N=4 SYM on real projective space and analytic conformal bootstrap

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Xinan Zhou (周稀楠)

Kavli Institute for Theoretical Sciences

University of Chinese Academy of Sciences

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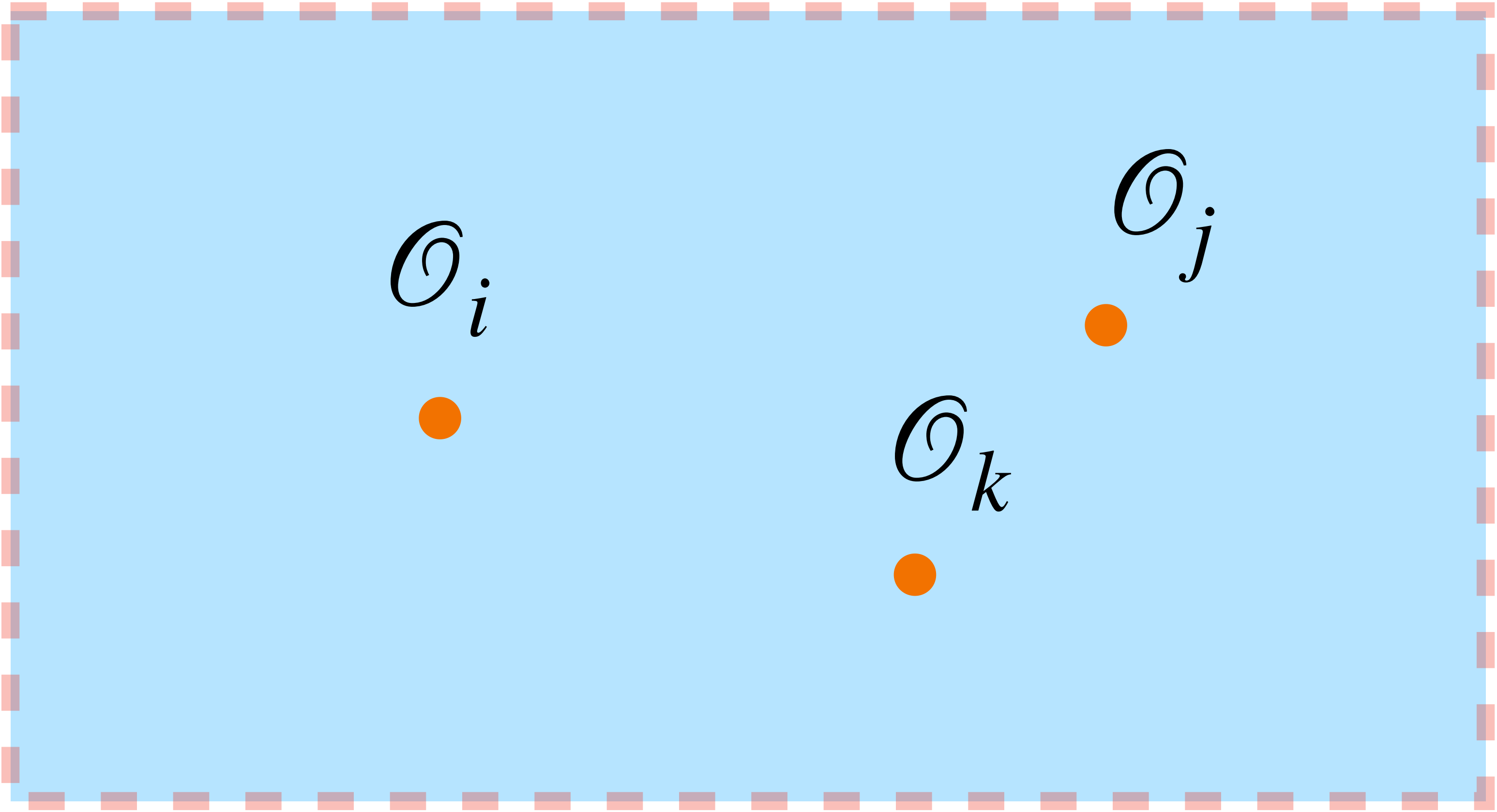
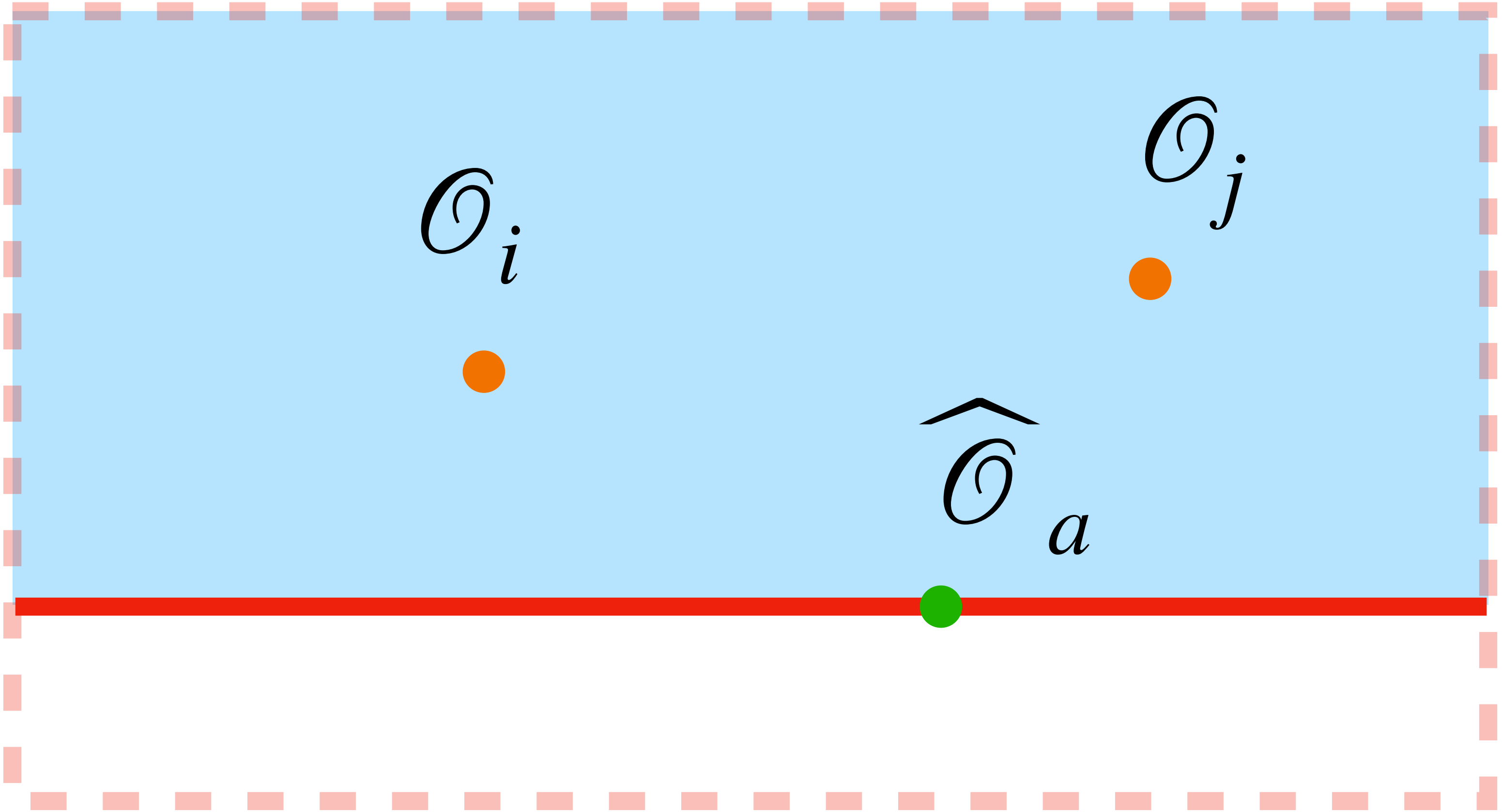
Theories on nontrivial backgrounds

