\{^3D_2\}} = 0.123(39)$

$a_t c_{D_s\bar{D}_s\{^1D_2\}} = 0.032(32)$

$a_t c_{\psi\omega\{^5S_2\}} \approx 0$

$a_t c_{D^*\bar{D}^*\{^5S_2\}} = 0.336(99)$

$a_t c_{\psi\phi\{^5S_2\}} \approx 0.$

- **Only one resonance pole between $\psi\omega$ and $D^*\bar{D}^*$**
- **Pole position:**

$$\sqrt{s_0} = \left[(3961 \pm 15) - \frac{i}{2}(65 \pm 15)\right] \text{ MeV}$$

- **Partial decay widths:** $\mathcal{B}_i = \frac{\Gamma_i}{\Gamma} = |c_i|^2 \frac{\rho_i(Re\ s_0)}{\sqrt{Re\ s_0}}$,

$$\Gamma_{D\bar{D}} = 26(12) \text{ MeV},$$
$$\Gamma_{D\bar{D}^*} = 22(14) \text{ MeV}$$
$$\Gamma_{D_s\bar{D}_s} = 2^{+3}_{-2} \text{ MeV}$$

- **Rapid turn-on of $D^*\bar{D}^*$ at threshold due to the large coupling $c_{D^*\bar{D}^*}$**

4. To summarize on the $(0, 2)^{++}$ channels

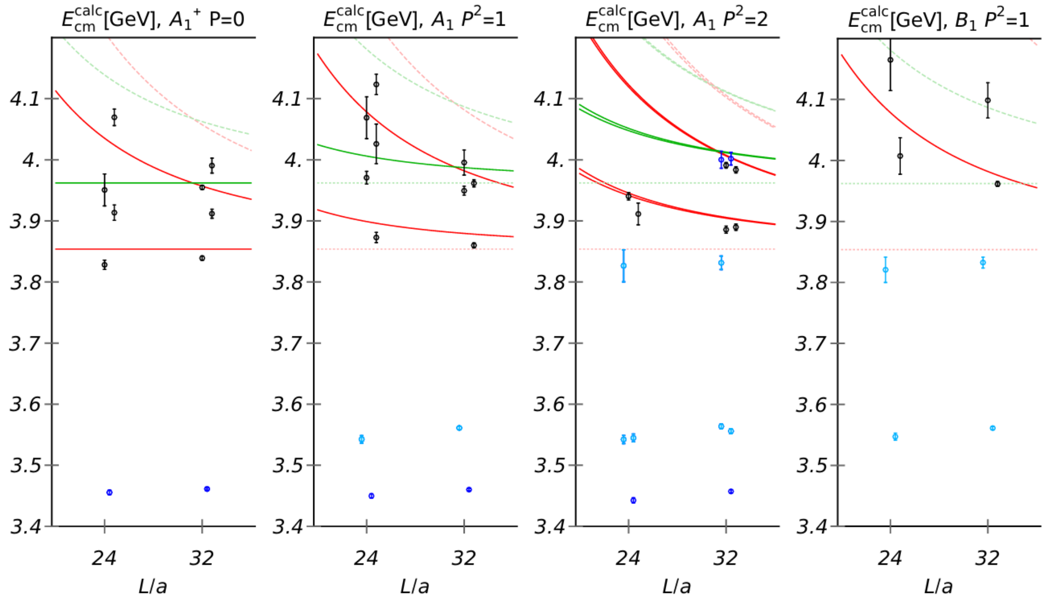
- The key observation: Only a single narrow resonance in each channel, lies above the $D_s\bar{D}_s$ threshold but slightly below $D^*\bar{D}^*$ threshold.
- For both channels, the resonance couples strongly to open-charm decay channels.
- Neither resonance has any significant couplings to closed-charm channels.
- There is no indication of any further states in the energy region below the $\psi\phi$ threshold.
- No large scattering amplitudes for closed-charm system composed of a charmonium and a light hadron.
- No additional states can be assigned to a tetraquark state.

