A Holographic Study of Multi-Magnetic Field Magnetohydrodynamics

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Based on [Arxiv 2408.04791]

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Contents

From AdS/CFT to Hydrodynamics

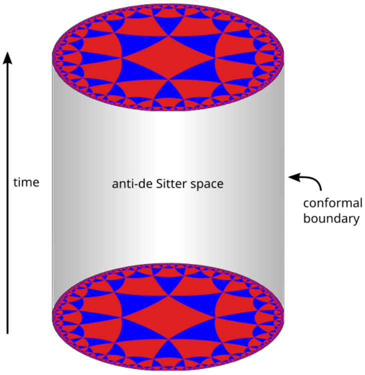

Figure 1: AdS/CFT[1]

- The generating Function: $Z_{gauge} = Z_{AdS}$
- Apply AdS/CFT to "real-world" such as the quark-gluon plasma (QGP), which can behave like a fluid.
- A famous result is the KSS (Kovtun, Son, Starinets) bound.[hep-th/0405231]

$$\frac{\eta}{s} > \frac{1}{4\pi}, \tag{1}$$

where η is the bulk viscosity, and s is the entropy density.

[1] https://en.wikipedia.org/wiki/AdS/CFT_correspondence

Motivations

- MHD (magnetohydrodynamics) is essential in diverse fields, including plasma physics [2], quantum chromodynamics (QCD) [1707.00795] and superfluidity [1808.01939].
- However, the inherent complexity of MHD poses significant challenges for traditional analytical methods and numerical simulations when calculating these transport coefficients [1505.02783].
- Gauge/gravity duality offers a novel perspective for calculating transport coefficients [hep-th/0104066].
- The KSS bound doesn't hold anymore with the presence of strong magnetic fields [1707.04182].
- MHD with one magnetic field has been studied before [1610.07392, 1707.04182], and we want to generalize these studies.

[2] P. M. Bellan, Fundamentals of Plasma Physics. Cambridge University Press, 2006.

Previous Works

For MHD with only one magnetic field:
[S. Grozdanov, D. M. Hofman, N. Iqbal [1610.07392]]

- The symmetry associated with the $U(1)$ current $j^{\nu}_{el} \sim \nabla_{\mu} F^{\mu\nu}$ is a gauge symmetry, which is redundant.
- The generalized global symmetries are associated with the charge current tensor $J^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.
- The conservation equations are $\nabla_{\mu} T^{\mu\nu} = H^{\nu}_{\rho\sigma} J^{\rho\sigma}$ and $\nabla_{\mu} J^{\mu\nu} = 0$.
- The structure of the first-order stress-energy tensor can be given by the positivity of the divergence of the entropy current $\nabla_{\mu} S^{\mu} > 0$.

[S. Grozdanov, N. Poovuttikul [1707.04182]]

- The KSS bound $\eta/s = 1/4\pi$ has been broken.
- The equation of motion can be solved by the Wronskian method.
- They also solved the resistivity using the Wronskian method, but we doubt the correctness because we cannot find a solution regular everywhere.

Contents

Notations & Symbols

Hydrodynamics in 4 dimension

- $u^\mu = \{1, 0, 0, 0\}$: velocity. $h^{(i)\mu} = \delta^{i\mu}$: The direction of the magnetic-field.
- γ the metric on the boundary.
- We use the english letters $i, j, k, a, b, \cdots$ as the spacial direction $\{x_1, x_2, x_3\}$.
- Sometimes we also use $\{1, 2, 3\}$ as a simplification of $\{x_1, x_2, x_3\}$.

Gravity in 5 dimension

- We use the greek letters $\mu, \nu, \rho, \sigma, \alpha, \beta, \cdots$ indicate the coordinates in the bulk $\{t, r, x_1, x_2, x_3\}$.