- Studying theories on nontrivial backgrounds allows us to access information which is otherwise invisible in infinite flat space.

- An example: CFTs with a conformal boundary



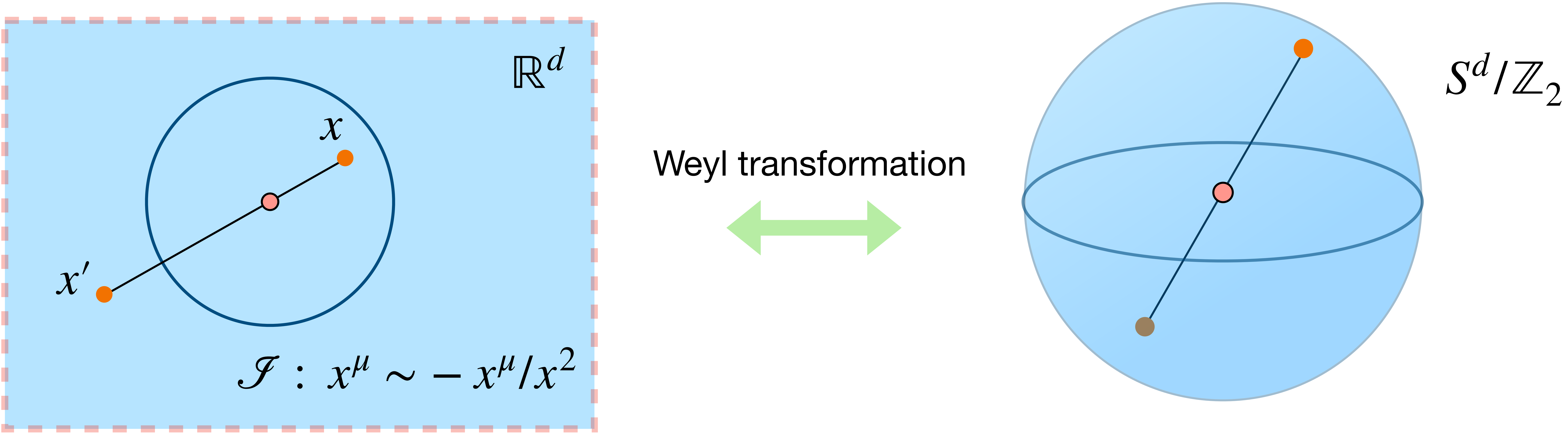
CFT data: $\{\Delta_i, C_{ijk}\}$



Extended CFT data: $\{\Delta_i, C_{ijk}, \widehat{\Delta}_a, \widehat{C}_{abc}, \widehat{b}_{ai}\}$

