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Bulk reconstruction and fine structure of entanglement

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with Y.-y. Lin, Y. Sun, C. Yu, J. Zhang, F. Chen, X. Fang, J.-C. Jin, L. h. Mo, Y. Zhou, P. Ye
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Outline

- Bulk reconstruction in AdS/CFT
- Surface growth approach
- Surface growth, bit threads and EoP
- Entanglement contour, fine structure
- Hyperfine Rényi entropy
- Conclusions

Bulk reconstruction in AdS/CFT

The AdS/CFT correspondence and the more general gauge/gravity duality provide a novel connection between different theories, one is a higher dimensional gravitational theory, another is a quantum field theory without gravity on the boundary.

The key equation in the AdS/CFT correspondence is

$$Z_{\mathrm{AdS}}[\phi_0(\vec{x})] = Z_{\mathrm{CFT}}[\phi_0(\vec{x})] = \left\langle \exp \int d^4x O(\vec{x})\phi_0(\vec{x}) \right\rangle$$

Important properties:

field/operator duality, **strong/weak duality**.

From the bulk to boundary--studying the strongly coupled systems

From the boundary to the bulk--an emergent picture of gravity

Bulk reconstruction--from the boundary to the bulk

bulk matter fields:

using the boundary operators to construct the bulk matter fields.

Banks, Douglas, Horowitz, Martinec, th/9808016;

Hamilton, Kabat, Lifschytz, and Lowe, th/0606141.

$$\phi(z, x) = \int dx' K(x'|z, x) \phi_0(x').$$

bulk local field ⟺ *boundary nonlocal operators*

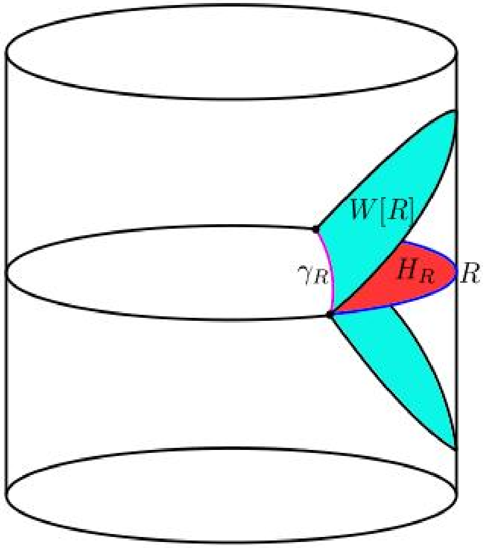
Entanglement wedge reconstruction

Headrick, Hubeny, Lawrence, Rangamani, 2014;

Dong, Harlow, Wall, 2016

$$\mathcal{W}_{\mathcal{E}}[\mathcal{A}] := \tilde{D}[\mathcal{R}_{\mathcal{A}}] \, .$$

subregion-subregion duality.



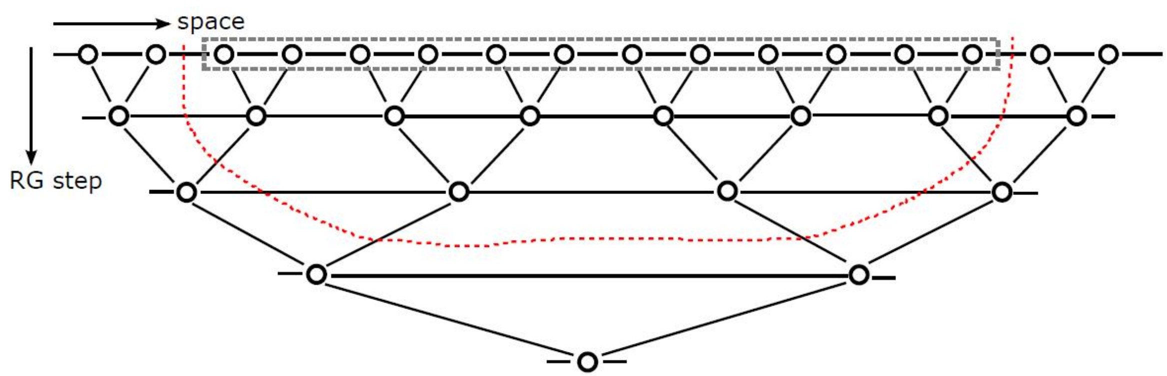
It's more difficult to construct the bulk geometry and the gravitational dynamics from the boundary CFT.

bulk geometry and gravitational dynamics:

using MERA tensor networks to construct the AdS geometry

Swingle, 0905.1317; 1209.3304;
Qi, 1309.6282;
Almheiri, Dong, Harlow, 1411.7041;
Pastawski, Yoshida, Harlow, Preskill, 1503.06237;
Hayden, Nezami, Qi, Thomas, Walter, Yang, 1601.01694;
Bhattacharyya, Gao, Hung, Liu, 1606.00621;
Gan and Shu, 1705.05750;
Ling, Xiao and Wu, 1907.01215;
Dai, Sun and Sun, 1912.02070;
Chen, Liu and Hung, 2102.12022
...

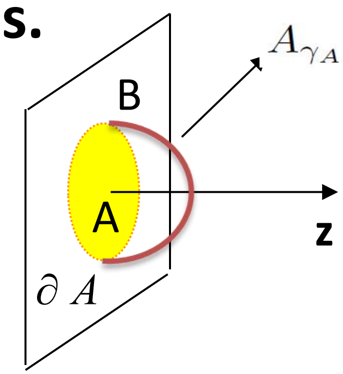
$$ds^2 = L^2\left(\frac{dr^2 + dx^2}{r^2}\right)$$



The holographic entanglement entropy plays crucial role in constructing bulk gravity from boundary quantum fields.

Ryu and Takayanagi 2006

$$S_A = \frac{A_{\gamma_A}}{4G_{d+1}}$$

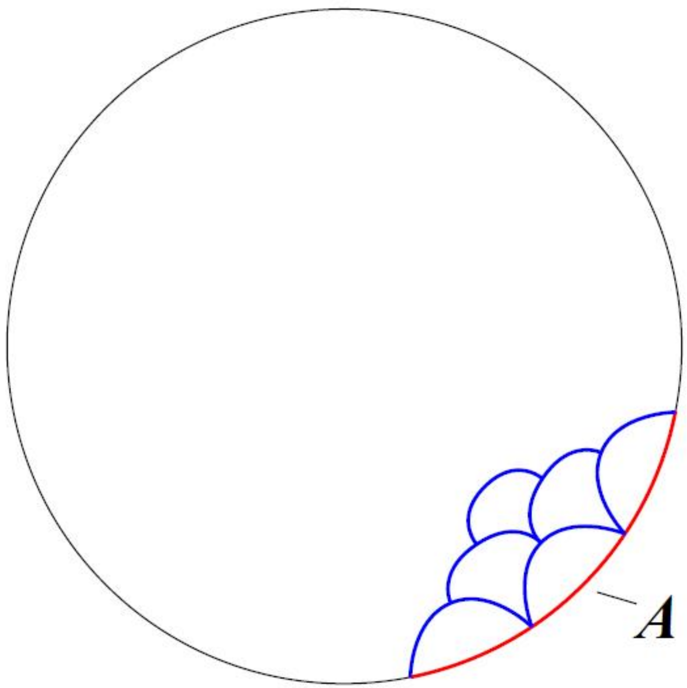


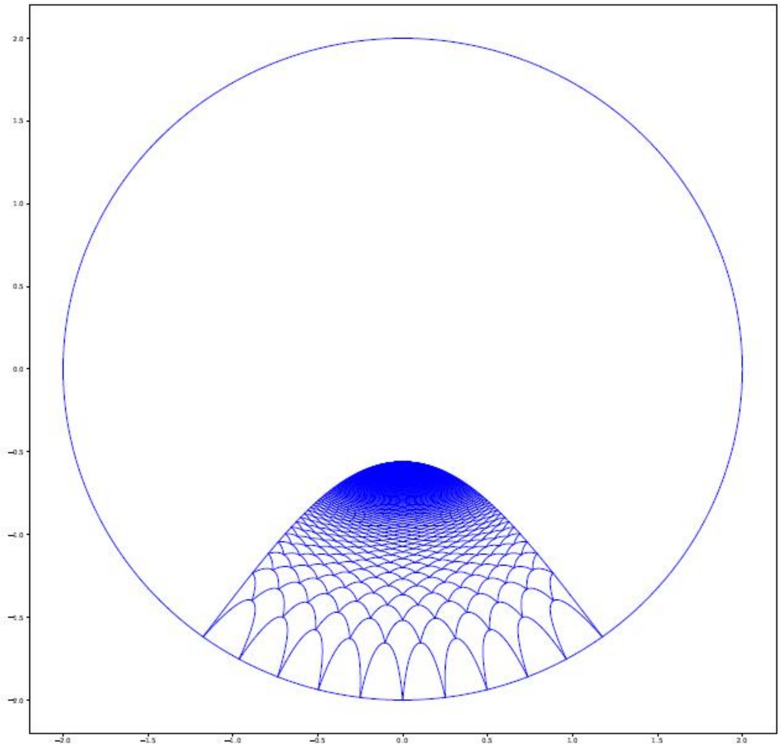
The surface growth approach

Y-y Lin, **JRS**, Y Sun, 2010.01907； C Yu, F-Z Chen,Y-y Lin, **JRS**, Y Sun, 2010.03167

Is there a direct and more efficient way to construct the bulk geometry and matter fields?

Besides, it is interesting to find a more refined structure in the subregion-subregion duality, such as how a given region in the entanglement wedge is dual to a boundary region?