5. Comparison with a previous lattice QCD study

(S. Prelovsek et al., JHEP 06 (2021) 035)

- Two volumes at $\boldsymbol{m_\pi = 280\ \mathrm{MeV}}$, only open-charm scattering channels are considered.
- The operator set includes $\bar{c}c$ operators and ($D\bar{D}$, $D_s\bar{D}_s$) operators with different relative momenta.
- Less finite volume energy levels.

	$\lvert\vec{P}\rvert^2$	$\Lambda^{(P)C}$	O	N_{ops}		$\lvert\vec{P}\rvert^2$	$\Lambda^{(P)C}$	O	N_{ops}
O_h	0	A_1^{++}	$\bar{c}c$	7	Dic$_4$	1	B_1^{+}	$\bar{c}c$	11
			$D(0)\bar{D}(0)$	2				$D(2)\bar{D}(1)$	1
			$D(1)\bar{D}(1)$	1	Dic$_2$	2	A_1^{+}	$\bar{c}c$	28
			$D_s(0)\bar{D}_s(0)$	2				$D(2)\bar{D}(0)$	2
			$D^*(0)\bar{D}^*(0)$	2				$D(1)\bar{D}(1)$	2
			$J/\psi(0)\omega(0)$	2				$D(2)\bar{D}(2)$	1
Dic$_4$	1	A_1^{+}	$\bar{c}c$	17				$D(3)\bar{D}(1)$	1
			$D(1)\bar{D}(0)$	2				$D_s(2)\bar{D}_s(0)$	1
			$D(2)\bar{D}(1)$	1				$D_s(1)\bar{D}_s(1)$	1
			$D_s(1)\bar{D}_s(0)$	2				$J/\psi(2)\omega(0)$	3
			$J/\psi(1)\omega(0)$	2					



$$\Lambda^P = A_1^+\,, \qquad |\vec{P}|^2 = 0\,, \qquad J^P[\lambda] = 0^+[0]\,,$$
$$\Lambda = A_1\,, \qquad |\vec{P}|^2 = 1\,, \qquad J^P[\lambda] = 0^+[0],\ 2^+[0]\,,$$
$$\Lambda = A_1\,, \qquad |\vec{P}|^2 = 2\,, \qquad J^P[\lambda] = 0^+[0],\ 2^+[0],\ 2^\pm[2]\,,$$
$$\Lambda = B_1\,, \qquad |\vec{P}|^2 = 1\,, \qquad J^P[\lambda] = 2^\pm[2]\,.$$

- ✓ A 0^{++} shallow bound state ($E_B \sim -4$ MeV) is observed right below the $D\bar{D}$ threshold

$$m - 2m_D = -4.0^{+3.7}_{-5.0}\,\mathrm{MeV}$$

- ✓ Single-channel analysis of $D_s\bar{D}_s(l=0)$ scattering

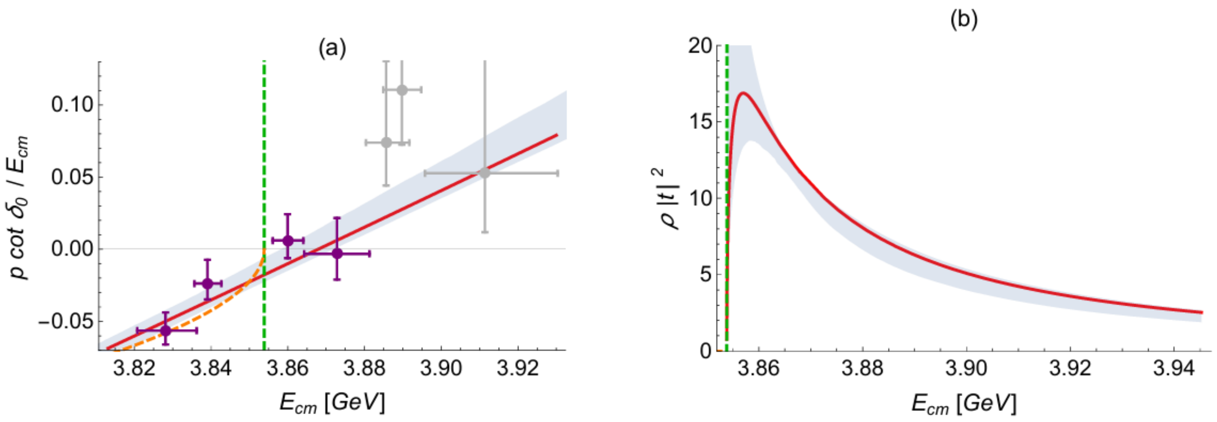
$$m - 2m_{D_s} = -6.2^{+3.8}_{-2.0}\ \mathrm{MeV}$$