The Background

The action

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(R + 12 - \frac{1}{4}\sum_{i=1}^{3}(F^i_{\mu\nu})^2\right)$$
$$+\frac{1}{\kappa^2}\int_{\partial_\Sigma} d^4x\sqrt{-\gamma}\left(K - 3 - \frac{1}{4}R[\gamma] + \sum_{i=1}^{3}\frac{1}{8}F^{(i)}_{\mu\nu}F^{(i)\mu\nu}\ln(r_c\Lambda)\right). \tag{2}$$

where Λ is the UV cut-off, the boundary is at $r = r_c \to \infty$. The solution is given in [1109.0471] as

$$ds_5^2 = -f dt^2 + \frac{dr^2}{f} + r^2\left(dx_1^2 + dx_2^2 + dx_3^2\right), \quad f = r^2 - \frac{r_+^4}{r^2} + \frac{B^2}{2r^2}\ln\frac{r_+}{r}$$
$$F^1 = B\,dx_2\wedge dx_3, \quad F^2 = B\,dx_3\wedge dx_1, \quad F^3 = B\,dx_1\wedge dx_2, \tag{3}$$

- The solution is invariant under the scale transformation

$$r\to\lambda r, \quad t\to t/\lambda, \quad x_i\to x_i/\lambda, \quad B\to\lambda^2 B, \quad r_+\to\lambda r_+. \tag{4}$$

Holographic Configuration

The generating function

$$W(g_{\mu\nu}, b_{\mu\nu}) = \left\langle \exp\left[i \int d^4x \sqrt{-g^{(0)}} \left(\frac{1}{2} T^{\mu\nu} g^{(0)}_{\mu\nu} + J^{\mu\nu} b^{(0)}_{\mu\nu} \right) \right] \right\rangle. \tag{5}$$

Transform $r = \frac{1}{\sqrt{u}}$ such that $u = 0$ is the boundary, and expanding g and b as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + u\, g^{(1)}_{\mu\nu} + \cdots , \quad B_{\mu\nu} = b^{(0)}_{\mu\nu} + b^{(1)}_{\mu\nu} \ln(u\Lambda^{-2}). \tag{6}$$

- The metric $g_{\mu\nu}$ satisfy the Dirichlet boundary condition, which means $g_{\mu\nu}|_{u\to 0} = g^{(0)}_{\mu\nu}$.
- The gauge fields $B_{\mu\nu}$ satisfy the mixed boundary condition, which means there are dynamic Maxwell fields on the boundary.

The one point functions are

$$\langle T^{\mu\nu}\rangle = -\frac{2\,i}{\sqrt{-g^{(0)}}}\frac{\delta \ln W}{\delta g^{(0)}_{\mu\nu}}, \quad \langle J^{\mu\nu}\rangle = -\frac{i}{\sqrt{-g^{(0)}}}\frac{\delta \ln W}{\delta b^{(0)}_{\mu\nu}} \tag{7}$$

Then we have the Brown-York tensor

$$\begin{aligned}\langle T_{\mu\nu}\rangle =& -\lim_{u\to 0}\frac{1}{\kappa^2}\frac{1}{u}\Bigg(K_{\mu\nu} - K\gamma_{\mu\nu} + 3\gamma_{\mu\nu} - \frac{1}{2}G_{\mu\nu}[\gamma] \\ & - \sum_{i=1}^{3}\frac{1}{4}\left(\mathcal{H}^{(i)}_{\mu\lambda}\mathcal{H}^{(i)\,\lambda}_{\nu} - \frac{1}{4}\gamma_{\mu\nu}\mathcal{H}^{(i)}_{\alpha\beta}\mathcal{H}^{(i)\,\alpha\beta}\right)\ln(u_c\Lambda^{-2})\Bigg), \\ \langle J^{(i)}_{\mu\nu}\rangle =& -\frac{1}{4\kappa^2u^2}\mathcal{H}^{(i)}_{\mu\nu}. \end{aligned} \tag{8}$$

where $G_{\mu\nu}[\gamma]$, $\mathcal{H}_{\mu\nu} = n^\alpha H_{\alpha\mu\nu}$, and n^α is the unit norm outward the boundary.

On the boundary

At the zeroth order, the stress-energy tensor and the charge current are

$$
\begin{aligned}
T^{\mu\nu}_{(0)} &= (\varepsilon + p)\, u^{\mu} u^{\nu} + p\, g^{\mu\nu} - \sum_{i=1}^{3} \mu^{(i)} \rho^{(i)}\, h^{(i)\mu} h^{(i)\nu}, \\
J^{(i)\mu\nu}_{(0)} &= 2\rho^{(i)}\, u^{[\mu} h^{(i)\nu]}, \quad i \in \{1, 2, 3\}
\end{aligned}
\tag{9}
$$

where ε is the energy density, p is the pressure, μ is the chemical potential, ρ is the charge density.

