Applying Noether's theorem to the pure AdS_3 gravity

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The talk is based on an ongoing work with
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- In this work, we revisit the approach with the covariant phase space formalism for the asymptotic symmetry analysis in the pure AdS_3 gravity
- We reformulate the approach into a version which is exactly in the framework of **Noether theorem**
- Specifically, we get the following two results:
    - First, we show that the asymptotic symmetries are indeed symmetries of the pure AdS_3 gravity **in the sense of Noether theorem**
    - Second, we compute the associated charges of the asymptotic symmetries with the expression of **Noether charge**, which reproduces the result from the ordinarily used approach with the covariant phase space formalism.

Quantum gravity

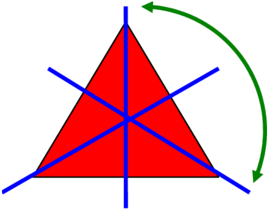
- Constructing quantum gravity is an important but difficult problem in physics
  - The singularity problem
  - The black hole informal paradox
  - The trans-planckian problem
- However, these problems are too difficult for me

A simpler set up

- To really have some understanding of quantum gravity, we consider a simpler setup and ask ourselves the following question
- Restricting to the effective field theory, low energy, and few body level, do we fully understand quantum gravity?
- There are still some unsolved problems:
  - Conserved charges
  - Diffeomorphism invariant observables
  - Solving the constraints in quantum level
  - Constructing the Hilbert space
  - Von-Neumann entropy
- Our focus: **how to properly define conserved charges**

Symmetries and conserved charges

- **Symmetries and conserved charges** are important topics in physics
- **Noether's theorem**

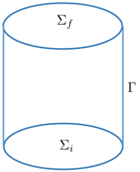


- Applications in classical mechanics, quantum mechanics, classical field theory, quantum field theory

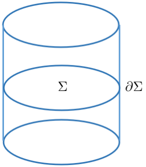
The difficulty in applying Noether's theorem to gravity

- However, **for gravitational system**, the application of Noether's theorem is **difficult**
- The difficulty is from the **constraints**

- **Difficulty I**
  - Noether's first theorem: $d\mathbf{j}_\xi = e.o.m$
  - $\int_{\Sigma_f} \mathbf{j}_\xi - \int_{\Sigma_i} \mathbf{j}_\xi = -\int_\Gamma \mathbf{j}_\xi$
  - For gravitational system, the spatial boundary term may not vanish



- **Difficulty II**
  - Noether's second theorem: $\mathbf{j}_\xi = e.o.m + d\mathbf{Q}_\xi$
  - The charge $\int_\Sigma \mathbf{j}_\xi = \int_{\partial\Sigma} \mathbf{Q}_\xi$
  - It requires a more careful treatment for the boundary effect



The covariant phase space formalism

- The more ordinarily used approach: the covariant phase space formalism

  Iyer, Lee, gr-qc/9403028

  Barnich, Brandt, hep-th/0111246

  Compere, 0708.3153

- Wide applications in the asymptotic symmetry analysis

  Barnich, Brandt, hep-th/0111246

  Compere, 0708.3153

  Strominger, 1703.05448

Our goal

- For the completeness of the framework, we still want to solve these problems in the framework of Noether theorem
- Our goal: reformulate the approach with the covariant phase space formalism into a version, which is **exactly in the framework of Noether's theorem**

- Previous attempts: finite systems with timelike or null boundaries

  Harlow, Wu, 1906.08616

  Shi, Wang, Xiu, Zhang, 2008.10551

  Chandrasekaran, Speranza, 2009.10739

- Key point: **treating the boundary effects systematically**
- This work: infinite systems with asymptotic boundaries
  More specifically, the pure AdS_3 gravity

Noether's theorem

- We illustrate Noether's theorem with a $0+1$ dimensional system
  $L = L(t; q_a, \dot{q}_a, \ddot{q}_a, ..., q_a^{(n)})$

The covariant phase space formalism

- To introduce Noether's theorem, we need to reformulate the $0+1$ dimensional system into the Hamiltonian formalism
- Here, we take use of the covariant phase space formalism
- Structures in the covariant phase space formalism:
  - Pre-phase space $\widetilde{\mathcal{P}}$: the set of solutions
  - Symplectic form ω: read out from the Lagrangian

