Non-extremal Island in de Sitter Gravity

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arXiv:2306.07575 with Taishi Kawamoto, Yu-ki Suzuki, Tadashi Takayanagi
arXiv:2407.21617 with Peng-Xiang Hao, Taishi Kawamoto, Tadashi Takayanagi

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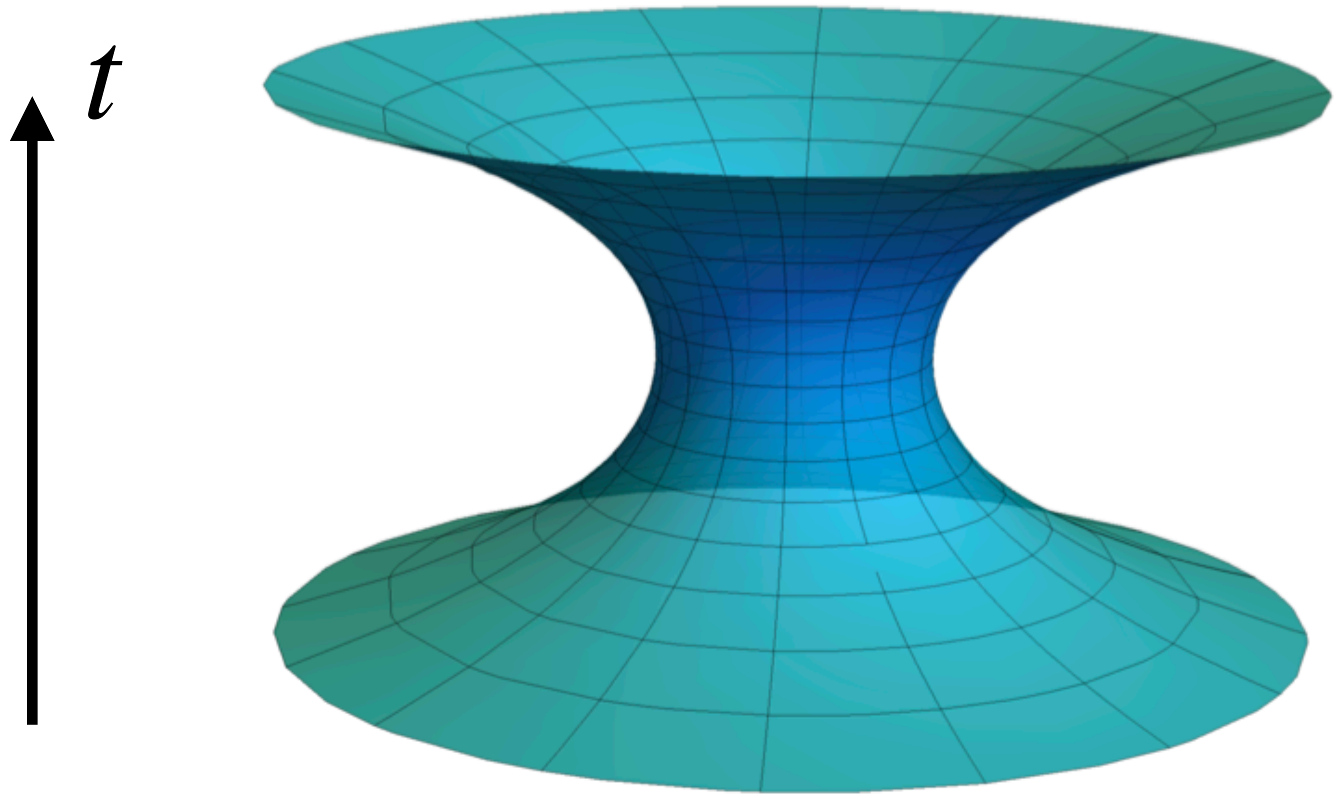
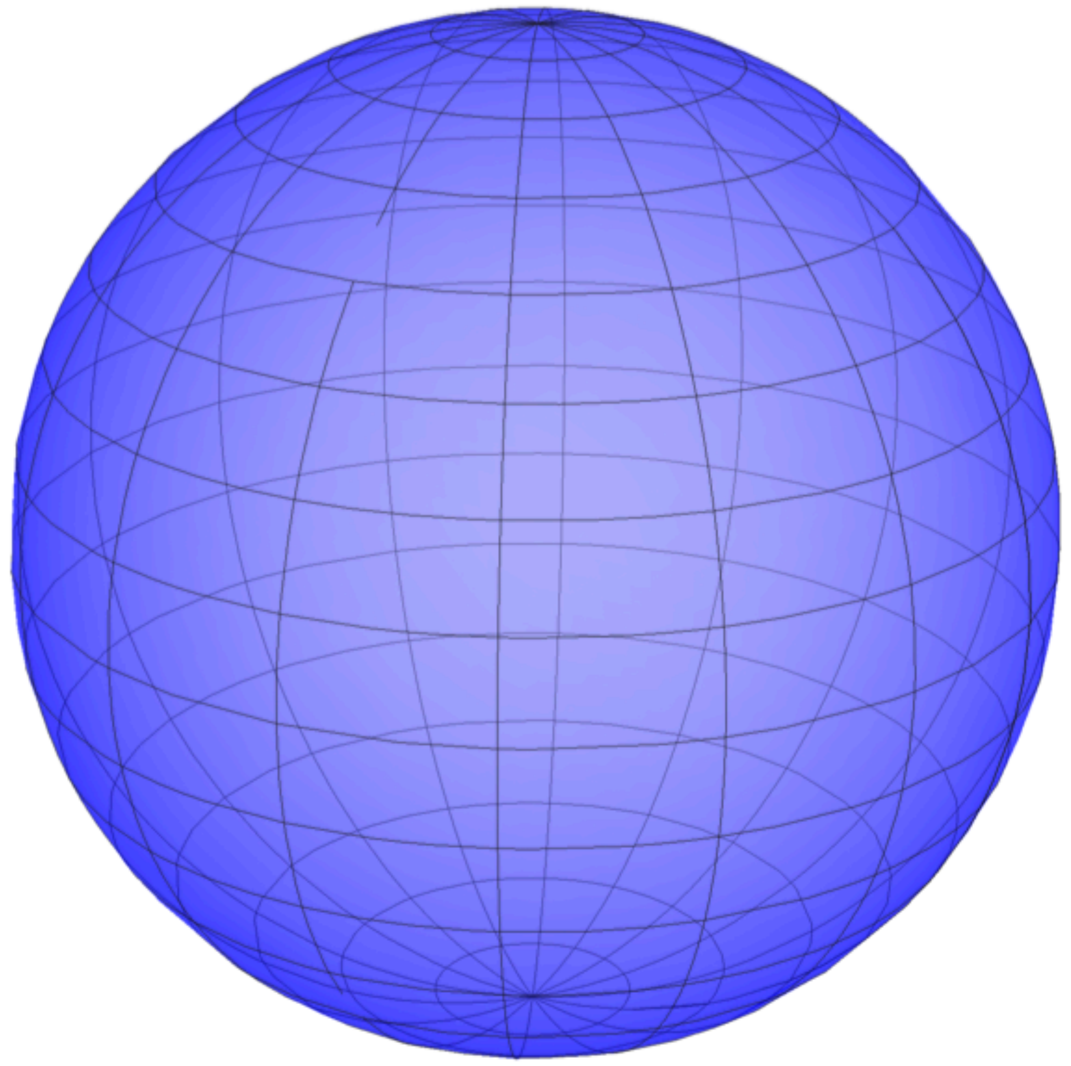
- 01. Introduction
- 02. de Sitter Holography
- 03. Problematic Extremal Islands in dS gravity
- 04. Resolution from Double Holography

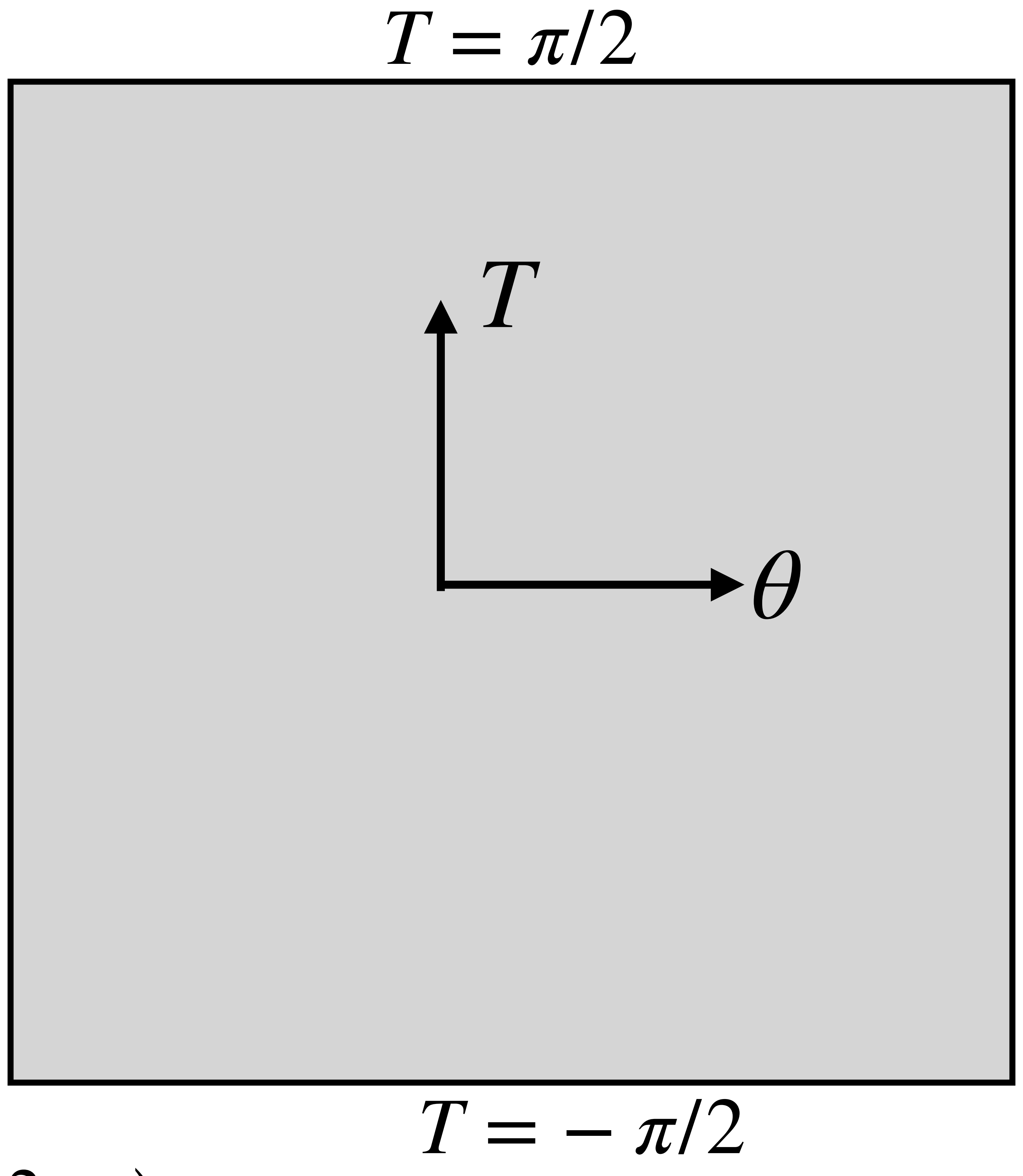
01. Introduction- de Sitter Space

01. Introduction: de Sitter Spacetime

$$ds^2 = L^2 \left(-dt^2 + \cosh^2 t \, (d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2) \right)$$

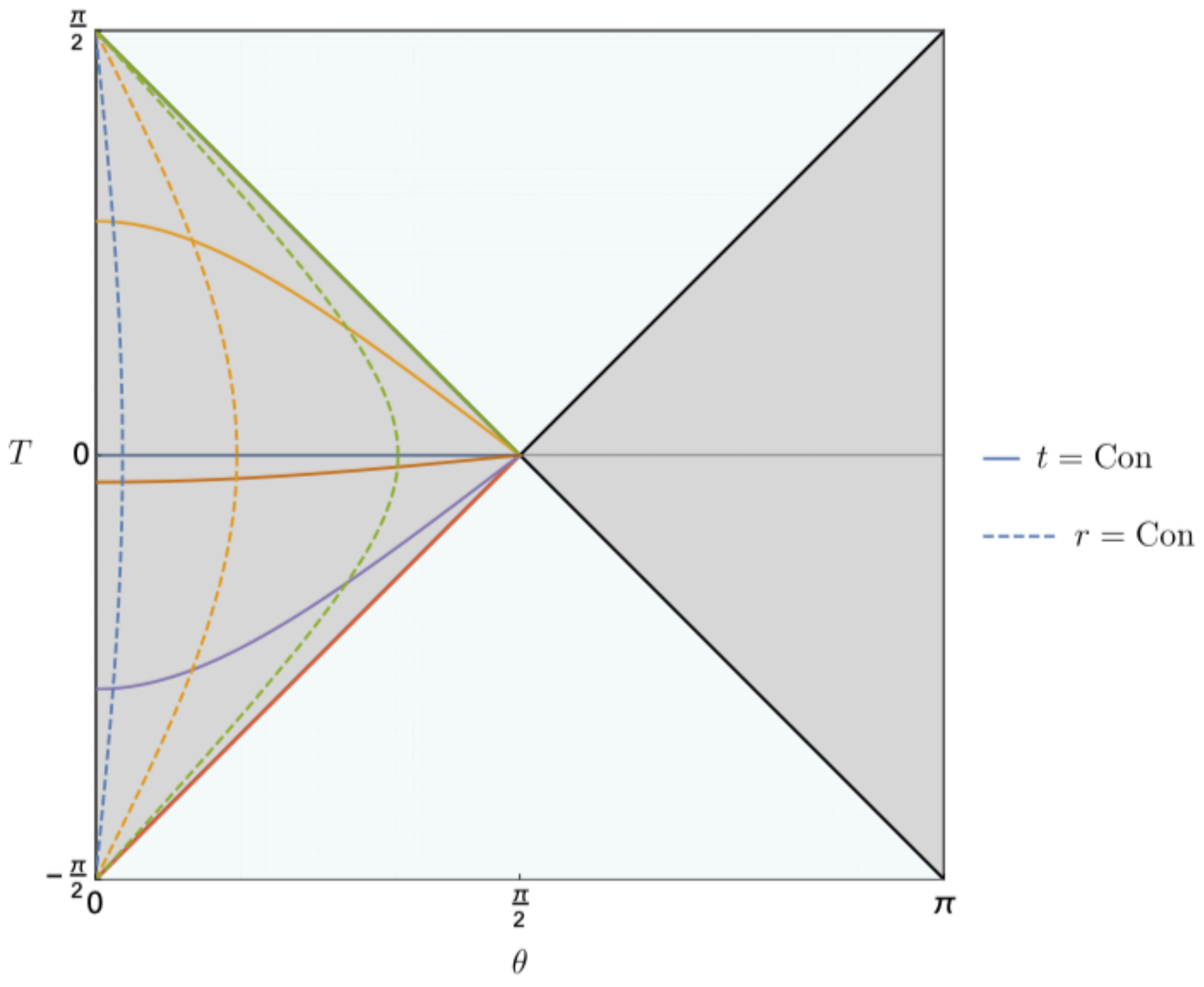






$$ds^2 = \frac{L^2}{\cos^2 T} \left(-dT^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2 \right)$$

Static Patch



$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{L^2}} + r^2 d\Omega_{d-1}^2 \, .$$

Hawking temperature of cosmological horizon

$$T = \frac{1}{2\pi L}$$

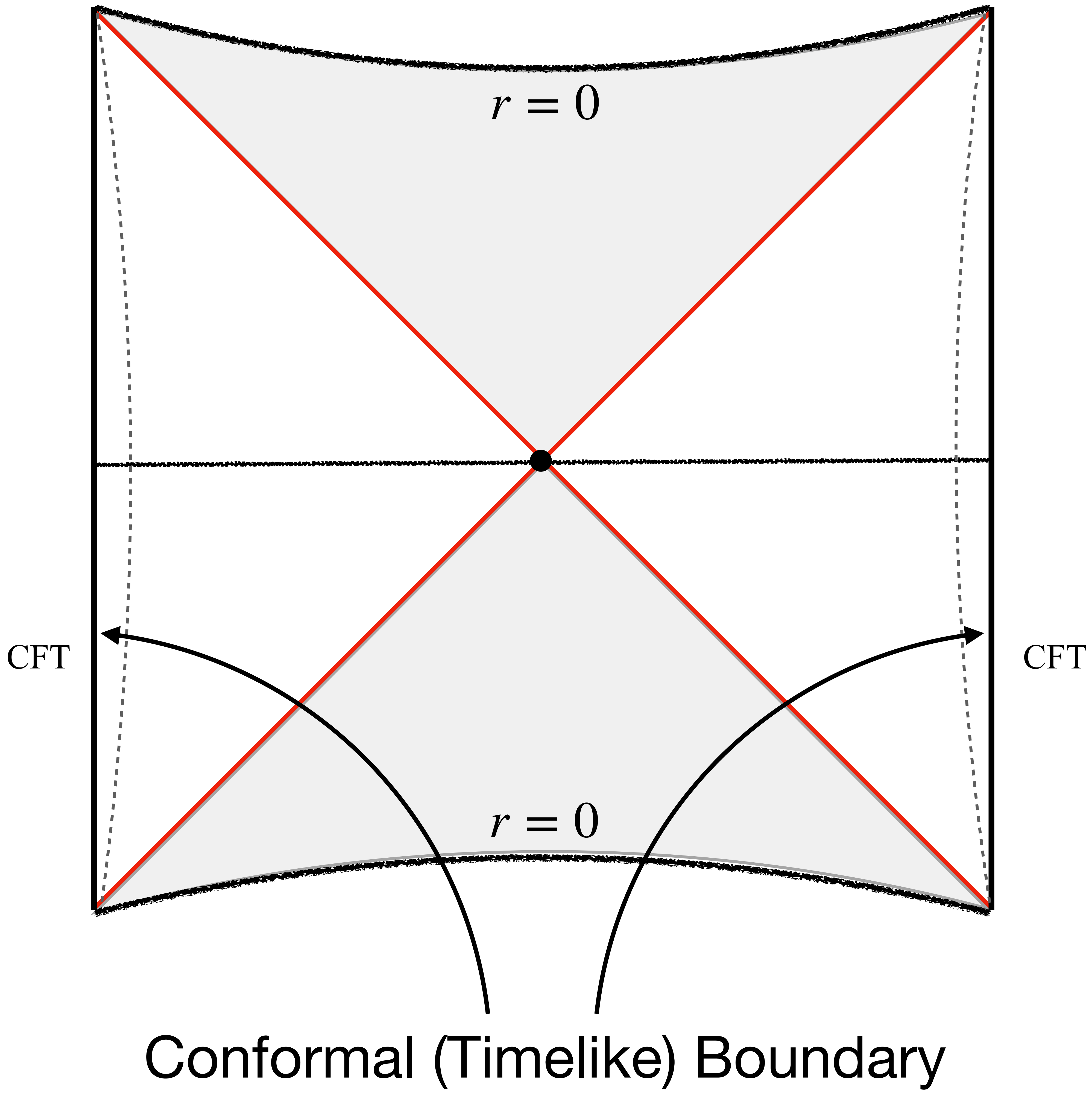
Entropy —> Bekenstein-Hawking formula

$$S_{\mathrm{dS}} = \frac{\mathrm{Area}}{4G_{\mathrm{N}}} = \frac{L^{d-1}\Omega_{d-1}}{4G_{\mathrm{N}}} \equiv N$$

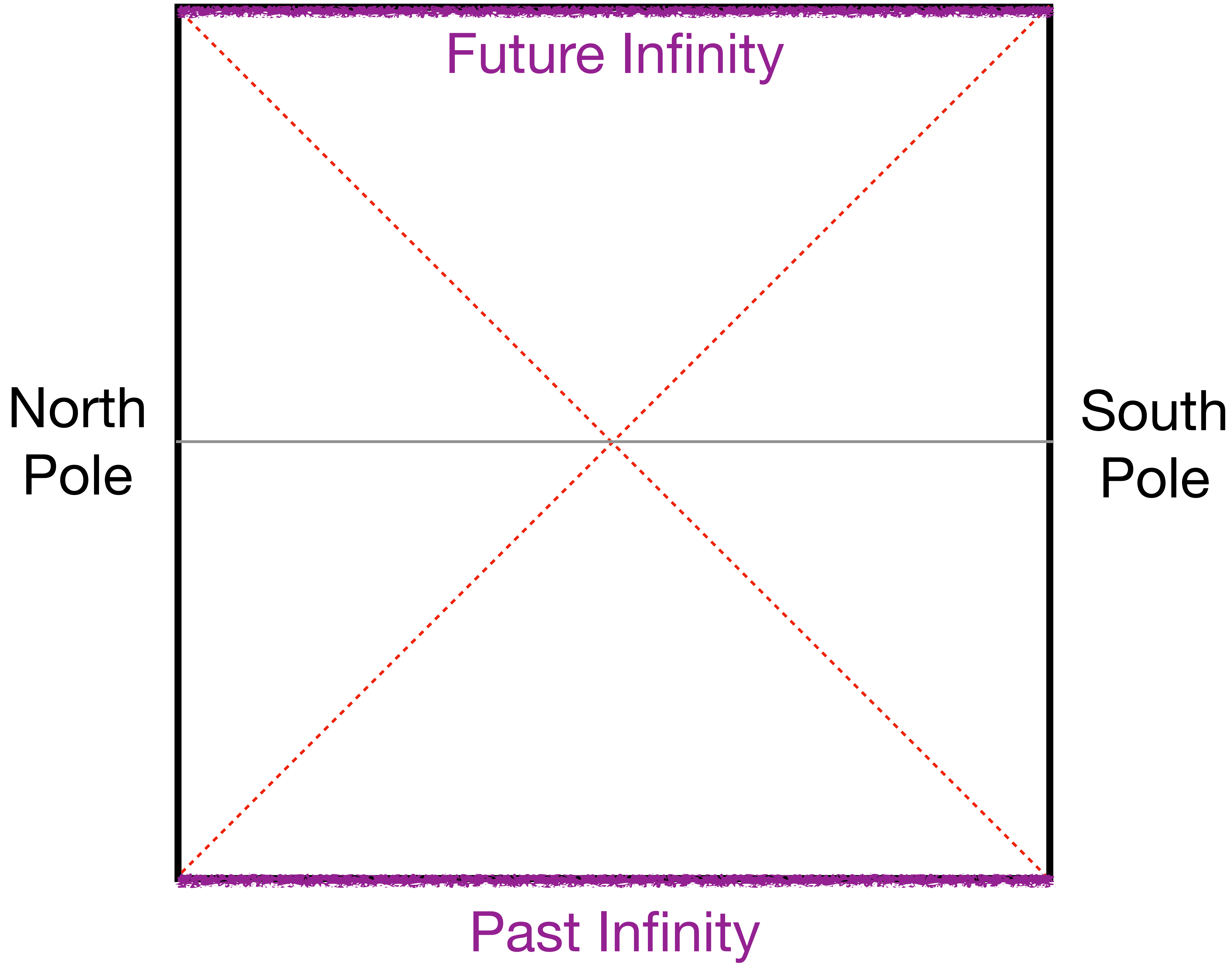
Maximal entropy? Finite Dimension of Hilbert Space?

