Fluid/Gravity Duality for Viscoelastic Crystals

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1) **Context, Motivation and Results**

2) **Hydrodynamics of Crystals**

3) **Fluid/Gravity Duality**

4) **Hydrodynamical Collective Modes**

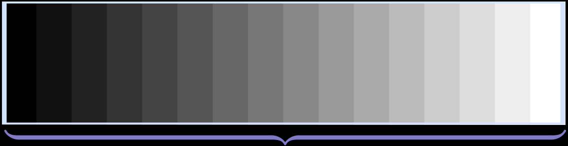
5) **Momentum Relaxation**

6) **Conclusion**

Context

When a material undergoes deformation:

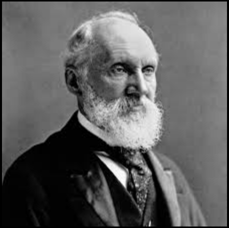
Elastic solid



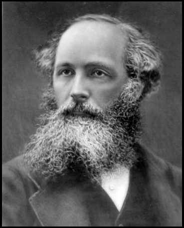
Viscous fluid

Viscoelastic materials

Kelvin-Voigt model



Maxwell model



Context

Hydrodynamic descriptions:

- [Eckart, 1948; Azeyanagi et al., '09, '10; Fukuma & Sakatani, '11a, '11b, '12] - geometrization of strain
- [Martin et al., 1972; Jähnig & Schmidt, 1972] - **SSB of spatial translations ⟹ crystals**

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Hydrodynamic descriptions:

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- [Martin et al., 1972; Jähnig & Schmidt, 1972] - **SSB of spatial translations** ➡ **crystals**

Holographic models with broken translations

explicitly [Vegh, '13; Davison et al., '14; Donos & Gauntlett, '14; Andrade & Withers, '14; etc.]

spontaneously [Baggioli & Pujolas, '15; Alberte et al., '16; etc.]

generated a renewed interest in a **complete hydrodynamic description**.

Motivation and Results

[Delacrétaz et al., '17]	[Grozdanov & Poovuttikul, '18]	[Armas & Jain, '19]
×	×	✓
[Ammon et al., '19; Donos et al., '19]		[Ammon et al., '20]

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Fluid-gravity duality ➡ **constitutive relations.** We find:

- agreement with [Armas & Jain, '19] for specific values of the transport coefficients
- spectrum of hydrodynamical collective modes (fixing inconsistencies of [Grozdanov & Poovuttikul, '18])
- range of parameters for which the theory is stable

Hydrodynamics of Crystals

Hydrodynamics of Crystals

Consider a crystal in a $(2+1)$ dimensions with pointlike lattice cores.

In the continuum limit, the crystal lattice can be described through a set of *surface forming* one-form fields:

[Armas & Jain, 2019]

$$e^a(x) = e^a_\mu(x) dx^\mu \quad \text{where} \quad a = 1, 2$$

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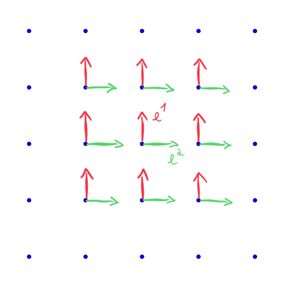
The matrix $e^a \cdot e^b$ contains essential physical information:

- diagonal elements encode compression along the directions of e^1 and e^2;
- off-diagonal encodes shear deformation.



Hydrodynamics of Crystals

Absense of topological defects (dislocations) $\Leftrightarrow de^a = 0$.

