Uncovering the thermodynamic origin of counterflow and coflow instabilities in miscible binary superfluids

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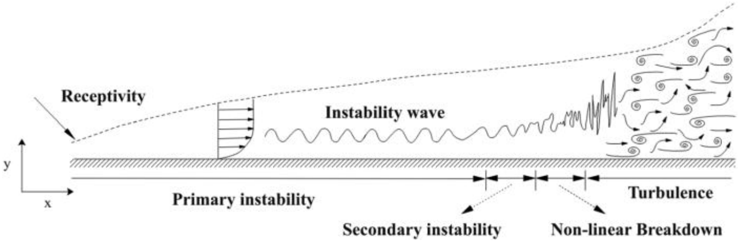
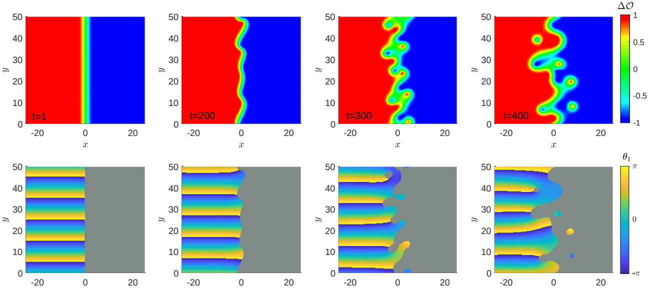
In both classical fluid systems and quantum superfluids with nonzero velocity, instabilities appear very generally. Such instabilities typically develop into turbulence in nonlinear stage. Understanding the origin and the final fate of these instabilities is crucial to understanding turbulence.
Binary superfluids are the most suitable platform to study these instabilities since there is no friction between the two components and one can study equilibrium states with nonzero relative velocity. Such an instability in miscible binary superfluids is often called counterflow instability.



Kumar, C. and A. Prakash (2021). Physics of Fluids 33(2): 024107.

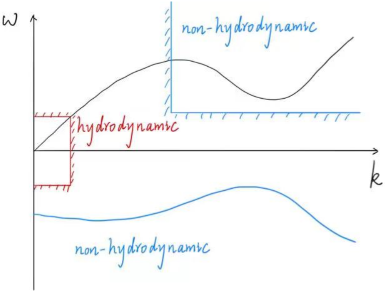
The counterflow instability of weakly interacting Bose-Einstein condensates (BECs) has been studied mostly relying on Gross-Pitaevskii equation (GPE), which is only valid in weak coupling limit, zero temperature, and does not include dissipation.

In order to go beyond the regime of validity of GPE, we employ a holographic binary superfluid model, which is intrinsically in the strong coupling limit, and naturally incorporates finite temperature and dissipation. Previously, we have studied interface instability of holographic immiscible binary superfluids. In this work we study instabilities in holographic miscible binary superfluids.



On the other hand, hydrodynamics gives an effective description of any interacting systems which is valid at suffciently long times and suffciently large distances. That is at the limit $k \to 0$ and $\omega \to 0$.

Recently, it is argued that the dynamical instability under linear perturbations of interacting systems in the hydrodynamic regime generally follows from local thermodynamic instability together with positivity of entropy production. In this work, we extend this result to binary superfluids, and verify it by utilizing both GPE and holographic model.



Hydrodynamics of homogeneous binary superfluids

In general, there can be two conserved U(1) charges in binary superfluids. Then the conservation equations are

$$\partial_t \epsilon + \partial_i j^i_\epsilon = 0, \quad \partial_t g^i + \partial_i \tau^{ji} = 0, \quad \partial_t n_I + \partial_i j^i_I = 0, \quad (I = 1, 2), \tag{1}$$

where ϵ, g^i and n_I are energy, momentum and charge densities, j_ϵ and j_I are energy and charge currents, and τ is the spatial stress tensor. From gauge invariance, we also have the Josephson relations

$$\partial_t \varphi_I + \mathbf{v}^{\mathbf{n}} \cdot \boldsymbol{\partial} \varphi_I + \mu_I = 0, \tag{2}$$

where μ_I is chemical potential. In this work we work in grand canonical ensemble, where chemical potential is fixed.