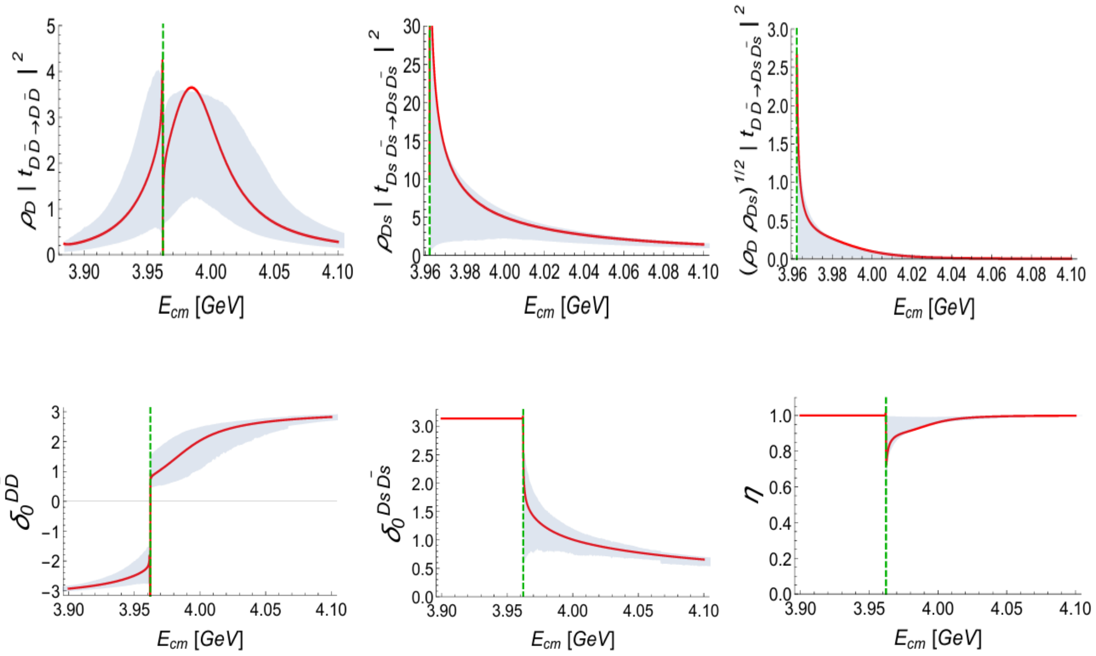
- ✓ A narrow resonance and a wider resonance appear around the $D_s\bar{D}_s$ threshold, through coupled $D\bar{D} - D_s\bar{D}_s$ channel analysis

$$\frac{(\tilde{K}^{-1})^{l=0}}{E_{\rm cm}} = \begin{pmatrix} a_{11} + b_{11}E_{\rm cm}^2 & a_{12} \\ a_{12} & a_{22} + b_{22}E_{\rm cm}^2 \end{pmatrix} \equiv \begin{pmatrix} \frac{m_{J0}^2 - E_{\rm cm}^2}{g_{J0}^2} & a_{12} \\ a_{12} & a_{22} + b_{22}E_{\rm cm}^2 \end{pmatrix}$$