Our crystal has two 1-form global symmetries:

- 2-form currents $\boxed{J_a \equiv \star e^a} \Rightarrow d\star J_a = 0$
- topological conserved charges

$$Q^a[\Sigma] = \int_\Sigma \star J_a$$

where Σ is a 1-dimensional surface

Hydrodynamics of Crystals

$$\boxed{T^{\mu\nu}} = T^{\nu\mu} \qquad\qquad \boxed{J_a^{\mu\nu}} = -J_a^{\nu\mu}$$

background (non-dynamical) fields

metric $g_{\mu\nu}$

gauge fields $b^a_{\mu\nu}$

$(h^a \equiv db^a)$

Sourced conservation equations:

[Grozdanov & Poovuttikul, '18]

$$\nabla_\mu J_a^{\mu\nu} = 0$$

$$\nabla_\mu T^\mu_\nu + \frac{1}{2}\sum_{a=1}^{2} h^a_{\nu\rho\sigma} J_a^{\rho\sigma} = 0$$

Constitutive relations are required to close the system!

Hydrodynamics of (isotropic) Crystals

Constitutive relations involve the following (leading) transport coefficients:

Viscous
- hydrostatic
  - thermodynamic pressure
  - entropy s
  - heat capacity
- dissipative
  - shear viscosity η
  - bulk viscosity ζ

Elastic
- hydrostatic
  - lattice pressure P_L
  - shear modulus $\mathcal{G}$ and bulk modulus $\mathcal{B}$
  - "thermal expansion coefficient"
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[Grozdanov & Poovuttikul, '18]

$$S[G, B^a] = \int d^4x\sqrt{-G}\left[R[G] + 6 - \frac{1}{2}dB_a \wedge *dB_a\right] + \text{GHY} + S_{\text{bdy}}$$

Gauge invariance of the 2-form B^a is the bulk realisation of the 1-form global symmetry at the boundary

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$$ds^2 = \frac{dr^2}{r^2\mathcal{F}} + r^2\left[-\mathcal{F}dt^2 + dx^2 + dy^2\right]$$

$$dB^a = (6m)dr \wedge dt \wedge dx^a$$

where

$$\mathcal{F} = 1 - \frac{m^2}{2r^2} - \left(1 - \frac{m^2}{2r_h^2}\right)\frac{r_h^3}{r^3}$$

[Andrade & Withers, '14]

$m \leftrightarrow$ density of lattice cores in homogeneous, isotropic crystal

Fluid/Gravity Duality

Let γ be the induced metric near the conformal boundary and $H_a := dB_a$.

$$S_{\text{bdy}} = \underbrace{\int d^3x\sqrt{-\gamma}\left(\frac{H^a_{r\mu\nu}H_a^{r\mu\nu}}{4} - 4\right)}_{\equiv\, S_{counterterm}} \underbrace{- \mathcal{C}\int d^3x\sqrt{-\gamma}\frac{H^a_{r\mu\nu}H_a^{r\mu\nu}}{4r}}_{\equiv\, S_{deformation}}$$

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in the dual boundary theory

$$S_{deformation} = \int d^3x \sqrt{-g}\frac{J^a_{\mu\nu} J_a^{\mu\nu}}{4} \quad \text{(relevant)}$$

Fluid/Gravity Duality

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Solve radial evolution eqs (linearised) perturbatively in gradients (ω, k)

r-constants of motion	Holographic Dictionary $\longleftrightarrow$	expectation values of currents $\oplus$ sources

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Impose ingoing b.c. at the horizon

Holographic ↓ Dictionary

Constitutive Relations

Fluid/Gravity Duality

$$\delta T^{xy} \approx \frac{\mathcal{G}}{m}\left(\delta J_1^{ty} + \delta J_2^{tx}\right) + \frac{\eta}{m}\partial_t\left(\delta J_1^{ty} + \delta J_2^{tx}\right)$$

$$\delta T^{xx} - \delta T^{yy} \approx 2\frac{\mathcal{G}}{m}\left(\delta J_1^{tx} - \delta J_2^{ty}\right) + 2\frac{\eta}{m}\partial_t\left(\delta J_1^{tx} - \delta J_2^{ty}\right)$$

$$\delta T^{xx} + \delta T^{yy} \approx \delta T^{tt} + m\mathcal{C}\left(\delta J_1^{tx} + \delta J_2^{ty}\right)$$