01. Introduction: de Sitter Spacetime

AdS Black Hole



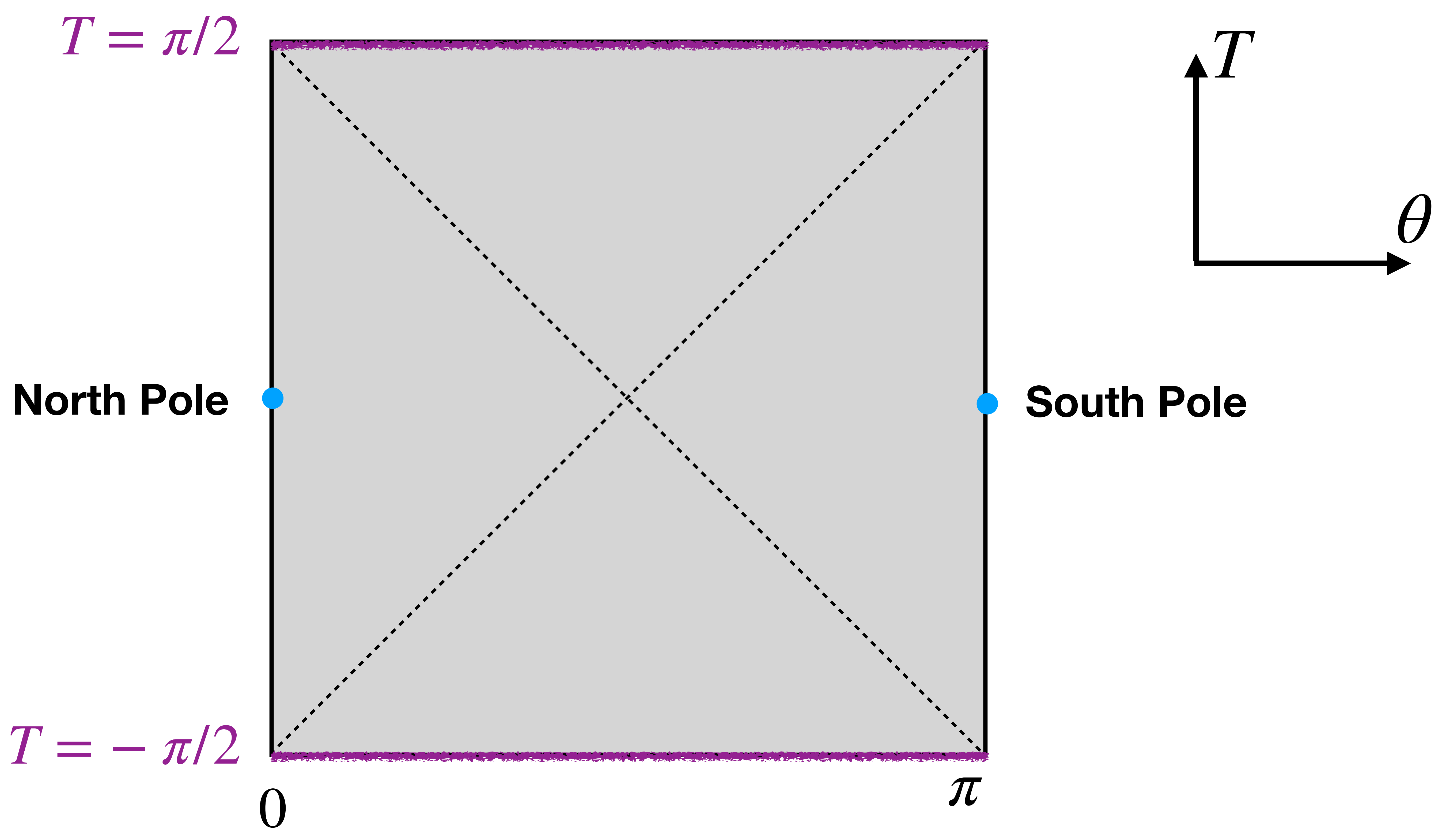
dS_{d+1}



02. de Sitter Holography

dS/CFT Correspondence

[Strominger 2001]

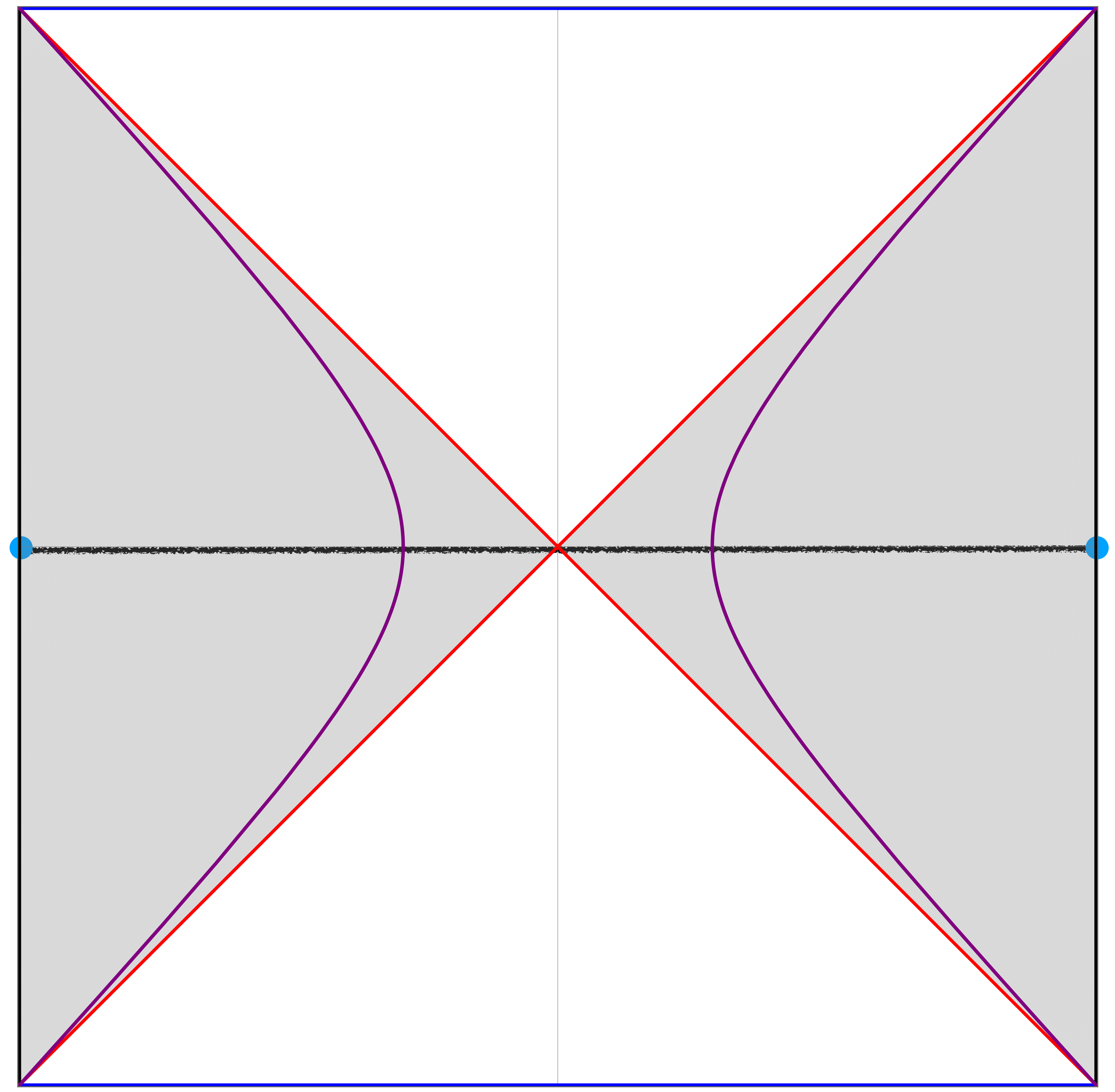


"Analytical continuation" of AdS/CFT correspondence

dS/CFT correspondence: gravity in asymptotically dS_{d+1} spacetime
a d-dimensional CFT living on the spacelike boundary at timelike future infinity of dS.

Static Patch Holography

[Parikh, Verlinde 2004]
[Alishahiha, Karch, Silverstein, Tong 2004]
[Banks, Fiol and Morisse 2006]
[Susskind 2021]
[Shaghoulian 2021]
[Narovlansky, Verlinde 2023]

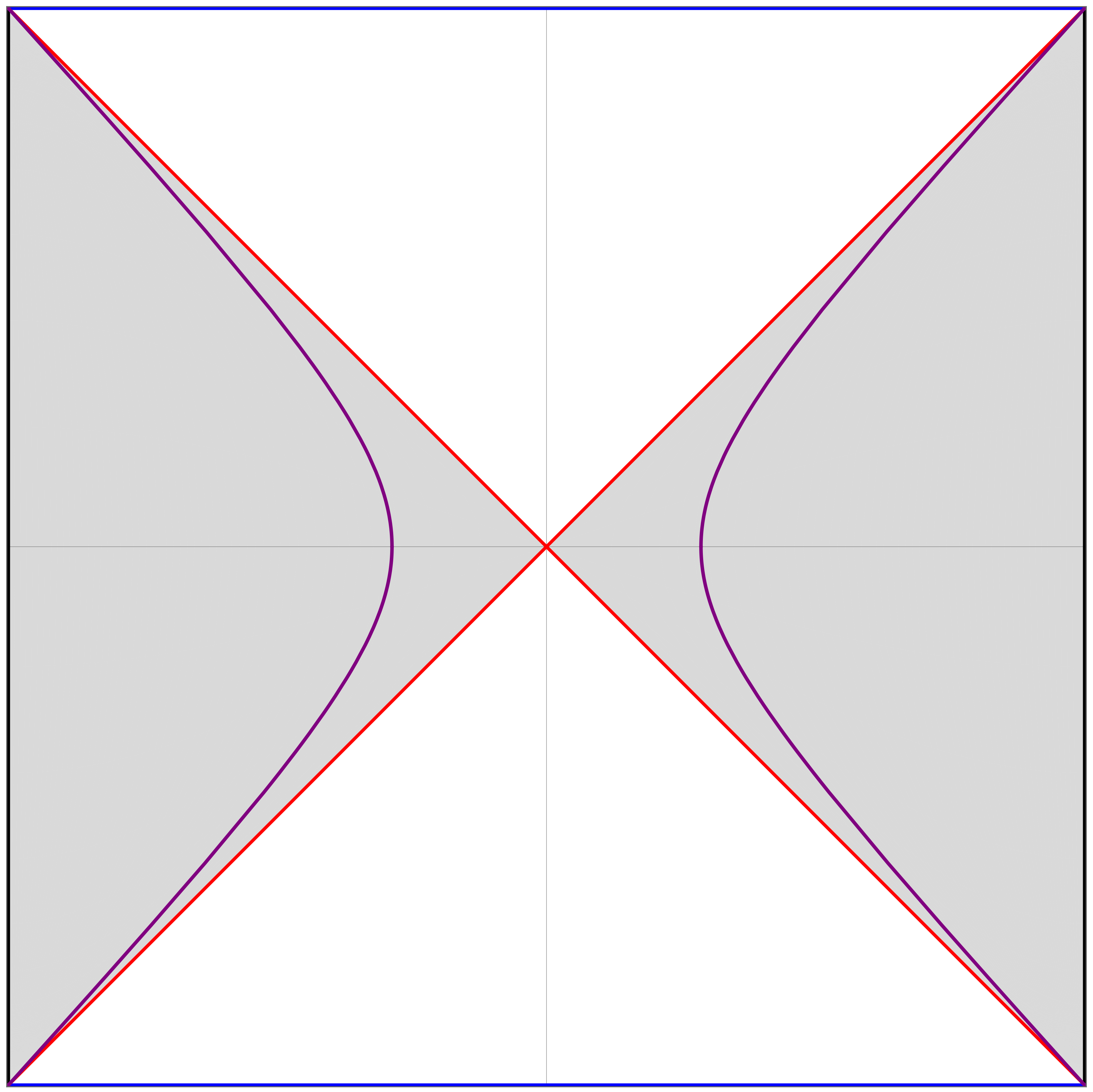


holographic degrees of freedom of de Sitter space

↓

Stretched Horizon

Static Patch Holography



De Sitter version of (H)RT formula **[Susskind 2021]**

$$S_{\mathrm{EE}} = \min \max \left(\frac{\mathrm{Area}}{4G_N} \right) \quad ?$$

entanglement entropy between the pode/antipode static patch

$$S_{\mathrm{EE}} = \frac{\text{Horizon Area}}{4G_N} = S_{\mathrm{dS}}$$

Different results: Replica trick + Holographic two-point function
[To appear, with Yu-ki Suzuki]

A Half de Sitter Holography

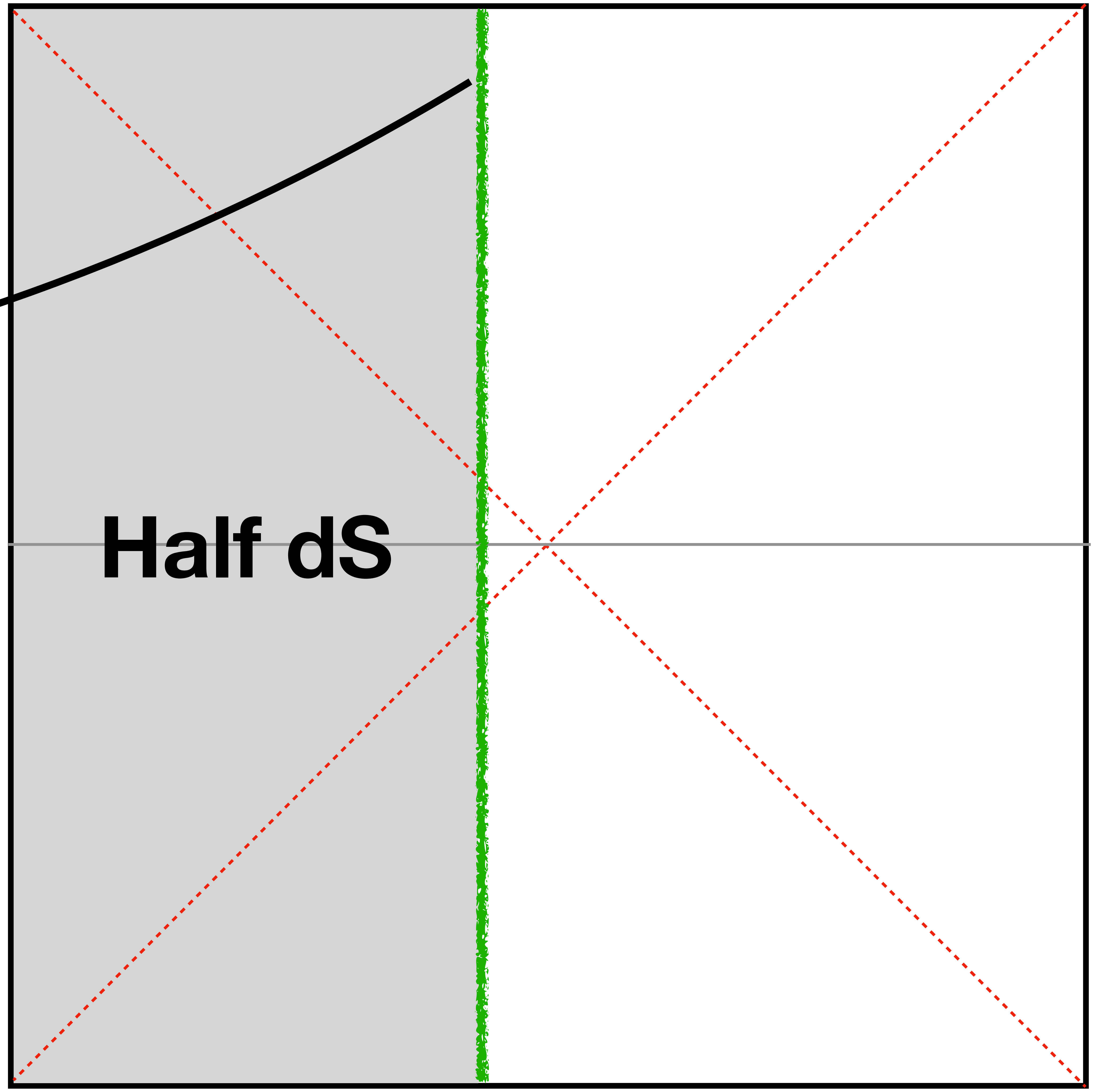
(H)RT formula in a half dS spacetime

Violation of (strong) subadditivity!