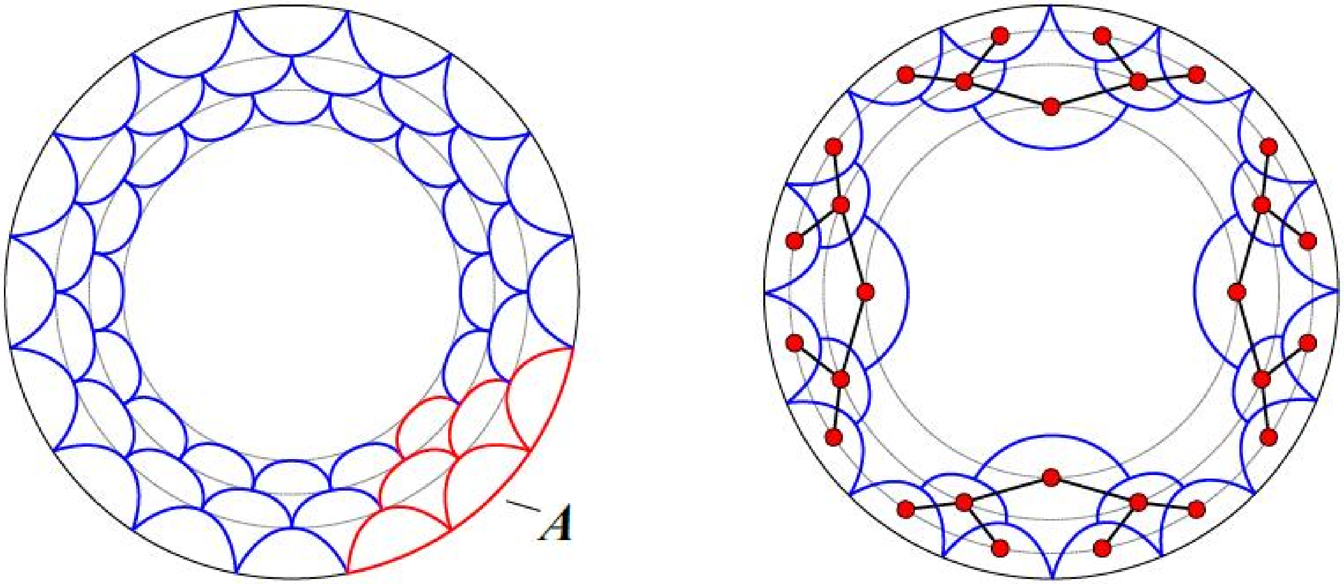




The surface growth approach from tensor networks

Y-y Lin, **JRS**, Y Sun, 2010.01907

Motivated by **Huygens' principle of wave propagation**, we proposed a **novel surface growth scheme** to reconstruct the bulk geometry, which can be explicitly realized with the help of the **surface/state correspondence** and the **one shot entanglement distillation** method.



(a) (b)

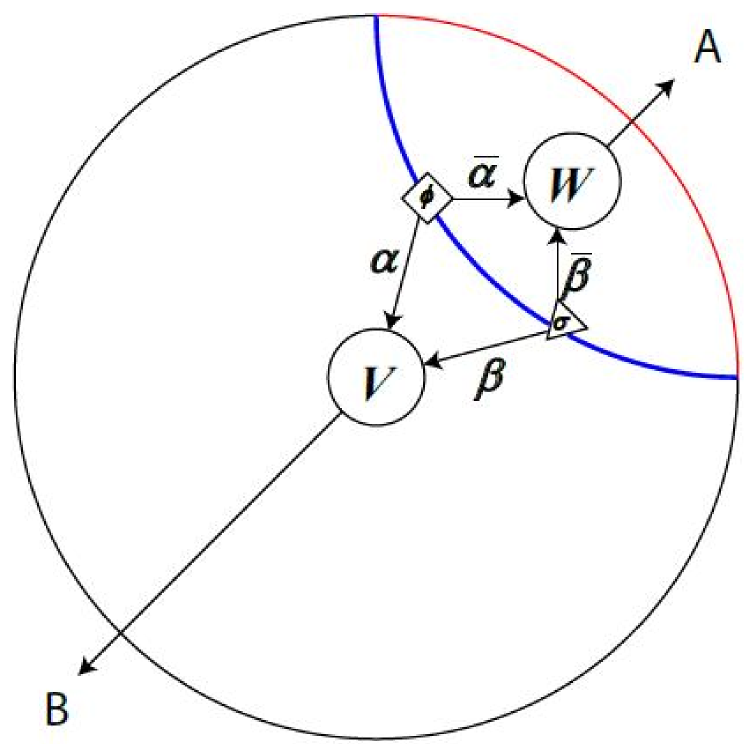
One shot entanglement distillation (OSED)

Bao, Penington, Sorce, Wall, 1812.01171

In quantum information theory, $|\varphi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{A^c}$,

$$|\varphi\rangle^{\otimes m} \approx (W \otimes V)\left[\frac{1}{\sqrt{D}} \sum_{i=0}^{e^{S(A)m-O(\sqrt{m})}} \left|i\bar{i}\right\rangle \otimes \sum_{j=0}^{e^{O(\sqrt{m})}} \sqrt{p_j}\left|j\bar{j}\right\rangle\right],$$

a single holographic state $|\Psi\rangle$ can play the role of $|\varphi\rangle^{\otimes m}$, which has a tensor representation

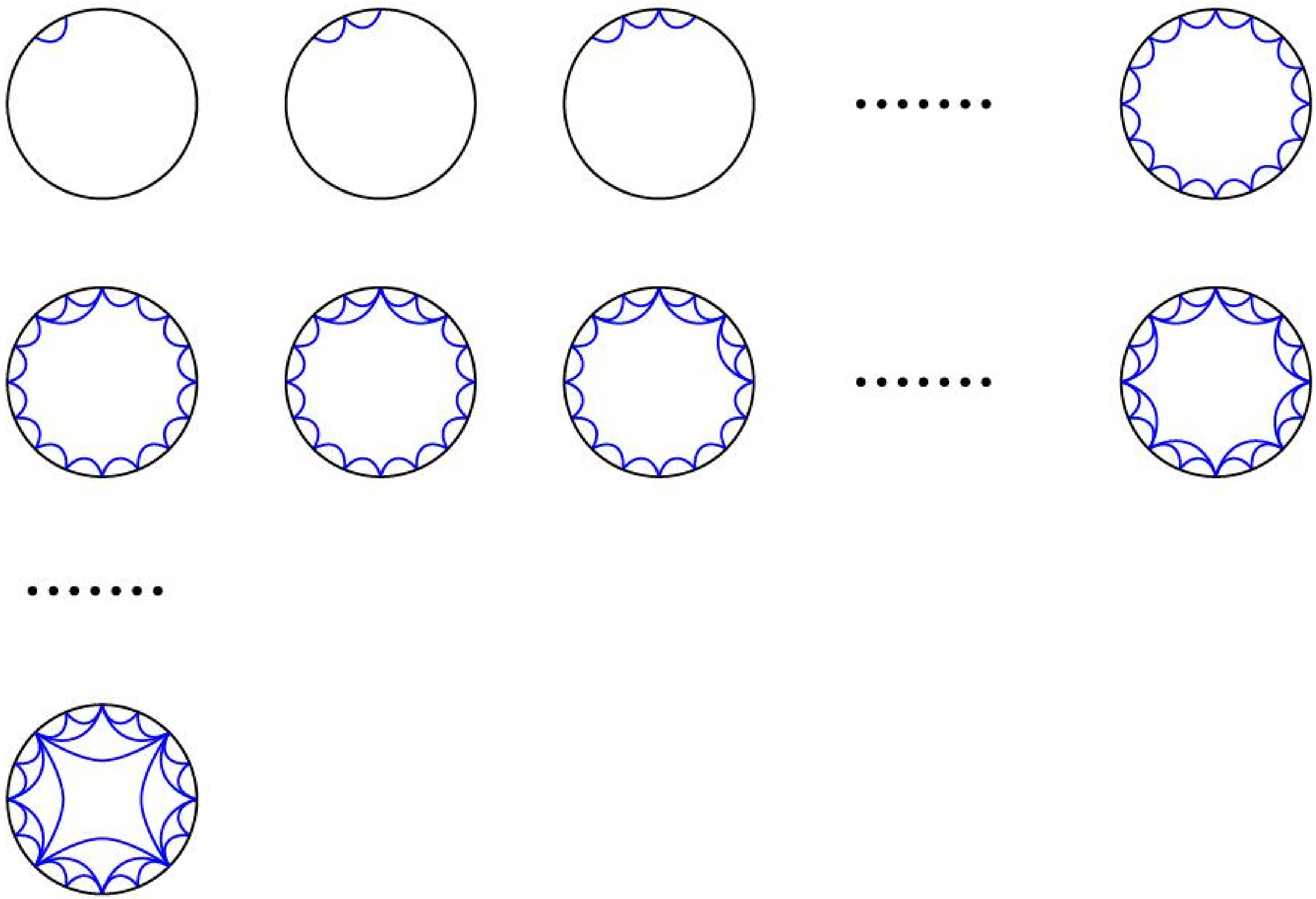


$$\Psi^{AB} = V^B_{\beta\alpha} W^A_{\bar{\beta}\bar{\alpha}} \phi^{\alpha\bar{\alpha}} \sigma^{\beta\bar{\beta}} = (V \otimes W)(|\phi\rangle \otimes |\sigma\rangle)$$

