Linear dynamical stability and the laws of thermodynamics

Blaise Goutéraux

Center for Theoretical Physics,
CNRS, Ecole polytechnique, Institut Polytechnique de Paris, Palaiseau, France

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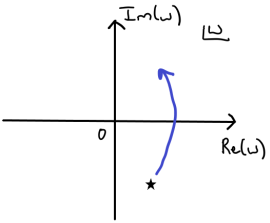
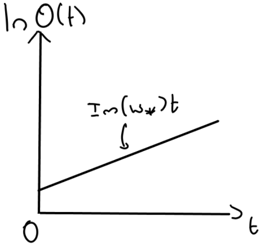
References

- *Linear dynamical stability and the laws of thermodynamics*, with Eric M., [arXiv:2407.07939].
- *Uncovering the thermodynamic origin of counterflow and coflow instabilities in miscible binary superfluids*, with Y. An and L. Li, [arXiv:2411.01972].
- *Hydrodynamics and instabilities of relativistic superfluids at finite superflow*, with D. Areán, E. Mefford and F. Sottovia, [arXiv:2312.08243].
- *Beyond Drude transport in hydrodynamic metals*, with A. Shukla, [arXiv:2309.04033].
- *Thermodynamic origin of the Landau instability of superfluids*, with Eric M. and F. Sottovia, [arXiv:2212.10410].
- Ongoing work with Y. An, L. Li, M. Sanchez-Garitaonandia and V. Ziogas.

Linear dynamical instabilities





- Linear dynamical instability: **pole** of the retarded Green's functions crossing to the **upper half frequency plane**.

$$\{G^R[\omega_\star(k), k]\}^{-1} = 0\,, \qquad \mathrm{Im}(\omega_\star) > 0 \Rightarrow \langle \mathcal{O}_A(t)\rangle \sim e^{\mathrm{Im}\omega_\star t} \text{ grows}$$

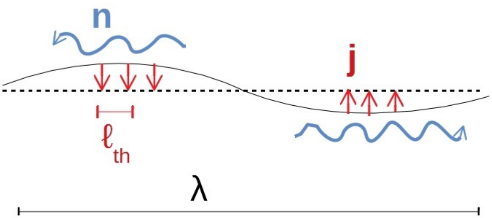
- Occurs when eigenvalue of static susceptibility matrix **changes sign** (local thermodynamic instability) provided 2^{nd} law $\Delta S \geq 0$ holds

$$\chi_{AB} = \frac{\delta O_A}{\delta s_B}$$

Scope of result

- Applies to (super)fluids, magnetohydro, approximate symmetries, etc. in the **thermodynamic limit**.
- Holographic 2^{nd} order large N phase transitions.
- Not covered: 1^{st} order phase transitions, finite N 2^{nd} order phase transitions (stochastic fluctuations), curved spaces (finite size).

Hydrodynamics

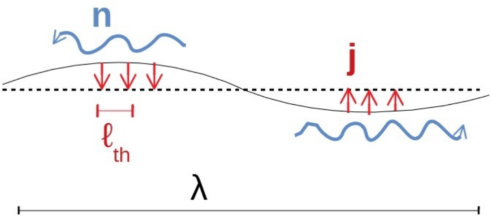


- Effective theory valid at times and lengths **long** compared to local equilibration scales, eg $\omega, k \lesssim T$.

- On such scales, only operators O^A protected by **global symmetries** survive, gives rise to **conservation laws** for their densities (ϵ, n, g^j):

$$\partial_t \langle O^A \rangle + \partial_i \langle j^{iA} \rangle = 0\,, \qquad \langle \ldots \rangle = \frac{1}{Z} \mathrm{Tr}\left[\ldots e^{\beta H} \right]$$