Non-local field theory on the timelike boundary?

arXiv:2306.07575

with Taishi Kawamoto, Yu-ki Suzuki, Tadashi Takayanagi



Goal: To Probe a Half de Sitter Space by a Bath System

arXiv:2407.21617

With Peng-Xiang Hao, Taishi Kawamoto, Tadashi Takayanagi

Background- Island Formula

A correct formula for
the fine grained entropy of Hawking radiation

Island Formula: $S_{\mathrm{EE}}(\hat{\rho}_A) = \mathrm{Min}_I \left[\mathrm{Ext}_I \left(\frac{\mathrm{Area}[\partial \Sigma_I]}{4G_{\mathrm{N}}} + S_{\mathrm{Bulk}} \left(\Sigma_A \cup \Sigma_{\mathrm{Island}} \right) \right) \right]$

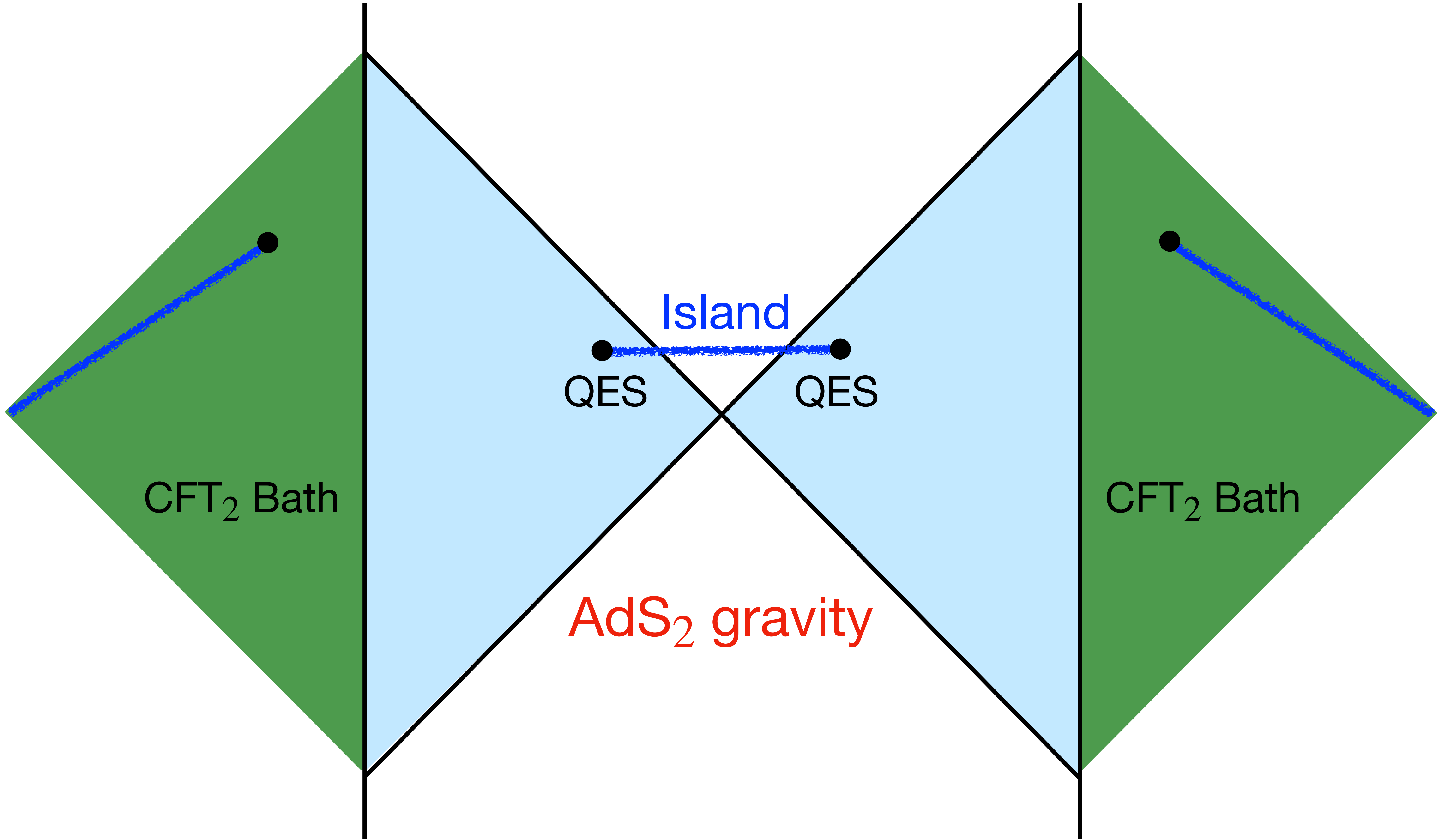
Disconnected Region

Background- Island Formula

The simplest model with Islands

An information paradox for the eternal black hole [arXiv:1910.11077]

Island Formula: $S_{\rm EE}(\hat{\rho}_A) = {\rm Min}_I \left[{\rm Ext}_I \left(\frac{{\rm Area}[\partial \Sigma_I]}{4G_{\rm N}} + S_{\rm Bulk}\left(\Sigma_A \cup \Sigma_{\rm Island}\right) \right) \right]$



Question: Does Island formula apply to dS gravity?

03: Extremal Islands in dS Gravity

03: Extremal Islands in dS gravity

A Half de Sitter

Gravitational region

Bath in de Sitter

non-gravitational region

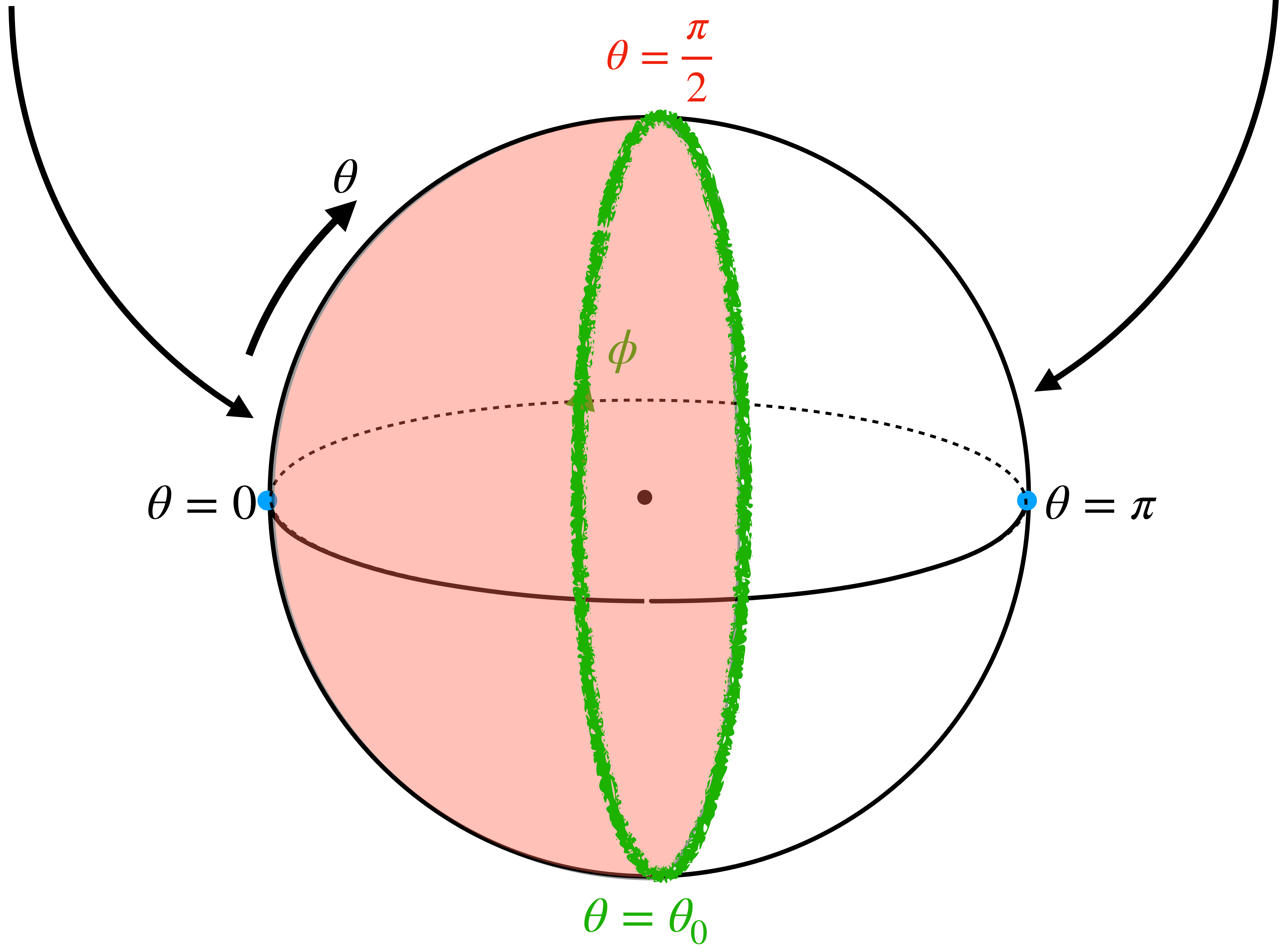
03: Extremal Islands in dS gravity

A Half de Sitter

Gravitational region

Bath in de Sitter

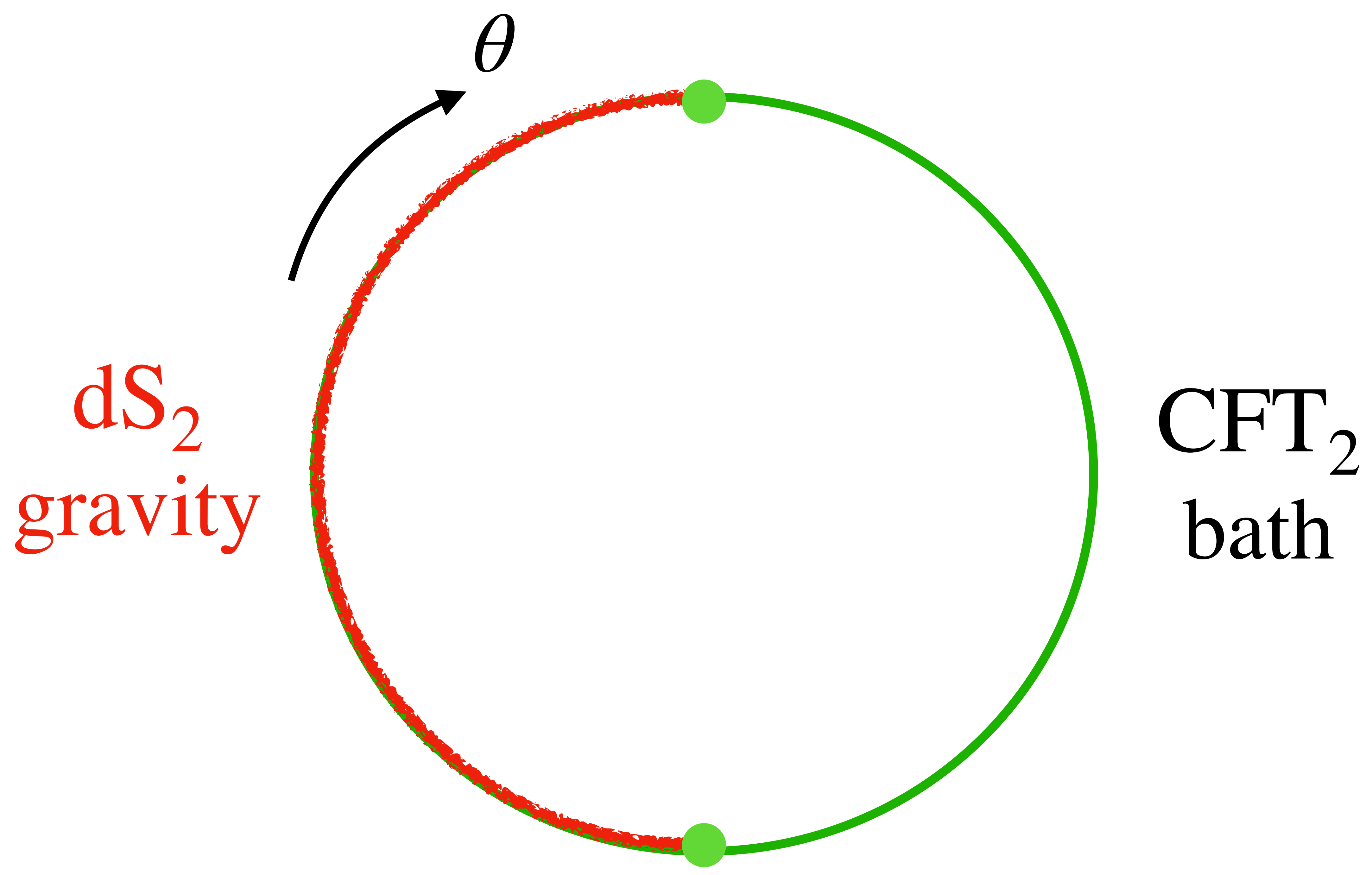
non-gravitational region

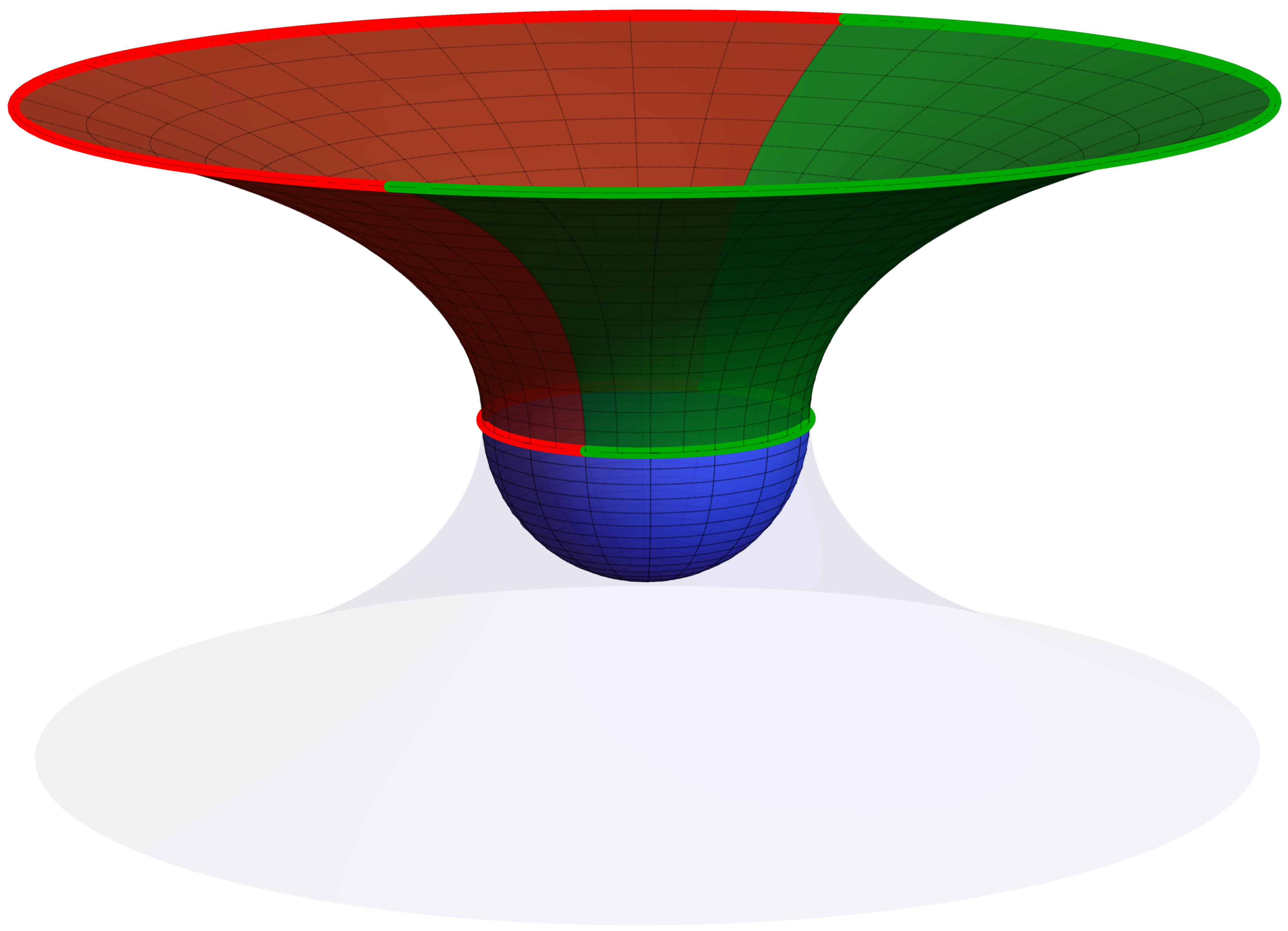


03: Extremal Islands in dS gravity

Simplest Model: Half dS_2 gravity + Half dS_2 bath

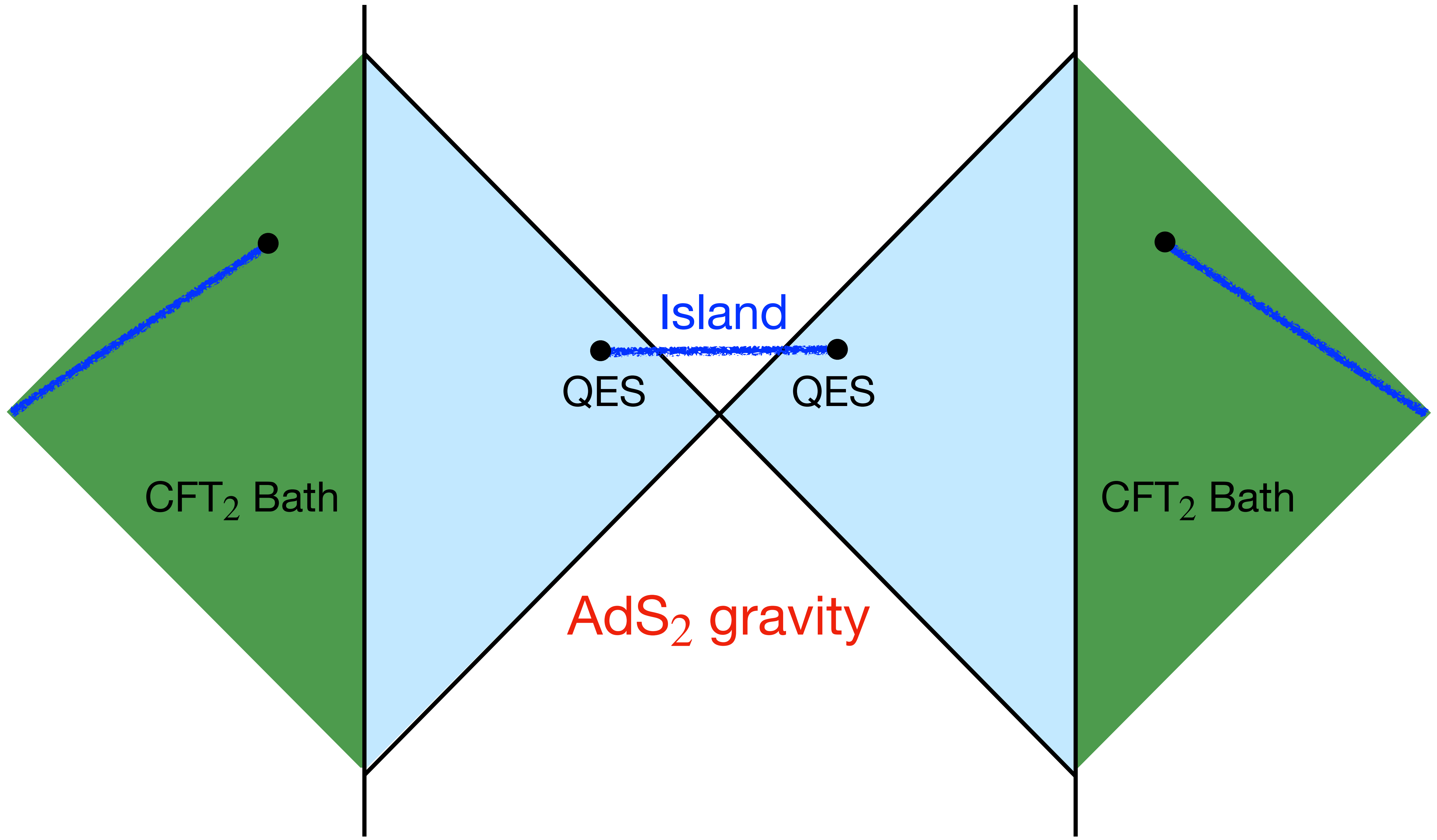
$$ds^2 = L^2\left(-dt^2 + \cosh^2 t\, d\theta^2\right) = \frac{L^2}{\cos^2 T}(-dT^2 + d\theta^2)$$