- $\mathcal{G}$ is the shear modulus and η is the shear viscosity.
- **bulk viscosity ζ vanishes.**

Given a regular solution Θ to $\partial_r\left(r^4\mathcal{F}\partial_r\Theta\right) = m^2\Theta$,

$$\mathcal{G} = m^2\mathcal{C} - \lim_{r\to\infty}\frac{r^2\partial_r\left(r^3\partial_r\Theta\right)}{\Theta}$$

$$\eta = \frac{r_h^2\Theta(r_h)^2}{\lim_{r\to\infty}\Theta^2}$$

Fluid/Gravity Duality

$$\delta J_b^{xy} \epsilon^{ba} \approx \frac{m \delta T^{ta}}{P_L + Ts} - \sigma \frac{\partial_t \delta T^{ta}}{m} + \sigma \frac{P_L + Ts}{2m^2} \partial_a \left[\beta \delta T^{tt} + \left(1 + \beta \frac{P_L + Ts}{m} \right) \left(\delta J_1^{tx} + \delta J_2^{ty} \right) \right]$$

where β is a complicated hydrostatic function involving the bulk modulus, the heat capacity and the "thermal expansion coefficient".

$$P_L = m^2 (\mathcal{C} - r_h)$$

$$\sigma = \frac{-\left(m^2 - 6r_h^2\right)^2}{\left(m^2(2\mathcal{C} - 3r_h) + 6r_h^3\right)^2}$$

$$\beta = -\frac{2m}{r_h} \frac{m^2 + 6r_h^2}{\left(m^2 - 6r_h^2\right)^2}$$

Hydrodynamical Collective Modes

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$$\text{transverse sound: } \omega = \pm\mathcal{V}_\perp k - i\frac{\Gamma_\perp}{r_h}k^2 + O(k^3)$$

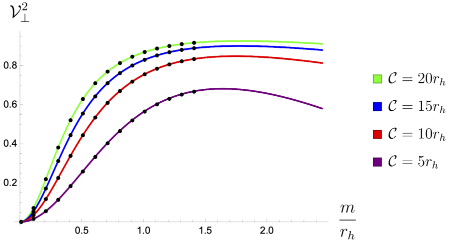
$$\text{longitudinal sound: } \omega = \pm\mathcal{V}_\parallel k - i\frac{\Gamma_\parallel}{r_h}k^2 + O(k^3)$$

$$\text{crystal diffusion: } \omega = -i\frac{\mathcal{D}_\parallel}{r_h}k^2 + O(k^3)$$

$$T = \frac{6r_h^2 - m^2}{8\pi r_h}$$

Hydrodynamical Collective Modes

Transverse speed^2
of sound
(as a function of $\frac{m}{r_h}$
for different $\mathcal{C}$ values)



The black dots come from the quasinormal modes of [Grozdanov & Poovuttikul, '18].

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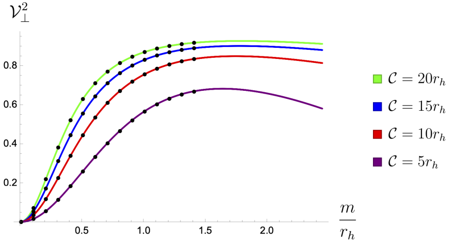

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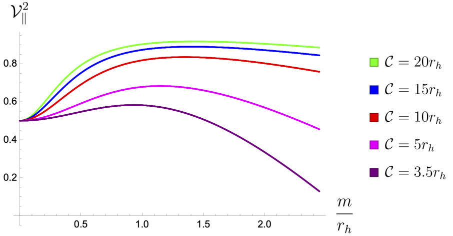
Longitudinal speed^2 of sound