Hydrodynamics

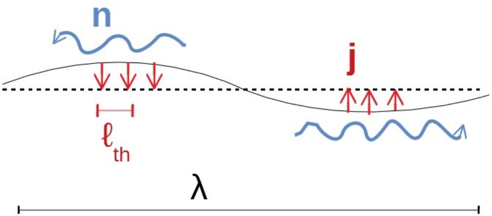


$$\partial_t \langle O^A \rangle + \partial_i \langle j^{iA} \rangle = 0$$

- Spatial fluxes $j^{Ai} = (\epsilon^i, j^i, \tau^{ij})$ are **fast operators** and decay locally.
- Constitutive relations for spatial fluxes: **gradient expansion**

$$\langle j^{Ai} \rangle = \sum_{n \geq 0} c^{AB} D^i_{(n)} \langle O^B \rangle \qquad D^i_{(n)} \sim (\nabla^i)^n$$

Hydrodynamic diffusion



$$\partial_t \langle n \rangle + \partial_i \langle j^i \rangle = 0$$

- **Constitutive relation** for spatial flux

$$\langle j^i \rangle = -\sigma_0 \left(\partial^i \mu - E^i \right) + O(\ell_{eq}^2 \partial^2)$$

- In thermal equilibrium, **thermal timelike Killing vector** β^μ s.t. $T \equiv 1/\sqrt{-\beta^2}$ + gauge trafos $A_\mu \mapsto A_\mu + \nabla_\mu \Lambda$

$$\delta_{\mathcal{B}} A_\mu = \mathcal{L}_{\beta^\nu} A_\mu + \partial_\mu \Lambda = 0\big|_{eq} \quad \Rightarrow \quad (A_t = \mu, A_x = 0)\big|_{eq} \quad \Rightarrow (E^i = \partial^i \mu)\big|_{eq}$$

- $\Rightarrow \sigma_0 \delta_{\mathcal{B}} A_i$ is an **out-of-equilibrium** gradient correction.

Hydrodynamic diffusion

$$\partial_t \langle n \rangle + \partial_i \langle j^i \rangle = 0\,, \quad \langle j^i \rangle = -\sigma_0 \left(\partial^i \mu - E^i \right) + O(\ell_{eq}^2 \partial^2)$$

- **Expand** around equilibrium $(n, \mu) = (\mu_0, n_0) + \delta(\mu, n) e^{-i\omega t + ikx}$
- **Solve** $(\delta n, \delta \mu)$ in terms of E^i using $\delta n = \chi_{nn} \delta \mu$.
- Compute **retarded Green's function**

$$G^R_{nn}(\omega, k) \equiv \frac{\delta n}{\delta A_t} = \frac{i\sigma_0 k^2}{\omega + iD_n k^2}\,, \qquad D_n = \sigma_0 / \chi_{nn}\,.$$

- **Diffusion** pole $\omega = -iD_n k^2$ in LHP if $\chi_{nn} > 0$ (**thermodynamic stability**) and $\sigma_0 \geq 0$ (**positivity of entropy production**):

$$\Delta \equiv T\partial_t s = \sigma_0 \left(\partial^i \mu - E^i \right) \left(\partial_i \mu - E_i \right) \geq 0$$

Neutral relativistic fluids in offshell formalism

$$\nabla_\mu T^{\mu\nu} = 0$$

- **Offshell** entropy current [Bhattacharyya'13,'14; Haehl, Loganayagam & Rangamani'14,'15]:

$$\Delta \equiv \nabla_\mu (s u^\mu + \tilde{s}^\mu) + \beta_\nu \nabla_\mu T^{\mu\nu} \geq 0$$

- **Constitutive relation** for stress-energy tensor (Landau frame)

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} + \tilde{T}^{\mu\nu} , \quad P^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu} , \quad u_\mu \tilde{T}^{\mu\nu} = 0$$

- Evaluate offshell entropy current:

$$\Delta = -\tilde{T}^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} , \qquad \delta_{\mathcal{B}} g_{\mu\nu} = 2\nabla_{(\mu}\beta_{\nu)} = 0|_{\mathrm{eq}}$$