The divergence of the entropy current is

$$
\nabla_{\mu} S^{\mu} = -\left[T^{\mu\nu}_{(1)} \nabla_{\mu} \left(\frac{u_{\nu}}{T} \right) + \sum_{i=1}^{3} J^{(i)\mu\nu}_{(1)} \left(\nabla_{\mu} \left(\frac{h^{(i)}_{\nu} \mu}{T} \right) + \frac{u_{\sigma} H^{(i)\sigma}_{\ \ \mu\nu}}{T} \right) \right] \tag{10}
$$

where T is the temperature.

$$\nabla_\mu S^\mu = -\left[T^{\mu\nu}_{(1)}\nabla_\mu\left(\frac{u_\nu}{T}\right) + \sum_{i=1}^{3} J^{(i)\mu\nu}_{(1)}\left(\nabla_\mu\left(\frac{h^{(i)}_\nu \mu}{T}\right) + \frac{u_\sigma H^{(i)\sigma}{}_{\mu\nu}}{T}\right)\right] \tag{11}$$

- $\nabla_\mu S^\mu > 0$ gives the form of a positive-definite quadratic, and then 1st-order transport coefficients can be given.

So the stress-energy tensor and the charge current are

$$T^{\mu\nu}_{(1)} = -\sum_{a,b=1}^{3} \zeta_{ab} P^{\mu\nu}_a P^{\alpha\beta}_b \nabla_\alpha u_\beta - 2\sum_{\substack{a,b=1 \\ a\neq b}}^{3} \eta_{ab} P^{\mu\alpha}_a P^{\nu\beta}_b \nabla_{(\alpha} u_{\beta)}, \tag{12}$$

$$\tilde{J}_{ab(1)} = -r_{(ab)} T\tilde{S}^{(ab)} - r_{[ab]} T\tilde{S}^{[ab]}. \tag{13}$$

$$S^a{}_{bc} = h^\mu_b h^\nu_c \left(\nabla_{[\mu}\left(\frac{h^a_{\nu]}\mu}{T}\right) + \frac{u_\sigma H^{a\sigma}{}_{\mu\nu}}{T}\right), \tag{14}$$

where $\tilde{J}_{ab(1)} = \frac{1}{2}\epsilon_{acd} J^{cd}_{b(1)}$, $\tilde{S}^{ab} = \frac{1}{2}\epsilon^{acd} S^b{}_{cd}$.

0th-order Transport Coefficients

0th-order transport coefficients can be read from the stress-energy tensor as

$$\begin{aligned} T_{tt} &= \epsilon = \frac{3}{2\kappa^2}\left(\frac{1}{u_+^2} + \frac{1}{4}B^2\ln(u_+\Lambda^{-2})\right), \\ T_{aa} &= p - \mu^a\rho^a = \frac{1}{2\kappa^2}\left(\frac{1}{u_+^2} - \frac{B^2}{2} + \frac{1}{4}B^2\ln(u_+\Lambda^{-2})\right) \\ J^a_{ta} &= -\rho^a = \frac{1}{4\kappa^2}B. \end{aligned} \tag{15}$$

- $T^{\mu\nu}$ have a non-vanishing trace, which indicate the fluids are non-conformal.
- They satisfy the thermodynamic equation

$$\varepsilon = Ts - p + \sum_{i=1}^{3}\mu_e^i\rho_e^i + \sum_{i=1}^{3}\mu_m^i\rho_m^i. \tag{16}$$

1st-order Fluids

To solve transport coefficients at the 1st order, we add perturbations to the fluids like

$$g_{\mu\mu} = g^{(0)}_{\mu\nu} + \delta h_{\mu\nu}(t) = \eta_{\mu\nu} + \int d\omega\, e^{-i\omega t} \delta h_{\mu\nu}(\omega). \tag{17}$$

Plug them into the stress-energy tensor and the charge current, we have

$$\begin{aligned} T_{\mu\nu} = & (\epsilon + p)\delta_{\mu t}\delta_{\nu t} + p g^{(0)}_{\mu\nu} - \sum_{a=1}^{3} \mu^a \rho^a \delta_{\mu a}\delta_{\nu a} \\ & + \int d\omega e^{-i\omega t} \Bigg[(\epsilon + p)\left(-\delta_{\mu t}\delta h_{\nu t} - \delta_{\nu t}\delta h_{\mu t}\right) + p\ \delta h_{\mu\nu} - \sum_{a=1}^{3} \mu^a \rho^a \left(\delta_{\mu a}\delta h_{\nu a} + \delta_{\nu a}\delta h_{\mu a}\right) \\ & + i\omega \frac{1}{2} \sum_{a,b}^{3} \zeta_{ab}\delta_{\mu a}\delta_{\nu a}\delta h_{bb} + i\omega \sum_{a \neq b}^{3} \eta_{ab}\delta_{\mu a}\delta_{\nu b}\delta h_{ab} \Bigg]. \end{aligned} \tag{18}$$

