Generalized Fefferman-Graham gauge and boundary Weyl structures

Gabriel Arenas-Henriquez
Yau Mathematical Sciences Center, Tsinghua University

In collaboration with Felipe Diaz (ITMP, Moscow) and David Rivera-Betancour (ITMP, Moscow)

arxiv:2411.12513

Gauge Gravity Duality 2024, Sanya, Nov 30-Dec 4, 2024

Motivation

- AdS/CFT correspondence, in the saddle point approximation, relates a gravitational action with some boundary conditions to the generating functional of a conformal field theory

$$-I_{\mathrm{grav}}[\phi \to \phi_0] = W_{\mathrm{CFT}}[\phi_0]$$

Maldacena 97; Gubser, Klebanov, Polyakov 98; Witten 98

- The sources in the CFT correspond to the boundary conditions of fields in the AdS space.

- In the case of the metric field $g_{\mu\nu}$, its boundary value $g_{(0)ij}$ corresponds to the background of the CFT.

- An asymptotically AdS space can always be written in the Fefferman-Graham (FG) gauge

$$ds^2 = \frac{\ell^2}{z^2}dz^2 + \frac{\ell^2}{z^2}\left(g_{(0)ij}(x) + z g_{(1)ij}(x) + z^2 g_{(2)ij}(x) + \cdots\right) dx^i dx^j$$

Fefferman, Graham 85

where the boundary is located at $z = 0$.

- The boundary of AdS, $g_{(0)ij}$, is not unique.

- It is defined up to a conformal representative ω, $g_{(0)} = \omega^2 \eta$

- Therefore, there is a family of conformally equivalent metrics $[g_{(0)}]$ which serve as a source of the stress tensor of the boundary CFT.

- In many cases, one can choose the boundary to be flat.

- What happens when $g_{(0)}$ is curved? → Conformal invariance is broken! Marchetto, Miscioscia, Pomoni 2023

  Weyl invariance remains

- Weyl invariance provides kinematic constraints on the building blocks for correlators of the field theory. Parisini, Skenderis, Withers 2023

- How to incorporate Weyl covariance in the holographic dictionary?

- In the FG gauge, a bulk Weyl rescaling must be accompanied by a boundary transformation to preserve the gauge

$$z \to z' = \frac{z}{\mathscr{B}(x)} \,, \quad x^i \to x'^i = x^i + \xi^i(z, x)$$

- Subleading terms in the expansion are not invariant under a Weyl rescaling!

- Fixing $g_{(0)}$ automatically breaks Weyl symmetry

Main goal

- Incorporate Weyl structures in the holographic dictionary promoting the boundary manifold to be a *Weyl manifold*
- For this, we will use three-dimensional AdS gravity
- We expect to be able to describe holographic field theories in non-trivial backgrounds with Weyl symmetry

Generalized Fefferman-Graham gauge

- It is possible to relax the gauge allowing non-vanishing g_{zi} components:

$$ds^2 = \ell^2 \left(\frac{dz}{z} - a_i dx^i\right)^2 + h_{ij} dx^i dx^j = \frac{\ell^2 dz^2}{z^2} - 2\frac{\ell^2}{z} a_i dx^i dz + \gamma_{ij} dx^i dx^j$$

$$\gamma_{ij} := h_{ij} + \ell^2 a_i a_j$$

Grumiller, Riegler 16; Ciambelli, Leigh 22

- This metric is preserved under Weyl rescaling

$$z \to z' = \frac{z}{\mathcal{B}(x)} , \qquad x \to x' = x$$

- Upon the following transformation relations

$$h_{ij}(z,x) \to h'_{ij}(z',x) = h_{ij}(\mathcal{B}(x)z, x) , \quad a_i(z,x) \to a'_i(z',x) = a_i(\mathcal{B}(x)z, x) - \partial_i \ln \mathcal{B}(x)$$

- We can consider the following power expansion

$$h_{ij} = \frac{\ell^2}{z^2} \sum_{k\geq 0} \left(\frac{z}{\ell}\right)^k h_{ij}^{(k)} \,, \qquad a_i = \sum_{k\geq 0} \left(\frac{z}{\ell}\right)^{k-1} a_i^{(k-1)}$$

