# Holgraphy and Pseudospectra

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D. Arean, D. Fariña, K.L. JHEP 12 (2023) 187 • e-Print: 2307.08751 [hep-th]
D. Fariña, K.L., P. Romeu, P. Saura'Bastida arXiv:2407.06104 [hep-th]
D. Arean, D. Fariña, K.L Front.in Phys. 12 (2024) 1460268 • arXiv: 2407.04372 [hep-th]

EXCELENCIA SEVERO OCHOA

*Gauge/Gravity 24, Sanya, 2.12.2024*

# Pseudospectra and the AdS/CFT correspondence

➔ Normal Modes

➔ Black Holes and Quasinormal modes

➔ AdS(/CFT)

➔ Pseudospectra

➔ Pseudospectra of QNMs and CMMs

➔ Summary and Outlook

# Normal Modes

Eigen modes of string:

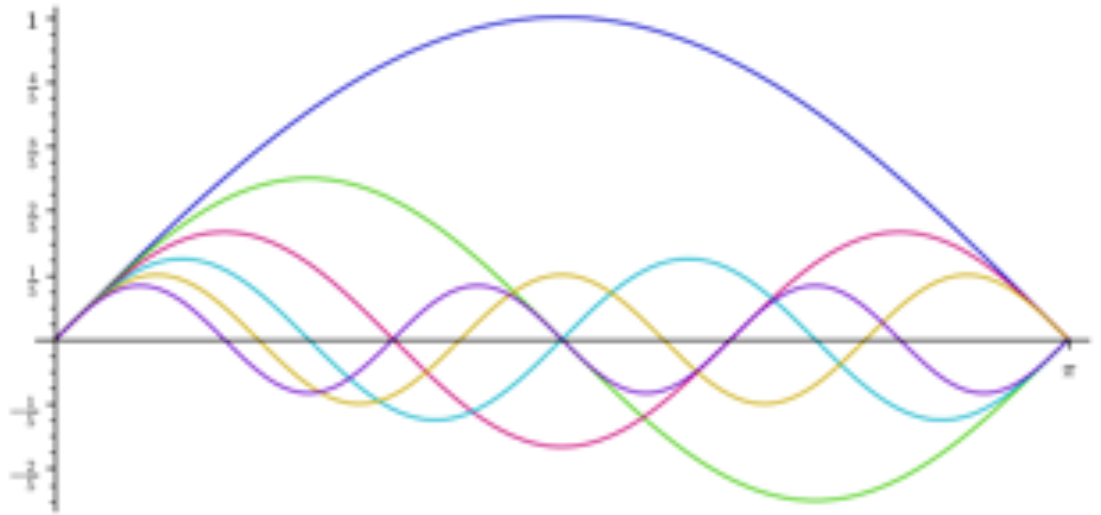$$\frac{d^2 \Phi(x)}{dx^2} + \lambda \Phi(x) = 0$$

Boundary conditions:

$$\Phi(0) = \Phi(\pi) = 0$$

Hermitian operator:

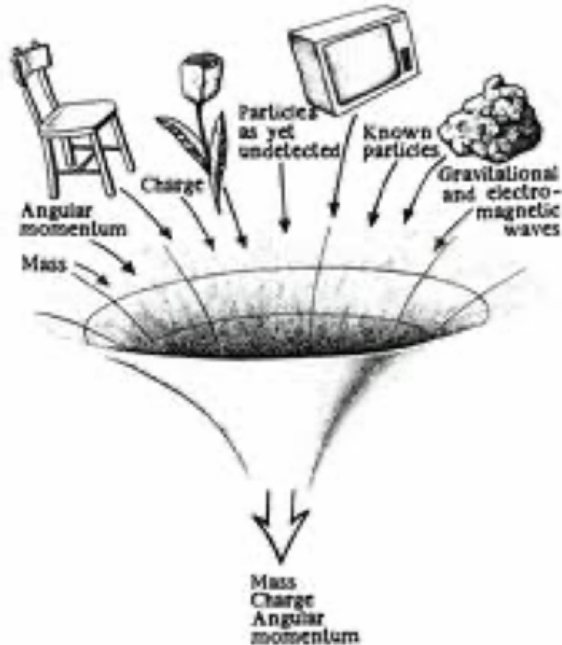$$\mathcal{L} = \frac{d^2}{dx^2} \quad , \quad \mathcal{L}^\dagger = \mathcal{L}$$

$$\langle \Psi, \Phi \rangle = \int_0^\pi dx \, \bar{\Psi} \Phi \qquad \langle \Psi, \mathcal{L}\Phi \rangle = \langle \mathcal{L}\Psi, \Phi \rangle$$
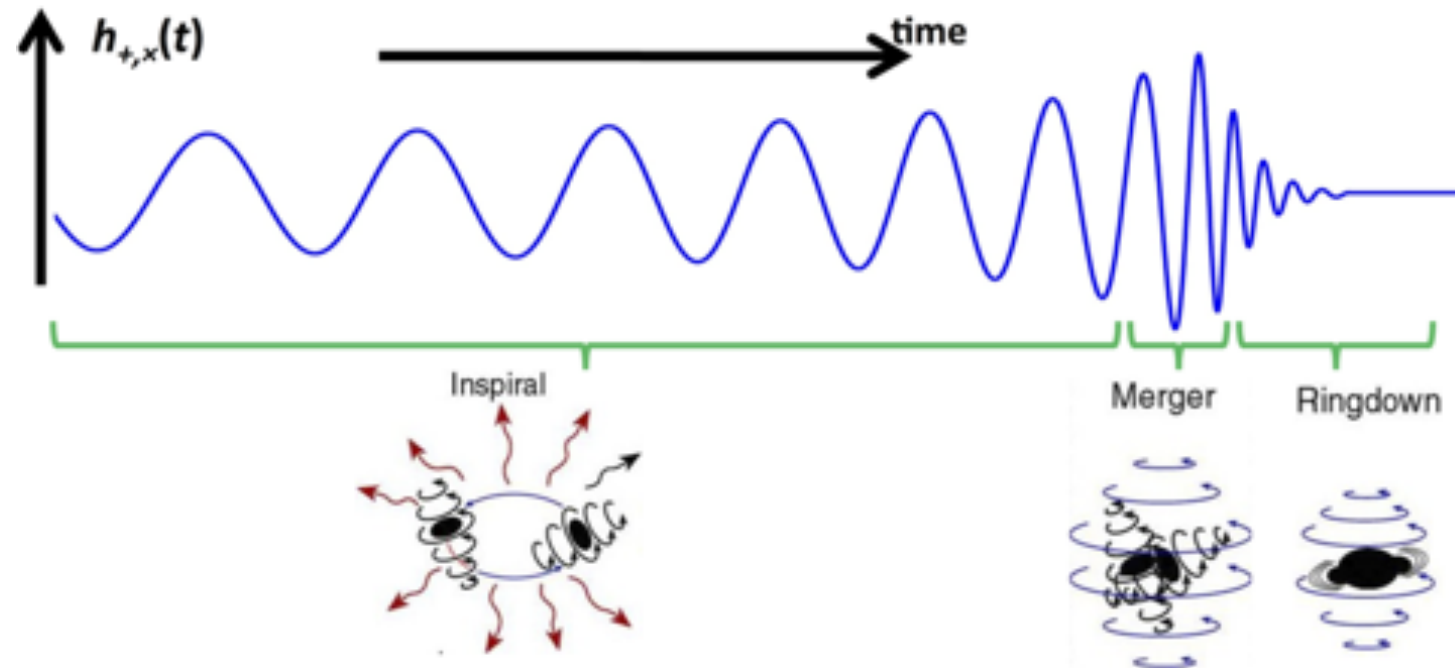