Then the first law of thermodynamics is

$$\begin{aligned} d\epsilon &= Tds + \mu_I dn_I + \mathbf{v^n} d\mathbf{g} + \mathbf{h_I} d\mathbf{v^s_I} + N d(\mathbf{v^s_1} \cdot \mathbf{v^s_2}) \,, \\ &= Tds + \mu_I dn_I + \mathbf{v^n} d\mathbf{g} + \bar{\mathbf{h}}_\mathbf{I} d\mathbf{v^s_I} , \end{aligned} \tag{3}$$

where s is the entropy density, $\mathbf{v^n}$ is the normal fluid velocity, $\mathbf{v^s_I}$ is superfluid velocity and $\mathbf{h_I} = n^s_I(\mathbf{v^s_I} - \mathbf{v^n})$ with n^s_I being the superfluid charge density. The cross term in the first line is the energy change due to relative motion between the two components of superfluids, and $\bar{\mathbf{h}}_\mathbf{I} = \mathbf{h_I} + N\sigma^{IJ}\mathbf{v^s_J}$ with $\sigma^{IJ} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then from positivity of entropy production

$$T\partial_t s + T\partial_i (s v^{ni} + \tilde{j}^i_s / T) \equiv \Delta \geq 0, \tag{4}$$

we can find the constitutive relations for the currents in equilibrium state compatible with parity and time-reversal invariance.

One can linearize equations of motion around equilibrium state by perturbing the thermodynamic quantities and sources

$$\begin{aligned} O_A =& O_{A0} + e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\delta O_A\,, \\ s_A =& s_{A0} + e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\delta s_A\,. \end{aligned} \tag{5}$$

They are related by static susceptibility matrix

$$\chi_{AB} = \delta O_A/\delta s_B = \delta^2 F/\delta s_A \delta s_B, \tag{6}$$

where $F = -T\ln Z$ is the thermal free energy. The linearized equations of motion in Fourier space would be

$$(-i\omega + ikv^n)\delta O_A + M_{AB}(k)\delta s_B = (-i\tilde{\omega} + M(k)\cdot\chi^{-1})\delta O = 0. \tag{7}$$

Then the spectrum of modes will be given by solving

$$\det(-i\tilde{\omega} + M \cdot \chi^{-1}) = 0. \tag{8}$$

The onset of a dynamical instability occurs when the imaginary part of $\tilde{\omega}$ becomes positive. This can happen when $\det(M \cdot \chi^{-1}) = \det(M)/\det(\chi)$ vanishes and changes sign. It can be shown that $\det(M) > 0$. Therefore an instability can only occur when $\det(\chi)$ diverges and changes sign. Such instability from divergence of susceptibility matrix is obviously thermodynamic.

For simplicity, we set $\mathbf{v^n} = 0$ from now on, which is the case for holographic superfluids in the probe limit, and also for the zero temperature superfluids modeled by GPE. We also keep the temperature fixed since we want to focus on the effects from superfluid velocities. In our holographic model, there is only one U(1) conserved charge n for the binary superfluid since there is only one U(1) gauge field in the bulk.[1] Then the thermodynamic quantities and sources we need to consider are just $\{n_I = (n, n),\ \mathbf{v^s_I}\}$ and $\{\mu_I = (\mu, \mu),\ \bar{\mathbf{h}}_\mathbf{I}\}$. In this case, the equations of motion are

$$\partial_t n + \partial_i (j^i + \tilde{j}^i) = 0, \quad \partial_t v_I^{si} + \partial_i (\mu_I + \tilde{\mu_I}) = 0. \tag{9}$$

[1] This is equivalent to setting the imbalance charge density $n_- = n_1 - n_2$ identically to zero. This can straightforwardly be relaxed if needed.

By solving

$$(-i\tilde{\omega}+M(k)\cdot\tilde{\chi}^{-1})\delta O=0, \tag{10}$$

we can find two sound modes and one diffusive mode

$$\omega_{\pm}=v_{\pm}k+i\Gamma_{\pm}k^2+\mathcal{O}(k^3),\quad \omega_0=i\Gamma_0k^2+\mathcal{O}(k^3), \tag{11}$$

with

$$\begin{aligned} v_{\pm}=&\frac{-\chi_{n+}\pm\sqrt{\chi_{n+}^2+\chi_{++}\chi_{nn}}}{\chi_{nn}},\\ \Gamma_0=&-\frac{(\chi_{++}\chi_{--}-\chi_{+-}^2)(\delta_2(v_-^s)^2+\delta_1)}{4\chi_{++}}. \end{aligned} \tag{12}$$