Theories on real projective space

- Real projective space is defined as



- Simplest non-orientable manifold in even d.
- Possible for any QFT with time reversal symmetry.
- Many studies in condensed matter and high energy physics in relation to subtle anomalies related to time reversal symmetry.

$\mathcal{N}=4$ SYM on $\mathbb{RP}^4$

- **$\mathcal{N}=4$ SYM: "hydrogen atom" for high energy physics**
  - Integrable, amenable to susy localization, prime example of AdS/CFT.
  - Can these be extended beyond flat space?

- **$\mathcal{N}=4$ SYM on $\mathbb{RP}^4$: Rigid and minimal**
  - Here we want to preserve half of supersymmetry [Wang '20].
  - Unlike BCFT with 1/2-BPS boundary condition, there are no new d.o.f. and no choice of boundary conditions.
  - We can, however, choose to gauge (or not to gauge) charge conjugation when we identify the operators under $\mathbb{Z}_2$

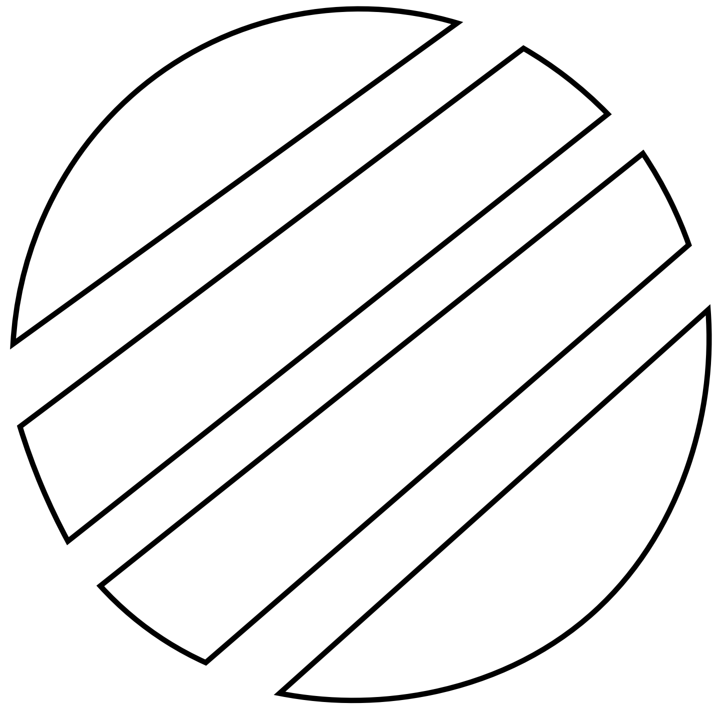
$$\mathcal{O}(x) \to \mathcal{O}(x')$$

Charge conjugation $\tau : g \to g^* \in SU(N) \quad (T^a)^m{}_n \to -(T^a)^n{}_m$

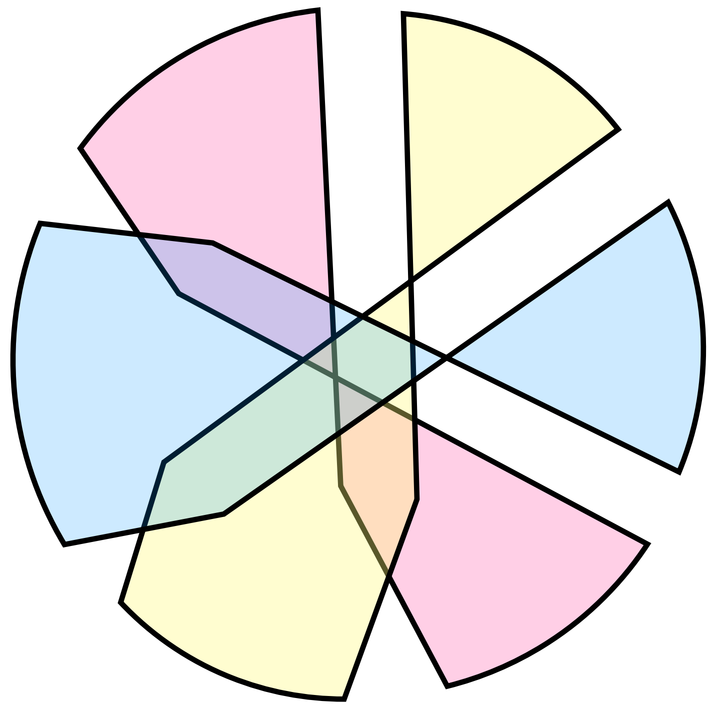
To gauge or not to gauge...