- Parametrize **first-order** corrections:

$$\Rightarrow \quad \tilde{T}^{\mu\nu} = \tilde{N}^{(\mu\nu)(\alpha\beta)} \delta_{\mathcal{B}} g_{\alpha\beta}$$

Linear dynamical stability of relativistic fluids

$$\tilde{T}^{\mu\nu} = \tilde{N}^{(\mu\nu)(\alpha\beta)} \delta_{\mathcal{B}} g_{\alpha\beta} \,, \qquad \Delta = -\delta_{\mathcal{B}} g_{\mu\nu} \tilde{N}^{(\mu\nu)(\alpha\beta)} \delta_{\mathcal{B}} g_{\alpha\beta}$$

- **Two independent transport coefficients** η, $\zeta \geq 0$

$$\tilde{T}^{\mu\nu} = -\frac{\eta}{2}\left(P^{\mu\alpha}P^{\nu\beta} - \frac{1}{d}P^{\mu\nu}P^{\alpha\beta}\right)\delta_{\mathcal{B}} g_{\mu\nu} - \frac{\zeta}{2d}P^{\mu\nu}P^{\alpha\beta}\delta_{\mathcal{B}} g_{\alpha\beta}$$

- Both are **out-of-equilibrium** terms: $\delta_{\mathcal{B}} g_{\mu\nu} = 2\nabla_{(\mu}\beta_{\nu)} = 0|_{\mathrm{eq}}$.

- By construction $\tilde{N}^{(\mu\nu)(\alpha\beta)}$ is **symmetric** and must be **positive definite**: only **out-of-equilibrium, dissipative** transport coefficients.

- 2 sound modes and 1 diffusion mode:

$$\omega = \pm\sqrt{\frac{\epsilon + p}{Tc_v}}k - i\frac{(\eta+\zeta)}{\epsilon + p}k^2 \,, \quad \omega = -\frac{i\eta k^2}{\epsilon + p} \,, \quad \chi_{AB} = \mathrm{Diag}\{Tc_v, \epsilon + p\}$$

- All modes are stable (lhp) if $\chi_{AB} > 0$.

Linear dynamical stability more generally?

- At first sight, seems straightforward to show by computing **location of poles** and **divergence of entropy current**.
- Nevertheless, **intractable** for less simple cases, eg:
  - non-boost invariant fluids [de Boer, Hartong, Obers, Sybesma & Vandoren'17,'20; Novak, Sonner & Withers'19; Armas & Jain'20] (neutral: 16 coefficients; charged: 29 coefficients);
  - relativistic superfluids (14 coefficients, [Bhattacharya, Bhattacharyya, Minwalla & Yarom'11]).
- Further, in some cases some transport coefficients appear in the modes but do not affect their stability.

Offshell formalism

$$\partial_t \langle O^A \rangle + \partial_i \langle j^{iA} \rangle = 0\,, \quad j^{Ai} = j^{Ai}_{id} + \tilde{j}^{Ai}$$

- Divergence of **offshell entropy current** + coupling to external sources $s_A = (g_{\mu\nu}, A_\mu, \ldots)$ [Armas & Jain'20].

$$\Delta = -\tilde{j}^A \delta_{\mathcal{B}} s_A = \delta_{\mathcal{B}} s_A \tilde{N}^{AB} \delta_{\mathcal{B}} s_B\,, \quad \tilde{N}^{AB} = \begin{cases} \tilde{N}^{AB}_{hs} & \Delta_{hs} = 0 \\ \tilde{N}^{[AB]}_{nhs,nd} & \Delta_{nhs,nd} = 0 \\ \tilde{N}^{(AB)}_{nhs,d} & \Delta_{nhs,d} > 0 \end{cases}$$

- $\tilde{N}_{hs}$: derivative corrections which do not vanish in **thermal equilibrium**, captured by correcting the hydrostatic pressure

$$p = p_{ideal} + \tilde{p} \quad \Rightarrow \quad O^A = O^A_{id} + \tilde{O}^A_{hs} \quad \Rightarrow \quad \chi_{AB} = \chi^{id}_{AB} + \tilde{\chi}^{hs}_{AB}$$