$$\Phi_n(x) = \sin(nx)$$
$$\lambda = n^2$$

# Quasi-Normal Modes

- Black Hole



➜ Black Holes no Hair

➜ Swallow everything

➜ Fate of a perturbation

  ➜ Either fall into BH

  ➜ Or radiate off to infinity

  ➜ Perturbation eventually dies off

# Quasi-Normal Modes

# Quasi-Normal Modes

How to compute Quasi Normal Modes:
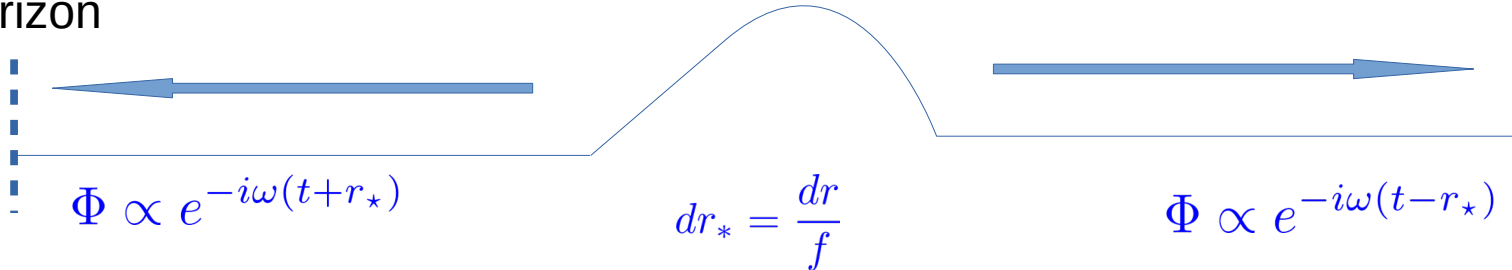
$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$\Phi = \phi(r)e^{-i\omega t}Y_{lm}(\Omega)$$

$$\phi'' + \frac{f'}{f}\phi + \left(\frac{\omega^2}{f^2} - \frac{l(l+1)}{r^2 f} + \frac{f'}{rf}\right)\phi = 0$$

"Outgoing" boundary conditions:

Horizon

[Vishveshwara]
[Press]

$$\Phi \propto e^{-i\omega(t+r_\star)} \qquad dr_* = \frac{dr}{f} \qquad \Phi \propto e^{-i\omega(t-r_\star)}$$

# Quasi-Normal Modes

Leaky boundary conditions lead to complex frequencies
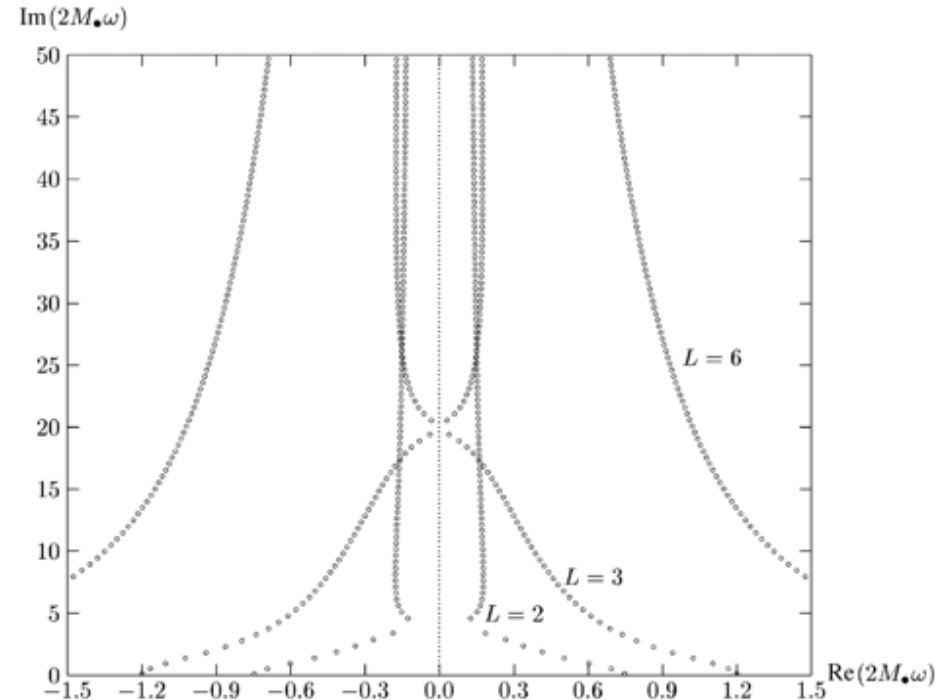
[Leaver, Prod. Roy. Soc. A402 (285)]

$$\omega_n = \Omega_n - i\Gamma_n$$

Oscillation          Damping

"Black Hole spectroscopy"

[Berti, Cardoso, Will, PRD 73 (2006)]

Reviews: [Nollert, "TOPICAL REVIEW", Class.Quant.Grav. 16 (1999) R159-R216 ]
[Berti, Cardoso, Starinets, Class.Quant.Grav. 26 (2009) 163001 ]

# Holographic QNMs

Field in AdS black hole:
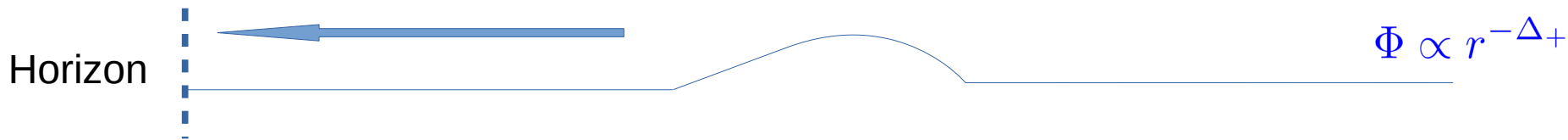$$\Phi(r,t,\vec{x}) = e^{-i\omega t + i\vec{k}.\vec{x}} \phi_{\omega,\vec{k}}(r)$$