- Physically, the two situations are very different. Let's consider 1pt functions

w/o gauging

$\frac{1}{N^{L/2}}$  $= N$

w gauging

$\frac{1}{N^{L/2}}$  $= 1$

SUGRA dual:

A new classical background asymptotic to $AdS_5 \times S^5/\mathbb{Z}_2$

A $\mathbb{Z}_2$ quotient of $AdS_5 \times S^5$ [Caetano, Rastelli '22]

- We consider the case of gauging charge conjugation because it's simpler and more interesting (preserving integrability [Caetano, Ratelli '22]).

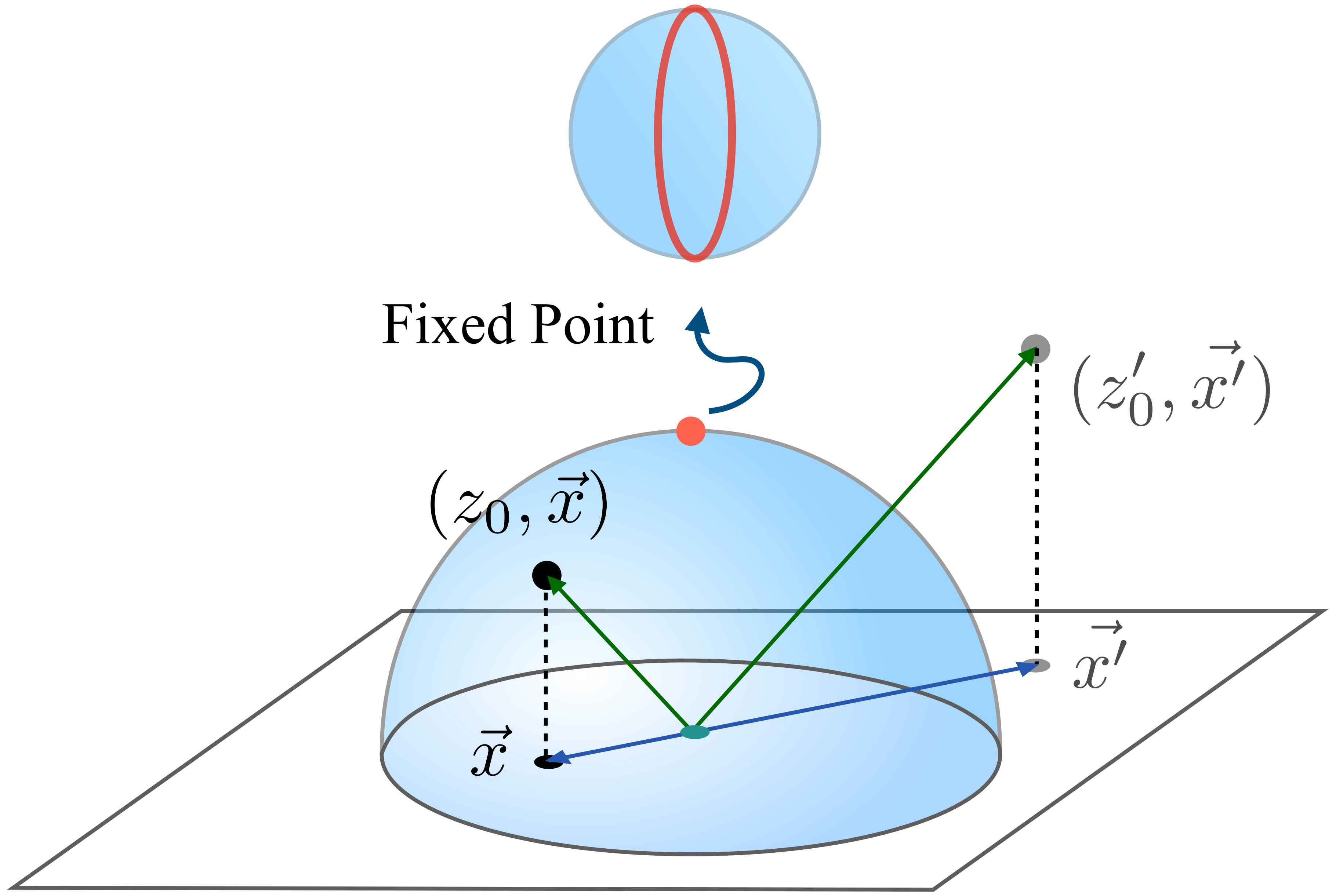
- Side comment: Constructing the dual from string theory is difficult because $\mathbb{Z}_2$ is a conformal isometry emerging only in the IR.

Quotient AdS

- The $\mathbb{Z}_2$ acts on AdS_5 as an in version with respect to a unit hemisphere

$$(z_0, \vec{z}) \to \left(\frac{z_0}{z_0^2 + \vec{z}^2}, -\frac{\vec{z}}{z_0^2 + \vec{z}^2} \right)$$

leaving the north pole invariant.