Reading out the symplectic form

- We now read out the symplectic form ω from the Lagrangian
- Variation of the Lagrangian
  $\delta L = E^a[t; q_a]\delta q_a + \frac{d}{dt}\theta[t; q_a, \delta q_a]$
- Symplectic potential: θ
  (a one-form of the set of configurations)
- Symplectic form: $\omega = \delta\theta$
  (a two-form of the set of configurations)
- The symplectic form of the pre-phase space: $\omega|_{\widetilde{\mathcal{P}}}$
  (The pull-back of ω to the pre-phase space $\widetilde{\mathcal{P}}$)

The symmetry

- We now introduce the notion of the symmetry. Here, we need to go back to Lagrangian formalism
- A symmetry in the frame of Noether's theorem:
  - an infinitesimal transformation;
  - it acts on all of the **configurations**;
  - its change on the Lagrangian is a total derivative, up to a configuration independent anomaly term
- Representation in a more rigorous level:
  - Symmetry: $X_\lambda = \int dt \delta_\lambda q_a(t) \frac{\delta}{\delta q_a(t)}$
  - The change of the Lagrangian under the symmetry
    $X_\lambda \cdot \delta L = \frac{d}{dt}\alpha_\lambda[t, q_a] + \beta_\lambda(t)$
    $X_\lambda \cdot \delta L$: acting the symmetry to the Lagrangian.
    $\frac{d}{dt}\alpha_\lambda$: the total derivative term.
    $\beta_\lambda(t)$: the anomaly term which is configuration independent

The Noether charge

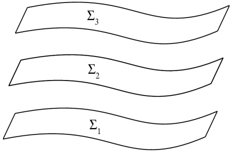
- **Noether charge:** $Q_\lambda = X_\lambda \cdot \theta - \alpha_\lambda$
  X_λ: the symmetry, a vector field of the set of configurations
  θ: the symplectic potential, a one-form of the set of configurations
  α_λ: the total derivative term appearing in $X_\lambda \cdot \delta L$

The Noether theorem

- **Noether theorem**
  - First, the Noether charge Q_λ, when evaluated for the set of solutions (namely under the on-shell condition) is time independent, up to a configuration independent anomaly term $\frac{d}{dt} Q_\lambda|_{\widetilde{\mathcal{P}}} = \beta_\lambda$.
  - Second, the symmetry indeed maps a solution to a solution $X \cdot E^a|_{\widetilde{\mathcal{P}}} = 0$
  - Third, the Noether charge Q_λ is indeed the charge conjugate to the symmetry X_λ.
    Namely, they satisfy the following Hamiltonian equation $X_\lambda \cdot \omega|_{\widetilde{\mathcal{P}}} = -\delta Q_\lambda|_{\widetilde{\mathcal{P}}}$

Generalization to the higher dimensional systems

- **Higher dimensional generalization: we view the spatial direction as the internal degrees of freedom; and we directly transplant the 0+1 dimensional result to the higher dimensional system**
  - The θ and α:
    $\delta S = \int d^{d+1}xE^a\delta\phi_a + \theta|_{\Sigma_f} - \theta|_{\Sigma_i}$
    $X \cdot \delta S = \alpha|_{\Sigma_f} - \alpha|_{\Sigma_i} + \beta|_M$



Two approaches for the Noether charge

- Two approaches of the Noether charge
  - Approach I: with the Noether charge, $Q = X \cdot \theta - \alpha$
  - Approach II: with the Hamiltonian equation: $X \cdot \omega|_{\widetilde{\mathcal{P}}} = -\delta Q|_{\widetilde{\mathcal{P}}}$
- The first approach: exactly what we learn in classical mechanics, $H = p\dot{q} - L$
- The second approach: the widely used one in gravity
- **Our goal: compute the charge in gravity with the first approach, namely with the expression of the Noether charge**

The application to the AdS$_3$ gravity

- We now apply Noether's theorem to the the pure AdS$_3$ gravity
- More specific targets: studying the asymptotic symmetries and their associated charges

The strategy

- The strategy:
  - Define the pure AdS$_3$ gravity in the Lagrangian formalism
  - Take a variation of the action
    Read out the symplectic potential θ
  - Act the asymptotic symmetries to the action
    Read out α, β
  - Compute the Noether charge with $Q = X \cdot \theta - \alpha$
- Technical issue: a systematical treatment of the boundary effect

Boundary effect I: the definition of the theory

- The boundary effect in the definition of the theory
  - we need to adopt proper asymptotic boundary conditions
    $$g_{zz} = \frac{1}{z^2} + \mathcal{O}(z^0)$$
    $$g_{za} = \mathcal{O}(\tfrac{1}{z})$$
    $$g_{ab} = \frac{1}{z^2} g^{(0)}_{ab} + \mathcal{O}(z^0)$$
  - we need to take holographic renormalization for the action **in the off-shell level** de Haro, Skenderis, Solodukhin, hep-th/0002230