- $\tilde{N}_{nhs}$: **out-of-equilibrium** corrections $\propto \delta_{\mathcal{B}} s_A \simeq \partial_i s_A$

$$\Rightarrow \Delta = \delta_{\mathcal{B}} s_A \tilde{N}^{(AB)}_{nhs,d} \delta_{\mathcal{B}} s_B \geq 0 \quad \begin{cases} \tilde{N}_{nhs,d} & \textbf{positive-definite} \\ \tilde{N}_{nhs,nd} & \text{unconstrained} \end{cases}$$

Sketch of stability proof [2407.07939]

$$\Delta = -\tilde{j}^A \delta_{\mathcal{B}} s_A = \delta_{\mathcal{B}} s_A \tilde{N}^{AB} \delta_{\mathcal{B}} s_B \,, \quad \tilde{N}^{AB} = \begin{cases} \tilde{N}^{AB}_{hs} & \Delta_{hs} = 0 \\ \tilde{N}^{AB}_{nhs,nd} & \Delta_{nhs,nd} = 0 \\ \tilde{N}^{AB}_{nhs,d} & \Delta_{nhs,d} > 0 \end{cases}$$

- Linearize eoms and Fourier transform:

$$-i\omega \delta O^A + M^{AB}(q) \chi^{-1}_{BC} \delta O^C = 0 \,, \quad \begin{cases} M^{AB} = iqN^{(AB)}_{id,hs} + q^2 \tilde{N}^{AB}_{nhs} \,, \\ \chi_{AB} = \chi^{id}_{AB} + \tilde{\chi}^{hs}_{AB} \end{cases}$$

- Poles are given by the **eigenvalues** $\tilde{m}^{(A)}$ of $M \cdot \chi^{-1}$

$$\det \left[-i\omega + M(q) \cdot \chi^{-1} \right] = 0 \quad \Rightarrow \quad \omega^{(A)}_{\star} = -i\tilde{m}^{(A)}$$

Sketch of stability proof [2407.07939]

- Poles are given by the **eigenvalues** $\tilde{m}^{(A)}$ of $M \cdot \chi^{-1}$

$$\det\left[-i\omega + M(q)\cdot\chi^{-1}\right] = 0 \quad \Rightarrow \quad \omega_\star^{(A)} = -i\tilde{m}^{(A)}$$

- Positivity of $\tilde{N}$ implies positivity of real part of eigenvalues of M.

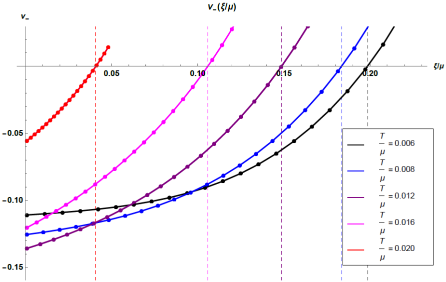
$$\Delta = \delta_{\mathcal{B}} s_A \tilde{N}^{AB} \delta_{\mathcal{B}} s_B \geq 0\,, \quad M^{AB} = iqN^{(AB)}_{id,hs} + q^2 \tilde{N}^{AB}_{nhs}$$

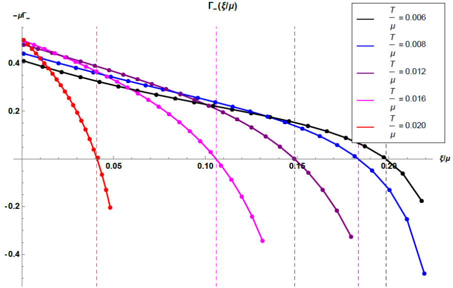
- Positivity of χ (local thermodynamic stability) $+\ \Delta \geq 0$ (2nd law)

$$\Rightarrow \quad \mathrm{Re}\tilde{m} \geq 0 \quad \Rightarrow \quad \mathrm{Im}\omega_\star < 0$$

- If χ has a **negative eigenvalue**, pole in upper half plane: **linear dynamical instability** caused by **thermodynamic instability**.