- But the $\mathbb{Z}_2$ also acts on the internal S^5

$$(\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6) \to (\Phi_1, \Phi_2, \Phi_3, -\Phi_4, -\Phi_5, -\Phi_6)$$

- Fixed locus $S^2 \subset S^5$: an O1 orientifold. We need the orientifold because the 5-form flux would vanish unless we also reverse worldsheet orientation (τ in the field theory).

As defects

- Let me also digress a bit and make a comment that the system can be more generally viewed as an example of holographic defects.

- A p-dimensional defect in the CFT is dual to a $(p+1)$-dimensional submanifold in AdS. A familiar example is WL in the fundamental representation which is dual to an AdS_2 string worldsheet.

- Recently, there has been a lot of progress in studying holographic defect correlators both at tree level [Gimenez-Grau ‘23, Chen, Gimenez-Grau, XZ ’23] and at loop level [Chen, Gimenez-Grau, Paul, XZ ’24].

- The 0-dimension fixed point in AdS can be viewed as a defect with dimension -1.

In this talk…

- As mentioned, there is no "derivation" of the AdS dual. This means that the construction is necessarily bottom-up and involves some unfixed ingredients.

- This may seem that we do not even have a starting point for doing calculations. However, this is not a problem for bootstrap approaches.

- Concretely, we will consider 2pt functions of 1/2-BPS operators (super gravitons) in the strong coupling limit.

- Although the details of the SUGRA effective Lagrangian are not known, we will show that it is possible to obtain all tree-level 2pt functions using analytic bootstrap techniques.

Kinematics

- We focus on super gravitons (1/2-BPS operators)

$$\mathcal{O}_p(x,Y) = \mathcal{N}_p \,\mathrm{tr}(\Phi^{i_1}(x)\dots\Phi^{i_p}(x))Y_{i_1}\dots Y_{i_p}$$

with $SO(6)$ null vectors $Y \cdot Y = 0$.

- The $\mathbb{Z}_2$ quotient breaks half of super symmetry

$$PSU(2,2|4) \to OSp(4|4) \begin{cases} \text{conformal} & SO(4,1) \subset SO(4,2) \\ \text{R-symmetry} & SO(3)\times SO(3) \subset SO(6) \end{cases}$$

$\hookrightarrow$ $Y = (\vec{u},\vec{v}) \quad \bar{Y} = (\vec{u},-\vec{v})$

- 1pt function can be non vanishing when p is even

$$\langle\!\langle \mathcal{O}_p \rangle\!\rangle = a_p \frac{(Y\cdot\bar{Y})^{\frac{p}{2}}}{(1+x^2)^p}$$

a_p is new CFT data

Kinematics

- 2pt functions are partially fixed

$$\langle\!\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\rangle\!\rangle = \frac{(Y_1\cdot\bar{Y}_1)^{\frac{p_1}{2}}(Y_2\cdot\bar{Y}_2)^{\frac{p_2}{2}}}{(1+x_1^2)^{p_1}(1+x_2^2)^{p_2}}\mathcal{G}_{p_1p_2}(\eta;\sigma,\bar{\sigma})$$

polynomials in σ and $\bar{\sigma}$ of degree $p_m = \min\{p_1,p_2\}$

up to a function of three cross ratios

$$\eta = \frac{x_{12}^2}{(1+x_1^2)(1+x_2^2)} \qquad \sigma = \frac{Y_1\cdot Y_2}{(Y_1\cdot\bar{Y}_1)^{\frac{1}{2}}(Y_2\cdot\bar{Y}_2)^{\frac{1}{2}}} \qquad \bar{\sigma} = \frac{Y_1\cdot\bar{Y}_2}{(Y_1\cdot\bar{Y}_1)^{\frac{1}{2}}(Y_2\cdot\bar{Y}_2)^{\frac{1}{2}}}$$

- Crossing symmetry

$$\mathcal{G}_{p_1p_2}(\eta;\sigma,\bar{\sigma}) = \mathcal{G}_{p_1p_2}(1-\eta;\bar{\sigma},\sigma)$$

- Scf Ward identities (analytically continue BCFT [Liendo, Meneghelli ‘16])