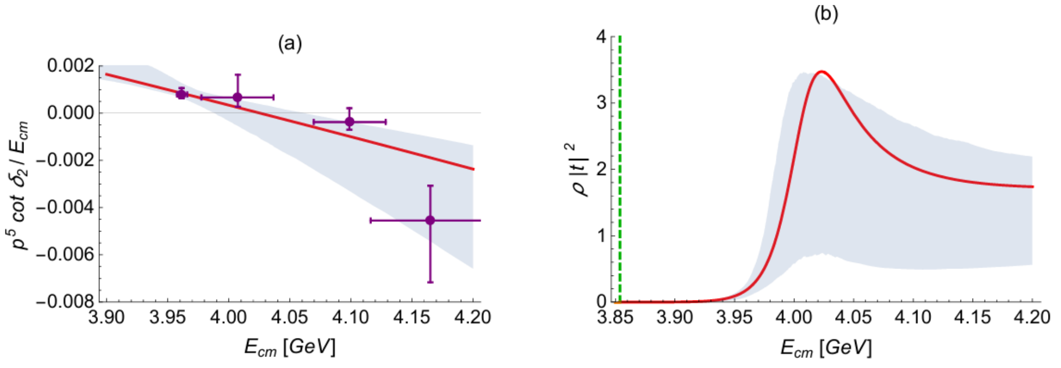
$$\chi'_{c0}:\ \ m - M_{av} = 880^{+28}_{-20}\,\mathrm{MeV},\quad \Gamma = 58\ {}^{+6}_{-11}\ \mathrm{MeV}$$

$$\chi_{c0}^{D_s\bar{D}_s}:\ \ E_{\rm cm}^p - 2m_{D_s} = (-0.2\ {}^{+0.16}_{-4.9})\ -\frac{i}{2}\ (0.27\ {}^{+2.5}_{-0.15})\,\mathrm{MeV}$$





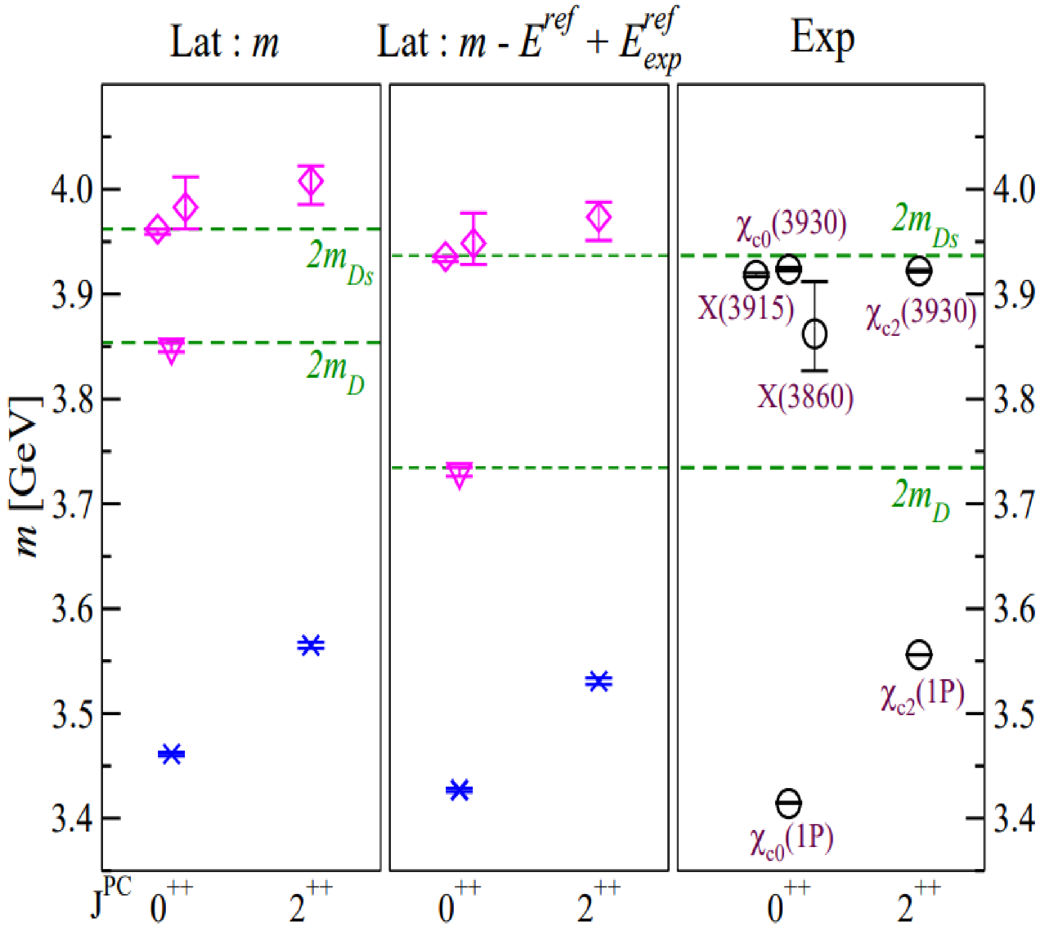
✓ Single-channel analysis of $D\bar{D}(l=2)$ scattering