Boundary condition Horizon:
$$\phi_{\omega,\vec{k}} \propto e^{-i\omega(t+r_*)}$$

Boundary condition boundary:
$$\phi \approx A(\omega,\vec{k})r^{-\Delta_-}(1+\ldots) + B(\omega,\vec{k})r^{-\Delta_+}(1+\ldots)$$

Retarded Green's function:
$$G_R(\omega,\vec{k}) = K\frac{B(\omega,\vec{k})}{A(\omega,\vec{k})}$$

[Horowitz, Hubeny],
[Birmingham, Sachs, Solodhukin]
[Kovtun, Son, Starinets]

Horizon

$$\Phi \propto r^{-\Delta_+}$$

# Holographic QNMs

Example: Scalar field in BTZ black hole

$$G_R(\omega, k) = \frac{(\omega^2 - k^2)}{4\pi^2} \left[ \psi \left( 1 - i\frac{\omega - k}{4\pi T} \right) + \psi \left( 1 - i\frac{\omega + k}{4\pi T} \right) \right]$$

$$\omega_n = \pm k - i4(n+1)$$    Exact spectrum of QNMs !

In general no exact solution, e.g. scalar in AdS$_5$:

$$\phi'' + \left( \frac{5}{r} + \frac{f'}{f} \right) \phi' + \frac{\omega^2 - f^2 \vec{k}}{r^2 f} \phi = 0$$
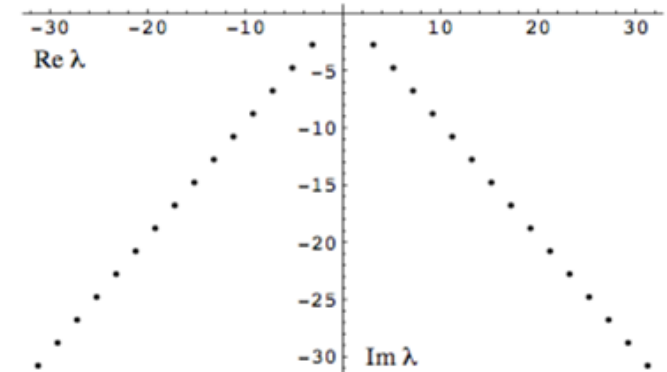
"Christmas tree"



Fig. 1: The lowest 15 quasinormal frequencies in the complex $\lambda$-plane for $q = 0$.
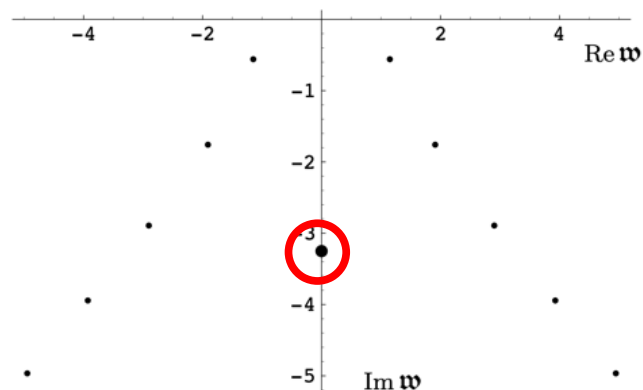
[Starinets]

# Holographic QNMs

Gauge fields: new ingredient gauge symmetry: conserved current $\quad \partial_\mu J^\mu = 0$

$$\frac{d}{dt} Q = 0 \qquad Q = \int d^3 x J^0$$

2 channels:
- Transverse is like scalar
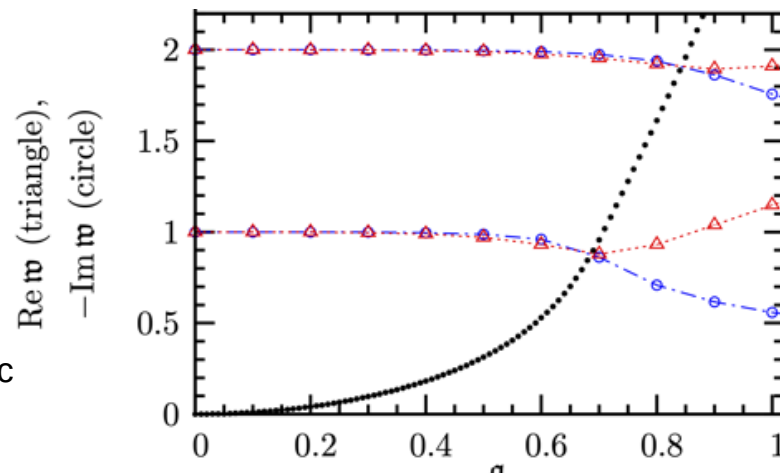- Longitudinal new: diffusion

[Son, Starinets]

[Amado, Hoyos, K.L., Montero]



$$\omega_{\text{hydro}}(k) = -iDk^2$$

$$\omega = i \sum_{n=1}^{\infty} a_i k^{2n} \quad \text{Problematic Causality!}$$

# Holographic QNMs

Metric: 3 channels:
- Tensor (Spin 2) is like scalar $\delta g_{xy}$
- Vector (Spin 1) is like vector → Shear channel $\delta g_{tx}, \delta g_{zx}$
- Scalar (Spin 0) new → Sound channel $\delta g_{tt}, \delta g_{zz}, \delta g_{tz}$

Shear = momentum diffusion $\qquad \omega(k) = -i\frac{\eta}{\epsilon + p}k^2 + O(k^2)$

$$\eta = \frac{1}{4\pi}$$

Universality of shear viscosity

[Policastro,Son,Starinets]
[Kovtun,Son,Starinets]

Sound = energy conservation $\qquad \omega(k) = \pm v_s k - i\gamma k^2 + O(k^3) \qquad v_s^2 = \frac{\partial \epsilon}{\partial p} = 1 d = \frac{1}{d}$

Hydrodynamics: $\qquad T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} - \eta P_T^{\mu\alpha}P_T^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{1}{d}\eta_{\alpha\beta}\partial.u)$

$\partial_\mu T^{\mu\nu} = 0$

# Holographic QNMs

Superconducing QNMs:

- Holographic "Superconductor"
- Spontaneous U(1) symmetry breaking
- Goldstone mode = 2$^{nd}$ sound

[Gubser]
[Hartnoll, Herzog, Herzog]

$$S = \int d^4x \sqrt{-g} \left[ R + \Lambda - \frac{1}{4}F^2 - |D\Phi|^2 - m^2|\Phi|^2 \right]$$