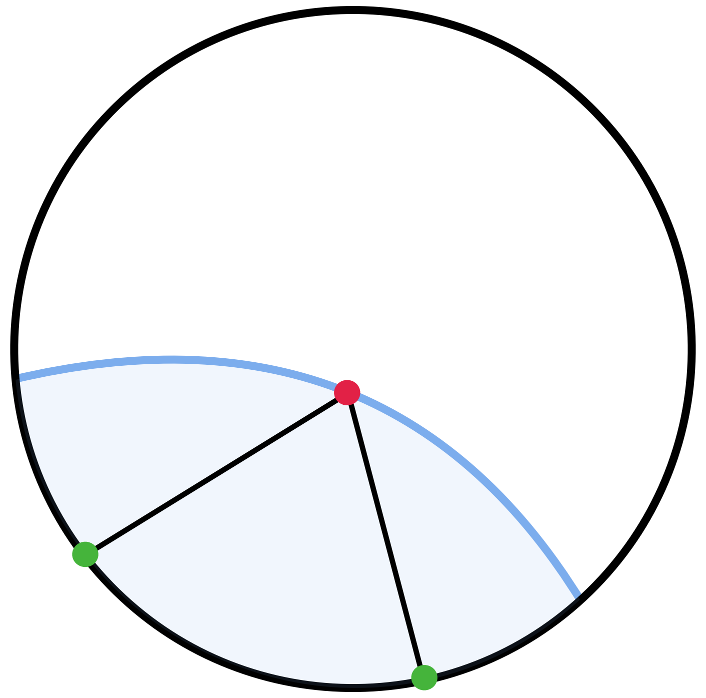
$$\left(\partial_{w_1}+\frac{1}{2}\partial_z\right)\mathcal{G}_{p_1p_2}(z;w_1,w_2)\Big|_{w_1=z} = 0$$

$$\eta = -\frac{(z-1)^2}{4z}$$

$$\sigma = \frac{(1-w_1)(1-w_2)}{4\sqrt{w_1w_2}} \qquad \bar{\sigma} = \frac{(1+w_1)(1+w_2)}{4\sqrt{w_1w_2}}$$

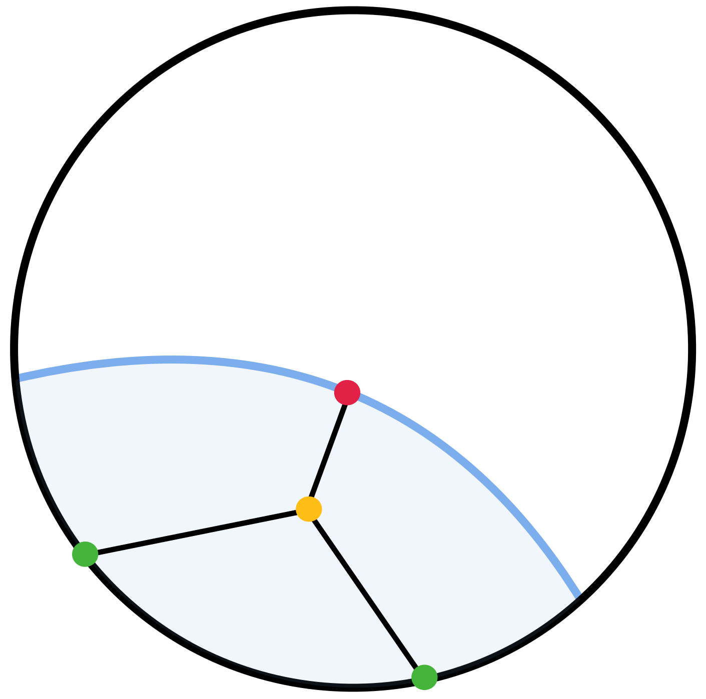
Witten diagrams

- The O1 orientifold induces vertices localized at a point in AdS.
- At tree level, i.e., $\mathcal{O}(1/N\sqrt{\lambda})$, we encounter two types of Witten diagrams



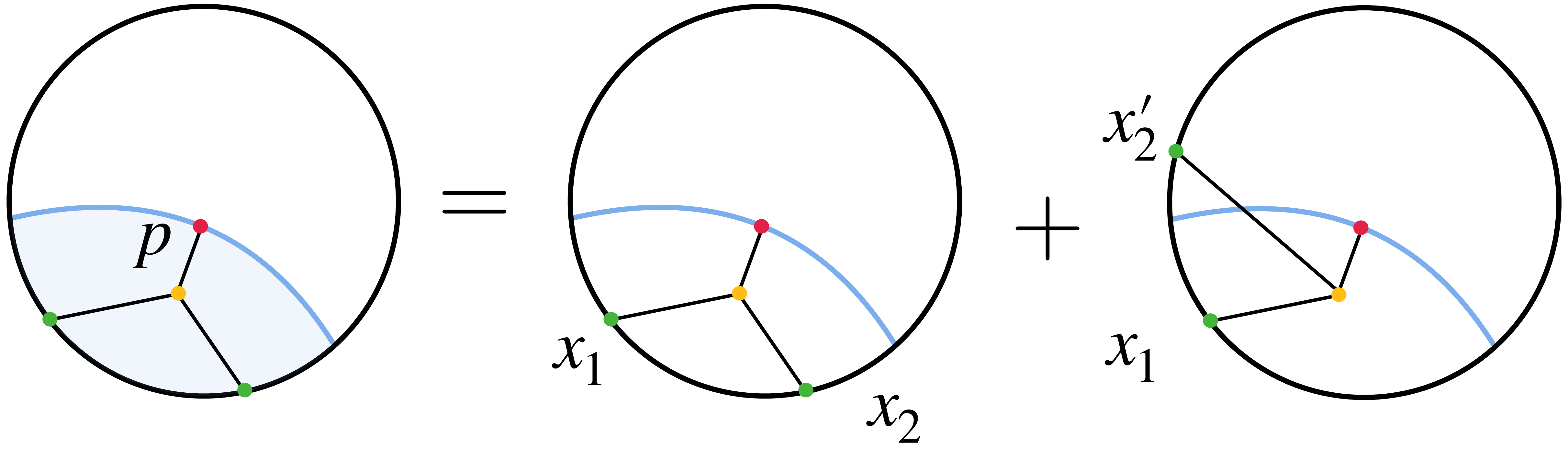
contact Witten diagram

$$\mathcal{W}_{con} = 1$$



exchange Witten diagram

- Method of images [Giombi, Khanchandani, XZ '20]