The criterion for onset of instability is given by

$$\det(\tilde{\chi})=-\chi_{nn}/(\chi_{++}\chi_{--}-\chi_{+-}^2) \tag{13}$$

diverges, which is

$$\chi_{++}\chi_{--}-\chi_{+-}^2=0. \tag{14}$$

For the special cases $v_+^s=0$ and $v_-^s=0$, we have $\chi_{+-}=0$, since $\chi_{+-}=\delta^2F((v_+^s)^2,(v_-^s)^2)/\delta v_+^s\delta v_-^s\sim v_+^sv_-^s$. Therefore in these cases, the criterion for onset of instability is given by $\chi_{++}=0$ or $\chi_{--}=0$.

Counterflow instability from GPE

The coupled GPEs are given as (capital Latin indices are not summed here)

$$i\partial_t \Psi_I = (-\frac{1}{2m_I}\nabla^2 - \mu_I + g_I|\Psi_I|^2 + g_{IJ}|\Psi_J|^2)\Psi_I, \quad (I, J = 1, 2, \quad I \neq J)\,, \tag{15}$$

which describes weakly interacting binary superfluids without dissipation and normal fluid component. The condensate wave functions are described by $\Psi_J(x,y,t) = \sqrt{n_J^s(x,y,t)}e^{i\theta_J}(\mathbf{r},t)$ where n_J^s and θ_J are the particle density and phase of the J-th component of the binary superfluids. The latter is nothing but the Goldstone field φ_J for the J-th superfluid component.

It was shown that the countersuperflow becomes unstable and quantized vortices are nucleated as the relative velocity exceeds a critical value, leading to isotropic quantum turbulence consisting of two superflows. For simplicity, let's consider the case where $m_1 = m_2 = m$, $g_1 = g_2 = g$ and $\mu_1 = \mu_2 = \mu$. We find the criterion for the onset of instability is given by

$$\chi_{++}|_{v_+^s=0, v_-^s=v_c} = \frac{2\mu - mv_c^2}{g + g_{12}} - \frac{2mv_c^2}{g - g_{12}} = 0\,, \tag{16}$$

which gives

$$v_c = \sqrt{(g - g_{12})n_+^s/2m}\,. \tag{17}$$

This is exactly the critical velocity derived from Bogoliubov-de Gennes analysis, with $n_+^s = 2n$.

Holographic miscible binary superfluids

Matter Lagrangian of holographic miscible binary superfluids is

$$\begin{aligned}\mathcal{L} =& -(\mathcal{D}_\mu\Psi_1)^*\mathcal{D}^\mu\Psi_1 - m_1^2|\Psi_1|^2 - (\mathcal{D}_\mu\Psi_2)^*\mathcal{D}^\mu\Psi_2 \\ &- m_2^2|\Psi_2|^2 - \frac{\nu}{2}|\Psi_1|^2|\Psi_2|^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}.\end{aligned} \tag{18}$$

In terms of the background, we utilize a planar AdS black hole described in Eddington-Finkelstein coordinates:

$$ds^2 = \frac{L^2}{z^2}(-(1-(z/z_h)^3)dt^2 - 2dtdz + dx^2 + dy^2). \tag{19}$$

For simplicity, we set $L = z_h = 1$, $m_1^2 = m_2^2 = -2$, $e_1 = e_2 = 1$, and adopt the radial gauge $A_z = 0$. The asymptotic expansions for A_μ and Ψ_i near the AdS boundary are:

$$\begin{aligned}A_\mu =& a_\mu + b_\mu z + \mathcal{O}(z^2), \\ \Psi_I =& (\Psi_I)_0 z + (\Psi_I)_1 z^2 + \mathcal{O}(z^3).\end{aligned} \tag{20}$$

Stationary configuration for miscible binary superfluids is homogeneous in spatial directions. Assuming the superfluid velocity is along y direction, we choose the following ansatz

$$\Psi_I = z\phi_I(z)e^{i\Theta_I(z,y)}, \quad A_t = A_t(z)\,, \quad V_{Iy}(z) \equiv \partial_y\Theta_I - A_y\,. \tag{21}$$