$$\begin{aligned} J^a_{\mu\nu} = & -\rho^a(\delta_{\mu t}\delta_{\nu a} - \delta_{\nu t}\delta_{\mu a}) \\ & + \int d\omega e^{-i\omega t} \left[\rho^a \left(\delta_{\nu a}\delta h_{\mu t} - \delta_{\mu a}\delta h_{\nu t} - \delta_{\mu t}\delta h_{\nu a} + \delta_{\nu t}\delta h_{\mu a}\right)\right. \\ & + \frac{1}{4} i\omega \epsilon_{\alpha\mu\nu} \left(\left(r_{(\alpha a)} + r_{[\alpha a]}\right) \epsilon^{\alpha mn} \delta B^a_{mn} + \left(r_{(\alpha a)} - r_{[\alpha a]}\right) \epsilon^{amn} \delta B^\alpha_{mn} \right). \end{aligned} \tag{19}$$

The Kubo Formula

Notice that 1st-order transport coefficients appear only in the imaginary part, we have Kubo formula like

$$\eta_{ab}=\lim_{\omega\to 0}\frac{\mathrm{Im}G^{TT,ab}_{ab}(\omega)}{-\omega},\zeta_{aa}=\lim_{\omega\to 0}\frac{\mathrm{Im}G^{TT,aa}_{aa}(\omega)}{-\omega},\quad \zeta_{ab}=\lim_{\omega\to 0}\frac{\mathrm{Im}G^{TT,bb}_{aa}(\omega)}{-\omega},$$
$$r_{(ab)}=\lim_{\omega\to 0}\frac{\mathrm{Im}G^{\tilde{J}\tilde{J},(ab)}_{ab}(\omega)}{-\omega},r_{[ab]}=\lim_{\omega\to 0}\frac{\mathrm{Im}G^{\tilde{J}\tilde{J},[ab]}_{ab}(\omega)}{-\omega}, \tag{20}$$

where the retarded Green's function are like

$$G^{TT,\lambda\sigma}_{\mu\nu}=-2\frac{\delta T_{\mu\nu}}{\delta h_{\lambda\sigma}},\quad G^{\tilde{J}\tilde{J},(ab)}_{ab}=-\frac{\delta\tilde{J}_{ab(1)}}{\delta\tilde{b}_{(ab)}},\quad G^{\tilde{J}\tilde{J},[ab]}_{ab}=\frac{\delta\tilde{J}_{ab(1)}}{\delta\tilde{b}_{[ab]}}. \tag{21}$$

1st-order Gravity

The perturbations are added to the bulk

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \delta g_{\mu\nu}(t), \quad B_{\mu\nu} \longrightarrow B_{\mu\nu} + \delta B_{\mu\nu}(t), \tag{22}$$

The Brown-York tensor

$$\begin{aligned} &\delta T_{tt} = \lim_{u\to 0}\frac{1}{\kappa^2}\frac{1}{u}(\delta {g^{1}}'_{1} + \delta {g^{2}}'_{2} + \delta {g^{3}}'_{3}), \quad \delta T_{aa} = \lim_{u\to 0} -\frac{1}{\kappa^2}\frac{1}{u}(\delta {g^{b}}'_{b} + \delta {g^{c}}'_{c} + \delta {g^{t}}'_{t}) \\ &\delta T_{12} = \lim_{u\to 0}\frac{1}{\kappa^2}\frac{1}{u}\delta {g^{1}}'_{2}, \quad \delta T_{13} = \lim_{u\to 0}\frac{1}{\kappa^2}\frac{1}{u}\delta {g^{1}}'_{3}, \quad \delta T_{23} = \lim_{u\to 0}\frac{1}{\kappa^2}\frac{1}{u}\delta {g^{2}}'_{3}, \end{aligned} \tag{23}$$

$$\delta J_{(a)12} = \lim_{u\to 0}\frac{1}{2\kappa^2}u\delta {B^{a}_{12}}', \quad \delta J_{(a)13} = \lim_{u\to 0}\frac{1}{2\kappa^2}u\delta {B^{a}_{13}}', \quad \delta J_{(a)23} = \lim_{u\to 0}\frac{1}{2\kappa^2}u\delta {B^{a}_{23}}'. \tag{24}$$

Shear Viscosity η

The related equation of motion is

$$\delta {g^1}_2'' + \left(-\frac{1}{u} + \frac{F'}{F}\right) \delta {g^1}_2' + \frac{\omega^2 - B^2 u F}{4uF^2} \delta {g^1}_2 = 0, \tag{25}$$

which can be solved by the Wronskian method [1707.04182].