$$\mathcal{W}_p = \sum_{k=\frac{p-p_1-p_2}{2}}^{-1} \frac{\Gamma(1-\frac{d}{2}+p)\Gamma(k+p_1)\Gamma(k+p_2)}{\Gamma(\frac{p+p_{12}}{2})\Gamma(\frac{p-p_{12}}{2})} \times \frac{\eta^k}{\Gamma(\frac{2+2k-p+p_1+p_2}{2})\Gamma(\frac{2-d+2k+p+p_1+p_2}{2})}$$

$$\mathcal{W}_\Delta^{\mathbb{Z}_2}(\eta) = \mathcal{W}_\Delta(\eta) + \bar{\mathcal{W}}_\Delta(\eta) \qquad \bar{\mathcal{W}}_\Delta(\eta) = \mathcal{W}_\Delta(1-\eta)$$

Only scalars can be exchanged.

Analytic bootstrap

- We start with an ansatz

$$\mathcal{A}_{p_1p_2} = \mathcal{A}_{p_1p_2,e} + \bar{\mathcal{A}}_{p_1p_2,e} + \mathcal{A}_{p_1p_2,c}$$

Exchange part:

$$\mathcal{A}_{p_1p_2,e}(\eta,\sigma,\bar{\sigma}) = \sum_X \lambda_X h_{R_X}(\sigma,\bar{\sigma})\mathcal{W}_{\Delta_X}(\eta) \qquad \bar{\mathcal{A}}_{p_1p_2,e}(\eta,\sigma,\bar{\sigma}) = \mathcal{A}_{p_1p_2,e}(1-\eta,\bar{\sigma},\sigma)$$

The exchange part is determined by selection rules

- R-symmetry selection rule
- Non-extremal: $p < p_1 + p_2$ etc

fields	s_p	ϕ_p	t_p
$SU(4)$ irrep	$[0,p,0]$	$[2,p-4,2]$	$[0,p-4,0]$
Δ	p	$p+2$	$p+4$

R-symmetry polynomials $h_X(\sigma,\bar{\sigma})$ can be obtained from solving the R-symmetry Casimir equation, which can be identified with the Casimir equation for 4pt conformal blocks in 3d [Dolan, Osborn ‘03].

Analytic bootstrap

Contact part:

$$\mathcal{A}_{p_1 p_2, c} = \sum_{a=0}^{p_m} \sum_{b=0}^{p_m - a} \delta_{a+b-p_m, \text{even}} c_{ab} \sigma^a \bar{\sigma}^b$$

We require contact vertices have no derivatives, but we include all R-symmetry structures.

- We can explicitly evaluate the ansatz and it is a rational function in the cross ratios.
- We then plug the ansatz into the suerpconformal Ward identities and then try to solve for the unknown coefficients.

Analytic bootstrap

- Let us implement this strategy starting from small values of $p_m = p_2$.
  - $p_m = 2$: only s_{p_1} can be exchanged

$$\mathcal{G}_{p_1 2} = \mu_1 \frac{\sigma\bar{\sigma}}{\eta(1-\eta)} + \mu_0$$

  - $p_m = 3$:

The fields which can be exchanged are $\{X\} = \{s_{p_1-1}, s_{p_1+1}\}$. We find everything is fixed up to an overall constant. In particular, the ratio of exchange contributions are fixed

$$\frac{\lambda_{s_{p_1-1}}}{\lambda_{s_{p_1+1}}} = \frac{p_1-1}{p_1+1}$$

Recall that the parameter λ has the meaning of 3pt times 1pt

$$\lambda_{s_p} = C_{p_1 p_2 p} a_p \qquad C_{p_1 p_2 p} = \sqrt{p_1 p_2 p}$$ [Lee, Minwalla, Rangamani, Seiberg ’98]

This gives

$$a_p = C\sqrt{p}$$

Analytic bootstrap

- $p_m \geq 4$:

The other two fields t and ϕ start to appear in the exchanges. But using the result from $p_m = 3$, superconformal Ward identities fix all parameters up to the trivial additive constant solution which only exists for p_m even.