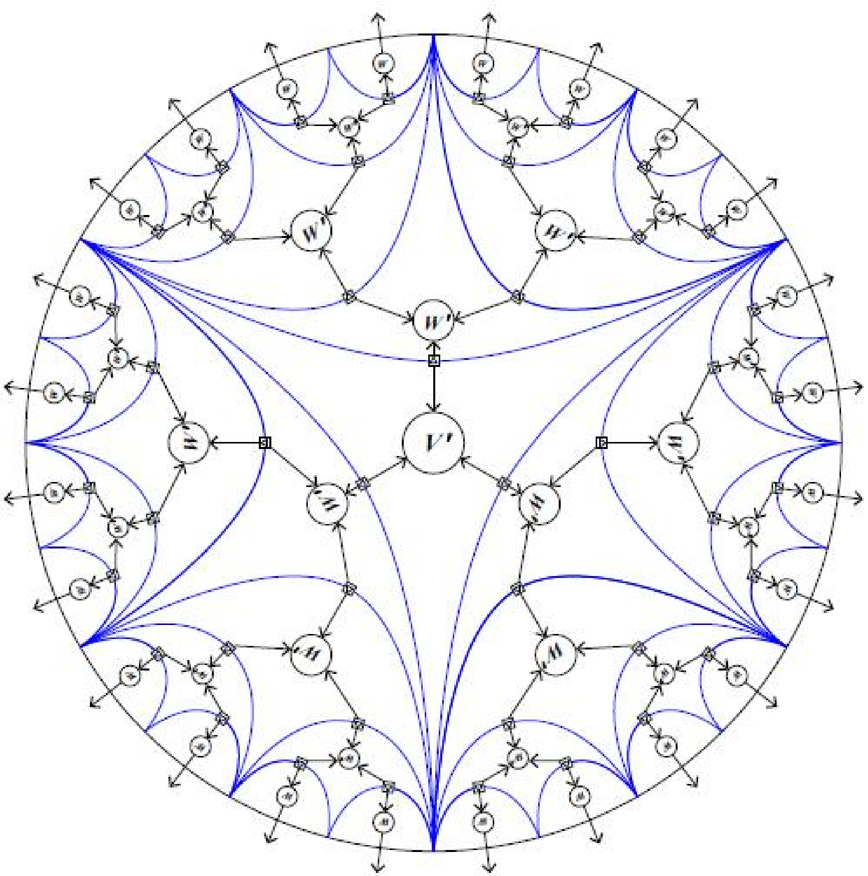
$$|\phi\rangle = \sum_{m=0}^{e^{S-O(\sqrt{S})}} |m\bar{m}\rangle_{\alpha\bar{\alpha}},$$

$$|\sigma\rangle = \sum_{n=0}^{e^{O(\sqrt{S})}} \sqrt{\tilde{p}^{avg}_{n\Delta}}|n\bar{n}\rangle_{\beta\bar{\beta}}.$$

Surface growth scheme--a special case

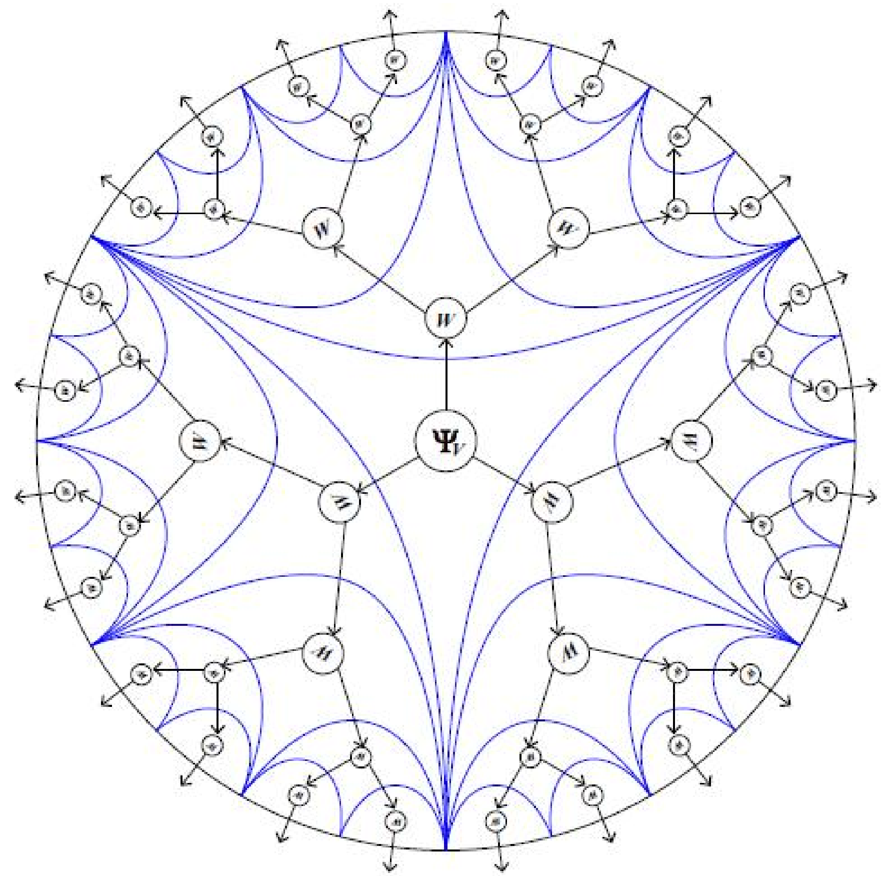


the final surface growth picture corresponds the OSED TN



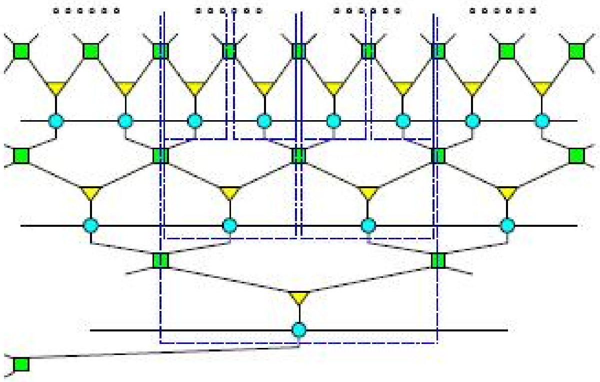
$$|\Psi\rangle^{k\mathrm{th}} = V'^{k\mathrm{th}}_{M} \prod_{i=1}^{N/2^{k-1}} \prod_{a=1}^{k} W'^{a\mathrm{th}}_{i} \, |\phi\rangle^{a\mathrm{th}}_{i} \, |\sigma\rangle^{a\mathrm{th}}_{i}$$

which can be identified with the MERA-like TN as

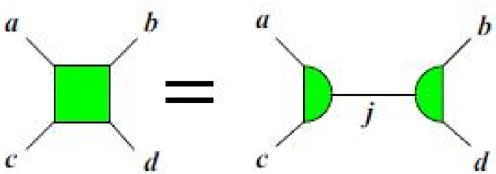


$$|\Psi\rangle^{k\mathrm{th}} = \Psi_{V}{}^{k\mathrm{th}}_{M} \prod_{i=1}^{N/2^{k-1}} \prod_{a=1}^{k} W^{a\mathrm{th}}_{i}.$$

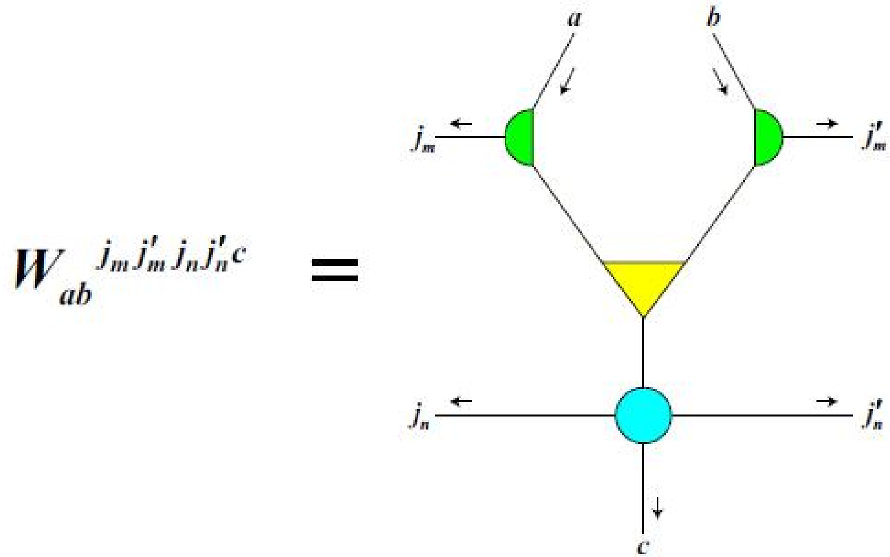
the RT surface in the MERA-like tensor network, and the expression of W tensor is



(a)



(b)



(c)

$$u^{abcd} = t^{acj} t^{bd}{}_{j},$$

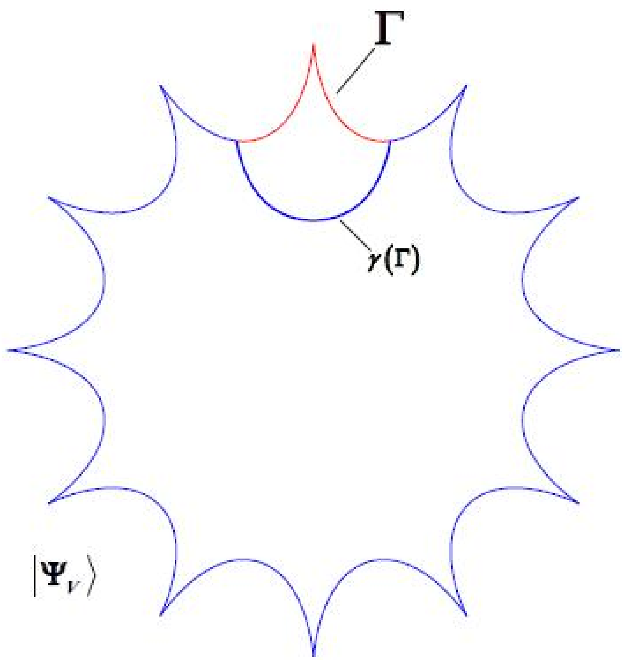
$$W = W_{ab}^{j_m j_m^{'} j_n j_n^{'} c}$$