- The ansatz is $\delta {g^1}_2(\omega, u) = \mathbf{h}^{(-)}(u) + \alpha(\omega)\mathbf{h}^{(+)}(u) + \mathcal{O}(\omega^2)$

$\mathbf{h}$ satisfy

$$\mathbf{h}^{(+)}(u) = \mathbf{h}^{(-)}(u) \int du' \frac{W(u')}{(\mathbf{h}^{(-)}(u'))^2}, \quad W(u) = \exp\left[-\int du' \left(-\frac{1}{u'} + \frac{F'(u')}{F(u')}\right)\right] = \frac{u}{F(u)}, \tag{26}$$

where $W(u)$ is the Wronskian.

- The in-falling boundary condition near the horizon read

$$\delta {g^1}_2(u) \sim (u_+ - u)^{-\frac{i\omega}{4\pi T}} \tilde{g}(u) \sim \mathbf{h}^{(-)}(u_+) \left(1 - \frac{i\omega}{4\pi T} \ln(u_+ - u)\right) \tag{27}$$

Finally, the result is

$$\eta_{ab} = \frac{1}{\kappa^2} \frac{1}{2u_+^{3/2}} \left[\frac{\mathbf{h}^{(-)}(u_+)}{\mathbf{h}^{(-)}(0)} \right]^2 , \tag{28}$$

where $\left[\frac{\mathbf{h}^{(-)}(u_+)}{\mathbf{h}^{(-)}(0)} \right]^2$ can be solved numerically.

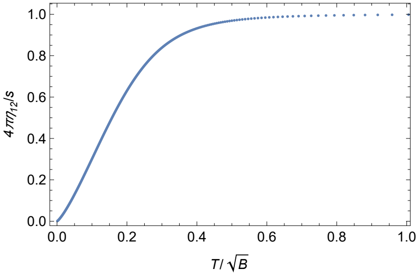

Figure 2: The profile of the scaling invariant viscosity to entropy density ratio $4\pi\eta_{[12]}/s$.

- The KSS bound $\eta/s > 1/4\pi$ has been broken.

Bulk Viscosity ζ

The equations are

$$Z'' + \left(\frac{F'}{F} - \frac{1}{u}\right) Z' + \frac{\omega^2 - B^2 uF}{4uF^2} Z = 0, \quad Y' - \left(\frac{F'}{2F}\right) Y = 0, \tag{29}$$

where

$$Z = \delta g^1{}_1 - \delta g^2{}_2 \text{ or } \delta g^2{}_2 - \delta g^3{}_3, \quad Y = \delta g^1{}_1 + \delta g^2_2 + \delta g^3{}_3. \tag{30}$$

The solutions are

$$\zeta_{aa} = \frac{4}{3}\eta, \quad \zeta_{ab} = -\frac{2}{3}\eta. \tag{31}$$

- The trace of the stress-energy tensor satisfy $TrT_{\mu\nu} \sim \sum_{a,b}^{3} \zeta_{a,b} = 0$, suggests it is conformal ar the 1st order.

Resistivity—Symmrtrical Part $r_{(ab)}$

The equation

$$\delta\tilde{B}''_{(ab)} + \left(\frac{1}{u} + \frac{F'}{F}\right)\delta\tilde{B}'_{(ab)} + \frac{\omega^2}{4uF^2}\delta\tilde{B}_{(ab)} = 0, \quad a, b \in \{1, 2, 3\} \tag{32}$$

The result

$$r_{(ab)} = \frac{1}{4\kappa^2} u_+^{1/2}. \tag{33}$$

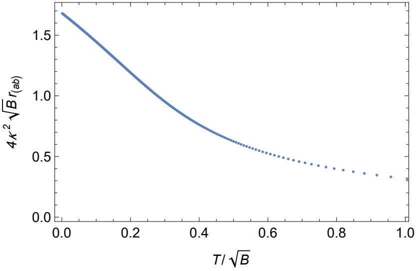

Figure 3: The profile of the scaling invariant resistivity $4\kappa^2\sqrt{B}r_{(ab)}$.