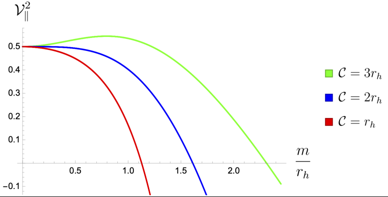


Hydrodynamical Collective Modes

For low $\mathcal{C}$,

$\mathcal{V}_\perp^2 < 0 \rightarrow Im(\omega) > 0$

(linear instability)



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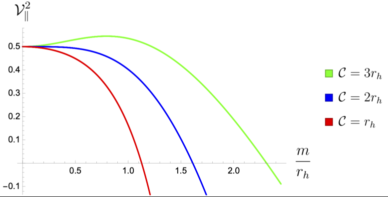
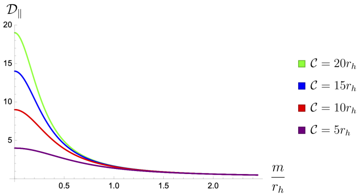
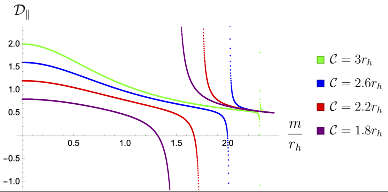


$$\frac{\mathcal{D}_\parallel}{r_h} = \sigma \frac{Ts + P_L}{2m^2}\left(\frac{1}{2\mathcal{V}_\parallel^2} + \beta \frac{Ts + P_L}{m}\right)$$



crystal diffusion constant



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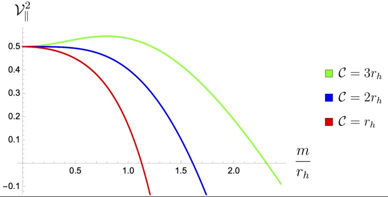
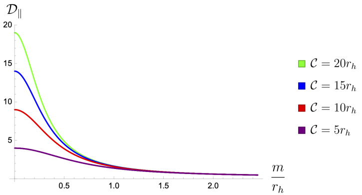


$$\frac{\mathcal{D}_\parallel}{r_h} = \sigma \frac{Ts + P_L}{2m^2} \left(\frac{1}{2\mathcal{V}_\parallel^2} + \beta \frac{Ts + P_L}{m} \right)$$



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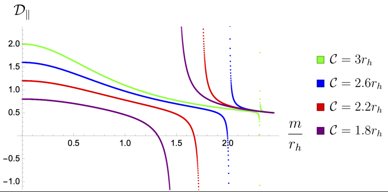

$\frac{\mathcal{C}}{r_h} > \frac{12-15\ln 3}{6\ln 3-8} \approx 3.18$ for the entire domain of $\frac{m}{r_h}$ to be safe in all modes!

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5) Momentum Relaxation ⟸

6) Conclusion

Momentum Relaxation

[Andrade & Withers, '14]:

$$S^{\Phi} = S_{grav} - \frac{1}{2}\int d^4x\sqrt{-G}\,F_a \wedge *F_a + S^{\Phi}_{\rm bdy}$$

where $F_a \equiv d\Phi_a$ and $S_{grav} = \int d^4x\sqrt{-G}\,(R+6) + \text{GHY}$.

$$\delta S^{\Phi} = \int d^3x\frac{\sqrt{-G}}{2}\left[\langle T^{\mu\nu}_{\Phi}\rangle\,\delta g_{\mu\nu} - 2\,\langle O^a\rangle\,\delta\phi_a\right]$$

Momentum Relaxation

$$S^B = S_{grav} - \frac{1}{2}\int d^4x\sqrt{-G}\,H_a \wedge *H_a + S^B_{\text{bdy}}$$