According to the holographic dictionary, we have $V_I|_{z=0} = v_I^s$.
Given the stationary solutions, we can calculate the free energy density and susceptibilities. The free energy is simply identified with on-shell bulk action with Euclidean signature times temperature. In the probe limit with sources of condensates set to 0, we can neglect the gravity part of the action, the Euclidean action is finite and no counter term is needed.

$$f = -\int dz\sqrt{-g}\mathcal{L}_{\text{on-shell}}. \tag{22}$$

To get $\chi_{AB} = \delta^2 f/\delta s_A \delta s_B$, we need to calculate the free energy density for different s_A and s_B (in this case, it's $f(\mu, v_+^s, v_-^s)$) and then extract the value of χ_{AB} numerically.

We can also study dynamical counterflow and coflow instability in miscible holographic binary superfluids using linear response theory. We turn on small perturbations on the stationary background

$$\Phi_i = (\Phi_{i0} + \delta\Phi_i)e^{i(v_i)_y y}, \quad A_t = A_{t0} + \delta A_t, \quad A_y = A_{y0} + \delta A_y \,, \tag{23}$$

and linearize the EoMs. We can express the bulk perturbation fields as

$$\begin{aligned} \delta\Phi_i &= u_i(z)e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}, \quad \delta\Phi_i^* = v_i(z)e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}, \\ \delta A_t &= a_t(z)e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}, \quad \delta A_y = a_y(z)e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}. \end{aligned} \tag{24}$$

For simplicity, in this work we only consider $\boldsymbol{k}\cdot\boldsymbol{x} = ky$. This results in a generalized eigenvalue problem

$$M_k u_k = i\omega_k B u_k, \quad u_k = \{u_1, v_1, u_2, v_2, a_t, a_y\}_k^{\mathrm{T}}. \tag{25}$$

Since $\delta\Phi \sim e^{-i\omega t}$, the stationary configuration is dynamical unstable whenever $\mathrm{Im}\omega_k > 0$.

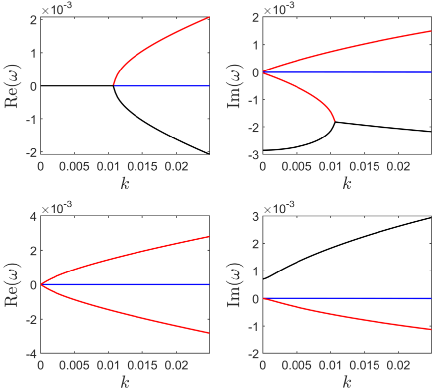


Figure 1: Spectrum of QNMs at low momentum for counterflow case. **Top**: Spectrum of QNMs for superfluid velocity $v_{y1} = 0.283$. **Bottom**: Spectrum of QNMs for superfluid velocity $v_{y2} = 0.314$. Red lines are sound modes, blue line is diffusive mode and black line is gapped mode. Relevant parameters are $T/T_c = 0.677$, $\nu = -0.2$. Note $v_{c1} < v_{y1} < v_{c2}$ and $v_{y2} > v_{c2}$. The unstable mode for v_{y1} is sound mode, while for v_{y2}, the unstable mode is the gapped mode. For reference, the two critical velocities at $T/T_c = 0.677$ and $\nu = -0.2$ are numerically found to be $v_{c1} = 0.251$, $v_{c2} = 0.308$.

The first critical velocity corresponds exactly to $\chi_{++} = 0$. Note that $\chi_{n+} \sim v_+^s$. Therefore $\chi_{n+} = 0$ for counterflow case. Then from

$$v_\pm = \frac{-\chi_{n+} \pm \sqrt{\chi_{n+}^2 + \chi_{++}\chi_{nn}}}{\chi_{nn}}, \tag{26}$$

the sound velocity is $v_s = \sqrt{\chi_{++}/\chi_{nn}}$. One can see the sound velocity should become pure imaginary when χ_{++} changes sign, which signals instability.

Above the second critical velocity, the system becomes unstable at $k = 0$, implying global instability. This is due to order competition between the two components. By calculating the difference of free energy density between the two phases $\Delta f = f_{\text{binary}} - f_{\text{single}}$, we find that Δf becomes positive exactly when $v_y > v_{c2}$.