- Note that the expansion for a_i starts at $\mathcal{O}(z^{-1})$: In the Chern-Simons formulation there are new chemical potentials associated with $a^{(-1)}$ Grumiller, Riegler 16

- One can check that all subleading terms in the expansion of a_i (and $a^{(-1)}$) transform covariantly under Weyl transformations, while the leading term $a_i^{(0)}$ transforms non-homogeneously

$$h_{ij}^{(k)} \to h'^{(k)}_{ij} = \mathcal{B}^{k-2} h_{ij}^{(k)},$$

$$a_i^{(k)} \to a'^{(k)}_i = \mathcal{B}^k a_i^{(k)}, \quad a_i^{(0)} \to a'^{(0)}_i = a_i^{(0)} - \partial_i \log \mathcal{B}$$

- The inhomogeneous transformation of $a_i^{(0)}$ resembles the one of a Weyl connection. To prove this, we need to check that $h_{ij}^{(0)}$ and $a_i^{(0)}$ define a **Weyl manifold**. Weyl 18;Folland 70; Hall 92;

- The leading term of the bulk Christoffel symbol expanded around $z = 0$ gives

$$\mathring{\Gamma}^m_{(0)ij} := \frac{1}{2}h^{mn}_{(0)}\left(\partial_i h^{(0)}_{nj} + \partial_j h^{(0)}_{ni} - \partial_n h^{(0)}_{ij}\right) - \left(a^{(0)}_i \delta^m_j + a^{(0)}_j \delta^m_i - a^{(0)}_n h^{mn}_{(0)} h^{(0)}_{ij}\right) \ .$$

- Then, we can define a unique-torsion-free connection such that

$$\mathring{\nabla}^{(0)}_k h^{(0)}_{ij} - 2a^{(0)}_k h^{(0)}_{ij} = 0 \ ,$$

- Where $\mathring{\nabla}^{(0)}_k$ is the Weyl covariant derivative constructed with $\mathring{\Gamma}^m_{(0)ij}$

- Then $h_{ij}^{k\neq 0}$ and $a_i^{k\neq 0}$ are Weyl tensors!

- The boundary metric

$$\gamma_{ij}^{(0)} \equiv \lim_{z\to 0} \frac{z^2}{\ell^2}\gamma_{ij} = h_{ij}^{(0)} + \ell^2 a_i^{(-1)} a_j^{(-1)} \ .$$

- Turning $a_i^{(-1)}$ leads to the Weyl-Fefferman-Graham gauge

Ciambelli, Leigh 19' ;
Jia, Karydas 21 ; Jia, Leigh, Karydas 23'
Ciambelli, Delfante, Ruzziconi, Zwikel 23'

- Solving Einstein's equations perturbatively requires specifying not only $\gamma_{ij}^{(0)}$, but also the entire asymptotic expansion of a_i!

- Now the odd coefficients of the boundary metric expansion $h_{ij}^{(k)}$ are non-vanishing

- For instance, at order $\mathcal{O}(z^{-2})$, the $zz-$component gives the normalization of the Weyl vector $a_i^{(-1)}$, i.e.,

$$a_i^{(-1)} a_j^{(-1)} \gamma_{(0)}^{ij} = 0 \ .$$

- At order $\mathcal{O}(z^{-1})$, the $ij-$component gives

$$h_{ij}^{(1)} = -2\ell^2 \hat{\nabla}_{(i} a_{j)}^{(-1)}$$

- At order $\mathcal{O}(1)$, the $zz-$component gives

$$\mathrm{Tr}\ h_{(2)} = -\frac{\ell^2}{2} \hat{\mathscr{R}}^{(0)} - \ell^2 h_{ij}^{(2)} a^i_{(-1)} a^j_{(-1)}$$

where $\hat{\mathscr{R}}^{(0)}$ is the Weyl-Ricci scalar.