Temperature and chemical potential: $4\pi T = f'(r_h)$

$$\mu = A_t(\infty) - A_t(r_h)$$

Unbroken phase T<T$_c$

Growing mode  T>T$_c$ $\Im(\omega) > 0$
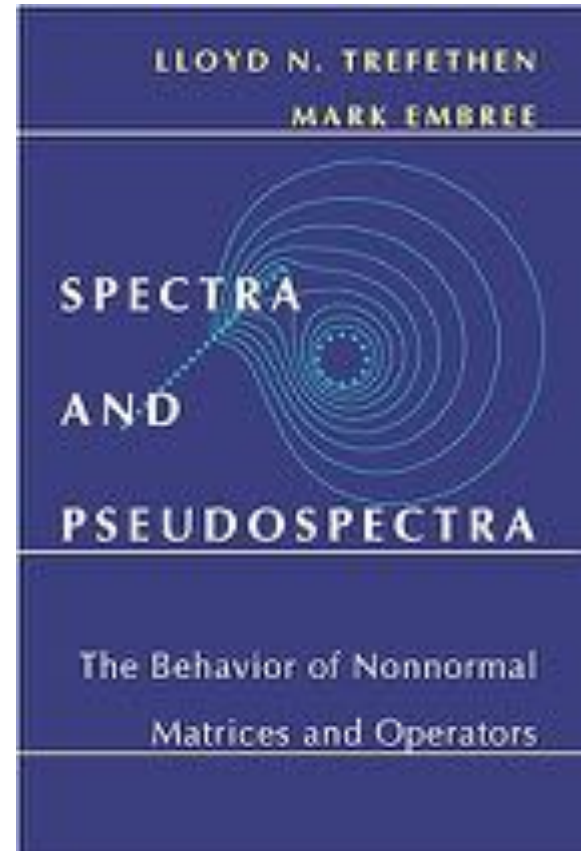
[Amado,Kaminski, K.L.]

# QNMs

**Quasinormal Modes are an essential ingredient to Gauge/Gravity duality!**

# Pseudospectra

- QNMs are eigenvectors of non-Hermitian operators

- No spectral theorem $\quad \mathcal{O} \neq \sum_n |n\rangle \lambda_n \langle n|$

- QNMs are not complete (only late time)

- Eigenfunctions are singular in Schwarzschild coords

- Robustness and physical significance in question

Near Horizon $\quad \phi(r) = e^{-i\omega r_*} = e^{-i\Omega r_* - \Gamma r_*} \quad , \quad r_* \to -\infty$

No Hilbert space interpretation

LLOYD N. TREFETHEN

MARK EMBREE

SPECTRA

AND

PSEUDOSPECTRA

The Behavior of Nonnormal

Matrices and Operators

# Pseudospectra

Resolvent: $\mathcal{R}(\mathcal{L}, z) = (\mathcal{L} - z)^{-1}$

Spectrum: $\sigma(\mathcal{L}) = \{z \in \mathbb{C} : "\mathcal{R}(\mathcal{L}, z) = \infty"\}$

Eigenvalues: $\mathcal{L}u_n = \lambda_n u_n$

Operator norm: $\|\mathcal{L}\| = \sup_{u \in H} \dfrac{\|\mathcal{L}u\|}{\|u\|}$     *(Max. SVD)*
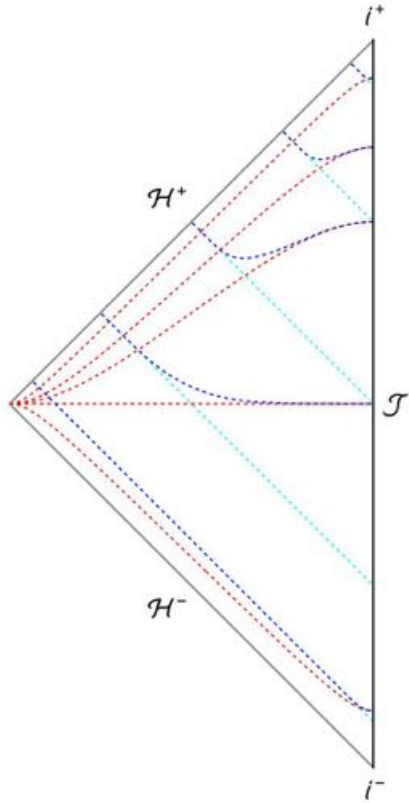
Definitions of **ε-Pseudospectra:**

1) Resolvent norm     $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C} : \|\mathcal{R}(\mathcal{L}, z)\| > 1/\epsilon\}$

2) Perturbation     $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C}, \exists \delta\mathcal{L}, \|\delta\mathcal{L}\| < \epsilon : z \in \sigma(\mathcal{L} + \delta L)\}$

3) Pseudo eigenvector     $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C}, \exists u^\epsilon : \|(\mathcal{L} - z)u^\epsilon\| < \epsilon\|u^\epsilon\|\}$

Theorem: The 3 definitions are equivalent

# Pseudospectra

How to deal with the QNM problems: chose better coordinates!



Schwarzschild coordinates (worst)

$$ds^2 = r^2 \left[ -f(r)dt^2 + d\vec{x}^2 \right] + \frac{dr^2}{r^2 f(r)} \qquad \phi(r) \propto (r - r_h)^{-i\omega/2}$$

Infalling Eddington-Finkelstein

$$dv = dt + \frac{dr}{r^2 f(r)}$$

$$ds^2 = r^2 \left[ -f(r)dv^2 + d\vec{x}^2 \right] + 2dvdr$$

$\phi(r)$ regular at $r = r_h$

"Regular"

$$\tau = v - (1 - r_h/r)$$

$$ds^2 = r^2 \left[ -fd\tau^2 + d\vec{x}^2 \right] + 2(1 - f)d\tau dr + (2 - f)\frac{dr^2}{r^2}$$

[Warnick]

# Pseudospectra

Condition number:
$$\kappa_i = \frac{||v_i||\,||u_i||}{|\langle v_i, u_i \rangle|}$$

Right eigenvector: $\mathcal{L}u_i = \lambda_i u_i$    Left eigenvector: $\mathcal{L}^\dagger v_i = \lambda_i^* v_i$

Perturbation: $||\delta\mathcal{L}|| = \epsilon$    Perturbed eigenvalue: $(\mathcal{L} + \delta\mathcal{L})u_i(\epsilon) = \lambda(\epsilon)u_i(\epsilon)$

$$|\lambda(\epsilon)_i - \lambda_i| \leq \epsilon\kappa_i$$

Def "small": Let $d_{min}$ be the minimal distance between disconnected regions in the spectrum.
$\delta\mathcal{L}$ , $||\delta\mathcal{L}|| = \epsilon$  is small if

$$\frac{\epsilon}{d_{min}} \ll 1$$

# Pseudospectra

We need a physically motivated norm: **Energy** !