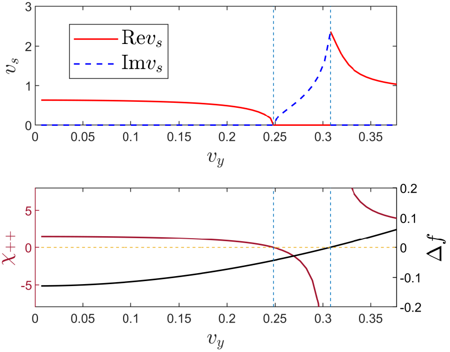


Figure 2: **Top**: Sound velocity of sound mode v_s for different superfluid velocities v_y in counterflow case. **Bottom**: Susceptibility χ_{++} and free energy density difference $\Delta f = f_{\text{binary}} - f_{\text{single}}$. The vertical dashed lines denote critical velocities v_{c1} and v_{c2}. Above v_{c1}, χ_{++} changes sign, the sound velocity becomes purely imaginary, and one sound mode becomes unstable. Above v_{c2}, Δf becomes positive. When $\Delta f > 0$, the binary superfluid phase is globally thermodynamically unstable, and the gapped mode becomes unstable with a positive imaginary part at $k = 0$. Relevant parameters are $T/T_c = 0.677$, $\nu = -0.2$.

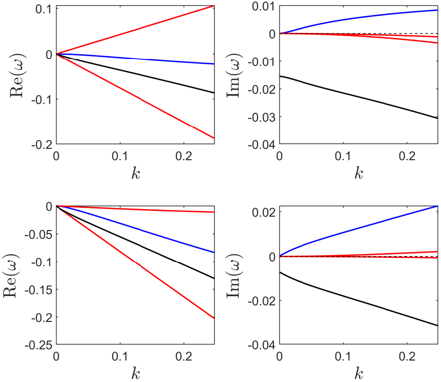


Figure 3: Spectrum of QNMs for coflow case. **Top**: Spectrum of QNMs for superfluid velocity $v'_{y1} = 1.257$. **Bottom**: Spectrum of QNMs for superfluid velocity $v'_{y2} = 2.199$. Red lines are sound modes, blue line is the diffusive mode and black line is the gapped mode. Relevant parameters are $T/T_c = 0.677$, $\nu = -0.2$. Note $v'_{c1} < v'_{y1} < v'_{c2}$ and $v'_{y2} > v'_{c2}$. For v'_{y1} only the diffusive mode is unstable, while for v'_{y2}, one of the sound mode also becomes unstable. For reference, the two critical velocities at $T/T_c = 0.677$ and $\nu = -0.2$ are numerically found to be $v'_{c1} = 0.691$, $v'_{c2} = 2.136$.

These two critical velocities are found to correspond to $\chi_{--} = 0$ and $\chi_{++} = 0$ correspondingly. For $v^s_- = 0$, from

$$\Gamma_0 = -\frac{(\chi_{++}\chi_{--} - \chi^2_{+-})(\delta_2(v^s_-)^2 + \delta_1)}{4\chi_{++}}, \tag{27}$$

the attenuation is $\Gamma_0 = -\chi_{--}\delta_1/4$. When χ_{--} changes sign, the diffusive mode becomes unstable.

And when χ_{++} changes sign, the sound velocity becomes negative. To see this, we expand the sound velocity and attenuation around $\chi_{++} = 0$. Without loss of generality, we assume $\chi_{n+} > 0$. Then one of the sound velocity and attenuation would be

$$\begin{aligned} v_s =& \delta\chi_{++}/2\chi_{n+}\,, \\ \Gamma =& -\frac{\delta\chi_{++}}{16}(\delta_1 + \delta_2(v^s_+)^2 + \frac{4\beta v^s_+}{\chi_{n+}} + \frac{4\sigma}{\chi^2_{n+}})\,. \end{aligned} \tag{28}$$

Negative v_s would signal excitation of modes with negative energy, and thus implies energetic instability.