Example 1: superfluids at finite superflow



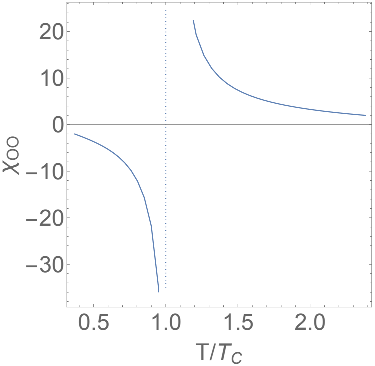


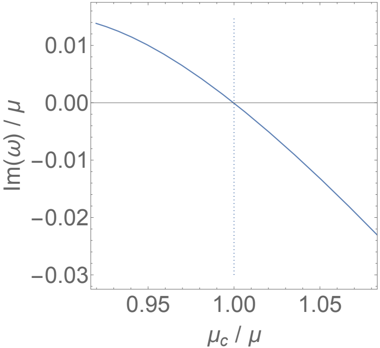
- Instabilities of superfluids at finite superflow fall into this category, [2212.10410,2312.08243]. Matches previous holographic studies [Amado, Arean, Jimenez-Alba, Landsteiner & Melgar'13; Lan, Liu, Tian & Zhang'20].

- The **Landau instability** of weakly-coupled superfluids does as well.

- See Yuping An's talk later today for the co-flow and counterflow instability of binary superfluids, [2411.01972].

Example 2: holographic 2[nd] order phase transitions





- Probe complex scalar in Reissner-Nordström background
- At T_c, $\chi_{OO} \equiv \delta O/\delta s_O$ **diverges and changes sign** + pole in upper half plane: linear dynamical instability.
- Stochastic fluctuations of O suppressed by large N: EFT of light mode [Herzog'10; Bhattacharya, Bhattacharyya & Minwalla'11; Donos & Kailidis'22]?

Example 3: Approximate symmetries

$$\partial_t \langle O^A \rangle + \partial_i \langle j^{iA} \rangle = \ell_{\rm eb} f^A \quad \Rightarrow \quad \Delta \supset -\ell_{\rm eb} f^A \delta s^A$$

- Symmetry-breaking currents can be expanded order by order in $\ell_{\rm eb}\delta s^A \equiv s^A - \bar{s}^a$ s.t. $\delta s^A = 0|_{\rm eq}$ + usual gradient expansion.
- **Enhance** the set of out-of-eq sources $\mapsto (\delta_{\mathcal{B}} s_A, \ell_{\rm eb}\delta s_A)$.
- **Update** the matrix N in Δ.
- **Model-independent algorithm** for hydrodynamics with approximate symmetries: magnetohydro, broken spatial translations, superfluids with vortices... Recover known results this way.

Example 3: Approximate spatial translations

$$\partial_t g^i + \partial_i \tau^{ji} = \ell_{\mathrm{eb}} f^{g^i} \quad \Rightarrow \quad \Delta \supset -\ell_{\mathrm{eb}} f^{g^i} \delta v^i$$

$$\begin{cases} f^{g^i} = & -\left(\gamma\delta^{ij} + \gamma_H \epsilon^{ij}\right) \ell_{\mathrm{eb}} \delta v^j - \ell_{\mathrm{eb}} \lambda_n \partial^i \mu - \ell_{\mathrm{eb}} \lambda_s \partial^i T + \dots \\ j^i = & n v^i + \ell_{\mathrm{eb}}^2 \lambda_n \delta v^i + \dots \\ s^i = & s v^i + \ell_{\mathrm{eb}}^2 \lambda_s \delta v^i + \dots \\ g^i = & \rho v^i + \ell_{\mathrm{eb}}^2 \lambda_g \delta v^i + \dots \end{cases}$$

- scale $\ell_{\mathrm{eb}}^2 \sim \partial$
- γ is a **dissipative, non-hydrostatic** transport coefficient: $\gamma \geq 0$
- γ_H, $\lambda_{n,s,g}$ are **non-dissipative, non-hydrodystatic** transport coefficients: sign unconstrained, contribute to modes but do not affect stability.