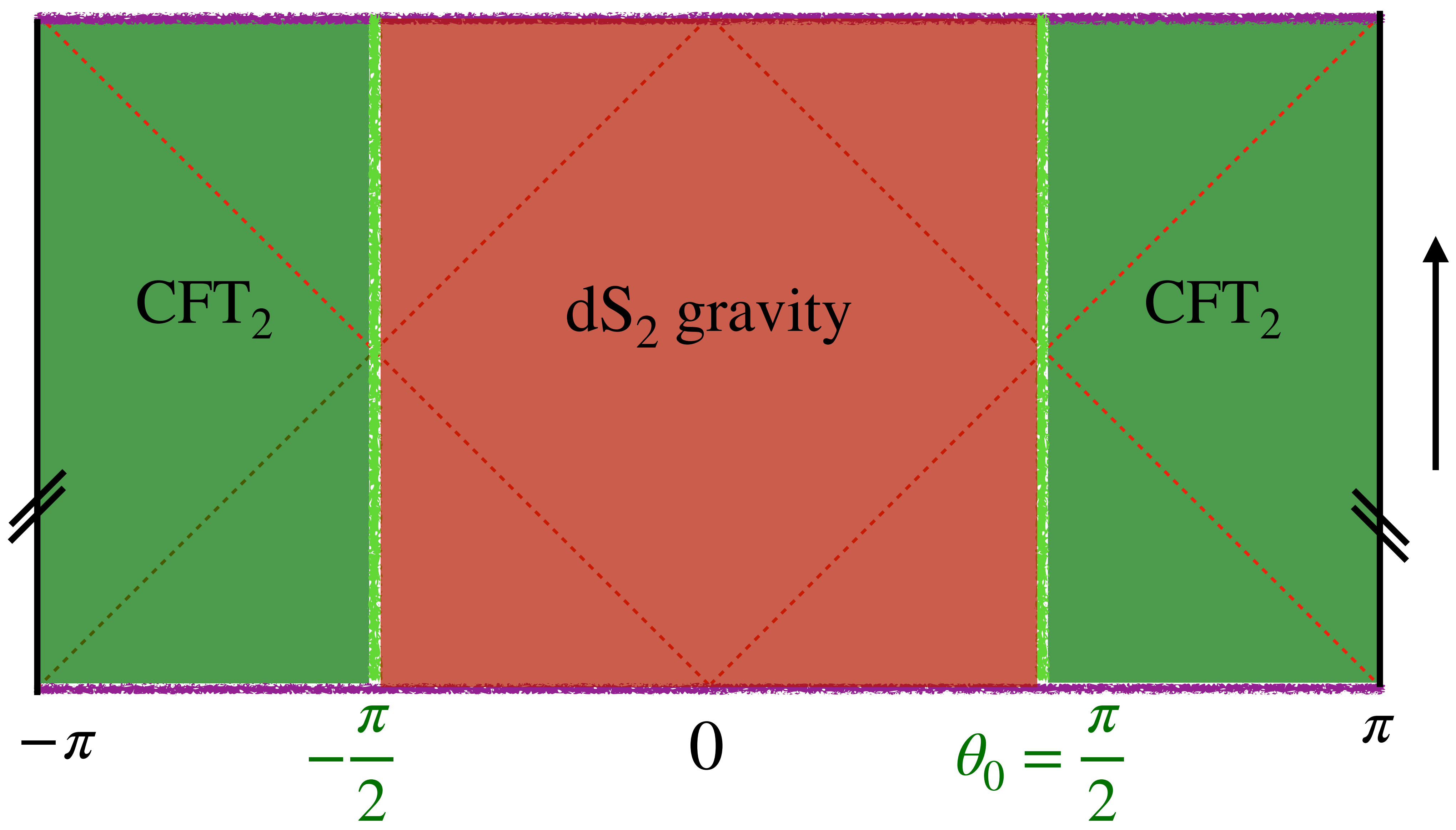


03: Extremal Islands in dS gravity

AdS+Bath



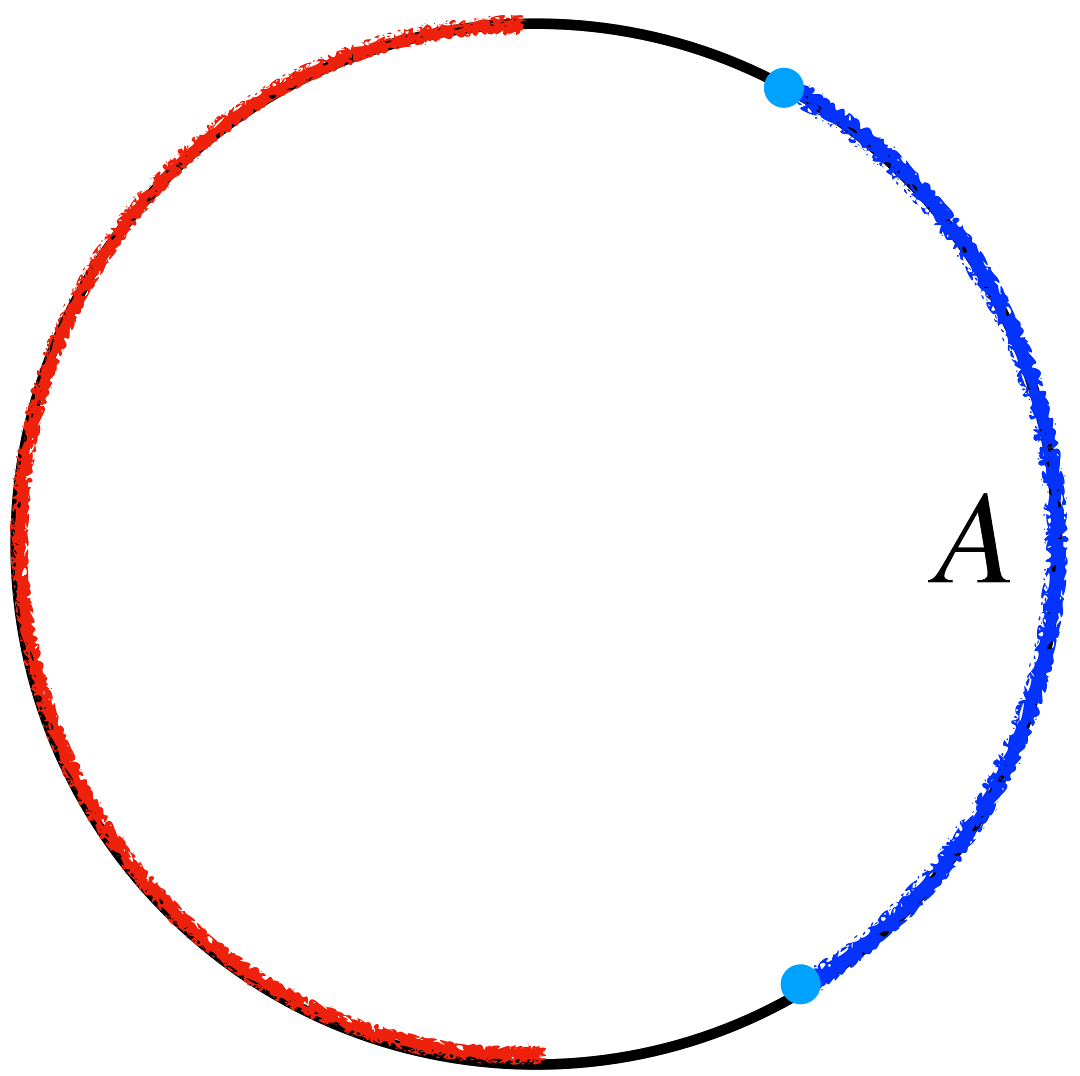
a half dS+Bath

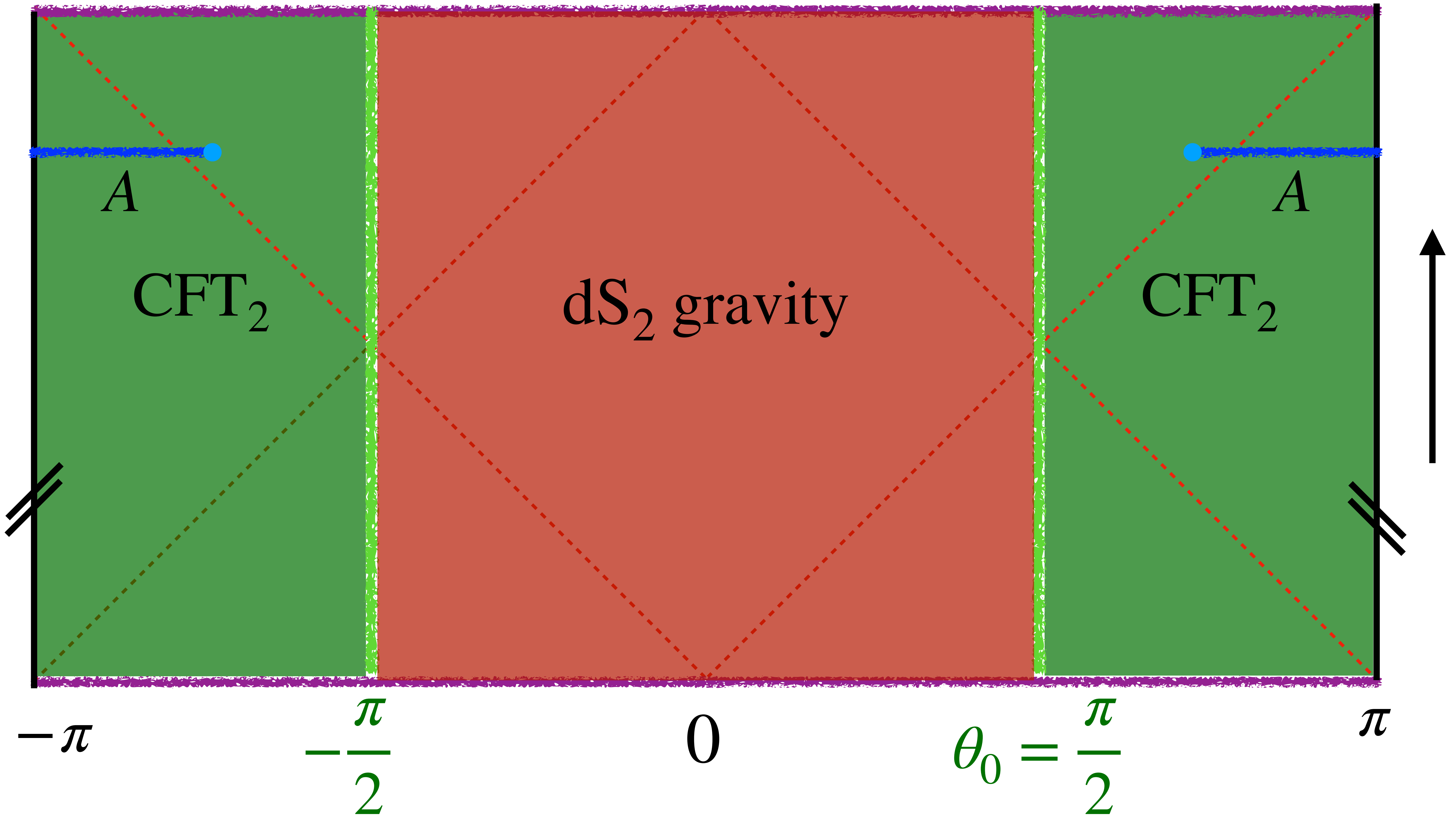


03: Extremal Islands in dS gravity

Simplest Model: Half dS_2 gravity + Half dS_2 bath (CFT_2)

Entanglement Entropy of a subsystem A: $S_{EE}(A) = \lim_{n\to 1} \frac{1}{1-n} \log \left\langle \sigma_n \left(T_1, \theta_1,\right) \tilde{\sigma}_n \left(T_2, \theta_2\right) \right\rangle$





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Simplest Model: Half dS_2 gravity + Half dS_2 bath (CFT_2)

Entanglement Entropy of a subsystem A: $S_{\text{EE}}(A) = \lim_{n\to 1} \frac{1}{1-n} \log \left\langle \sigma_n\left(T_A, \theta_1\right) \tilde{\sigma}_n\left(T_A, \theta_2\right) \right\rangle$

conformal dimension

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

$$ds^2 = \Omega^{-2} dz d\bar{z} = \frac{4}{(1+z\bar{z})^2} dz d\bar{z},$$

$$S_{\text{Bulk}} = \frac{c}{6} \log\left(\frac{\left(z_1 - z_2\right)\left(\bar{z}_1 - \bar{z}_2\right)}{\epsilon^2 \Omega_1 \Omega_2}\right) = \frac{c}{6} \log\left(\frac{2\left(\cos\left(T_1 - T_2\right) - \cos\left(\theta_1 - \theta_2\right)\right)}{\epsilon^2 \cos T_1 \cos T_2}\right)$$

Connected Phase (No Island Phase)

$$S_{\text{EE}}^{\text{con}}\left(t_0\right) = \frac{c}{3} \log \cosh t_0 + \frac{c}{3} \log\left(\frac{2}{\epsilon}\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|\right).$$

03: Extremal Islands in dS gravity

Island Formula: $S_{\mathrm{EE}}(\hat{\rho}_A) = \mathrm{Min}_I \left[\mathrm{Ext}_I \left(\frac{\mathrm{Area}[\partial \Sigma_I]}{4G_{\mathrm{N}}} + S_{\mathrm{Bulk}} \left(\Sigma_A \cup \Sigma_{\mathrm{Island}} \right) \right) \right]$

Area term → constant $\sim \log g$

Reduction from AdS3 —> 01. Einstein gravity, higher curvature gravity f(R) gravity, scalar-tensor theory: constant $\sim \log g$

Reduction from dS3 —> 02. dS JT gravity: $\Phi = \frac{\cos \theta}{\cos T} > 0$

Reduction from Extremal dS4 BH (Nariai solution) —> 03. Topological term + dS JT gravity

$$S = \frac{\Phi_0}{16\pi G} \int d^2x \sqrt{-g} R + \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi (R - 2),$$

$\Phi_0 + \Phi > 0$

03: Extremal Islands in dS gravity

Simplest Model： Half dS_2 gravity+ Half dS_2 bath (CFT_2)

Island Formula： $S_{\text{EE}}(\hat{\rho}_A) = \text{Min}_I \left[\text{Ext}_I \left(\frac{\Phi_0}{4G_N} + S_{\text{Bulk}} \left(\Sigma_A \cup \Sigma_{\text{Island}} \right) \right) \right] = \text{constant}$

$$\partial_{\theta_I} S_{\text{gen}} = 0 \longrightarrow \quad \cosh t_I \cosh t_A \sin\left(\theta_I - \theta_A\right) = 0,$$

$$\partial_{t_I} S_{\text{gen}} = 0 \longrightarrow \quad \cosh t_I \sinh t_A - \sinh t_I \cosh t_A \cos\left(\theta_I - \theta_A\right) = 0$$

Unique Solution: $\theta_I = -\pi + \theta_A, \quad t_I = -t_A$

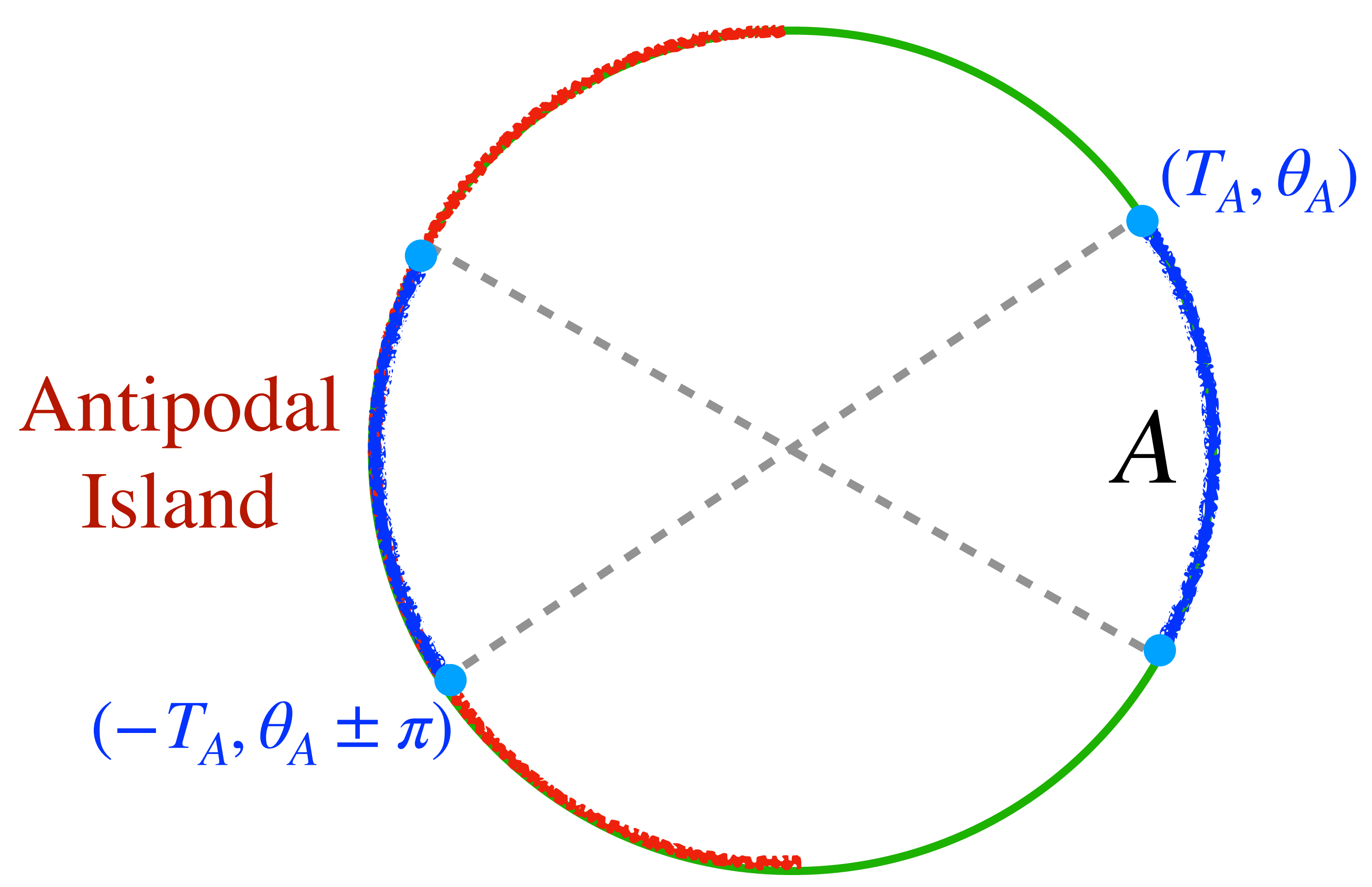
Antipodal Island

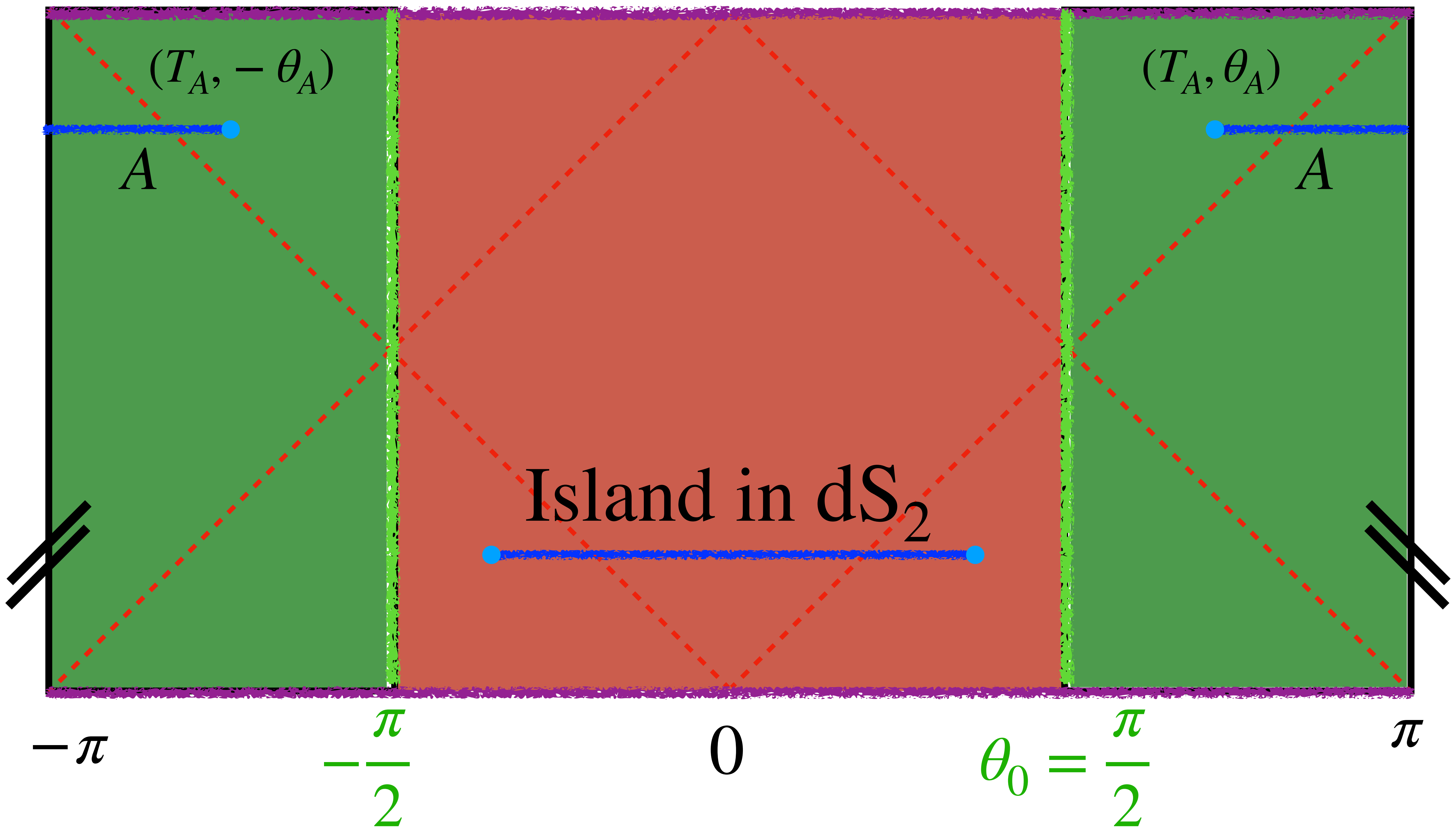
03: Extremal Islands in dS gravity

Simplest Model: Half dS_2 gravity + Half dS_2 bath (CFT_2)