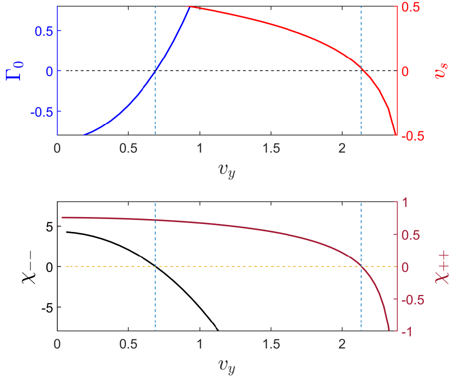


Figure 4: **Top**: Sound velocity v_s of the unstable sound mode and attenuation Γ_0 of the diffusive mode for different superfluid velocities v_y in coflow case. **Bottom**: susceptibilities χ_{++} and χ_{--}. The vertical dashed lines denote the critical velocities v'_{c1} and v'_{c2}, given by $\chi_{--} = 0$ and $\chi_{++} = 0$ respectively. Above v'_{c1}, the diffusivity changes sign and the diffusive mode becomes unstable. Above v'_{c2}, the sound mode also becomes unstable, with its velocity and attenuation both changing sign. Relevant parameters are $T/T_c = 0.677$, $\nu = -0.2$.

By checking the eigenfunctions numerically, we find that the sound modes correspond to

$$(u_1, v_1) = (u_2, v_2), \tag{29}$$

while the diffusive mode and the gapped mode correspond to

$$(u_1, v_1) = -(u_2, v_2). \tag{30}$$

This makes the physical meaning of the two kinds of coflow instability clear. For the first instability, perturbations of the two condensates are in the opposite phase, *i.e.* the two components have relative motion. For the second instability, perturbations are in the same phase, *i.e.* the two components move as a whole.

Universal scaling law of critical velocity

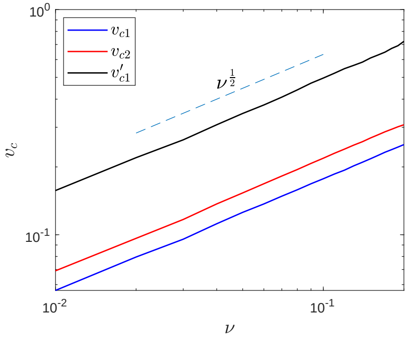


Figure 5: Two critical velocities v_{c1}, v_{c2} for counterflow case and the first critical velocity v'_{c1} for coflow case versus coupling strength ν at $T/T_c = 0.677$. All critical velocities scale as $\nu^{1/2}$. This result also holds at other temperatures. Similar scaling behavior also appears in GPE.

Nonlinear stage for counterflow and coflow instability

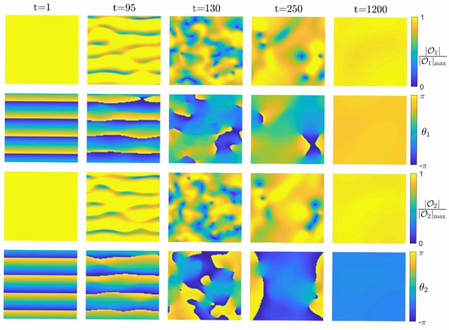


Figure 6: Nonlinear evolution of counterflow instability at $T/T_c = 0.677$, $\nu = -0.2$. Initial superfluid velocity is $v_y = 0.628$. Plotted region is [0,40]×[0,40]. From top to bottom, plotted values are $|\mathcal{O}_1|/|\mathcal{O}_1|_{\max}$, θ_1, $|\mathcal{O}_2|/|\mathcal{O}_2|_{\max}$ and θ_2. As time goes by, initial small perturbations grows exponentially. Then a dark soliton forms and vortex nucleation occurs. Due to strong dissipation within the system, vortices annihilate with anti-vortices. During this process, the kinetic energy of superfluids is also dissipated and the final state is homogeneous binary superfluids with velocity below critical velocity v_{c1}. Similar for coflow case.

Summary

In this work, we explore instabilities in binary superfluids with non-vanishing superflow, particularly focusing on counterflow and coflow instabilities.

- We derive a thermodynamic criterion for the onset of superflow instability in miscible binary superfluids through a hydrodynamic analysis.
- To verify this result, we analyze both GPE and a holographic binary superfluid model. We find agreement in both models. Except the one due to order competing via global thermodynamic instability, the others are caused by an eigenvalue of the free energy Hessian diverging and changing sign.
- We also observe that the critical velocities of these instabilities follow a general scaling law related to the interaction strength between superfluid components. The nonlinear stages of the instabilities are also studied by full time evolution.