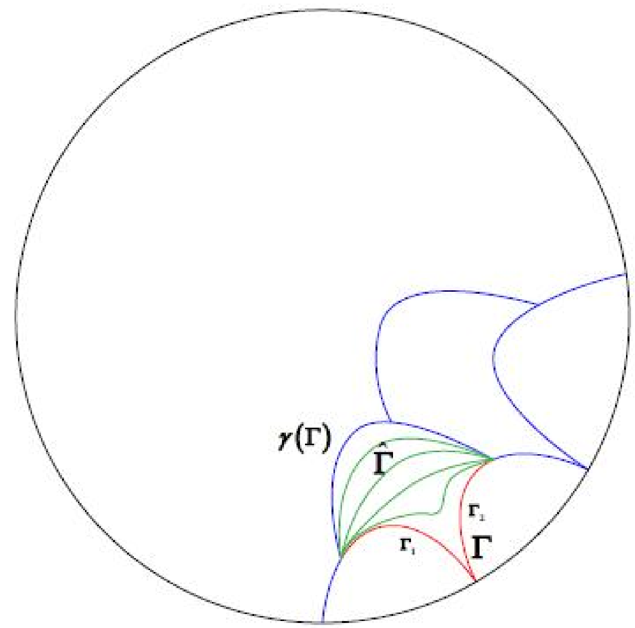
$$S_A = n \cdot \ln J = (4\log_2 J) \cdot \ln \frac{l}{\varepsilon}.$$

$$\frac{c}{3} = 4\log_2 J$$

$$ds^2 = L^2 \left[(\ln 2)^2 \cdot dv^2 + \left(2^{-v}\right)^2 \cdot dx^2 \right]$$

More general surface growth scheme





$$\Psi_V = W^{\Gamma}_{\bar{\beta}\bar{\alpha}} V^{\bar{\Gamma}}_{\beta\alpha} \phi^{\alpha\bar{\alpha}} \sigma^{\beta\bar{\beta}} = W^{\Gamma}_{\bar{\beta}\bar{\alpha}} (V \left|\phi\right\rangle \left|\sigma\right\rangle)^{\bar{\Gamma}\bar{\alpha}\bar{\beta}},$$

$$\Psi = \left(W^{1\mathrm{st}}\right)^N W^{\Gamma}_{\bar{\beta}\bar{\alpha}} (V \left|\phi\right\rangle \left|\sigma\right\rangle)^{\bar{\Gamma}\bar{\alpha}\bar{\beta}}.$$

Direct growth of bulk minimal surfaces

C Yu, F-Z Chen,Y-y Lin, **JRS**, Y Sun, 2010.03167

The surface growth scheme can also be directly checked by the growth of the bulk minimal surfaces layer by layer.

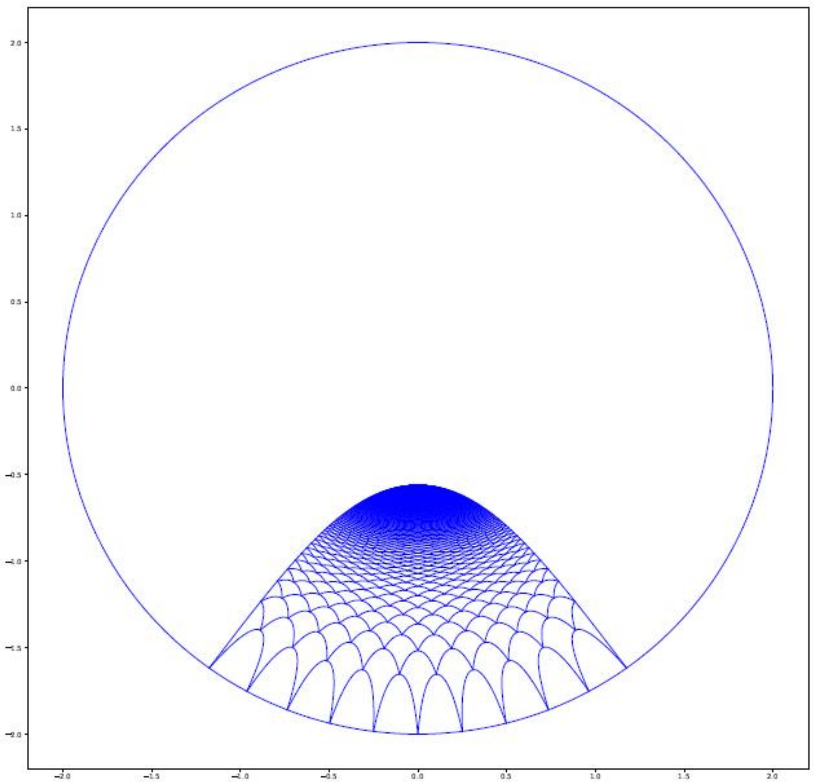
Pure AdS3 case

$$ds^2 = d\rho^2 + L^2 \left(-\cosh^2 \frac{\rho}{L} dt^2 + \sinh^2 \frac{\rho}{L} d\phi^2 \right),$$

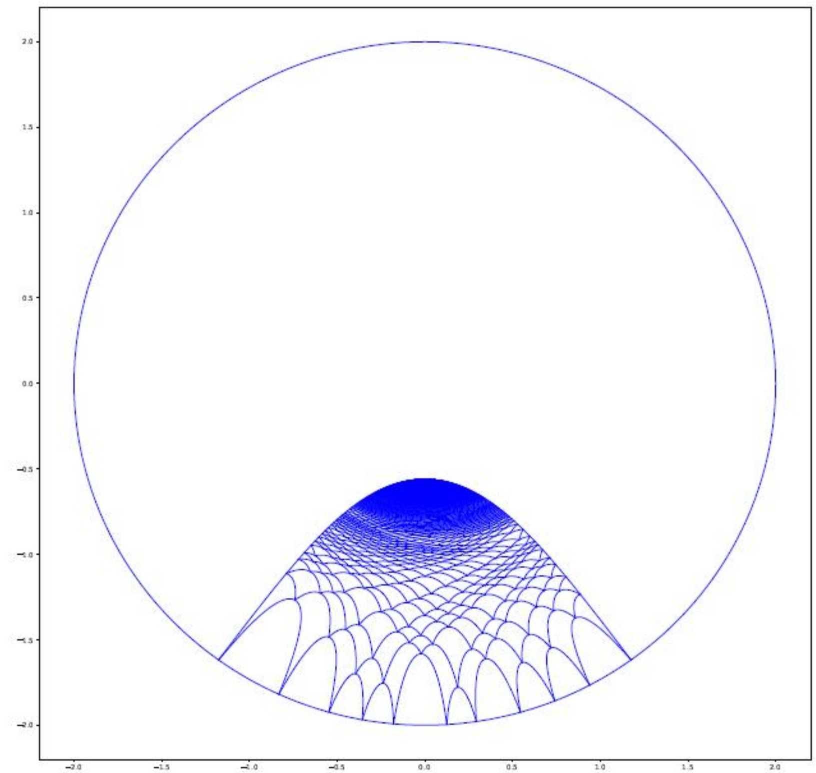
EoM of bulk minimal curve (geodesics) is

$$\begin{aligned}\phi \;=\; & \pm \arctan \left(\frac{\sinh^2 \tilde{\rho}}{\sinh \tilde{\rho}_*} + \cosh \tilde{\rho} \sqrt{\frac{\sinh^2 \tilde{\rho}}{\sinh^2 \tilde{\rho}_*} - 1} \right) \\ & \mp \arctan \left(\sinh \tilde{\rho}_* \right) + \phi_0,\end{aligned}$$

for given angular size of the subsystem, different radial cutoff corresponds to different turning position.



homogenous subregions, with each subregion has angle $\phi = \pi / 25$, the growing steps are 300.



inhomogenous subregions, with growing steps 300.

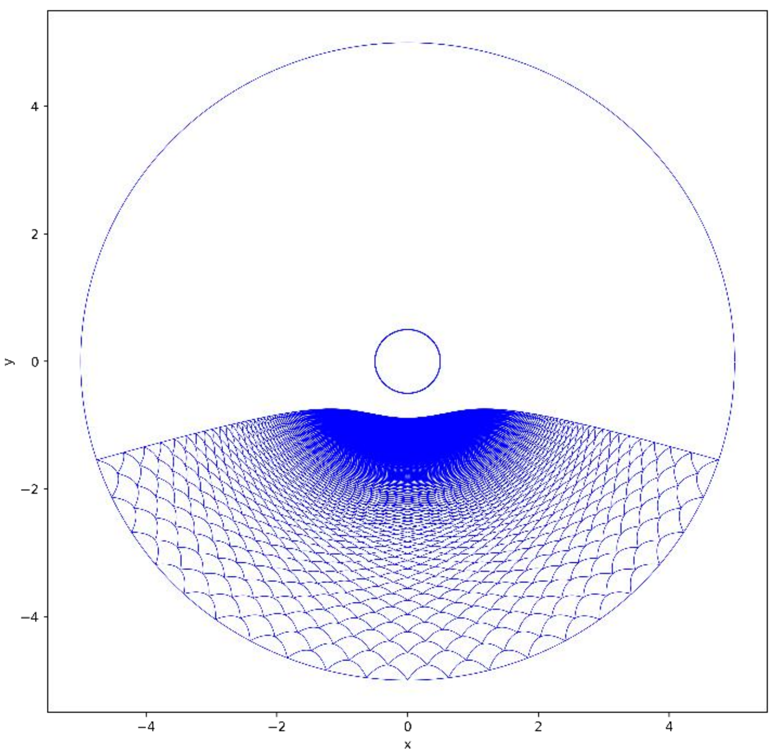
BTZ black hole case

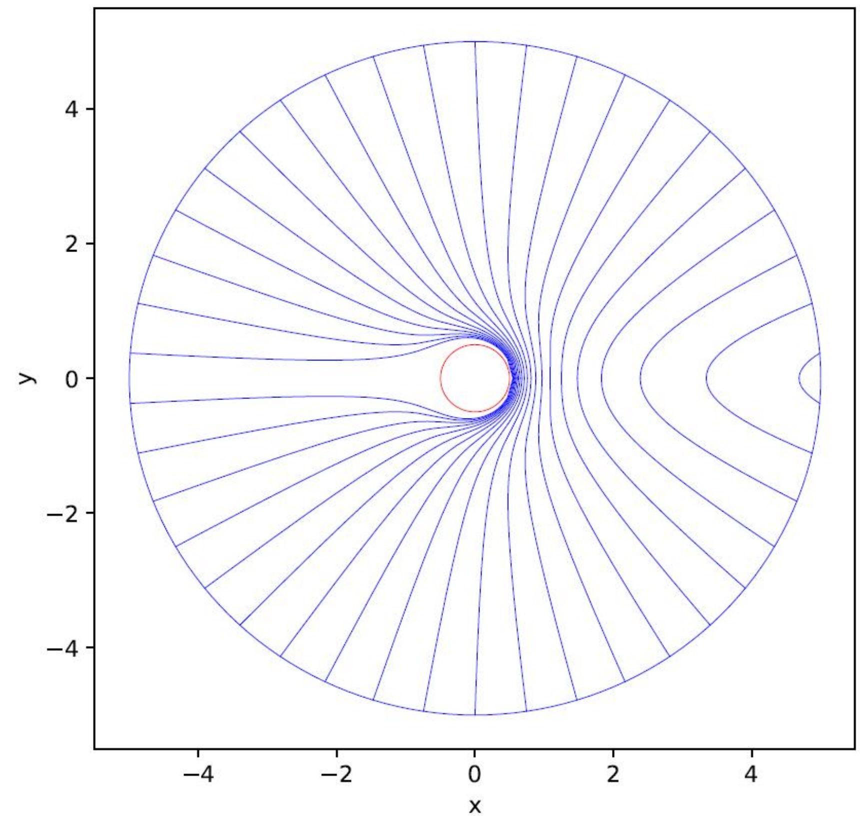
$$ds^2 = -\frac{r^2}{L^2}f(r)dt^2 + \frac{L^2}{r^2 f(r)}dr^2 + r^2 d\phi^2,$$