Island Formula: $S_{EE}(\hat{\rho}_A) = \text{Min}_I \left[\text{Ext}_I \left(\frac{\Phi_0}{4G_N} + S_{\text{Bulk}} \left(\Sigma_A \cup \Sigma_{\text{Island}} \right) \right) \right] = \text{constant}$

Unique Solution —> Antipodal Island:

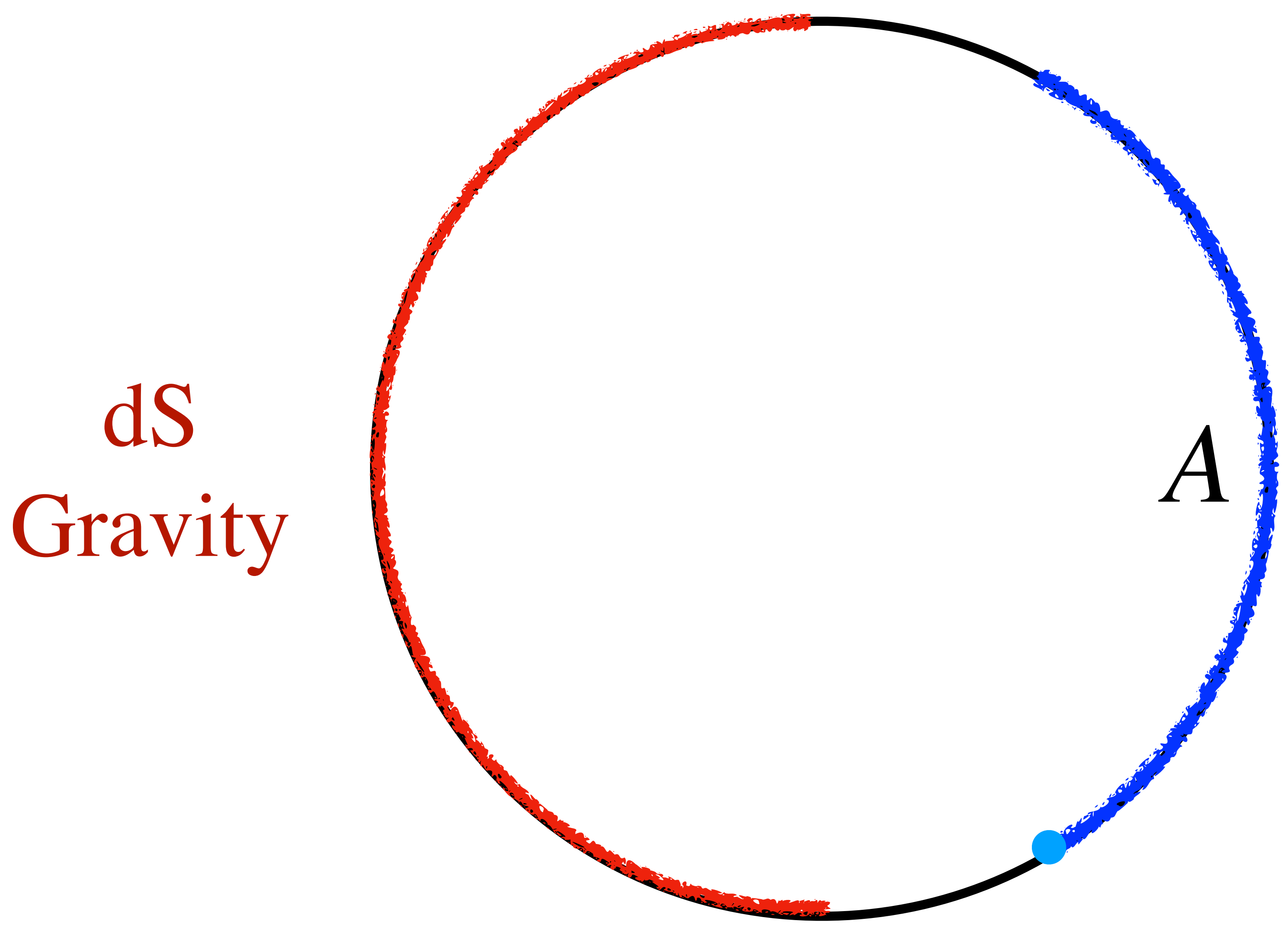




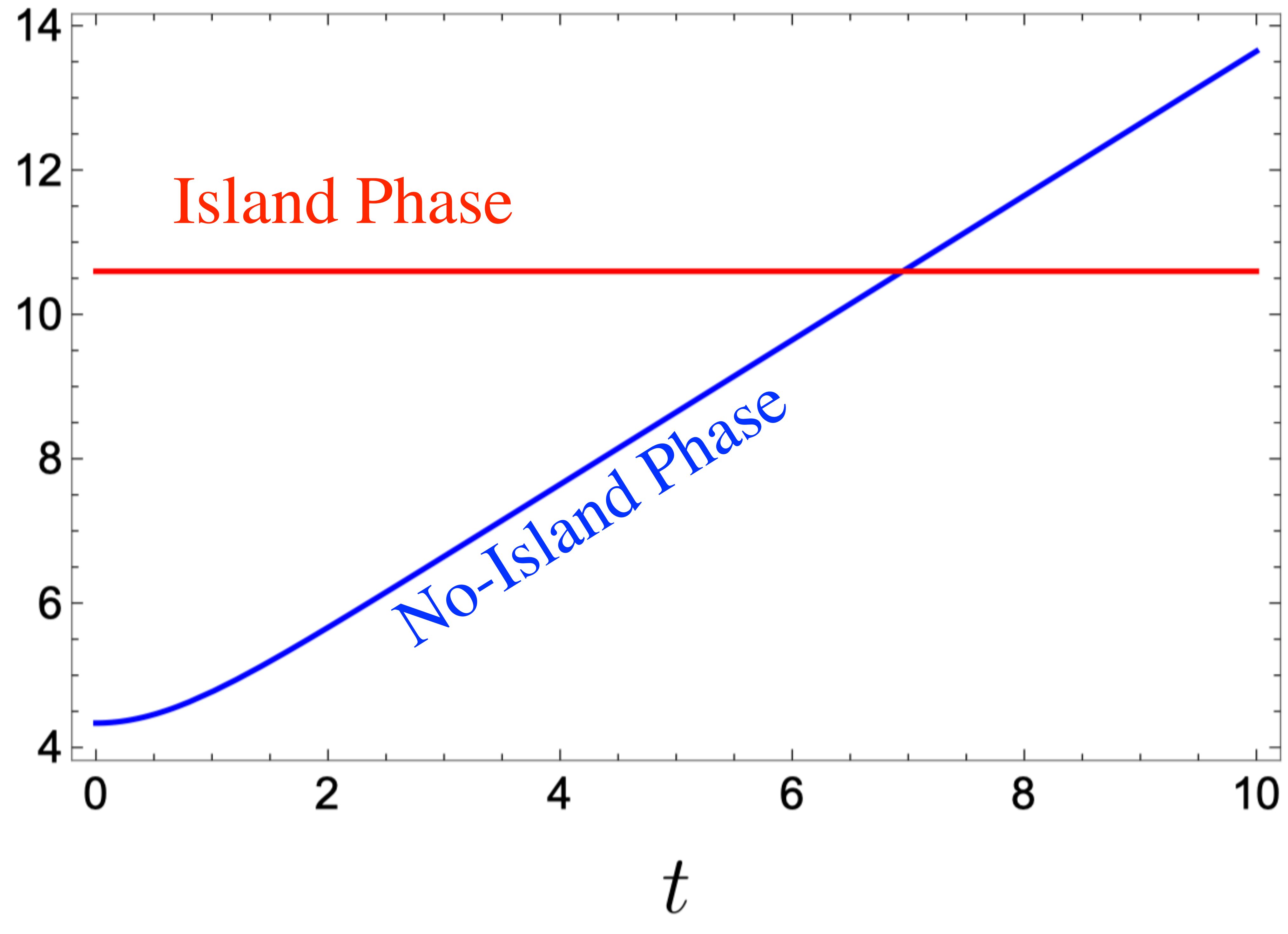
03: Extremal Islands in dS gravity

Simplest Model： Half dS_2 gravity+ Half dS_2 bath (CFT_2)

$$S_{\rm EE}\left(\hat{\rho}_A\right)\Big|_{\rm island} = S_0 + \frac{2c}{3}\log\frac{2}{\epsilon} = \tilde{S}_0 + \frac{c}{3}\log\left(\frac{4}{\epsilon}\right)$$



$$S_{\rm EE}(A) = ?$$



03: Problematic Extremal Islands in dS gravity

However, there are many reasons that "Island"/QES in dS is not physical!
violate SSA, violate entanglement wedge nesting…

It is a saddle point but it cannot dominate!

$$\frac{\partial^2 S_{\text{bulk}}}{\partial t_I \partial t_I} > 0$$ Local minimum in the Lorentzian timelike direction

$$\frac{\partial^2 S_{\text{bulk}}}{\partial \theta_I \partial \theta_I} < 0$$ Local maximum in the spatial direction

(Same conclusion for any positive dilaton Φ)

The quantum extremal surface in dS is a minimax surface!

?

04: Resolution from Double Holography

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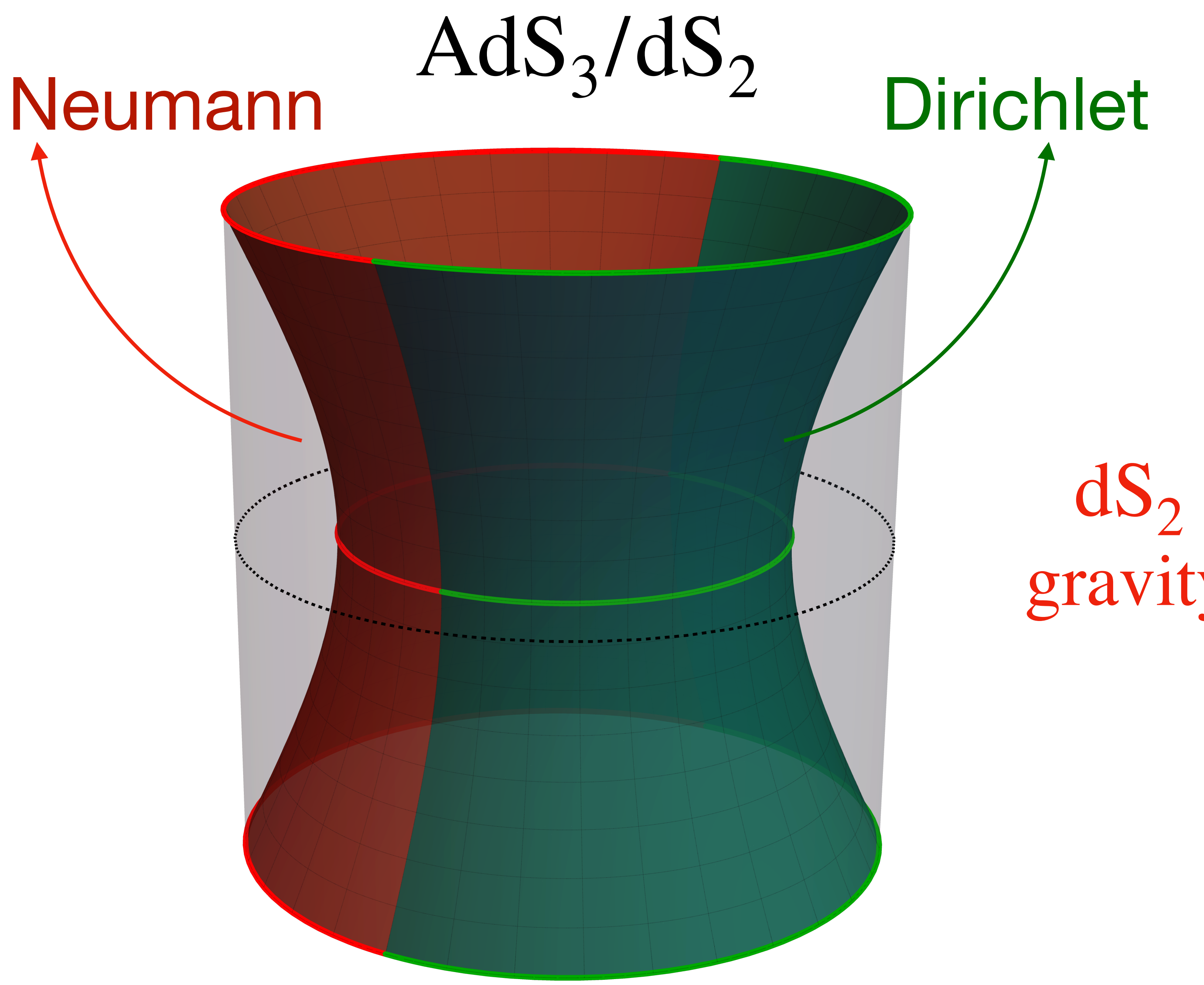
Simplest Model: Half dS_2 gravity + Half dS_2 bath

$$ds^2 = L^2\left(-dt^2 + \cosh^2 t\, d\theta^2\right) = \frac{L^2}{\cos^2 T}(-dT^2 + d\theta^2)$$

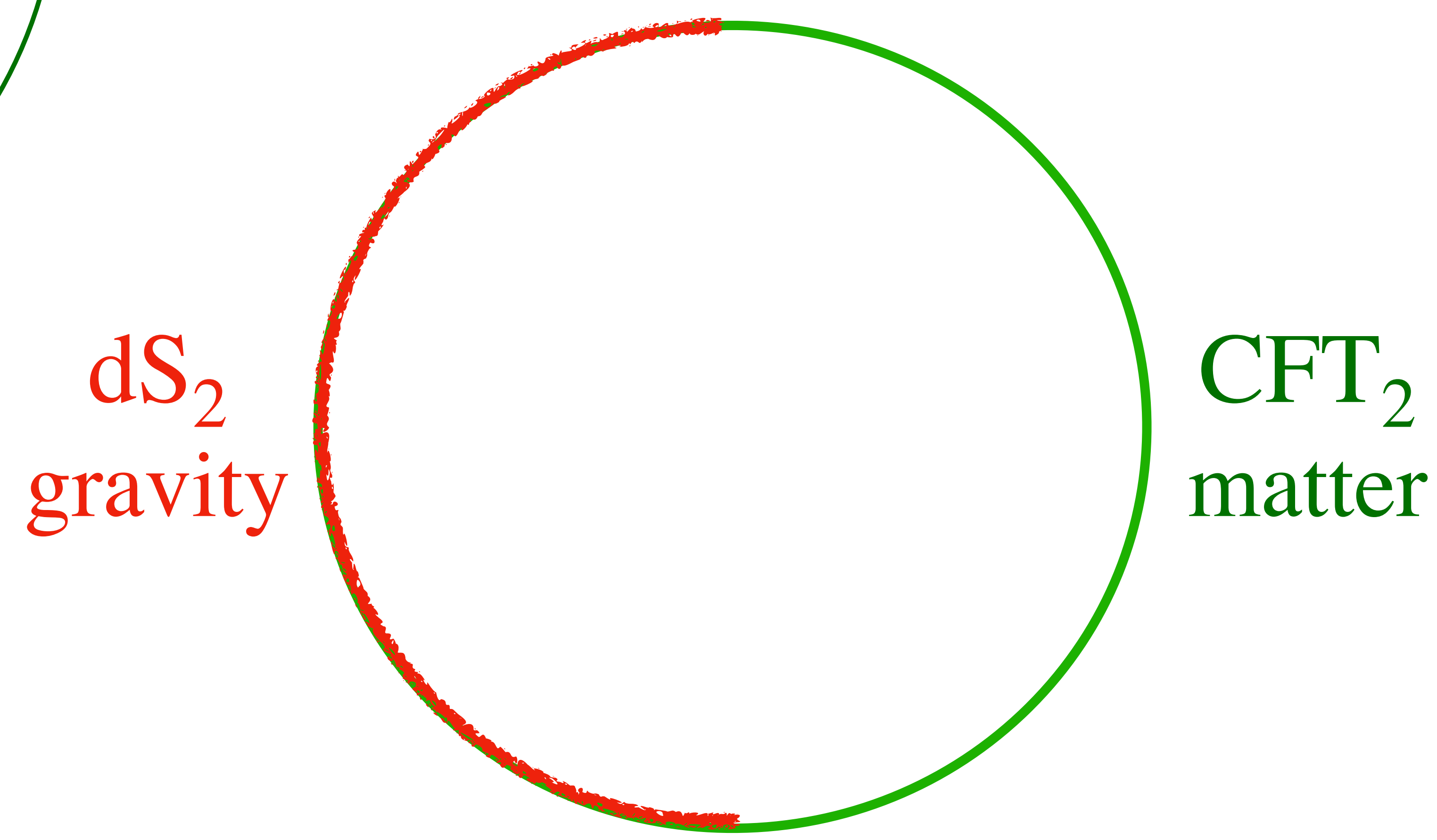
AdS_3/dS_2 Slicing: $$ds^2 = d\eta^2 + \sinh^2\eta\left(-dt^2 + \cosh^2 t d\theta^2\right)$$