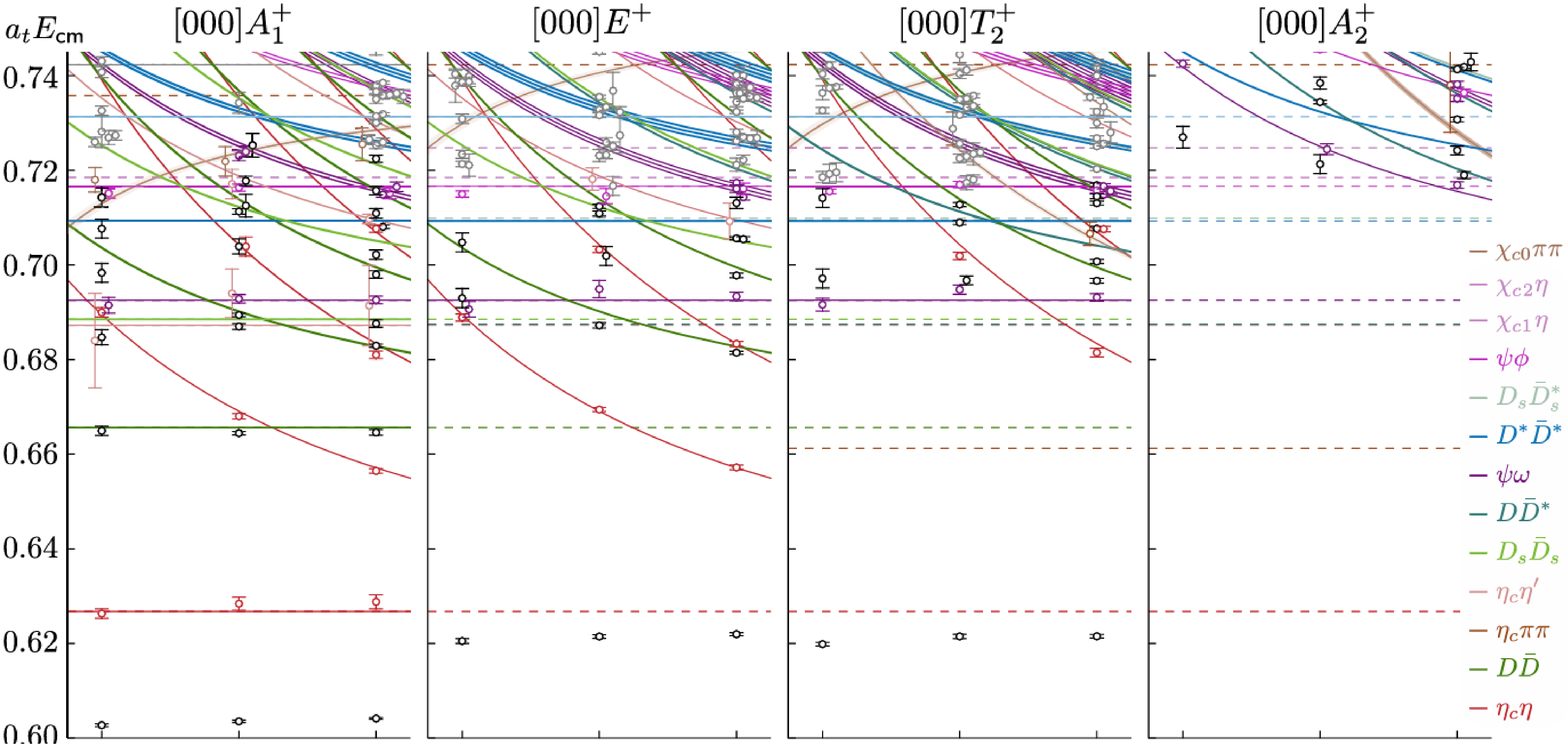
$$\chi_{c2}(3930): \quad E^p = (4.008\,^{+0.014}_{-0.022}) - \frac{i}{2}\,(0.064^{+0.032}_{-0.042})\ \text{GeV}$$