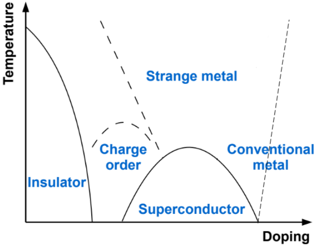
Example 3: Approximate spatial translations

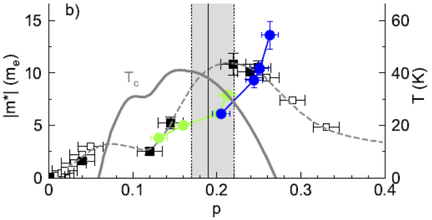
$$\partial_t g^i + \partial_i \tau^{ji} = \ell_{\rm eb} f^{g^i} \quad \Rightarrow \quad \Delta \supset -\ell_{\rm eb} f^{g^i} \delta v^i$$

$$\begin{cases} f^{g^i} = & -\left(\gamma\delta^{ij} + \gamma_H \epsilon^{ij}\right) \ell_{\rm eb} \delta v^j - \ell_{\rm eb}\lambda_n \partial^i \mu - \ell_{\rm eb}\lambda_s \partial^i T + \dots \\ j^i = & n v^i + \ell_{\rm eb}^2 \lambda_n \delta v^i + \dots \\ s^i = & s v^i + \ell_{\rm eb}^2 \lambda_s \delta v^i + \dots \\ g^i = & \rho v^i + \ell_{\rm eb}^2 \lambda_g \delta v^i + \dots \end{cases}$$

- Have been derived from hydro when translations are broken with a **scalar operator** [Goutéraux & Shukla'23] or a **periodic chemical potential** [Chagnet & Schalm'24], including with a magnetic field.
- Match to holographic results for **linear axion** models, [Davison & Goutéraux'15; Blake'15] and **holographic lattices** [Chagnet & Schalm'24].
- Disorder? [Andreev, Kivelson & Spivak'10; Lucas'15; Huang, Sachdev & Lucas '23, ...]

Example 3: Approximate spatial translations





- Out-of-equilibrium shifts of Drude weight and cyclotron frequency:

$$\sigma(\omega) = \sigma_0 + \frac{(n + \ell_{\rm eb}^2 \lambda_n)^2}{\gamma - i\omega(\rho + \ell_{\rm eb}^2 \lambda_g)}\,, \qquad \omega_c = \frac{n + 2\ell_{\rm eb}^2 \lambda_n}{\rho + \ell_{\rm eb}^2 \lambda_g} B$$

- Matches holography [Davison & Goutéraux'15; Blake'15; Chagnet & Schalm'24] and likely SYK-Yukawa models [Guo, Valentinis, Schmalian, Sachdev & Patel'23].

- Relevant for overdoped cuprates? Discrepancy between **thermodynamic** and **cyclotron** mass [Legros, Post, ... & Armitage'22].

Summary and outlook

- In hydro theories, $\left.\begin{array}{l}\textbf{thermodynamic stability}\\ \textbf{positivity of entropy prod}\end{array}\right\}$
  $\Rightarrow$ **linear dynamical stability**

- Conversely, **thermodynamic instab** $\Rightarrow$ **linear dynamical instab**.

- Proof covers fluids, superfluids, crystalline solids, weak translation breaking and magnetic fields...

  Possible extensions: anomalous transport, higher form symmetries, fractons. . . ?

- **Landau instability** of superfluids: **thermodynamic** instability.
  Extends to non-trivial equilibrium states: miscible binary superfluids.
  Immiscible binary superfluids [An, Li, Xia & Zeng'24; An, Li & Zeng'24]?

- **Model-independent** hydrodynamics with approximate symmetry.