tension of EOW brane

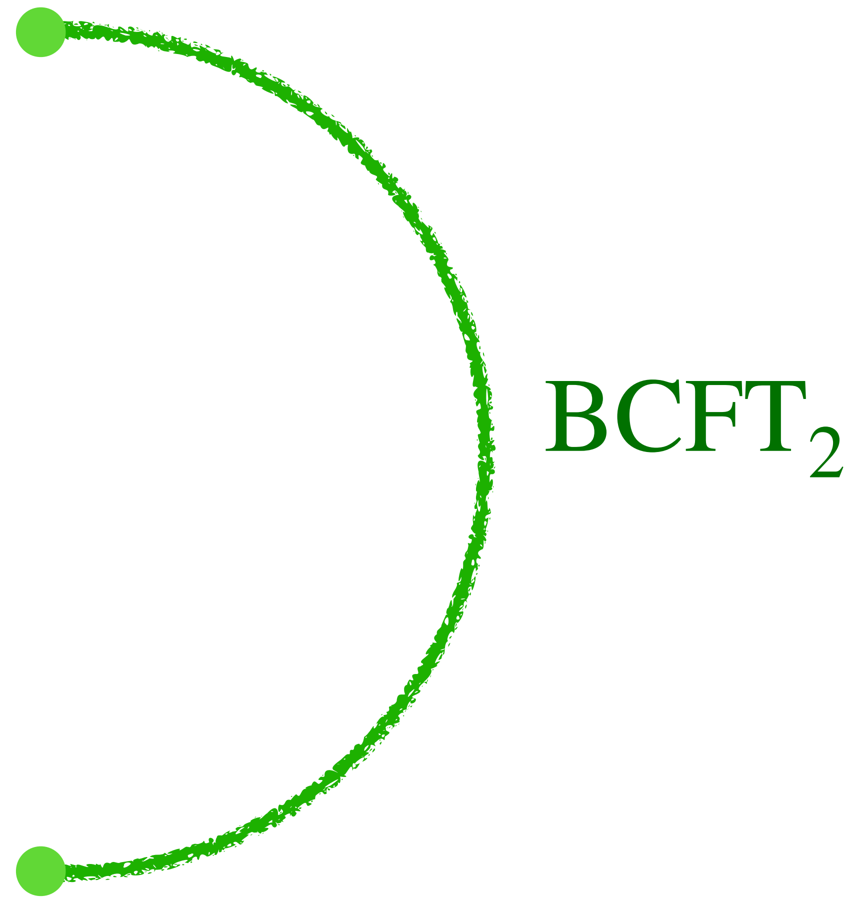
$$T = \frac{\coth \eta_b}{L_{AdS}}$$



Bulk Perspective



Brane Perspective



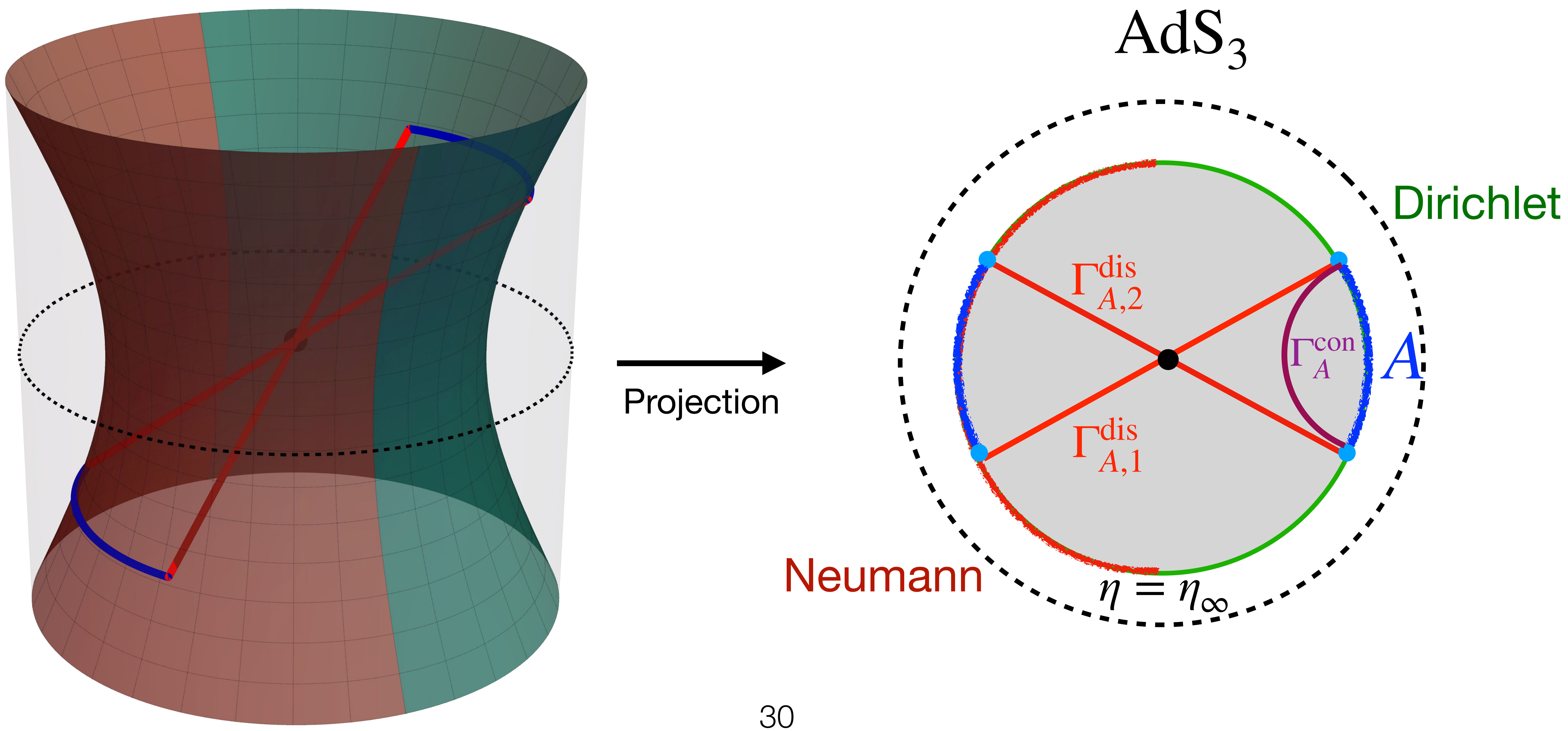
Boundary Perspective

04: Resolution from Double Holography

RT formula provides the same results as the Island formula:

$$S_A = \min\left\{S_A^{\text{con}}, S_A^{\text{dis}}\right\} = \min\left\{\frac{\text{Area}\left(\Gamma_A^{\text{con}}\right)}{4G_{\text{N}}}, \frac{\text{Area}\left(\Gamma_A^{\text{dis}}\right)}{4G_{\text{N}}}\right\},$$

Naive holographic entanglement entropy is problematic!



04: Resolution from Double Holography

Entanglement Entropy of a subsystem A: $S_{\mathrm{EE}}(A) = \lim_{n\to 1} \frac{1}{1-n} \log \left\langle \sigma_n\left(T_A, \theta_1\right) \tilde{\sigma}_n\left(T_A, \theta_2\right) \right\rangle$

$$\langle \sigma_n(t_{A_1},\theta_{A_1})\bar{\sigma}_n(t_{A_2},\theta_{A_2})\rangle_{\mathrm{BCFT}} = \max\begin{cases} \langle \sigma_n(t_{A_1},\theta_{A_1})\bar{\sigma}_n(t_{A_2},\theta_{A_2})\rangle \\ \langle \sigma_n(t_{A_1},\theta_{A_1})\rangle\langle\bar{\sigma}_n(t_{A_2},\theta_{A_2})\rangle \end{cases}$$

One-Point Function $\left\langle \sigma_n\left(T_0, \theta_1\right) \right\rangle$?

Advantage from the double holography:
The one-point function can be derived by using the bulk-to-boundary propagator:

$$\langle \mathcal{O}(\zeta) \rangle = \int_{\mathrm{EOW}} d^d\hat{x}\sqrt{h}K^{\Delta}_{\mathrm{Bb}}(X, Y(\hat{x}))$$

$$\left(\Box_g + m^2\right) \underline{G^{\Delta}_{\mathrm{BB}}(X,X')} = \frac{\delta(X-X')}{\sqrt{g}}\,.$$

Bulk-to-Bulk propagator

$$G^{\Delta}_{\mathrm{BB}}(X,X') = G^{\Delta}_{\mathrm{BB}}(\xi) = \frac{C_\Delta}{2^\Delta(2\Delta-d)}\xi^\Delta \cdot {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2}+1; \xi^2\right)$$

$$\xi = \frac{1}{\cosh d_{\mathrm{geodesic}}(X,X')}$$

04: Resolution from Double Holography

From (Euclidean) AdS3: $ds^2 = d\eta^2 + \sinh^2\eta\left(dt_{\mathrm{E}}^2 + \cos^2 t_{\mathrm{E}} d\theta^2\right)$

$$K_{\mathrm{Bb}}^{\Delta}\left(t_{\mathrm{E}b}, \eta_b, \theta_b; t_{\mathrm{E}1}, \eta_1 = \infty, \theta_1\right) = C_\Delta \left(\frac{1}{\cosh\eta_b - \sinh\eta_b\left(\cos\left(t_{\mathrm{E}1}\right)\cos\left(t_{\mathrm{E}b}\right)\cos\left(\theta_b - \theta_1\right) + \sin\left(t_{\mathrm{E}1}\right)\sin\left(t_{\mathrm{E}b}\right)\right)}\right)^{\Delta}$$

$$\langle \mathcal{O}(x)\rangle_{\mathrm{BCFT}} = \alpha \cdot \int_{\mathrm{EOW}} d^d x_b \sqrt{g}\, K_{\mathrm{Bb}}^{\Delta}\left(x; x_b\right).$$

Large dimension limit $\Delta_n \to \infty$

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

Geodesic approximation $$\left\langle \sigma_n\left(x_A\right)\right\rangle \approx \mathrm{const.} \times \int_{\mathrm{EOW}} d^2 x_b \sqrt{g}\, e^{-\Delta_n D\left(x_A; x_b\right)}$$

04: Resolution from Double Holography

From (Euclidean) AdS3: $ds^2 = d\eta^2 + \sinh^2\eta \left(dt_{\mathrm{E}}^2 + \cos^2 t_{\mathrm{E}} d\theta^2\right)$

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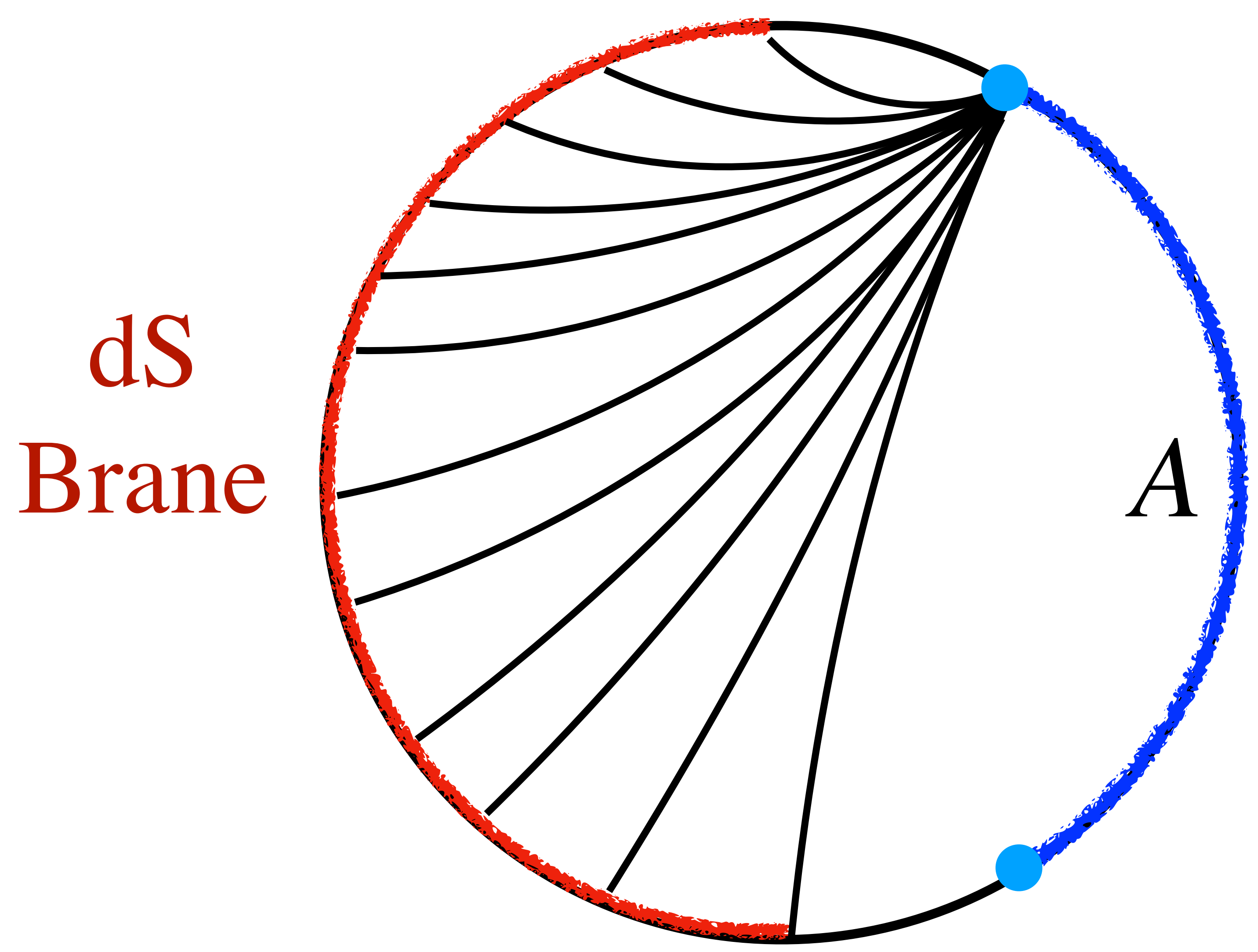
Geometric/Holographic Interpretation:



Large dimension limit $\Delta_n \to \infty$ (Geodesic approximation) ⟷ HRT surface

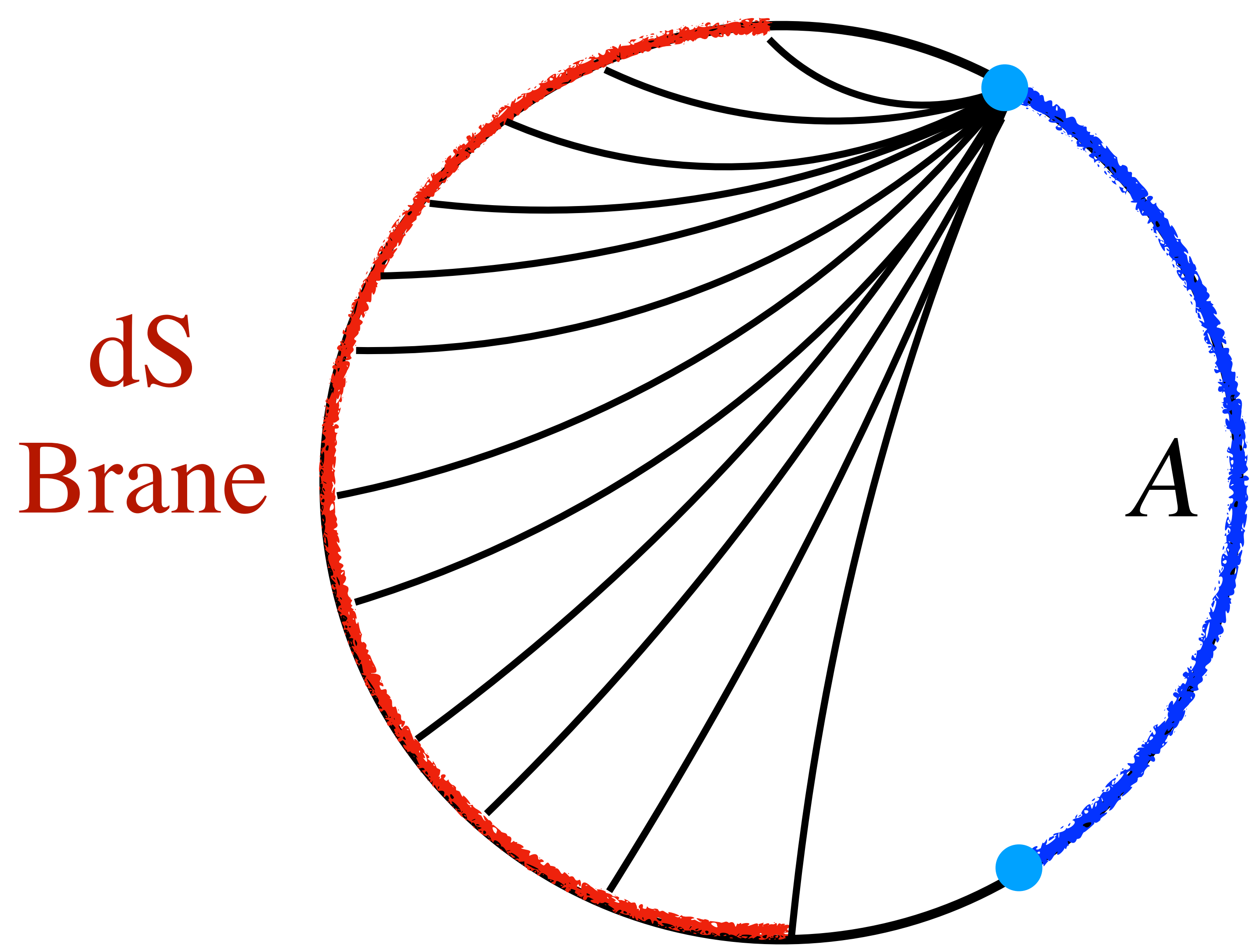
Saddle Point approximation ⟷ Minimization in Island formula

04: Resolution from Double Holography

From (Euclidean) AdS3: $ds^2 = d\eta^2 + \sinh^2\eta\left(dt_{\rm E}^2 + \cos^2 t_{\rm E} d\theta^2\right)$

$$\left\langle \sigma_n\left(x_A\right)\right\rangle \approx \text{const.} \times \int_{\rm EOW} d^2x_b\sqrt{g}\,e^{-\Delta_n D\left(x_A;x_b\right)}$$

Geometric/Holographic Interpretation:



Large dimension limit $\Delta_n \to \infty$ (Geodesic approximation) ⟷ HRT surface

Saddle Point approximation ? ⟷ Minimization in Island formula

04: Resolution from Double Holography

$$\langle \mathcal{O}(\zeta) \rangle = \int_{\text{dS brane}} d^d \hat{x} \sqrt{h} K^{\Delta}_{\text{Bb}}(X, Y(\hat{x}))$$

No saddle point approximation!

$$\left\langle \mathcal{O}\left(t_{\text{E}}, \theta\right) \right\rangle_{\text{dS}} \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt_{\text{Eb}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_b \cos\left(t_{\text{Eb}}\right) e^{-\Delta D\left(t_{\text{E}}, \theta; t_{\text{E}}, \theta_b\right)},$$

only one saddle point for the integral: $x_b^* = \left(\theta_b = \theta - \pi, t_{\text{Eb}} = -t_{\text{E}}\right)$

This saddle point is a local maximum!

$$\left.\frac{\partial^2 D\left(t_{\text{E}}, \theta; t_{\text{Eb}}, \theta_b\right)}{\partial t_{\text{Eb}}^2}\right|_{x_b = x_b^*} < 0, \quad \left.\frac{\partial^2 D\left(t_{\text{E}}, \theta; t_{\text{Eb}}, \theta_b\right)}{\partial \theta_b^2}\right|_{x_b = x_b^*} < 0.$$

04: Resolution from Double Holography

$$\langle \mathcal{O}(\zeta)\rangle = \int_{\text{dS brane}} d^d\hat{x}\sqrt{h}K^{\Delta}_{\text{Bb}}(X, Y(\hat{x}))$$

Dominated by the edge

$$\langle \sigma_n(t_{A_1},\theta_{A_1})\bar{\sigma}_n(t_{A_2},\theta_{A_2})\rangle_{\text{BCFT}} = \max\begin{cases}\langle \sigma_n(t_{A_1},\theta_{A_1})\bar{\sigma}_n(t_{A_2},\theta_{A_2})\rangle \\ \langle \sigma_n(t_{A_1},\theta_{A_1})\rangle\langle\bar{\sigma}_n(t_{A_2},\theta_{A_2})\rangle\end{cases}$$