Energycurrent: $J = t^\mu T_{\mu\nu} dx^\mu$ $\qquad\qquad E[\Phi] = \int_{\Sigma_t} \star J$

| | | |
|---|---|---|
| Schwarzschild: | $E = \dfrac{1}{2}\int d^3x dr \left[ r^2 f(\Phi')^2 + \dfrac{(\partial_t \Phi)^2}{r^2 f} + m^2\Phi^2 + \dfrac{(\vec{\partial}\Phi)^2}{r^2} \right]$ | $\dfrac{dE}{dt} = 0$ |
| Infalling EF: | $E = \dfrac{1}{2}\int d^3x dr r^3 \left[ r^2 f(\Phi')^2 + m^2\Phi^2 + \dfrac{(\vec{\partial}\Phi)^2}{r^2} \right]$ | $\dfrac{dE}{dv} = -r_h^3 \int_{r=r_h} (\partial_v \Phi)^2$ |
| Regular: | $E = \dfrac{1}{2}\int d^3x dr r^3 \left[ r^2 f(\Phi')^2 + m^2\Phi^2 + \dfrac{(\vec{\partial}\Phi)^2}{r^2} + (2-f)(\partial_\tau \Phi)^2 \right]$ | $\dfrac{dE}{d\tau} = -r_h^3 \int_{r=r_h} (\partial_\tau \Phi)^2$ |

# Pseudospectra

- We want to re-write the wave equation as a standard eigenvalue problem
- Only possible in regular coordinates

$$\psi = \partial_\tau \Phi \qquad \Psi = \begin{pmatrix} \Phi \\ \psi \end{pmatrix} \qquad \mathcal{L} = i \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \qquad \Longrightarrow \qquad \mathcal{L}\Psi = \omega\Psi$$

- Compactify radial coordinate $\qquad \rho = 1 - \dfrac{r_h}{r}$

$$L_1 = [f(\rho) - 2]^{-1} \left[ \frac{m^2 l^2}{(1-\rho)^2} + \mathbf{q}^2 - (1-\rho)^3 \left( \frac{f(\rho)}{(1-\rho)^3} \right)' \partial_\rho - f(\rho)\partial_\rho^2 \right]$$

$$L_2 = [f(\rho) - 2]^{-1} \left[ (1-\rho)^3 \left( \frac{f(\rho) - 1}{(1-\rho)^3} \right)' + 2\left( f(\rho) - 1 \right) \partial_\rho \right]$$

- Adjoint operator in energy norm $\qquad \mathcal{L}^\dagger = \mathcal{L} + \begin{pmatrix} 0 & 0 \\ 0 & -i\,\delta(\rho) \end{pmatrix} \qquad \dfrac{d}{d\tau}E = -\bar\psi\psi|_{\rho=0}$

AdS in IEF [Cownden, Pantelidou, Zilhao] e-Print: 2312.08352

# Pseudospectra

- No exact solutions → numerical methods
- Pseudospectral methods

- Differential operator becomes a (N+1)x(N+1) matrix D    $F'(\rho_j) = D_{jk}F(\rho_j)$

- Boundary conditions: delete rows and columns corresponding to $\rho = 1$
  regularity corresponds to no boundary condition at $\rho = 0$

- Resolvent norm becomes maximal svd    $||(\mathcal{L} - \omega\mathbf{1})^{-1}||_E \approx \inf(\mathrm{sv}((L - \omega\mathbf{1})_{N \times N})$

- Energy norm becomes a metric 2Nx2N matrix

$$E \approx \bar{u}_k^* G_E^{km} u_m \quad , \quad u \approx (\phi(\rho_j), \psi(\rho_j))^T$$

# Pseudospectra

A toy example: $A = \begin{pmatrix} -1 & 0 \\ -50 & -2 \end{pmatrix}$  Eigenvalues: $\lambda_1 = -1$ , $\lambda_2 = -2$

1. $\ell_2$ norm: $||u|| = [\bar{u}.u]^{1/2}$

$\kappa_1 = \kappa_2 = \sqrt{2501} \approx 50$

Contour maps of $\log ||A - \omega \mathbf{1}||$

1. G-norm: $||u||_G = [\bar{u}.G.u]^{1/2}$

$A^\dagger = [(G.A.G^{-1})^T]^* = A$

$G = \begin{pmatrix} 20000 & 50 \\ 50 & 1 \end{pmatrix}$

$\kappa_1 = \kappa_2 = 1$
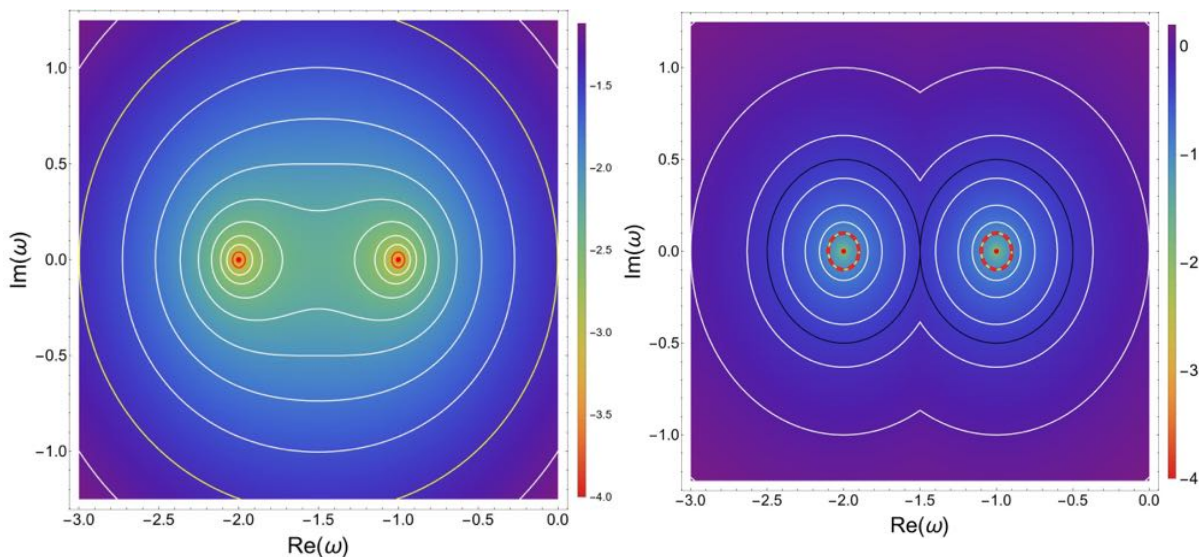
# Pseudospectra