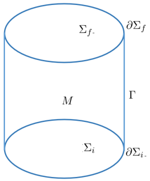
- We define the action as a limit of the regularized action $S = \lim_{\epsilon\to 0} S_\epsilon$
- The regularized action $S_\epsilon = \int_{M_\epsilon} \mathbf{L} + \int_{\Gamma_\epsilon} \boldsymbol{\ell}$
- The definition is finite **for all configurations**

Boundary effect II: the covariant phase space formalism

- The boundary effect in reformulating the theory into the covariant phase space formalism

  Harlow, Wu, 1906.08616

$$\delta S = \lim_{\epsilon\to 0}\left[\int_{M_\epsilon}\mathbf{E}^a\delta\phi_a + \int_{\Gamma_\epsilon}\mathbf{F}^a\delta\phi_a + \int_{\Sigma_{f,\epsilon}}\mathbf{\Theta} - \int_{\partial\Sigma_{f,\epsilon}}\mathbf{C} - \int_{\Sigma_{i,\epsilon}}\mathbf{\Theta} + \int_{\partial\Sigma_{i,\epsilon}}\mathbf{C}\right]$$

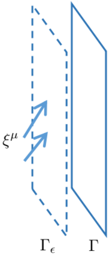


- The symplectic potential:

  $\theta = \lim_{\epsilon\to 0}\left[\int_{\Sigma_\epsilon}\mathbf{\Theta} - \int_{\partial\Sigma_\epsilon}\mathbf{C}\right]$

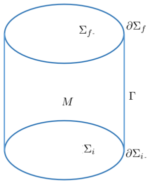
Boundary effect III: acting the asymptotic symmetry

- The boundary effect in applying the asymptotic symmetry to the action
- The diffeomorphism is not parallel to the cutoff surface



- It has the following two effects:
  - For bulk Lagrangian density $X_\xi \cdot \delta(\int_{M_\epsilon} \mathbf{L}) = \int_{M_\epsilon} \mathcal{L}_\xi \mathbf{L} = \int_{M_\epsilon} d(\xi \cdot \mathbf{L}) \supset \int_{\Gamma_\epsilon} \xi \cdot \mathbf{L}$
  - For boundary Lagrangian density $X_\xi \cdot \delta \mathbf{l} \neq \mathcal{L}_\xi \mathbf{l}$

- The change of the action under the asymptotic symmetry:
  $$X_\xi \cdot \delta S = \lim_{\epsilon\to 0}\left[\left(\int_{\Sigma_{f,\epsilon}} \xi\cdot \mathbf{L} - \int_{\partial\Sigma_{f,\epsilon}} \boldsymbol{\mu}_\xi\right) - \left(\int_{\Sigma_{i,\epsilon}} \xi\cdot \mathbf{L} - \int_{\partial\Sigma_{i,\epsilon}} \boldsymbol{\mu}_\xi\right) + \int_{\Gamma_\epsilon} \boldsymbol{\nu}\right]$$
  - The expressions of α_ξ, β_ξ
    $$\alpha_\xi = \lim_{\epsilon\to 0}\left[\int_{\Sigma_\epsilon} \xi\cdot \mathbf{L} - \int_{\partial\Sigma_\epsilon} \boldsymbol{\mu}_\xi\right]$$
    $$\beta_\xi = \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} \boldsymbol{\nu}_\xi$$
    β_ξ is configuration independent



- Two results:
  - The asymptotic symmetry is indeed a symmetry in the sense of Noether's theorem
  - Noether charge:
    $Q_\xi = X_\xi \cdot \theta - \alpha_\xi$
    $\quad = \int_{\partial\Sigma} \frac{1}{8\pi G}(-K^{\mu\nu} + K\gamma^{\mu\nu} - \gamma^{\mu\nu})\xi_\nu \epsilon^\Gamma{}_{\mu\mu_1} dx^{\mu_1}$
    consistent with the covariant phase space formalism and the boundary stress tensor

Discussions

- A new approach; open questions:
  - TMG
  - Kerr/CFT
  - Higher dimensional theories
  - Theories in asymptotic flat spacetime
- Boundary effects
- An off-shell definition of the pure AdS_3 gravity

Thanks

Thanks for your attention!