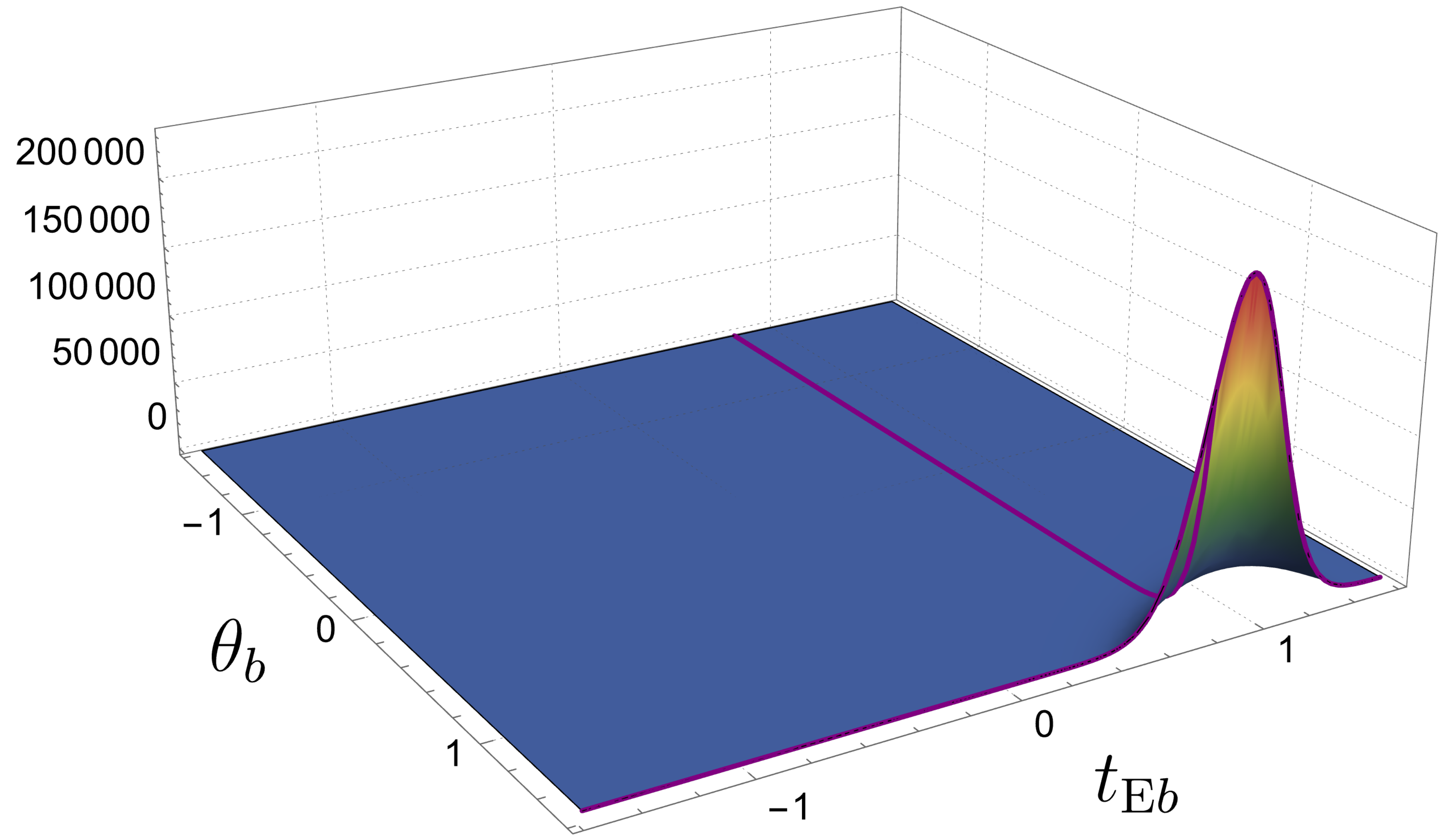
↓ Double Holography

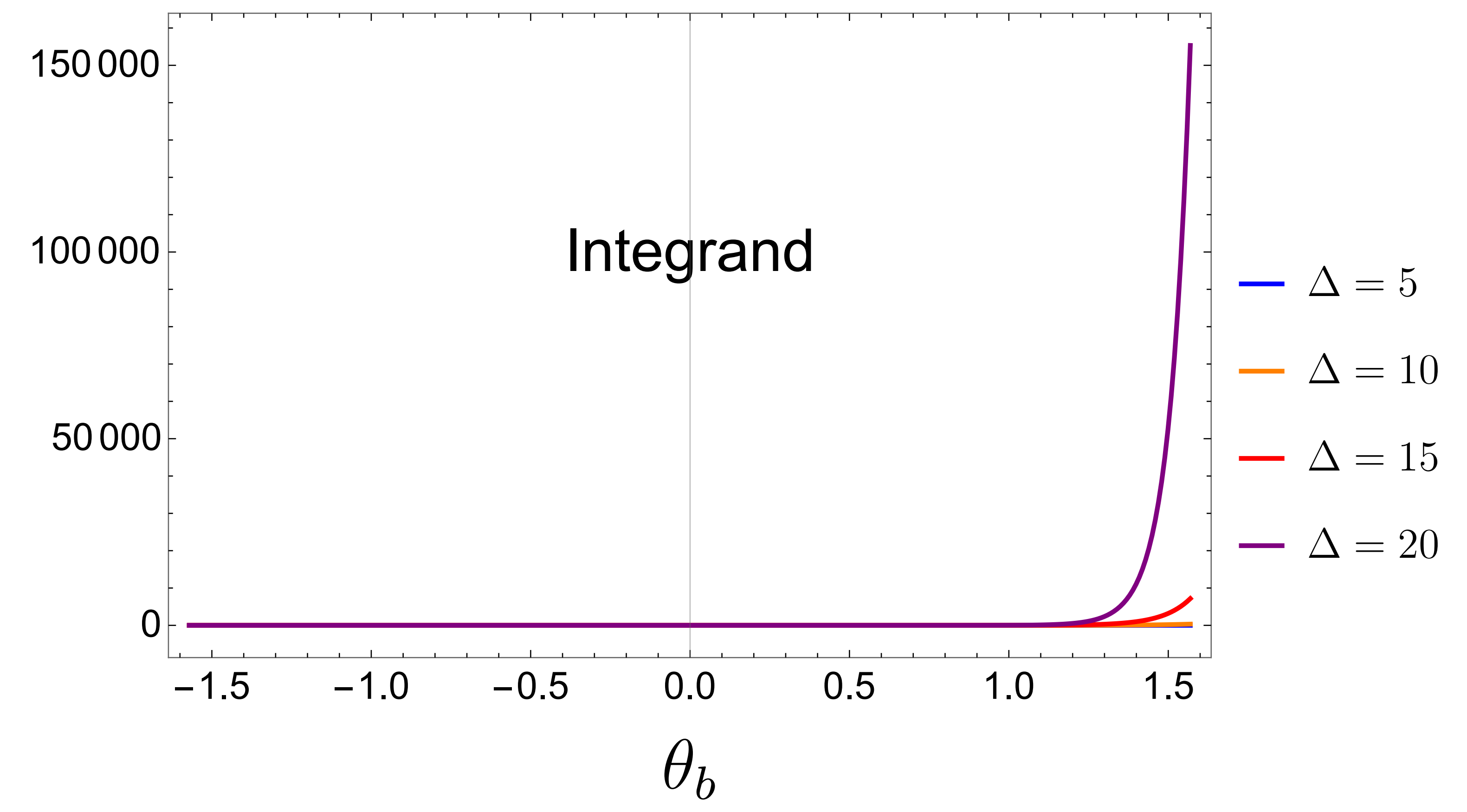
Leading Result from the integral:

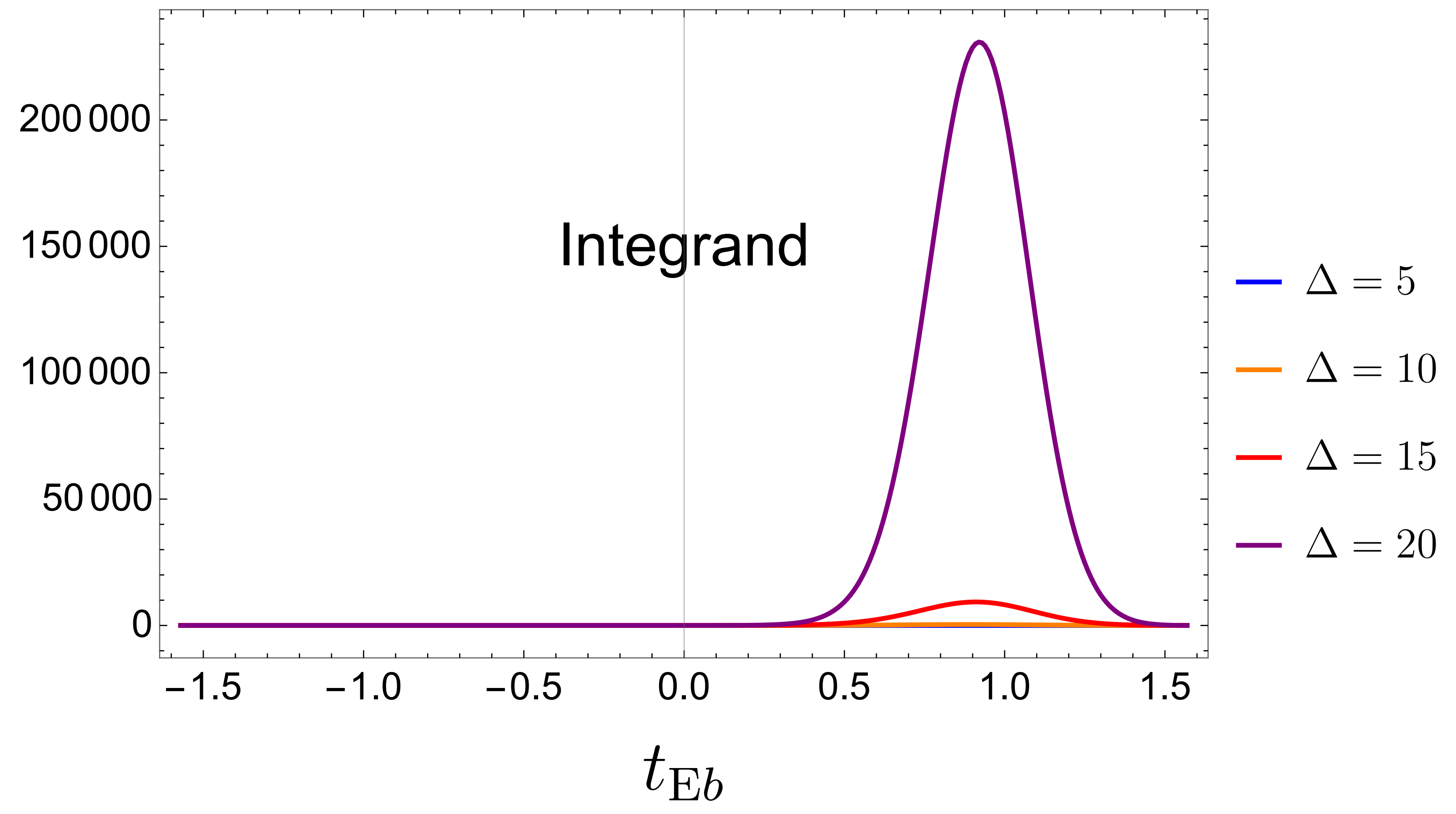
Entanglement Entropy: $S_{\mathcal{A}} = \min\begin{cases} S^{\text{con}}_{\mathcal{A}} = \frac{c}{3}\log\left(\frac{2\cosh t_A\cdot\sin\theta_A}{\epsilon}\right) \\ \tilde{S}^{\text{dis}}_{\mathcal{A}} = \frac{c}{3}\log\left(\frac{2}{\epsilon}\left(\cosh\eta_b - \sinh\eta_b\sqrt{\cosh^2 t_A\sin^2\theta_A - \sinh^2 t_A}\right)\right)\end{cases}$

04: Resolution from Double Holography

$$K^{\Delta}_{\mathrm{Bb}}\left(t_{\mathrm{E}b}, \eta_b, \theta_b; t_{\mathrm{E}1}, \eta_1 = \infty, \theta_1\right) = C_{\Delta}\left(\frac{1}{\cosh\eta_b - \sinh\eta_b\left(\cos\left(t_{\mathrm{E}1}\right)\cos\left(t_{\mathrm{E}b}\right)\cos\left(\theta_b - \theta_1\right) + \sin\left(t_{\mathrm{E}1}\right)\sin\left(t_{\mathrm{E}b}\right)\right)}\right)^{\Delta}$$







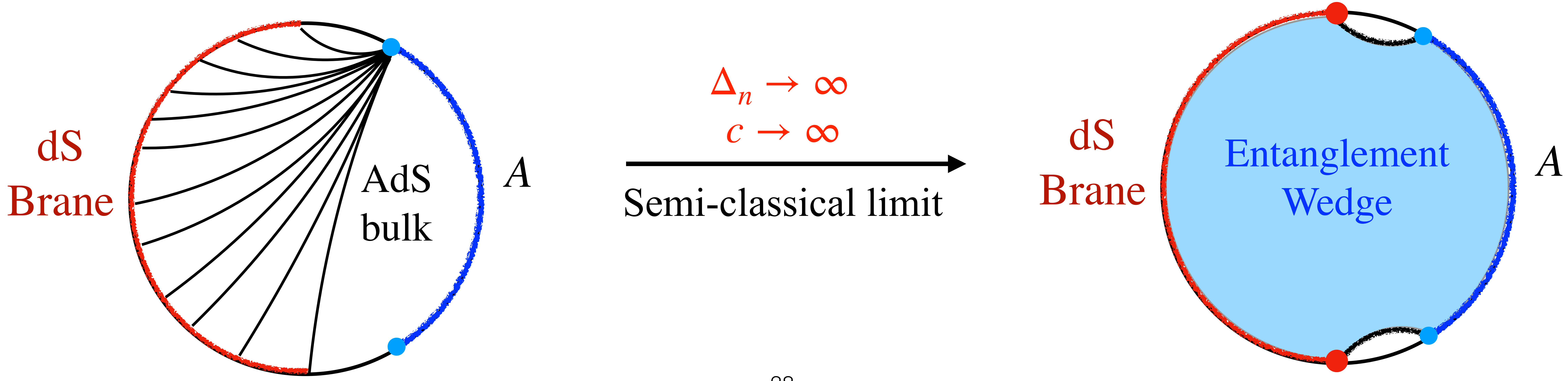
04: Resolution from Double Holography

To get the same results from holography

$$S_A = \min\left\{S_A^{\mathrm{con}}, \tilde{S}_A^{\mathrm{dis}}\right\} = \min\left\{\frac{\mathrm{Area}\left(\Gamma_A^{\mathrm{con}}\right)}{4G_{\mathrm{N}}}, \frac{\mathrm{Area}\left(\tilde{\Gamma}_A^{\mathrm{dis}}\right)}{4G_{\mathrm{N}}}\right\},$$

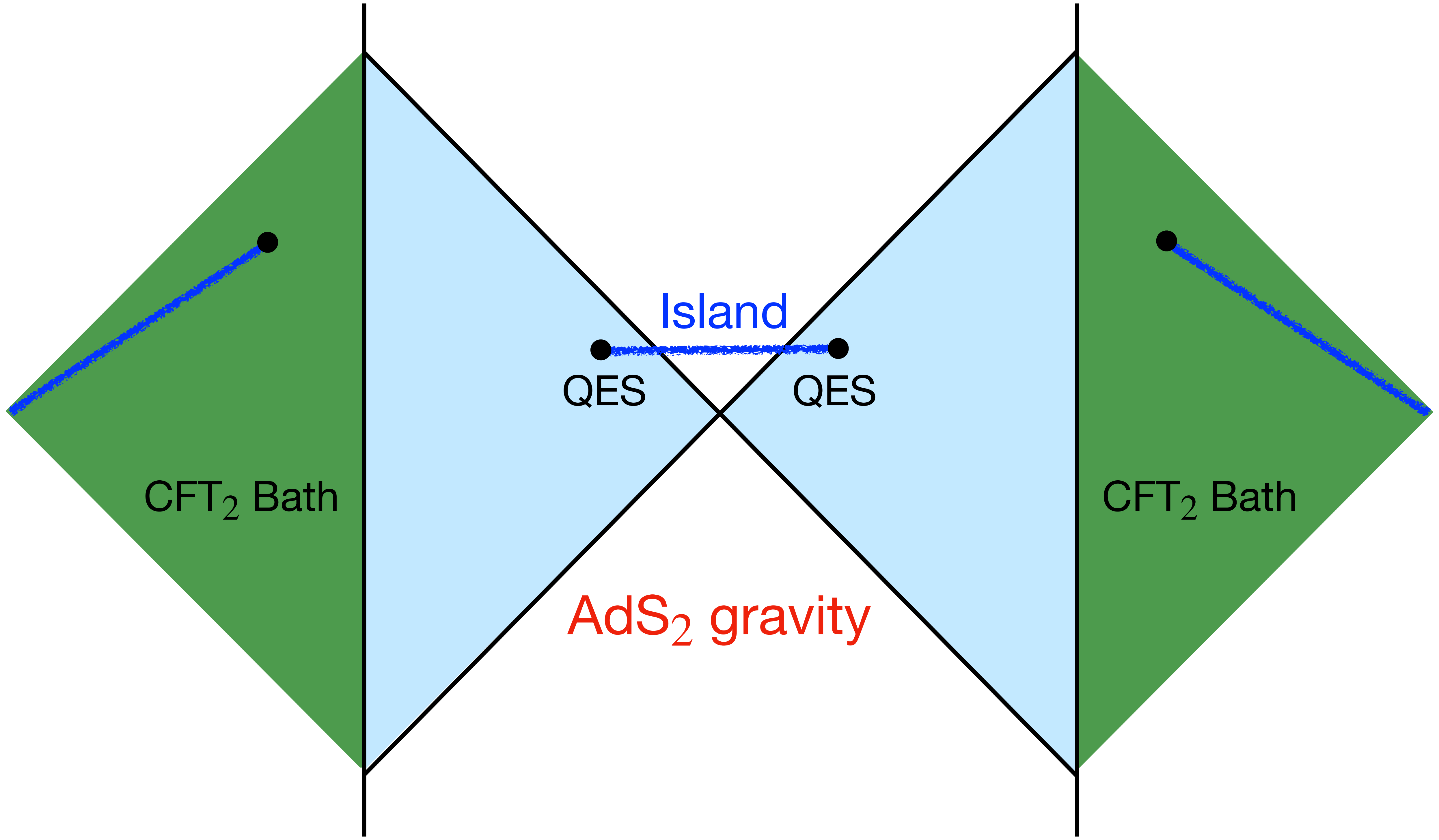
By anchoring the non-extremal surface at the edge of dS gravity region

Holographic dual: Non-extremal Island in dS



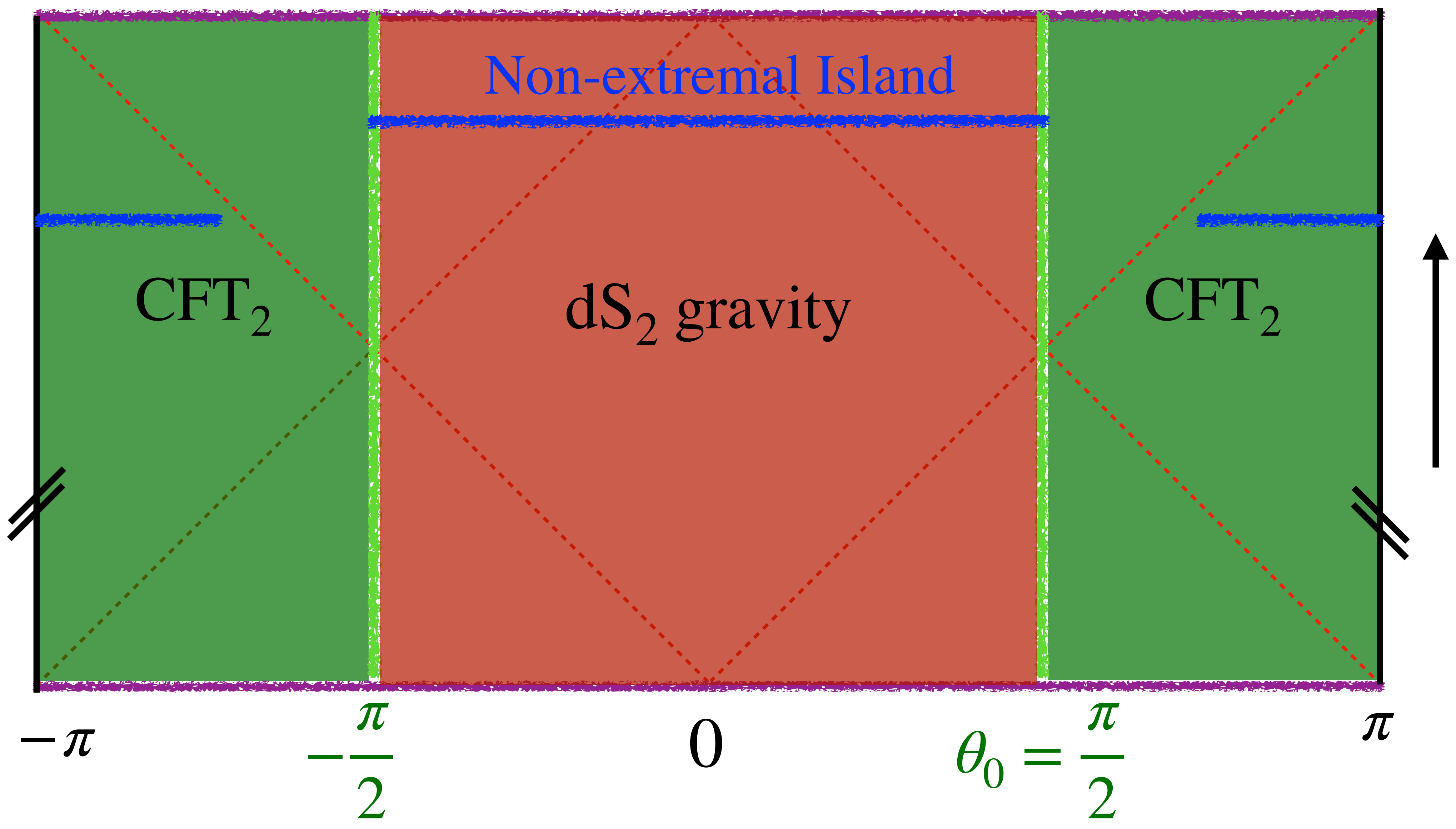
Holographic Suggestion: Non-extremal Island

AdS+Bath



Island Formula
(Quantum Extremal Surface)

dS+Bath



Non-extremal Island
(Non-extremal Surface)