EoM of bulk geodesics is

$$\pm(\phi - \phi_0) = -\frac{L}{r_h}\ln\left(\sqrt{1-\frac{r_h^2}{r^2}} - \sqrt{\frac{r_h^2}{r_*^2} - \frac{r_h^2}{r^2}}\right) + \frac{L}{r_h}\ln\sqrt{1-\frac{r_h^2}{r_*^2}},$$

where $r_* < r_1 < r_2$ and $\phi(r_1) < \phi_0 < \phi(r_2)$. Also, for given angular size of the subsystem, different radial cutoff corresponds to different turning position.



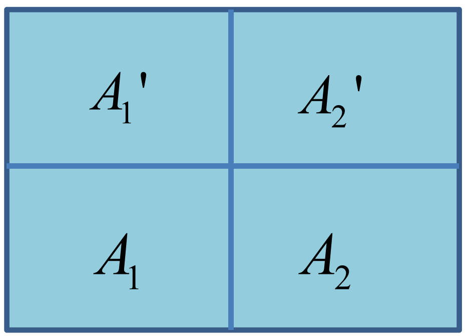


homogenous subregions, with each subregion has angle $\phi = \pi / 25$ cutoff surface is r_c=5, the growing steps are 1500.

minimal surfaces wich do not surround the black hole horizon, entanglement plateaux phenomenon.

Surface growth, bit thread and EoP

Dividing a mixed quantum system into two parts, a quantity used to describe correlations between A_1 and A_2 is called the **entanglement of purification** (EoP) $E_P(A_1 : A_2)$



Let $|\psi\rangle \in H_{A_1A_1'} \otimes H_{A_2A_2'}$ be a purification of the density matrix

$$\rho_{A_1A_2} = \mathrm{Tr}_{A_1'A_2'}|\psi\rangle\langle\psi|$$

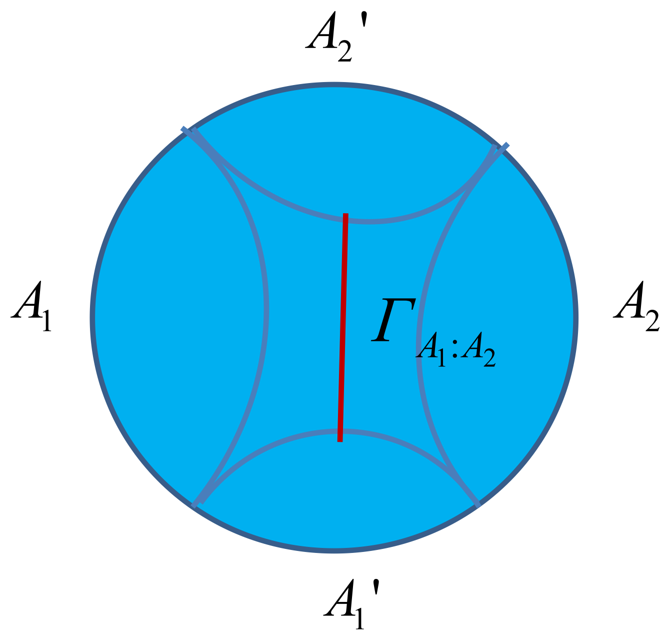
The EoP is defined as

$$E_P(A_1 : A_2) = \min_{|\psi\rangle_{A_1A_1'A_2A_2'}} S(A_1A_1'),$$

A holographic dual of EoP is

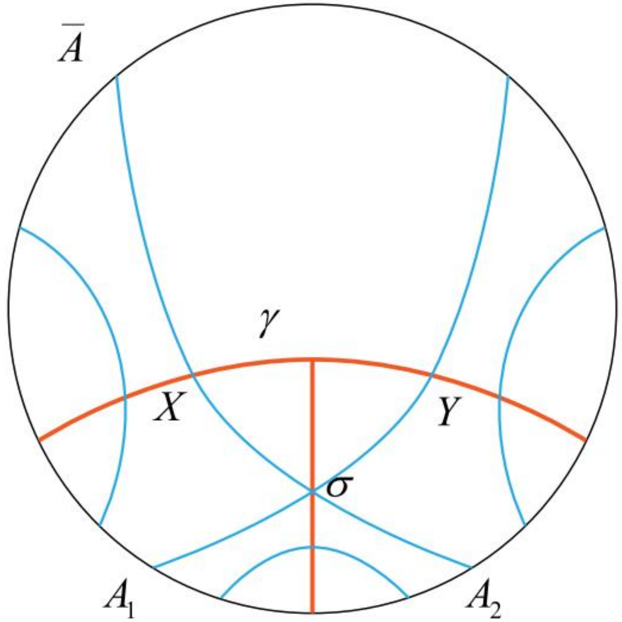
Takayanagi, Umemoto, 2018

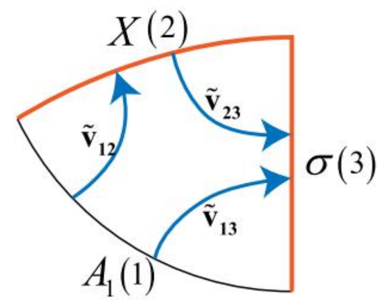
$$E_P(A_1 : A_2) = \mathrm{Area}(\Gamma_{A_1:A_2}).$$ area of EWCS



The holographic EoP gives more refined description of entanglement in the entanglement wedge, and it can be natually regarded as a surface growth process.

EoP from surface growth → OSED tensor network → new bit thread description of EoP





$$\text{Area}(\sigma) = S(XA_1),$$

$$\rho(\tilde{\mathbf{v}}_{12}) = \rho(\vec{v}_{\bar{A}XA_1}),$$

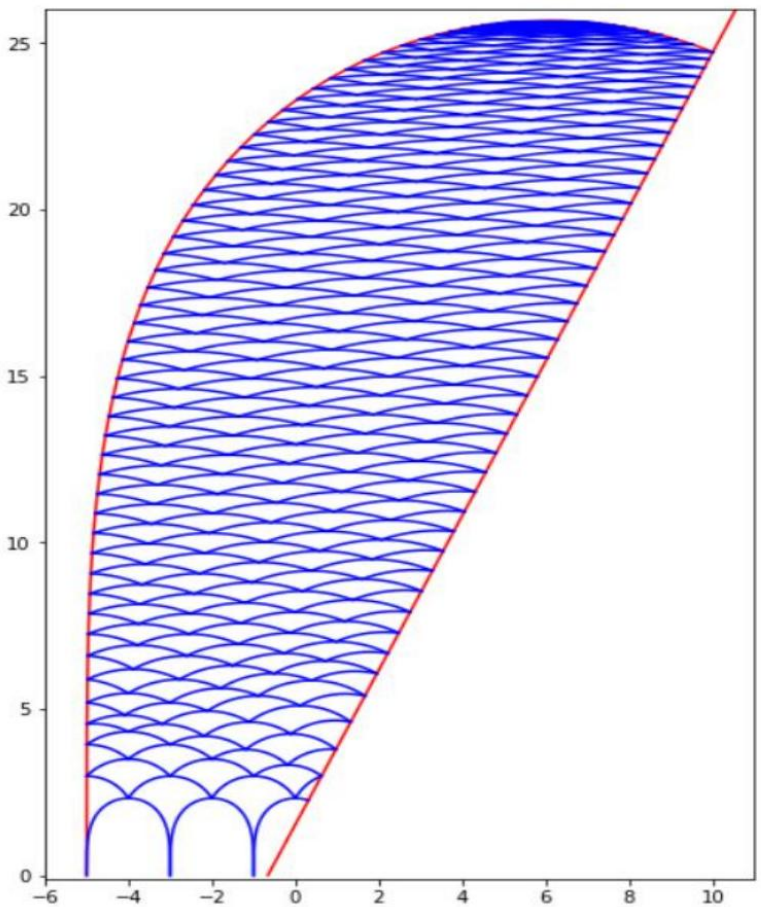
$$\rho(\tilde{\mathbf{v}}_{\mathbf{13}}) = \rho(\vec{v}_{\bar{A}Y\sigma A_1}) + \rho(\vec{v}_{A_1\sigma A_2}),$$