Harmonic Oscillator: $\quad -\dfrac{d^2\phi}{dx^2} + c\,x^2\,\phi = \omega\phi \qquad G_E = \int dx\,\bar\phi\phi$

c=1
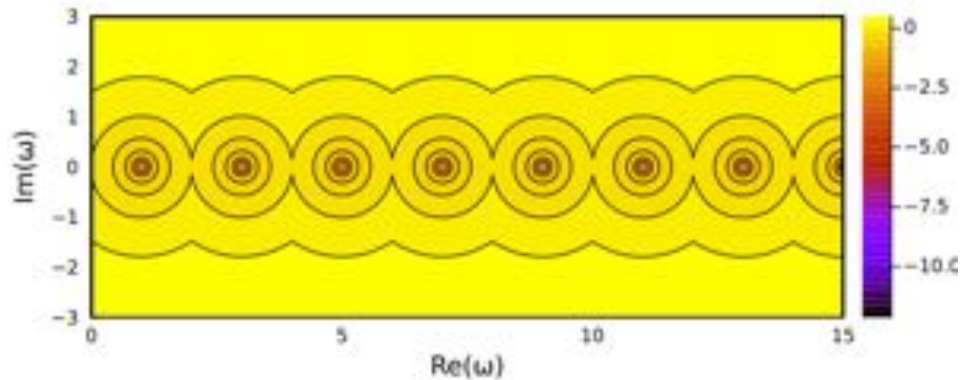
c=1+3i
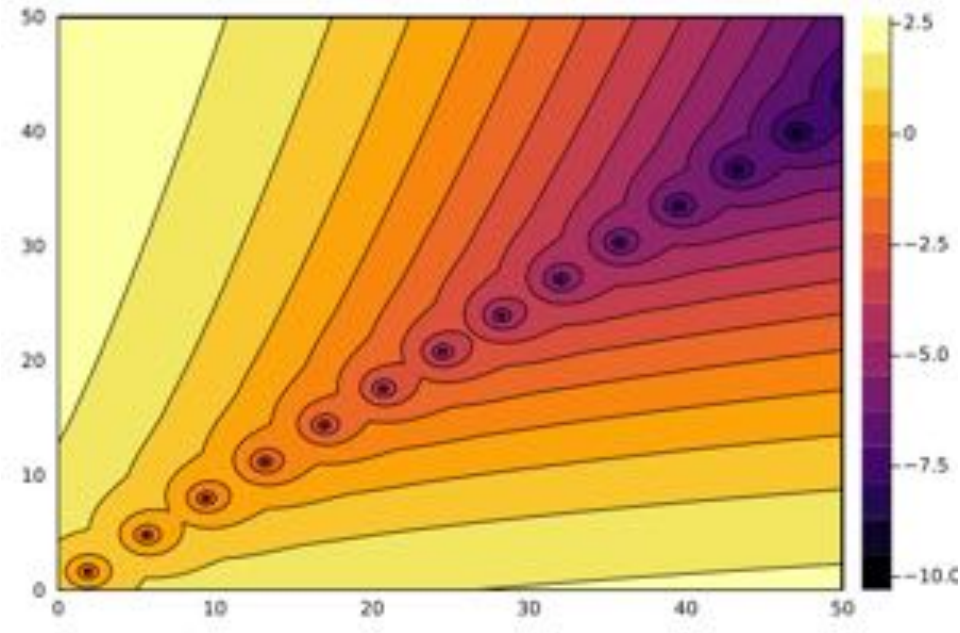
Pseudospectra of massless scalar in AdS$_5$:

"Selective" Pseudospectrum
Local random
Potential perturbations

$||V_{\mathrm{rand}}||$

- $10^{-1}$
- $10^{-3}$
- $10^{-5}$
- $10^{-7}$
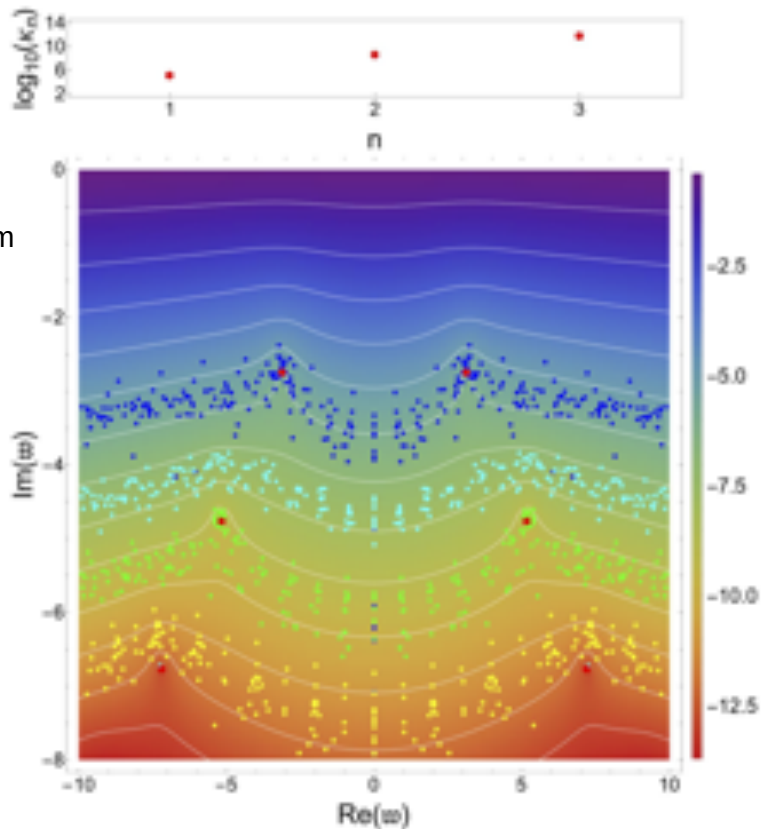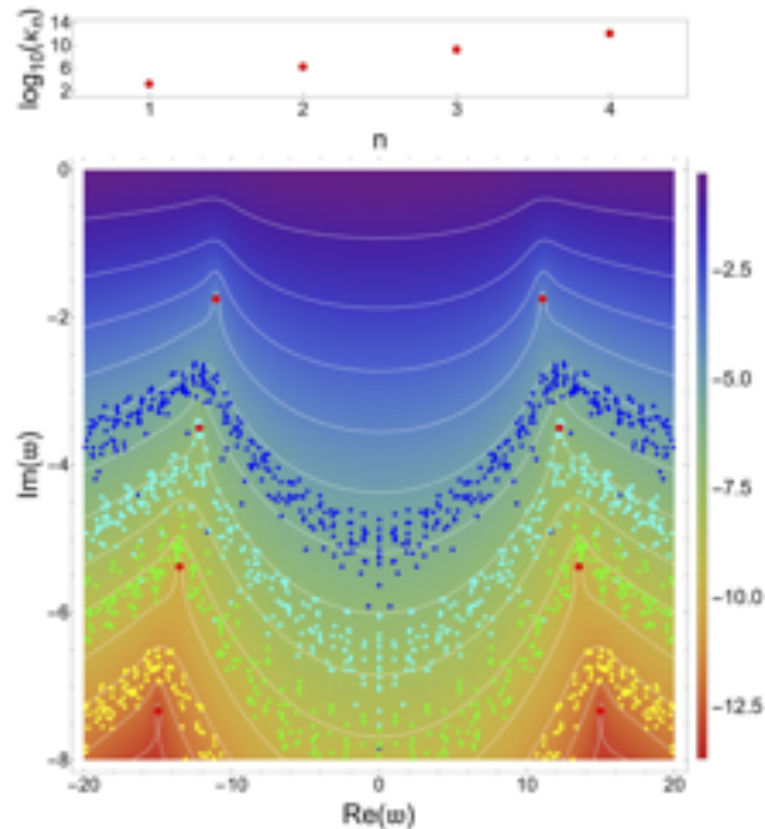