Resistivity—Anti-symmrtrical Part $r_{[ab]}$

The equation

$$\delta\tilde{B}''_{[ab]} + \left(\frac{1}{u} + \frac{F'}{F}\right)\delta\tilde{B}'_{[ab]} + \frac{\omega^2 - 2B^2uF}{4uF^2}\delta\tilde{B}_{[ab]} = 0. \quad a, b \in \{1, 2, 3\} \tag{34}$$

The equation is significantly different when $B = 0$ or not, so we discuss them separately.

- When $B = 0$

The equation is the same as the sase in $r_{(ab)}$, and the result is the same

$$r_{[ab]} = r_{(ab)}. \tag{35}$$

- When $B \neq 0$

The solutions of

$$\boldsymbol{b}_{[ab]}^{(\pm)''}(z) + (\frac{1}{z} + \frac{h'(z)}{h(z)})\boldsymbol{b}_{[ab]}^{(\pm)'}(z) - \frac{u_B^2}{2h(z)}\boldsymbol{b}_{[ab]}^{(\pm)}(z) = 0\ , \quad z = u/u_+, \quad u_B = u_+ b \tag{36}$$

are

$$\boldsymbol{b}_{[ab]}^{(-)}(z) = c(u_B^2 - 8 + 2u_B^2 \log z), \quad \boldsymbol{b}_{[ab]}^{(+)}(z) = \boldsymbol{b}_{[ab]}^{(-)}(z)\Big(\int_0^z \frac{dy}{yh(y)[\boldsymbol{b}_{[ab]}^{(-)}(y)]^2} + \gamma\Big)\ . \tag{37}$$

- Notice that we found the analytical solution of the equation, but the authors of [1707.04182] didn't, this may be due to the non-obvious of the log divergence in the numerical methods.

The result is

$$r_{[ab]} = \frac{c^2\gamma\sqrt{u_+}u_B^2\left(u_B^2-8\right)^2}{2\kappa^2\left(u_B^2(4\log\Lambda+1)-8\right)}\ , \qquad a,b=1,2,3\ . \tag{38}$$

where

$$\gamma = -\int_0^1 dz\Big(\frac{1}{zh(z)[\boldsymbol{b}_{[ab]}^{(-)}(z)]^2} - \frac{4}{c^2(u_B^2-8)^3(z-1)}\Big) \tag{39}$$

is a constant.

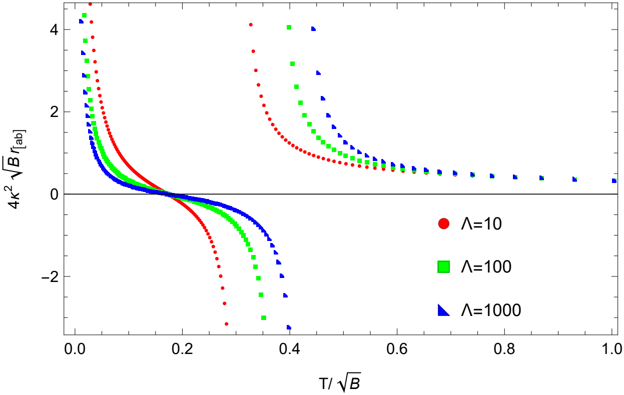

Figure 4: The profile of the scaling invariant resistivity $4\kappa^2\sqrt{B}r_{[ab]}$ as a function of the scaling invariant temperature $T_B = T/\sqrt{B}$.

Discussions and Outlook

Discussions

- The 1st-order transport coefficients become the form of a matrix.
- The resistivity can be separated into two parts, the symmetrical part and the anti-symmetrical part.
- The anti-symmetrical part of the resistivity cannot be solved by the same method as the others because there is no solution that is regular everywhere.
- For the equation of the resistivity, we found an analytical solution that suggests a log divergence near the boundary.
- At the 0th-order, the fluids is non-conformal. But if we consider only the 1st-order fluids, it becomes conformal, and we have non-zero ζ.

Outlook

- General dimensions
- Diffusion
- Other methods, such as fluids/gravity correspondence and EFT
- $\cdots$

There are so many things yet to be discovered.