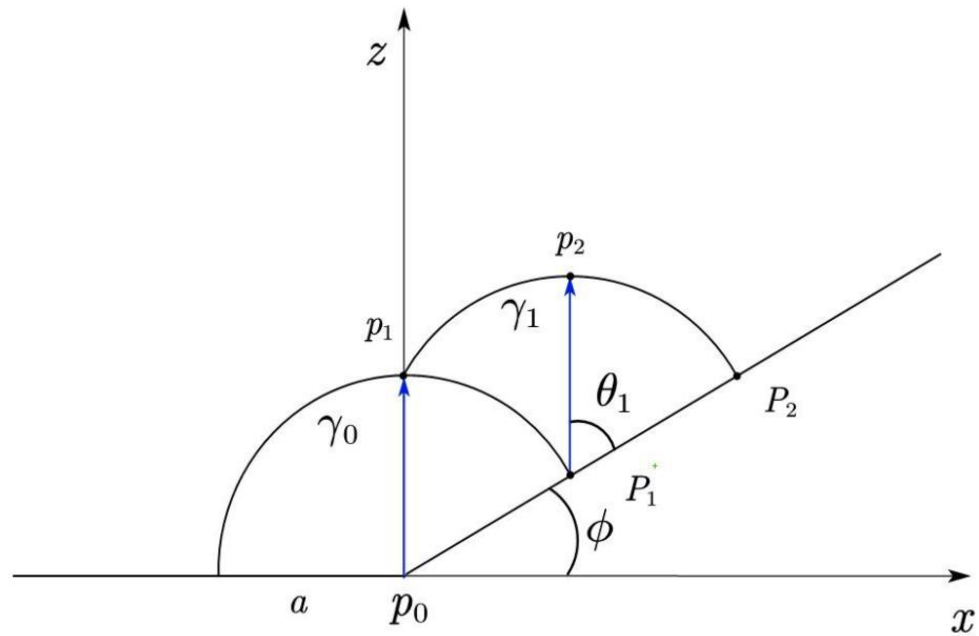
$$\rho(\tilde{\mathbf{v}}_{\mathbf{23}}) = \rho(\vec{v}_{\bar{A}X\sigma A_2}),$$ which gives $$S(2) + S(3) - S(1) = 2F(2)_{23}$$

EoP in surface growth in AdS/BCFT

X. Fang, F.-Z. Chen, **JRS**, 2403.12086

The EoP indicates a selection rule for surface growth in AdS/BCFT; gives more refined description for the entanglement wedge.





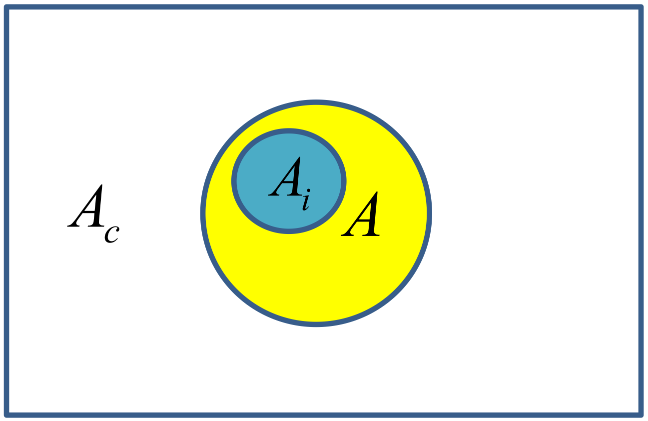
$$\cosh\frac{D_{p_0p_1}}{L} = 1 + \frac{r^2\cos^2(\theta+\phi) + (r\sin(\theta+\phi)-\epsilon)^2}{2\epsilon r\sin(\theta+\phi)},$$

then $\frac{d}{d\theta}\cosh\frac{D_{p_0p_1}}{L} = 0$ gives $\cos(\theta+\phi)=0$, i.e. $\theta+\phi=\frac{\pi}{2}$

Entanglement contour, fine structure

Chen, Vidal, 1406.1471

Entanglement contour (EC): a local function $s_A(x)$ trying to describe the **fine structure** the entanglement entropy in real space



$$S_A = \int_A s_A(x) dx$$

Explicit examples of EC: Gaussian states, CFT, partial entanglement entropy (PEE) , e.g., $s_A(A_i)$ of some subsystem of A

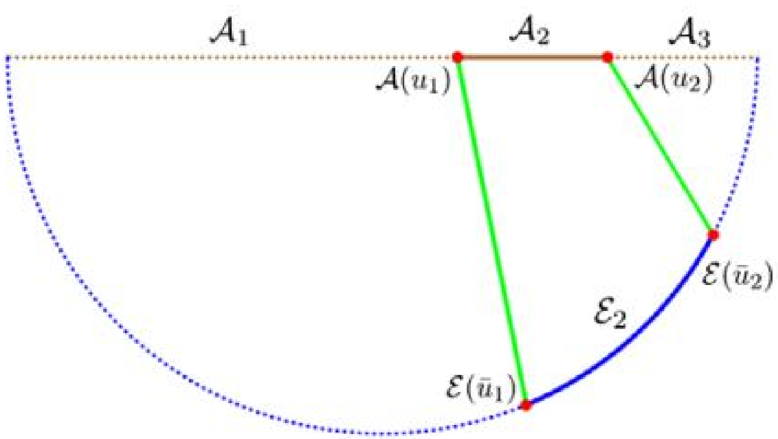
$$s_A(A_i) = \int_{A_i} s_A(x) dx$$

However, the bove requirements are not sufficient to uniquely determine the PEE in general.

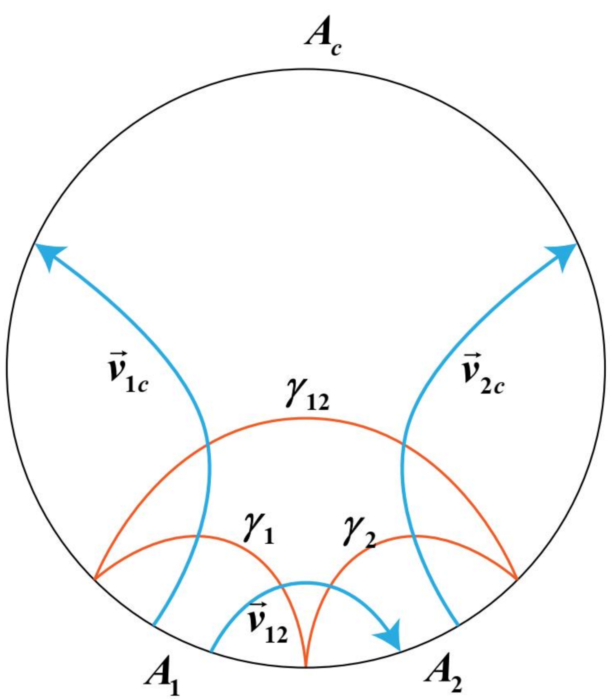
PEE proposal Q Wen, 1902.06905, 1803.05552; Kudler-Flam, MacCormack, Ryu, 1902.04654

$$s_A(A_2) = \frac{1}{2}\left(S_{12} + S_{23} - S_1 - S_3\right)$$

Holographically, PEE can be described by combination of **extremal surfaces** and the **bit threads**.



Q Wen, 1803.05552



Y-y Lin, **JRS**, J Zhang, 2105.09176

Hyperfine Rényi entropy

L H Mo, Y Zhou, **JRS**, P Ye, 2311.01997

A natural question is to ask what is the entanglement contour for the Rényi entropy?

We introduced a hyperfine structure for entanglement by exactly decomposing the Rényi contour into the contributions from particle-number cumulants in free fermion system.

$$\begin{array}{ccccc} S(A) & \Longrightarrow & s(i) & \Longrightarrow & h_{1;k}(i) \\ \uparrow n=1 & & \uparrow n=1 & & \uparrow n=1 \\ S_n(A)(\tilde{S}_n(A)) & \Longrightarrow & s_n(i)(\tilde{s}_n(i)) & \Longrightarrow & h_{n;k}(i)(\tilde{h}_{n;k}(i)) \end{array}$$

$$s_n(j) \equiv \sum_{k=1}^{\infty} h_{n;k}(j) = \sum_{k=1}^{\infty} \Big(\beta_k(n) C_k(j) \Big),$$

$$\beta_k(n) = \frac{2}{n-1}\frac{1}{k!}\left(\frac{2\pi i}{n}\right)^k \zeta\left(-k, \frac{n+1}{2}\right)$$ is nonzero for even *k*,

$C_k(j)$ is the density of cumulant on site *j*, it is a 2*k*-point function

$$C_k(j) \equiv (-i\partial_\lambda)^{k-1} \frac{\langle \exp(i\lambda \hat{N}_A)\hat{n}_j\rangle}{\langle \exp(i\lambda \hat{N}_A)\rangle}\Big|_{\lambda=0},$$

$\hat{n}_j$ is particle number operator on site *j*, and $\hat{N}_A$ is the particle number operator of *A*. λ is a real number.

The first nonzero term is

$$C_2(j) = \sum_i \langle \hat{n}_i \hat{n}_j\rangle - \langle \hat{n}_i\rangle\langle \hat{n}_j\rangle$$

Properties:

- additivity $h_{n;k}(i) + h_{n;k}(j) = h_{n;k}(i \cup j)$
- normalization $S_{n;k} = \beta_n(k) C_k$
- exchange symmetry $h_{n;k}(i) = h_{n;k}(j)$
- invariance under local unitary transformation
- post-measurement state entanglement

Application in lattice fermion model

We consider a Chern insulator model called Qi-Wu-Zhang model

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}} H(\mathbf{k}) \hat{c}_{\mathbf{k}},$$

$$H(\mathbf{k}) = (m + \cos k_x + \cos k_y)\sigma_z + \lambda(\sin k_x \sigma_x + \sin k_y \sigma_y)$$