(a) $m^2 l^2 = 0$, $\mathfrak{q} = 0$.

(b) $m^2 l^2 = 0$, $\mathfrak{q} = 10$.

# QNM-Pseudospectra

Zoom into first QNM:

Circle of stability

$$|\lambda(\epsilon)_i - \lambda_i| \leq \epsilon \kappa_i$$

- $10^{-1}$
- $10^{-3}$

Restricted spectral instability

Restricted spectral stability
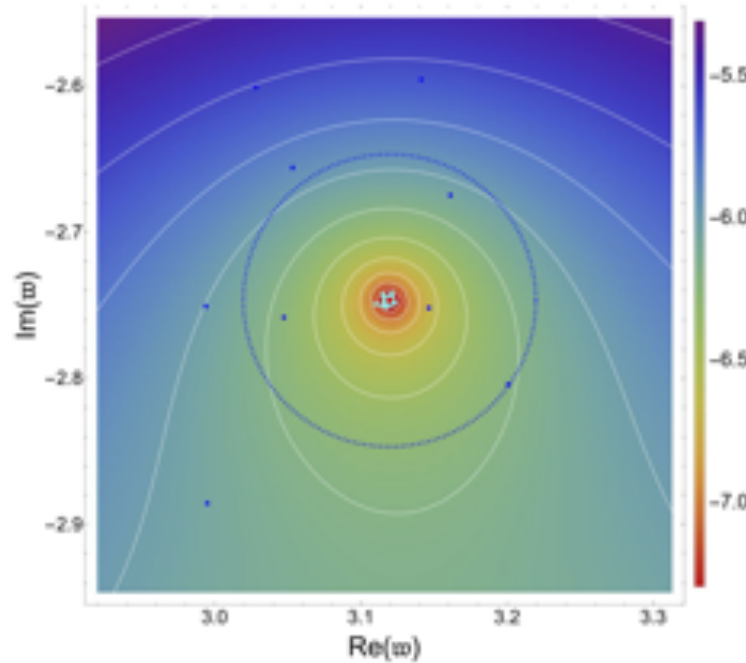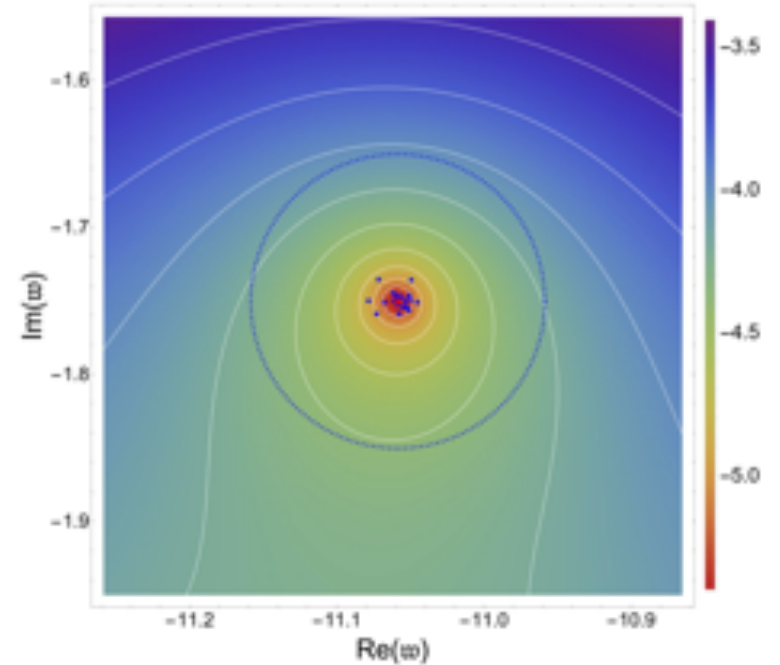
(a) $m^2 l^2 = 0$, $\mathfrak{q} = 0$.

(b) $m^2 l^2 = 0$, $\mathfrak{q} = 10$.