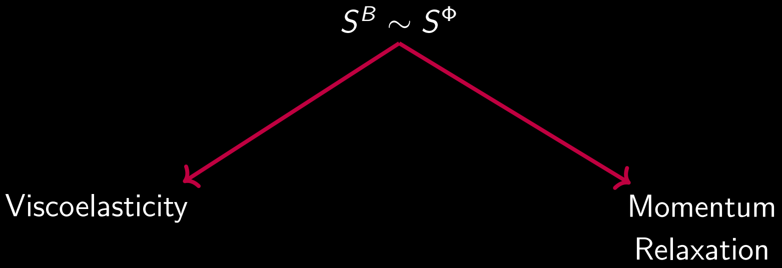
$$S^\Phi = S_{grav} - \frac{1}{2}\int d^4x\sqrt{-G}\,F_a \wedge *F_a + S^\Phi_{\text{bdy}}$$

$$H^a \to \epsilon^{ab} * F_b \;:$$

$$\frac{\delta S^B}{\delta G} = 0\,, \quad \frac{\delta S^B}{\delta B^a} = 0 \quad \text{and} \quad dH_a = 0$$

$$\frac{\delta S^\Phi}{\delta G} = 0\,, \quad \frac{\delta S^\Phi}{\delta \Phi^a} = 0 \quad \text{and} \quad dF_a = 0$$

Momentum Relaxation



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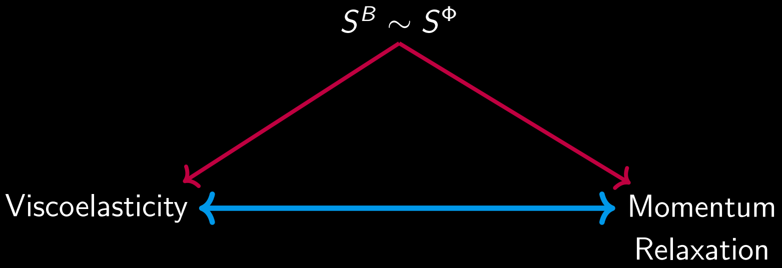

Momentum Relaxation

1 longitudinal diffusion mode

$$\delta O_1 \approx \frac{-m^2 \delta T_{\Phi}^{tx}}{\beta\sigma\left(P_L + sT\right)^2} + \frac{\partial_t \delta T_{\Phi}^{tx}}{m}$$

$$\omega \approx -ik^2 \frac{\beta\sigma\left(P_L + sT\right)^2}{2m^3}$$

Recalling the crystal diffusion mode

$$\frac{\mathcal{D}_{\parallel}}{r_h} = \frac{\beta\sigma\left(P_L + sT\right)^2}{2m^3} + O\left(\mathcal{C}^{-1}\right)$$

Momentum Relaxation

relaxation dispersion relation:

$$\omega = -i\Gamma_{rel} + O(k^2)\ , \quad \Gamma_{rel} \ll 1$$

$$\boxed{m \ll 1}$$

- Scalar theory: 1 transverse and 1 longitudinal relaxation mode with the same $\Gamma_{rel} = \Gamma^{\Phi}_{rel}$.
- Higher-form theory: 1 transverse and 1 longitudinal relaxation mode with the same $\Gamma_{rel} = \Gamma^{B}_{rel}$, $\boxed{\text{if } \mathcal{C} \gg 1.}$

$$\Gamma^{B}_{rel} \to \Gamma^{\Phi}_{rel} \quad \text{when} \quad \mathcal{C} \gg m^{-2}$$

Conclusion and Outlook

- slow perturbations of the charged (under 2-form gauge fields) black branes that we considered, obey the viscoelastic hydrodynamics of [Armas & Jain, '19]

- novel expressions for the **shear modulus**, **bulk viscosity**, **shear viscosity** and the elastic dissipative transport coefficient $\boldsymbol{\sigma}$

- lower bound on the deformation coupling $\mathcal{C}$

- what about: bulk solutions that are stable for low $\mathcal{C}$; anisotropic lattice pressure; etc?

- holographic dual of a crystal with dynamical dislocations

Thanks for your attention

谢谢