✓ One bound state pole, one narrow resonance, and one wider resonance observed in 0^{++} channel.

✓ One resonance observed in 2^{++} channel.



Finite volume energy levels in the rest frame of the system



V. Summary and perspectives

1^{++} channel:

- $X(3872)$ as a (virtual) $D\bar{D}^*$ bound state at unphysical m_π
- Hint at a resonance between 3.9 and 4.0 GeV, which may correspond to $\chi_{c1}(2P)$

- X(3940) : $I^G J^{PC} = ?^? ?^{??}$, $m_X = 3942^{+7}_{-6} \pm 6$ MeV, $\Gamma_x = 37^{+26}_{-15} \pm 8$ MeV, decays to $D\bar{D}^*$

- $\chi_{c1}(4010)$: $J^{PC} = 1^{++}$, $m_{\chi_{c1}} = 4012.5^{+3.6+4.1}_{-3.9-3.7}$ MeV, $\Gamma_x = 62.7^{+7.0+6.4}_{-6.4-6.6}$MeV, decays to $D\bar{D}^*$
  (LHCb, arXiv:2406.03156[hep-ex])

0^{++} channel:

- Unique resonance around 3.9-4.0 GeV: $m_{\chi_{c0}} = 3995(14)$MeV, $\Gamma_{\chi_{c0}} = 67(38)$ MeV
- Decay modes: $\Gamma_{D\bar{D}} = 23(13)$ MeV, $\Gamma_{D_s\bar{D}_s} = 28(26)$ MeV, $\Gamma_{\psi\omega} = 9^{+18}_{-9}$ MeV

- X(3915) (aka $\chi_{c0}(3915)$) : $I^G J^{PC} = 0^+ 0^{++}$, $m_X = 3922.1 \pm 1.8$ MeV, $\Gamma_x = 20 \pm 4$ MeV
  Decay channels: $D^+D^-, D_s^+D_s^-, \omega J/\psi, \gamma\gamma$, $\Gamma(D^+D^-)/\Gamma(D_s^+D_s^-) = 0.29 \pm 0.016$

2^{++} channel:

- Unique resonance around 3.9-4.0 GeV: $m_{\chi_{c2}} = 3961(15)$MeV, $\Gamma_{\chi_{c2}} = 65(15)$ MeV
- Decay modes: $\Gamma_{D\bar{D}} = 26(12)$ MeV, $\Gamma_{D\bar{D}^*} = 22(14)$ MeV, $\Gamma_{D_s\bar{D}_s} = 2^{+3}_{-2}$ MeV

- $\chi_{c2}(3930)$: $I^G J^{PC} = 0^+ 2^{++}$, $m_X = 3922.5 \pm 1.0$ MeV, $\Gamma_{\chi_{c2}} = 35.2 \pm 2.2$ MeV
  X(3915) decay channels: $D^+D^-, D^0\bar{D}^0, \gamma\gamma$, $\Gamma(D^+D^-)/\Gamma\left(D^0\bar{D}^0\right) = 0.74 \pm 0.43 \pm 0.16$

Thank you for your Attention!

Thanks!