Renormalized Action

- In AdS_3, the renormalized action in the FG gauge is

$$S_{\text{renFG}} = S_{\text{EH}} + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma}\left(K - \frac{1}{\ell}\right)$$

- In the gFG frame, the $g_{zi} \neq 0$ terms induce new subleading divergences at the boundary. Specifically, a straightforward computation shows that the GHY term is divergent.

- It is necessary to introduce a new covariant boundary term to deal with this divergence.

- Expanding the on-shell action asymptotically, we find that the necessary term is

$$I_{\text{gFG}} = I_{\text{EH}} + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma}\left(K - \frac{1}{\ell} + \bar{\nabla}_i n^i\right)$$

$$n^i = -\frac{\ell a^i}{\sqrt{1 - \ell^2 a_i a^i}}$$

- This new counterterm is a total derivative, so it is a cod-2 object, which has been referred to as *corner term*

Adami, Parvizi, Sheikh-Jabbari, Taghiloo, Yavartanoo 23'
Ciambelli, Delfante, Ruzziconi, Zwikel 23'

- It plays an important role in spacetimes with topological defects.

- The holographic quantum effective action is then

$$I_{\text{gFG}} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma_{(0)}}\left[\frac{\ell}{4}\hat{\mathscr{R}} - \frac{\ell}{2}h^{(2)}_{ij}a^i_{(-1)}a^j_{(-1)} + \ell a^{(1)}_i a^i_{(-1)} + \frac{\ell}{2}a^{(0)}_i a^i_{(0)} - \nabla^{(0)}_i\left(2\ell a^i_{(0)} + \ell^3 a^{(0)}_j a^j_{(-1)}a^i_{(-1)}\right)\right]$$

- There is an associated Weyl current

$$J^i = -\frac{1}{\sqrt{-\gamma_{(0)}}}\frac{\delta I_{\text{gFG}}}{\delta a^{(0)}_i} = -\frac{\ell}{8\pi G}a^i_{(0)}$$

Ciambelli, Delfante, Ruzziconi, Zwikel 23'

- The holographic stress tensor can be written as

$$\langle T_{ij}\rangle = \hat{T}_{ij} + \nabla^{(0)}_{(i}J_{j)} - \gamma^{(0)}_{ij}\nabla^{(0)}_l J^l - \frac{8\pi G}{\ell}\left(J_iJ_j - \frac{1}{2}\gamma^{(0)}_{ij}J^lJ_l\right) + \ell^2 a^l_{(-1)}\left(a^{(-1)}_{(i}\nabla^{(0)}_l J_{j)} - a^{(-1)}_{(i}\nabla^{(0)}_{j)}J_l\right)$$

where $\hat{T}_{ij}$ is the Weyl covariant part of the stress tensor and J^i is the Weyl current.

- The Weyl anomaly is then recovered as

$$\langle T^i_i\rangle = \hat{T}^i_i + \nabla_i J^i = \mathscr{A}$$

with

$$\mathscr{A} = -\frac{\ell}{16\pi G}\hat{\mathscr{R}}[\gamma_0]$$

Jia, Karydas 23

Applications: Accelerating Black Holes in 2+1

- Non-trivial boundary structure
- Intriguing thermodynamics

$$ds^2 = \frac{1}{\Omega^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2\right)$$

$$f(r) = \frac{r^2}{\ell^2} - m^2(1 - \mathscr{A}^2 r^2)\ , \quad \Omega = 1 + \mathscr{A} r \cosh(m\phi)$$

Astorino 11; GAH, Scoins, Gregory 22

- An appropriate coordinate transformation $\frac{1}{r} = z - \mathscr{A}\cosh(m\phi) + \zeta \ell z^2$ renders the line element to