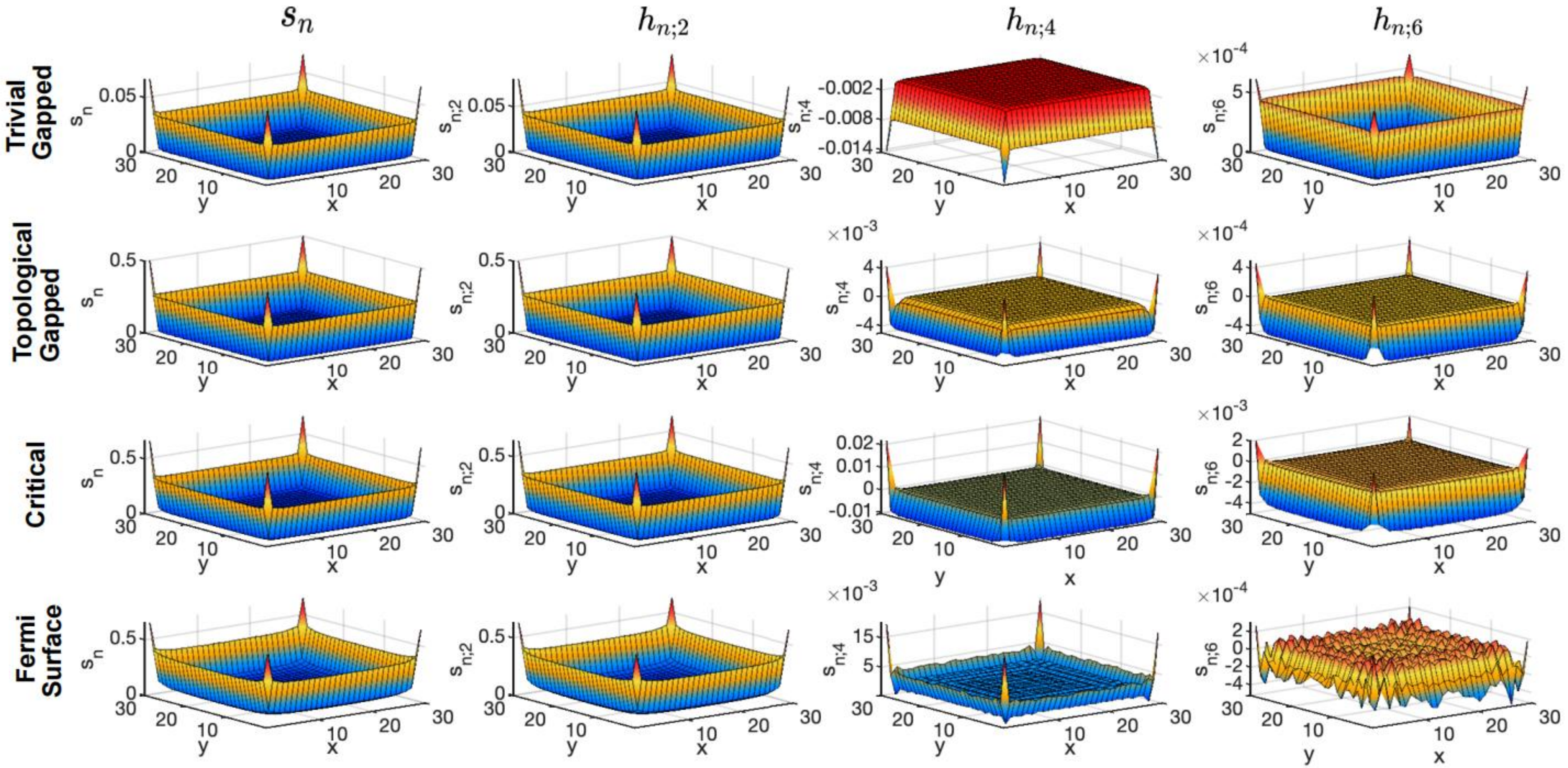
The energy gap closes at $m = \pm 2,$ forming Dirac point at

$$k_x = k_y = 0 \text{ and } k_x = k_y = \pi.$$

The energy gap closes at $m = 0,$ forming Dirac point at

$$k_x = 0, k_y = \pi \text{ and } \quad k_x = \pi, k_y = 0.$$

The topological properties of the electronic band structure are characterized by the Chern number.

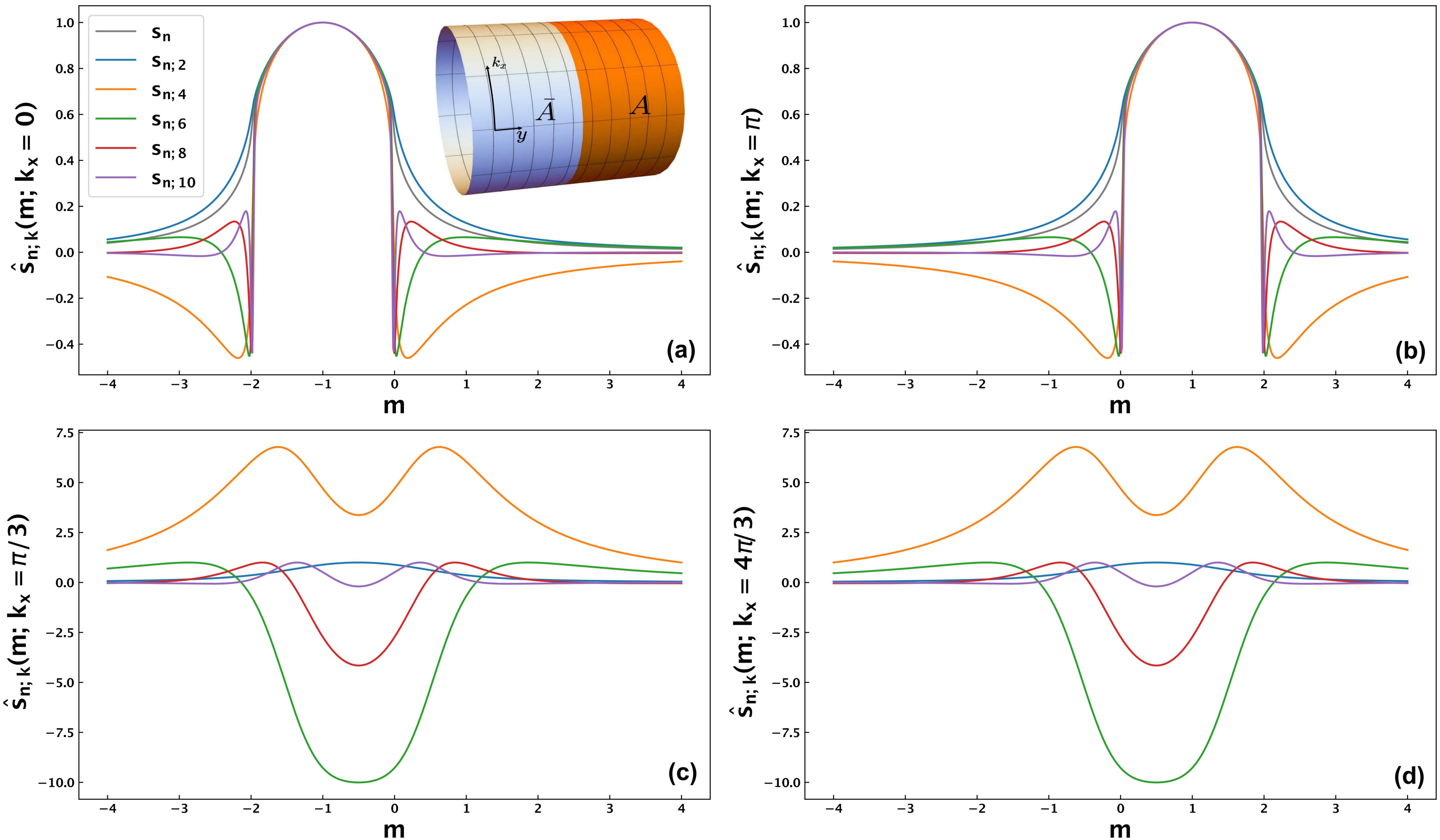


Distributions of RC and hyperfine RC for different energy spectrum:

k=2: dominant contribution; k>2

(trivial)gapped: ‘bowl’ shape; critical: corner vs hinge sites

Fermi surface: oscillation; topological: same as critical



$$\hat{h}_{n;k}(m;k_x) \equiv h_{n;k}(m;k_x) / \max h_{n;k}(m;k_x)$$

$k_x = 0, k_x = \pi$ emergence of scaling law

other momentum: randomly distrubted.

The emergence of scaling law within topological gap regions $m \in (-2,0) \cup (0,2)$, strongly suggest the existence of critical edge states and a fundamental 1/2 mode in the entanglement spectrum, which is the most entangled and correlated mode --**boundary EPR pair**.

The distribution properties of **hyperfine RC highlight the different features** of a mass gap, a critical Dirac cone, and a Fermi surface, and they **reveal an universal scaling behavior in the presence of topological edge states**.

Holographic realization of RC

Consider the boundary CFT is a *d*+1 dim Fermi gas,

$$\frac{S_n}{C_2} = \frac{(1+n^{-1})\pi^2}{6} + o(1)$$

then the hyperfine RC can be simplified as

$$\frac{s_n(x)}{h_{n;2}(x)} = 1 + o(1),$$

where

$$h_{n;2}(x) = \beta_k(n) \int_{y\in A} [\langle \hat{n}(x)\hat{n}(y)\rangle - \langle \hat{n}(x)\rangle\langle \hat{n}(y)\rangle] dy.$$

$$I_2(A_1, A_2) = 2\left(S(A_1) - \int_{x\in A_1} h_{1;2}(x) dx \right),$$

for Dirac fermion

$$
\begin{aligned}
&\langle \hat{n}(x)\hat{n}(y)\rangle - \langle \hat{n}(x)\rangle\langle \hat{n}(y)\rangle \\
&= - \langle \psi_R^\dagger(x)\psi_R(y)\rangle\langle \psi_R^\dagger(y)\psi_R(x)\rangle - \langle \psi_L^\dagger(x)\psi_L(y)\rangle\langle \psi_L^\dagger(y)\psi_L(y)\rangle \\
&= \frac{1}{2\pi^2}\frac{1}{(x-y)^2}.
\end{aligned}
$$

the dominant hyperfine structure is expressed as

$$
\begin{aligned}
h_{n;2}(x) &= \frac{(1+n^{-1})\pi^2}{6}\int_{-R+\epsilon}^{R-\epsilon}\frac{1}{2\pi^2}\frac{1}{(x-y)^2}dy \\
&= \frac{(1+n^{-1})}{12}\left(\frac{1}{R-x}+\frac{1}{R+x}\right).
\end{aligned}
$$

$$
S_n = \int_{-R+\epsilon}^{R-\epsilon} dx s_n(x) \approx \left[\left(1+n^{-1}\right)/6\right]\ln\frac{2R}{\epsilon}.
$$

which gives the central charge c=1.