$$\mathcal{G}_{p_1p_2} = \Big\{ \sum_{p\in\mathcal{S}} a_p C_{p_1p_2p} \big(h_{p,0}\mathcal{W}_p + \nu_p h_{p-4,2}\mathcal{W}_{p+2} + \rho_p h_{p-4,0}\mathcal{W}_{p+4} \big) + (\eta \to 1-\eta\,,\, \sigma \leftrightarrow \bar{\sigma}) \Big\}$$

$$+\mathcal{G}_{p_1p_2,\text{con}} + B_{p_1p_2}\delta_{p_m,\text{even}}$$

$$\mathcal{G}_{p_1p_2,\text{con}} = C \sum_{t=1}^{\lfloor \frac{p_m}{2} \rfloor} \left[\frac{4(-1)^{\lfloor \frac{p_m-1}{2} \rfloor} \sqrt{p_1} (\lfloor \frac{1+p_m}{2} \rfloor)_{\frac{p_{12}}{2}} (\frac{1+p_m}{2} - t)_t (\frac{2+p_m}{2} - t)_t \Gamma(\frac{3-p_1-p_2}{2})}{\Gamma(t)\Gamma(\frac{p_{12}}{2}+t)\sqrt{p_m}(\lfloor \frac{1+p_m}{2} \rfloor)_{\frac{1-p_1-p_2}{2}}} (\sigma+\bar{\sigma})^{p_m-2t} \right]$$

$$\nu_p = \frac{((p-2)^2 - p_{12}^2)(p^2 - p_{12}^2)}{16(p-3)(p-1)^2 p} \qquad \rho_p = \frac{(p-1)((p+2)^2 - p_{12}^2)(p^2 - p_{12}^2)}{16(p^2-4)p(p+1)^2}\nu_p$$

Comparing with SUGRA

- One can also try to compare with direct supergravity calculation. This provides checks and only fixes the ambiguity. Only the tension term of O1 is relevant

$$-T_{\rm O1}\int_{\rm S^2} e^{-\frac{\Phi}{2}}\sqrt{-\det g_{ab}^{\rm P.B.}}$$

- For s and t, only the fluctuation field $\pi = h^{\alpha}{}_{\alpha}$ is relevant

$$\pi(x,y) = \sum \pi_p^I(x) Y_p^I(y) \qquad \pi_p = 10 p s_p + 10(p+4) t_{p+4}$$

- The strategy is to expand in fluctuations and then integrate over the 2-sphere.
- At linear order, this gives rise to 1pt functions which fully agrees with the bootstrap calculation

$$a_p \propto \sqrt{p} \qquad\qquad a_{t_{p+4}} \propto \sqrt{\frac{(p+3)(p+4)(p+7)}{(p+1)(p+5)}}$$

Comparing with SUGRA

- At quadratic order and without derivatives, we can only write down π^2. It contributes only to the contact part

$$\Pi_{p_1p_2} = \frac{25\sqrt{p_1p_2}(p_1+1)(p_2+1)}{2}\sum_{p\in\mathcal{I}}\frac{4\pi}{(p_1+p_2+1)!!}\binom{p_1}{p}\binom{p_2}{p}p!(p_1-p-1)!!(p_2-p-1)!!(\sigma+\bar{\sigma})^p$$

- However, this does not exactly match the bootstrap result

$$\mathcal{G}_{p_1p_2,\text{con}} = C\frac{(p_1+p_2+1)(p_1+p_2-1)}{50\pi(p_1+1)(p_2+1)p_1p_2}\times(\Sigma\partial_\Sigma - p_1)(\Sigma\partial_\Sigma - p_2)\Pi_{p_1p_2} \qquad \Sigma = \sigma+\bar{\sigma}$$

- The mismatch does not prove bootstrap is wrong. Instead it indicates that there must exist some other subtle contributions to contact terms (e.g., from EOM). A similar mismatch was also observed for defect 2pt functions with WL [Gimenez-Grau '23] .

Comparing with SUGRA

- Although the naive SUGRA analysis fails, it correctly captures two important features
  - It depends only on the combination $\sigma + \bar{\sigma}$.
  - The correlator should be analytic in the power p.

- Note that the additive ambiguity term corresponds to $p = 0$ and is only present when p_m is even. The rest of the correlator is already analytic in p. This suggests the ambiguities should be set to zero!

- This way the bootstrap calculation completely determines all tree-level two-point functions.

Outlook

- We studied a cute model which has very simple results. It would be interesting to reproduce these results from other methods such as integrability [Caetano, Rastelli '22].
- An immediate generalization of the tree-level result is to go to loops, using AdS unitarity method [Aharony, Alday, Bissi, Perlmutter '16; Chen, Gimenez-Grau, Paul, XZ '24].
- We only looked at the SUGRA limit. But we can also go beyond it and consider stringy corrections.
  - Similar analysis has been carried out for 4pt functions or defect 2pt functions. The simpler model considered here offers an attractive alternative.
  - Use supersymmetric localization [Wang '20]. Integrated 2pt functions can be analyzed similar to defect 2pt functions [Pufu, Rodriguez, Wang '23; Dempsey, Pufu, Wang '24; Billo, Galvagno, Frau, Lerda '23, '24].
  - Study the flat-space limit [Alday, XZ '24] and combine worldsheet description?

Thank you!