GAH, Cisterna, Diaz, Gregory 23

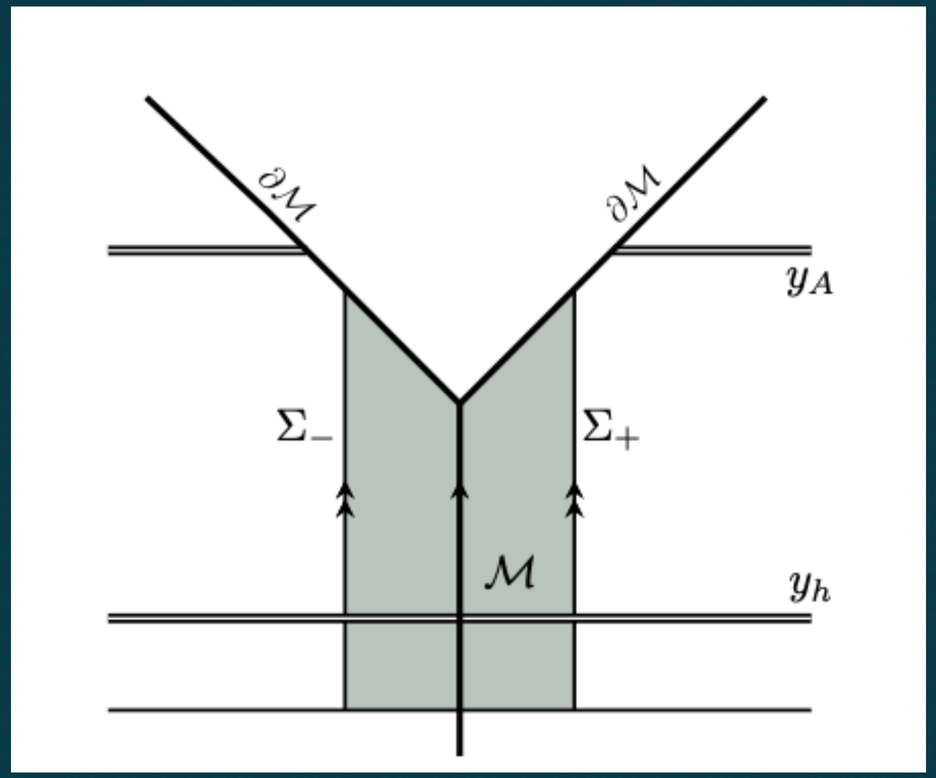
$$ds^2 = \ell^2 F^2 \left(\frac{dz}{z} - a_\phi(z,\phi)d\phi\right)^2 + h_{ij}(z,\phi)dx^i dx^j\ ,$$

$$F = F_{(0)}(x^i) + \frac{z}{\ell}F_{(1)}(x^i) + \dots \qquad a_\phi = \frac{m\mathscr{A}\sinh(m\phi)}{z} + 2m\mathscr{A}\ell\zeta\sinh(m\phi) + 4m\ell^2\mathscr{A}\sinh(m\phi)\zeta^2 z + \dots$$

$$J^i = \frac{\mathscr{A} m \ell^4}{2\pi G}\left(1 - \mathscr{A}^2 m^2 \ell^2 \sinh^2(m\phi)\right)\zeta\sinh(m\phi)\delta^i_\phi$$

- Euclidean action: accelerating BTZ black hole

$$I_{\text{ren}} = I_{\text{FG}} - \int_{\Sigma} d^2y \sqrt{\gamma} \left(\frac{1}{8\pi G} [\mathscr{K}]|_{-}^{+} + \mu \right)$$



- On-shell:

$$I_{\text{ren}} = \beta M - S$$

$$\mu = -\frac{m\mathscr{A}\sinh(m\pi)}{4\pi G}$$

GAH, Cisterna, Diaz, Gregory 23

- Instead, we evaluate I_{gFG} on-shell finding that

$$I_{\text{gFG}}\Big|_{\text{on-shell}} = I_{\text{ren}}\Big|_{\text{on-shell}} = \beta M - S$$

Final Comments

- Useful toolkit for studying dual field theories in non-trivial backgrounds where Weyl symmetry might provide important constraints on the correlator's building blocks.
- CFT correlators   Parisini, Skenderis, Withers 2023
- Extensions to higher dimensions? → 4D accelerating black holes
- Surface charges and asymptotic symmetry algebra
- Extension to asymptotically flat spaces   Grumiller, Merbis, Riegler 17
- Entanglement entropy
- Many open questions!

谢谢！