To find the holographic duality of the hyperfine RC, it's convenient to use the **refined Rényi entropy**

$$\widetilde{S}_n \equiv n^2 \partial_n \left(\frac{(n-1)}{n} S_n \right)$$

which is dual to the cosmic brane in AdS spacetime, and the tension of the brane is $T_n = (n-1)/(4nG)$

Dong, 1601.06788

Interesting, **the refined Rényi entropy is equivalent to the von Neumann entropy of a new density matrix** $\widetilde{\rho}_A = \hat{\rho}_A^n / \mathrm{tr}\, \hat{\rho}_A^n$

$$\begin{aligned}
\tilde{S}_n(\hat{\rho}_A) &= -n^2 \partial_n (\frac{1}{n} \log \mathrm{tr}\, \hat{\rho}_A^n) \\
&= \log \mathrm{tr}\, \hat{\rho}_A^n - n\partial_n \log \mathrm{tr}\, \hat{\rho}_A^n \\
&= -\,\mathrm{tr} \left(\frac{\hat{\rho}_A^n}{\mathrm{tr}\, \hat{\rho}_A^n} \right) \log \left(\frac{\hat{\rho}_A^n}{\mathrm{tr}\, \hat{\rho}_A^n} \right) \\
&= S \left(\frac{\hat{\rho}_A^n}{\alpha_n} \right)
\end{aligned}$$

Then we obtain **the refined Rényi contour**

$$\widetilde{s}_n(x) = n^2 \partial_n \left(\frac{(n-1)}{n} s_n(x) \right)$$

Furthermore, using the entanglement Hamiltonian, **the refined Rényi contour** can be expressed from the particle number fluctuation

$$\hat{\rho}_A = \sum_p a_p^2 |\psi_A^p\rangle\langle\psi_A^p| \; = e^{-\hat{K}_A}$$

Entanglement Hamiltonian: $\hat{K}_A = \sum_p - \ln a_p^2 |\psi_A^p\rangle\langle\psi_A^p|$

$$\tilde{\rho}_A^{(n)} = \sum_p \frac{a_p^{2n}}{\alpha_n} |\psi_A^p\rangle\langle\psi_A^p| \; = \frac{e^{-K_A^{(n)}}}{Z^{(n)}}$$

Entanglement Hamiltonian: $\hat{K}_A^{(n)} = n\hat{K}_A$ with $T = \dfrac{1}{n}$.

Then we obtain

$$\tilde{s}_n(x) \approx \tilde{h}_{n;2}(x) = \frac{\pi^2}{3n} \int_{y \in A} dy [\langle \hat{n}(x) \hat{n}(y) \rangle - \langle \hat{n}(x) \rangle \langle \hat{n}(y) \rangle].$$

Now using the HEE, the holographic duality for **Rényi entropy** is just the bulk extremal surface (RT surface) for **the refined Rényi entropy.**

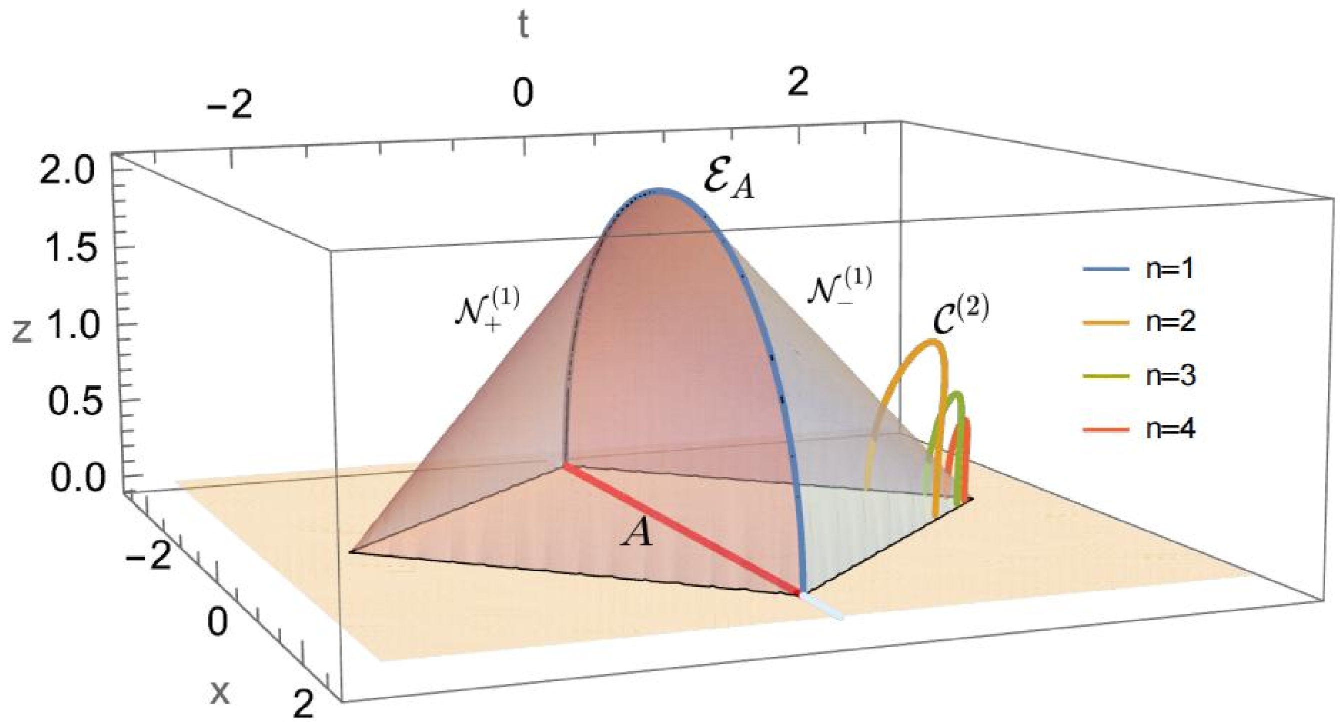
For AdS_3 case, using the Rindler method to map the extremal surface of subregion *A* to the horizon area entropy

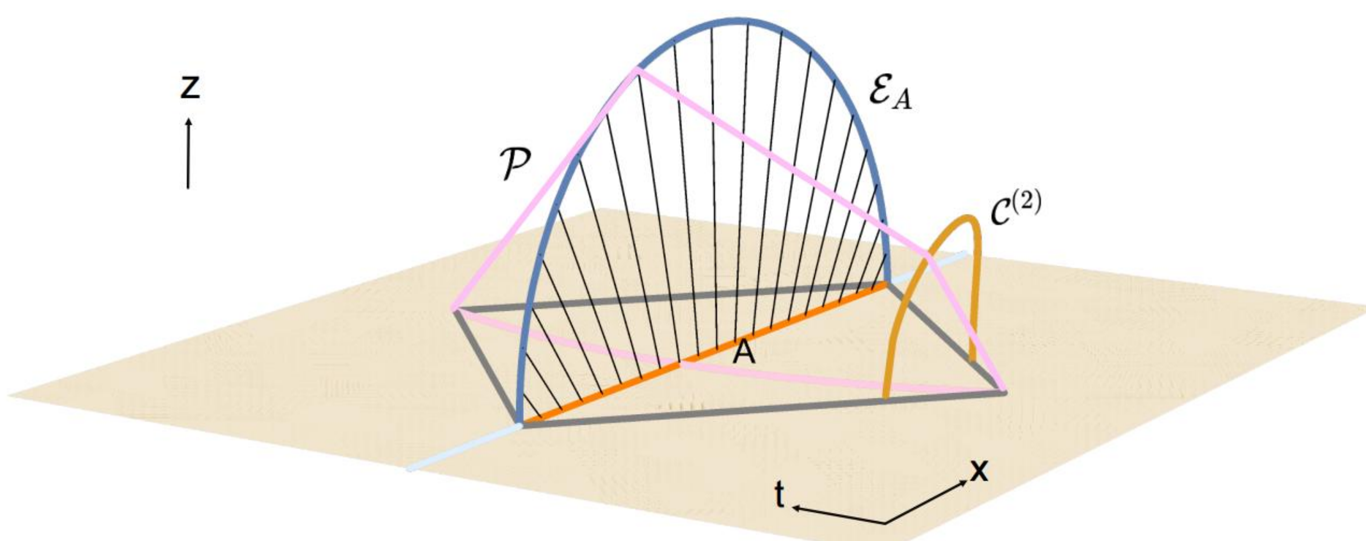
$$ds^2 = 2rdudv + \frac{dr^2}{4r^2}.$$

$$ds^2 = du^{*2} + r^{*2} du^* dv^* + dv^{*2} + \frac{dr^{*2}}{4(r^{*2} - 1)},$$

$$\mathcal{N}_+^{(n)}: \quad r = \frac{-2n^2}{l_u^2(n^2-2) + 4n^2uv + 2l_u\sqrt{l_u^2(1-n^2) - 4n^2uv + n^4(u+v)^2}}$$

$$\mathcal{N}_-^{(n)}: \quad r = \frac{-2n^2}{l_u^2(n^2-2) + 4n^2uv + 2l_u\sqrt{l_u^2(1-n^2) - 4n^2uv + n^4(u+v)^2}}$$





As n increases, $\mathcal{C}^{(n)}$ decreases.

For n>1, $\mathcal{C}^{(n)}$ are outside the entanglement wedge of the n=1 extremal surface (EE), which indicates the Rényi entanglement wedge can probe more information of the bulk spacetime.

Conclusions

*The surface growth approach provides an efficient and refined way to build the bulk geometry in the entanglement wedge far away from the boundary;

*By combining the surface growth approach and the bit threads, we give a new and more reasonable bit thread description for the holographic EoP;

*EoP gives more refined description in surface growth, and also provides a selection rule for it;

*We derive the hyperfine structure of Rényi contour from particle number cumulants for free fermions;

*The hyperfine Rényi contour shows many interesting features: such as can be used to characterized the topological edge states;

*The holographic duality of the hyperfine Rényi contour, the Rényi entanglement wedge give new tool to study the bulk reconstruction and more refined description for subregion-subregion duality;

*The connection between surface growth approach and the entanglement contour description requires further study.

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謝謝大家！