# QNM Pseudospectra

In asymptotically flat space:
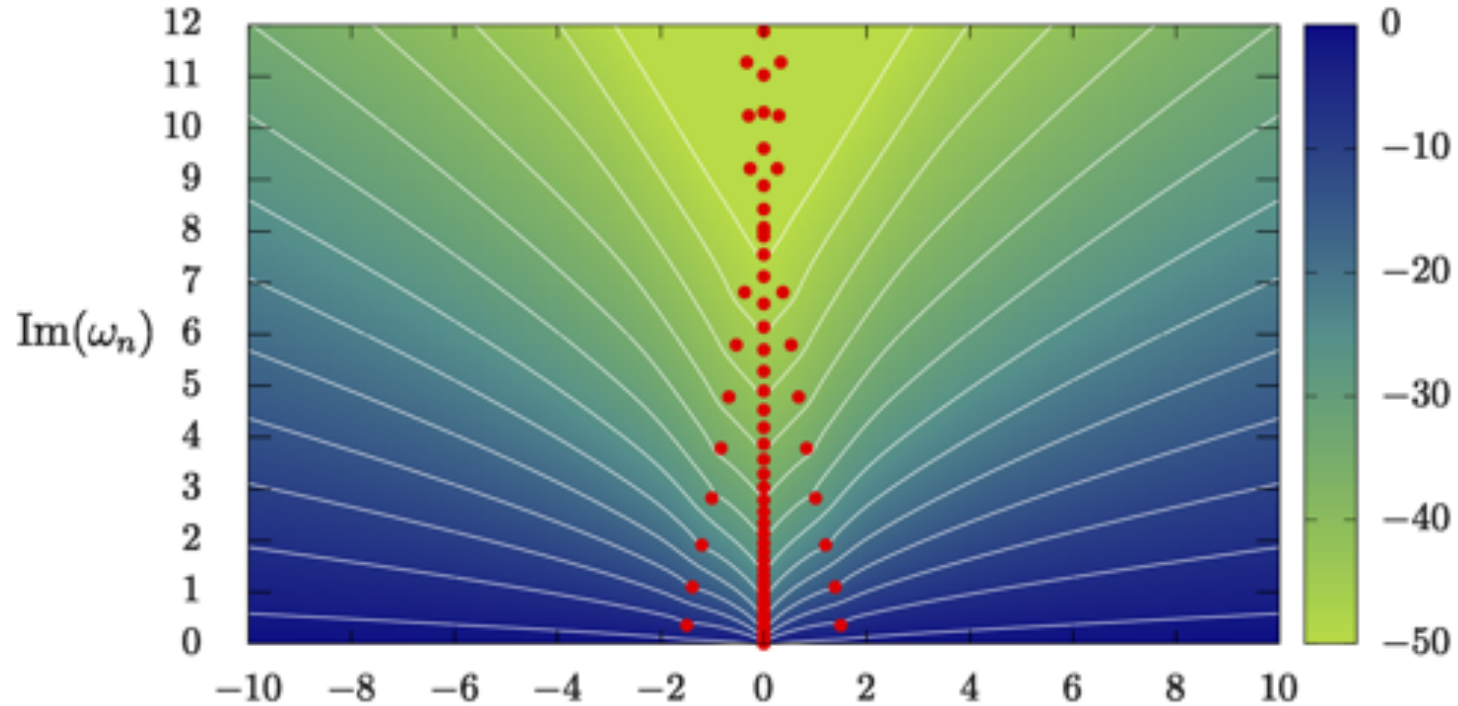


Review:

# QNM-Pseudospectra

Caveats:    Ingoing modes are integrable in the energy norm:

$$\phi_{\text{in}} \approx \rho^{i\omega} = \rho^{i\Omega}\rho^{\Gamma}$$

$$|\phi'_{\text{in}}|^2 \approx \rho^{2\Gamma-1} \qquad \Gamma = -\Im(\omega) > \frac{1}{2}$$

- Hilbert space of square integrable functions with energy norm
  In this sense the ingoing solutions belong to the spectrum
- Pseudospectrum in IEF $N \to$ infinity limit converges to this limit

- Math: Sobolev norm, Hilbert space $H^{(k)}$
  Physics: higher derivative theories!

- In regular coords resolvent seems to be divergent in general

- Finite N provides a natural cutoff $\to$ what is the optimal N?
  e.g stretched Horizon, Planck length away from horizon

- Universality? Holographic interpretation?
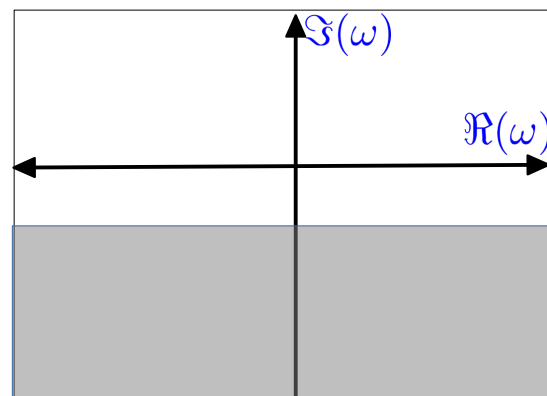
$$\|\Phi\|^2 = \int \sum_{m=0}^{k} |D^m \Phi|^2$$

[Warnick: CMP. 333 (2015) 2, 959-1035 •
e-Print: 1306.5760 [gr-qc]]

[Boyanov, Cardoso, Destounis,Jaramillo, Macedo]
e-Print: 2312.11998 [gr-qc]

# Pseudospectra of CMMs

$$G_R(\omega, \vec{k}) = K \frac{B(\omega, \vec{k})}{A(\omega, \vec{k})}$$

Look for poles with complex k and real ω

$$\Phi \propto e^{-i\omega t + ikx} e^{-\kappa|x|}$$

Absorbtion length: $\quad L = \dfrac{1}{\kappa}$

Norm = time averages energy density:

$$\bar{\varrho}_E[\phi] = \int \frac{d\rho}{(1-\rho)^3} \left\{ f \partial_\rho \phi \partial_\rho \bar{\phi} + z_h^2 \partial_3 \phi \partial_3 \bar{\phi} + z_h^2 (2-f) \omega^2 \phi \bar{\phi} + \frac{m^2 l^2}{(1-\rho)^2} \phi \bar{\phi} \right\}$$

In asymptotically flat space: Regge poles or Complex Angular Momentum

# Pseudospectra of CMMs
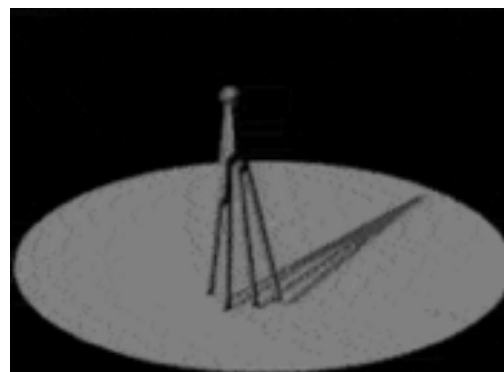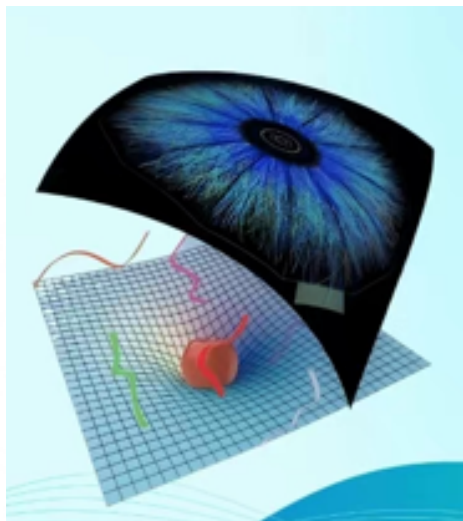
$\omega = 0$                    $\omega = 10\pi T$

# Summary

- **Quasinormal Modes are central to black hole physics**
- **Subject to spectral instability**
- **Choice of norm is important → Energy norm**
- **QNMs or continuous spectrum? → Norm dependence**
- **CMMs do not suffer from this problem, well defined**
- **Pseudospectra of CAMs for asymptotically flat black holes?**
- **Physical significance: Hammer vs. Antenna**

# QNMs vs CMMs



谢谢