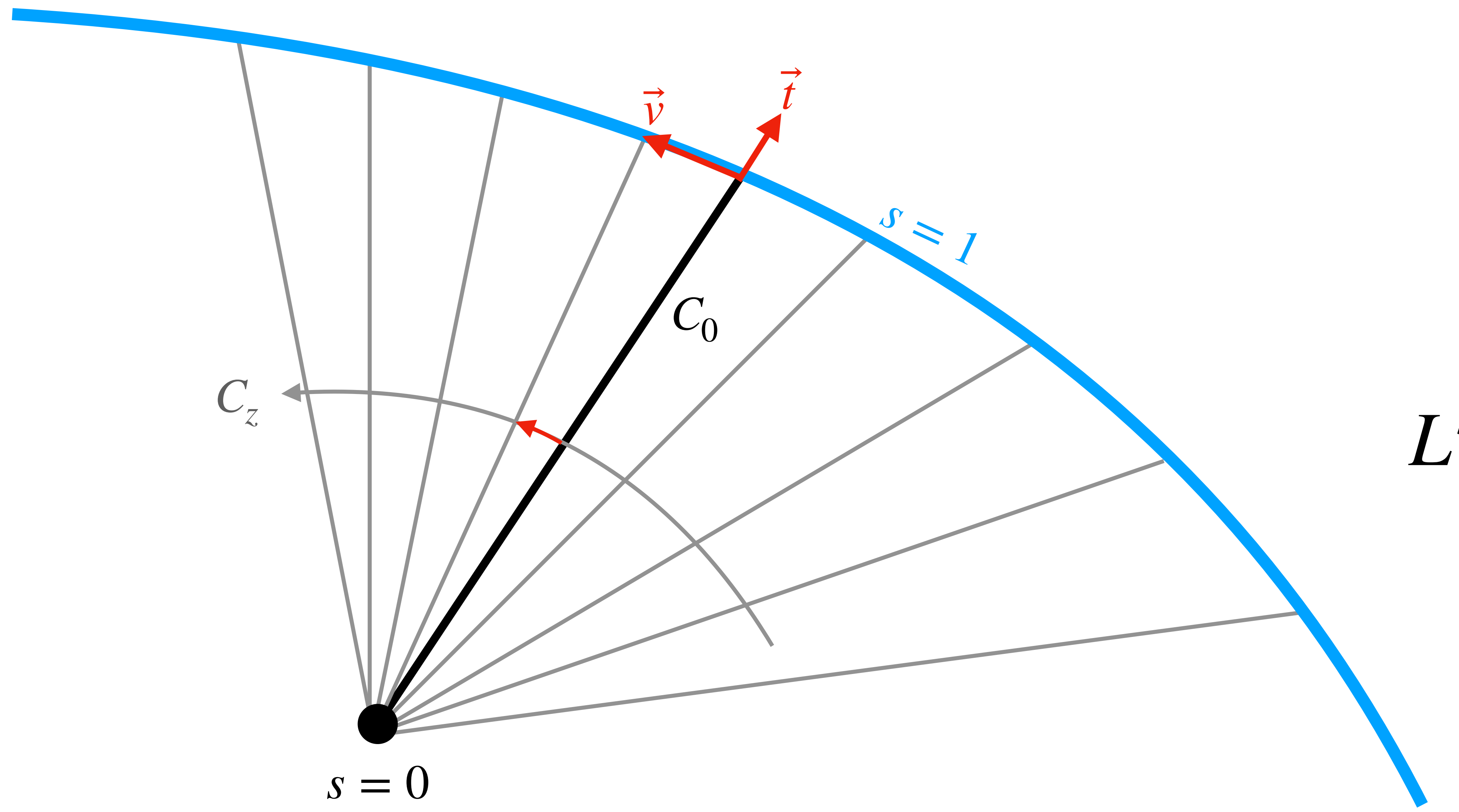
Question: Does Island formula apply to dS gravity?

NO!

Why is there no quantum extremal surface on dS gravity?

04: Resolution from Double Holography

Why is there no quantum extremal surface on dS gravity?



Synge's formula for the second variation of geodesics

$$L''\left(C_0\right) = -K(\mathbf{v},\mathbf{v}) + \frac{1}{L_0}\int_0^1 \left[\left\| \left(\nabla_{\mathrm{t}}\mathbf{v}\right) \right\|^2 - K_{\mathrm{set}}(\mathbf{v},\mathbf{t})\|\mathbf{v}\wedge\mathbf{t}\|^2 \right] ds$$

A geometric explanation from the bulk AdS:

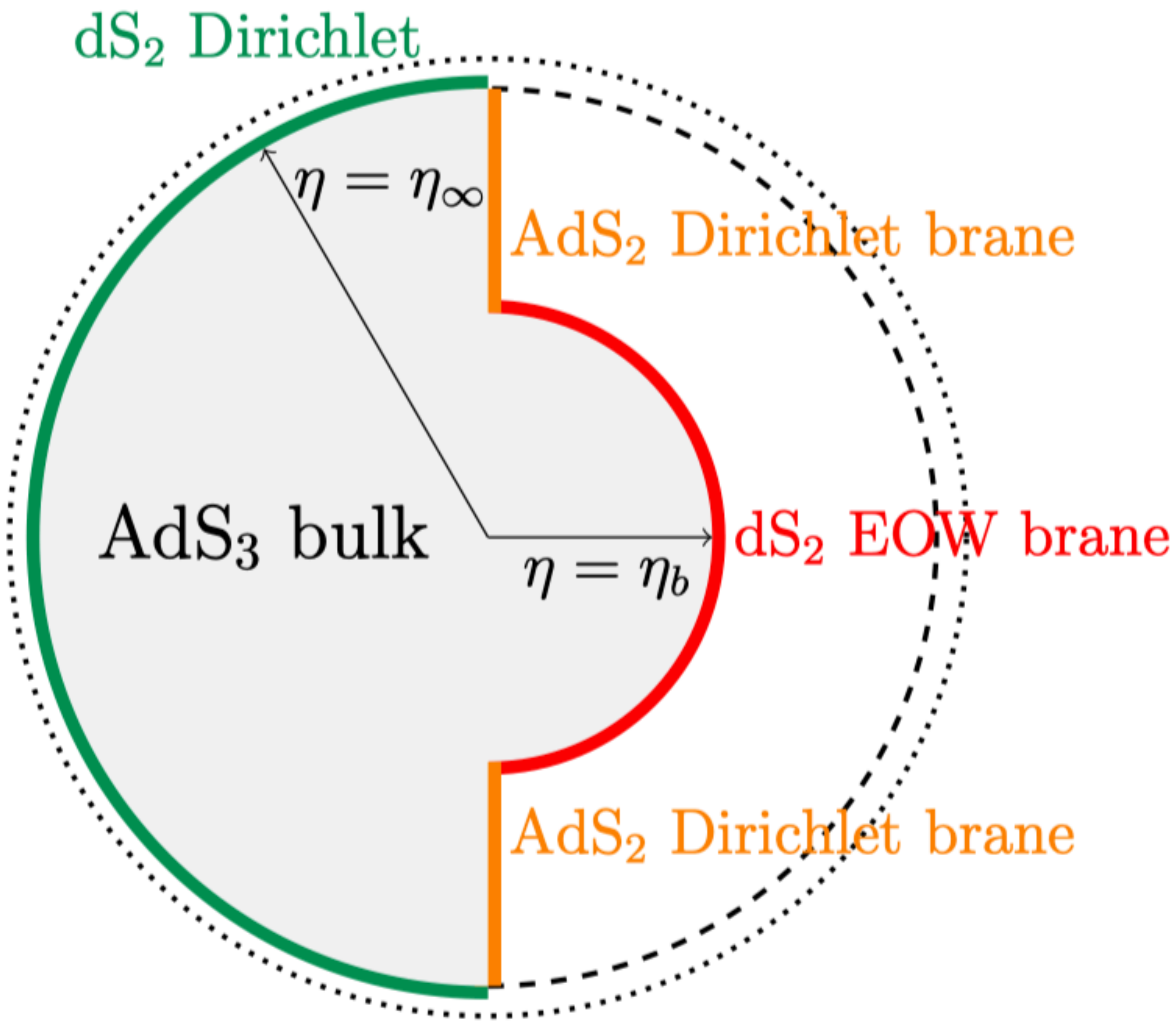
the (positive) extrinsic curvature of the dS brane is too large!

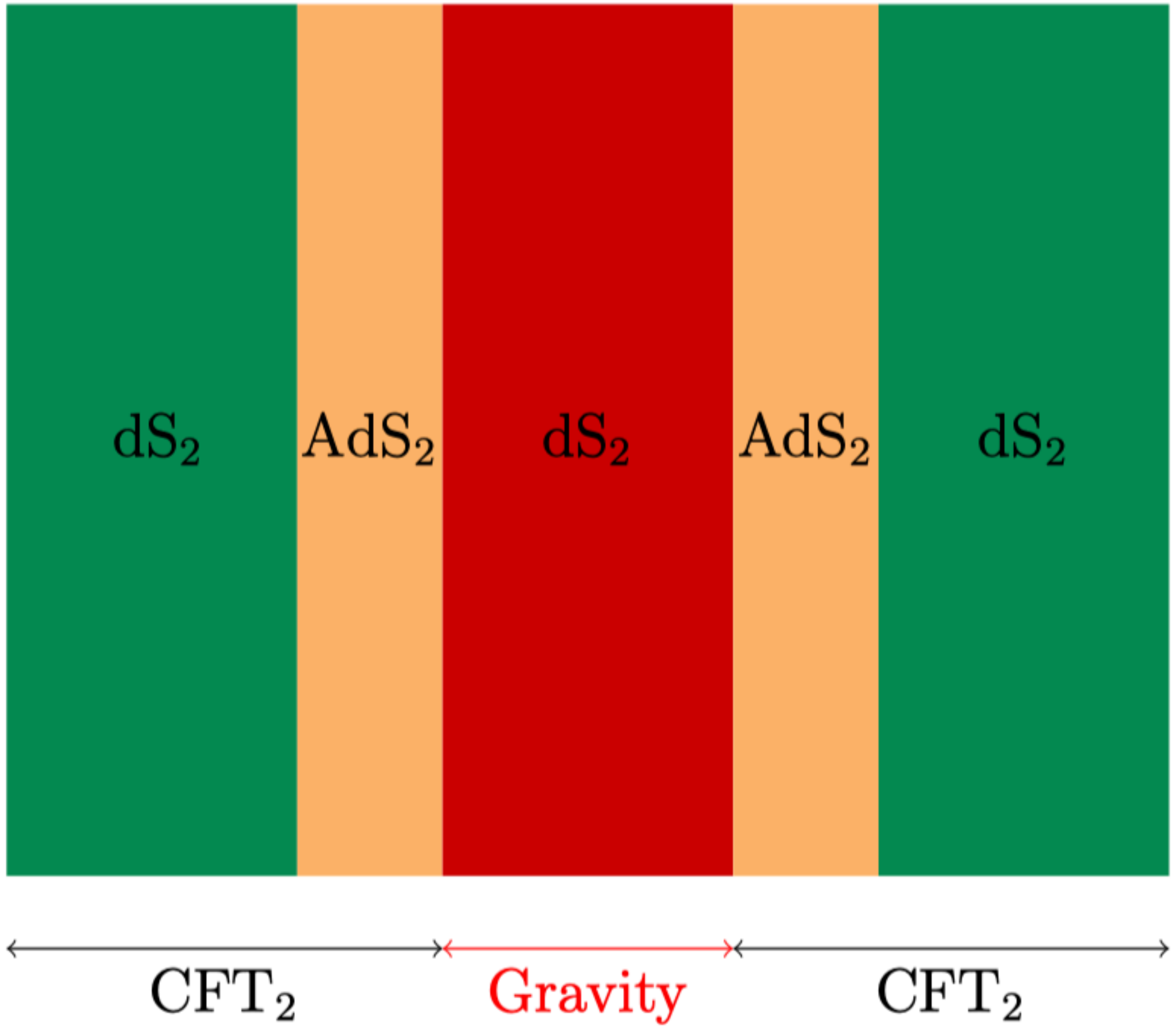
$$K(\mathbf{v},\mathbf{v})\Big|_{\text{dS brane}} \propto K\Big|_{\text{dS brane}} > \frac{d}{L_{\text{AdS}}}$$

Thanks for your attention!

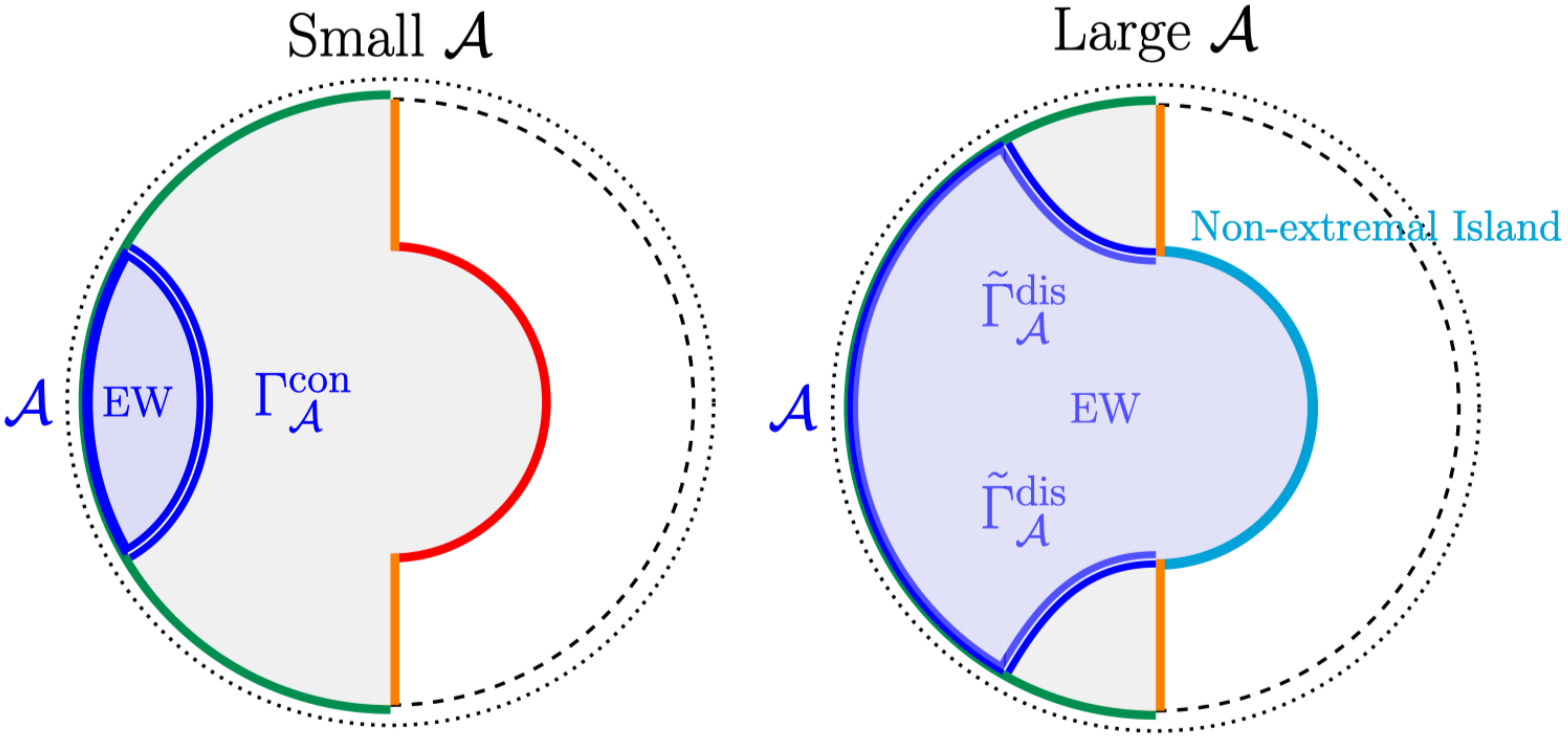
Extra Slices

Improved doubly holographic model





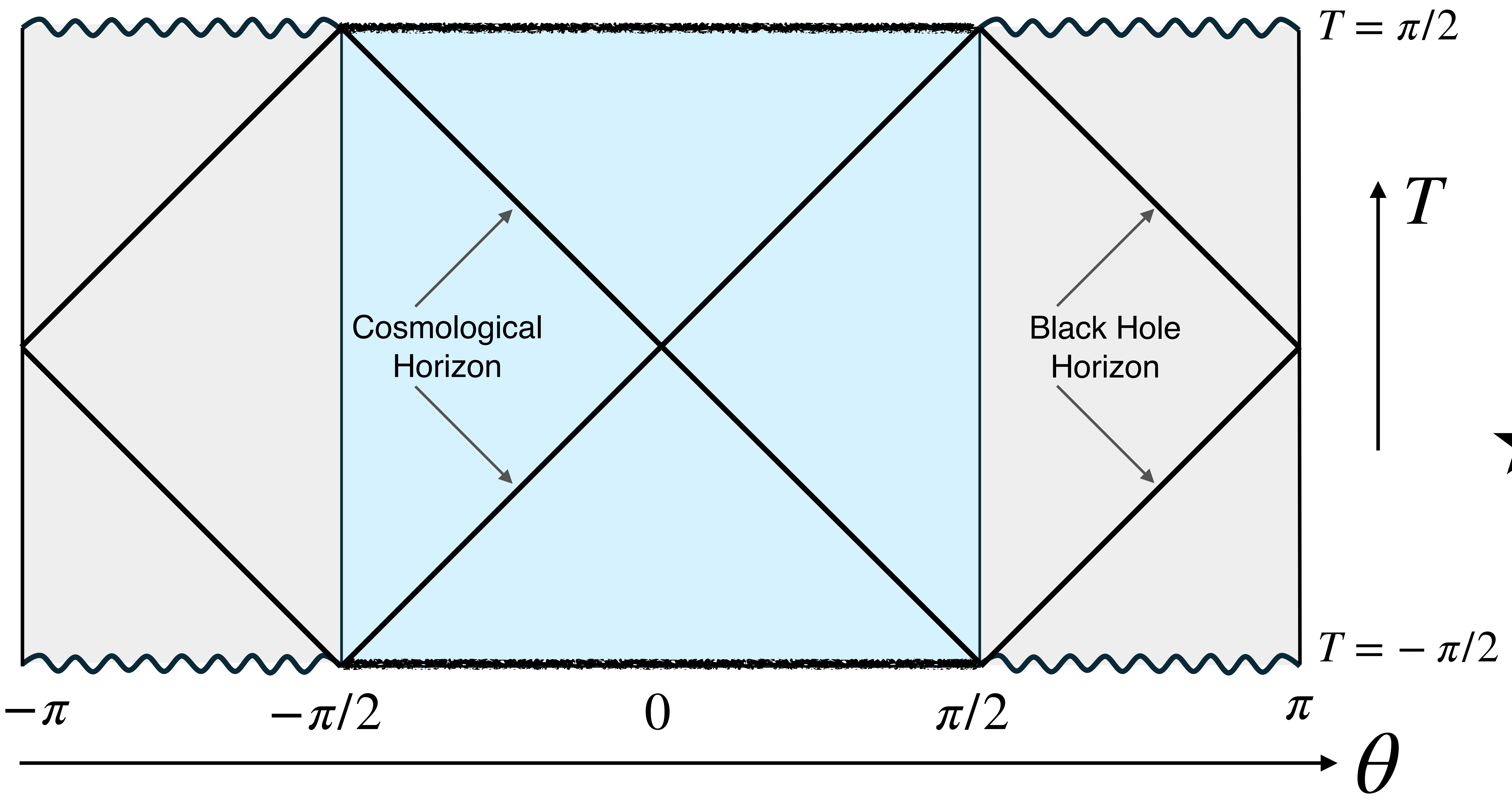
Improved doubly holographic model



Appendix: Extremal Islands in dS JT gravity?

$$S_{\mathrm{JT}'} = \frac{\Phi_0}{16\pi G_N^{(2)}} \int d^2x \sqrt{-g} R + \frac{1}{16\pi G_N^{(2)}} \int d^2x \sqrt{-g}\, \Phi \left(R - \frac{2}{L^2} \right) + \text{boundary terms}\,,$$

Dilation profile: $\Phi(T, \theta) = \Phi_r \dfrac{\cos\theta}{\cos T}$



★ QES could be locally minimal in the spatial direction only if

$$\Phi(T, \theta) < 0$$

Behind the black hole horizon

★ QES has to be close to the BH singularity

$$\frac{1}{\cos T_I} > \frac{c}{12\Phi_r} \gg 1$$