Dynamics of Entanglement Entropy in CFT_2: Einstein Equation

Haitang Yang

College of Physics, Sichuan University

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Joint work with Xin Jiang(蒋昕), Houwen Wu(伍厚文) and Peng Wang(王鹏).

Based on arXiv:2406.09033 (mixed state EE), arXiv:2410.19711 (Einstein equation from CFT_2) and arXiv:2411.18485 (ER=EPR)

Line of reasoning

Purpose: Demonstrate how Einstein equation arises in CFT_2.

Strategy:

1. Metric $\Leftrightarrow$ Geodesic. They can be derived from each other.
2. (Metric version) Einstein equation $\Rightarrow$ Geodesic version Einstein equation.
3. (Mixed state) RT formula: Geodesic $=$ Entanglement Entropy

   If both quantities are finite, thereby making this equality exact, we would have:
4. Geodesic version Einstein equation $\Rightarrow$ Entanglement Entropy Einstein equation.

Metric $\Leftrightarrow$ Geodesic

Given a metric, we know how to compute the geodesic distance between two points. On the other hand, the metric can also be extracted from geodesics. Consider a geodesic $x(\tau), \tau \in [0, t]$ connecting two points $x(\tau = 0)$ and $x'(\tau = t)$. $L(x, x')$ is the geodesic length. Recall geodesics are defined as curves whose tangent vector is parallel transported

$$\nabla_\tau \left(\sqrt{g_{ij} \dot{X}^i \dot{X}^j} \right) = \frac{d}{d\tau} \sqrt{g_{ij} \dot{X}^i \dot{X}^j} = 0.$$

The Synge's world function is a biscalar of x and x', defined as,

$$\begin{aligned} \sigma(x, x') \quad &:= \quad \frac{1}{2} L^2(x, x') = \frac{1}{2} \left[\int_0^t d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \right]^2 \\ &= \quad \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \bigg|_{\text{any point}} t^2 \\ &\xrightarrow{t \to 0} \quad \frac{1}{2} g_{\mu\nu} \Delta X^\mu \Delta X^\nu \end{aligned}$$

So, the metric is

$$g_{\mu\nu} = -[\sigma_{\mu\nu'}] := -\lim_{x' \to x} \partial_{x'^\nu} \partial_{x^\mu} \sigma(x, x'),$$

or

$$g_{\mu\nu} = [\sigma_{\mu\nu}] := \lim_{x' \to x} \nabla_{x^\mu} \partial_{x^\nu} \sigma(x, x').$$

where we defined the coincidence limit $[\Omega] := \lim_{x' \to x} \Omega(x, x')$ for any bitensor $\Omega(x, x')$.

Metric $\Leftrightarrow$ Geodesic

$$g_{\mu\nu} = -[\sigma_{\mu\nu'}] := -\lim_{x'\to x} \partial_{x'^\nu}\partial_{x^\mu}\sigma(x,x'),$$

or

$$g_{\mu\nu} = [\sigma_{\mu\nu}] := \lim_{x'\to x} \nabla_{x^\mu}\partial_{x^\nu}\sigma(x,x').$$

Note $\sigma_\mu := \partial_{x^\mu}\sigma$ is a vector with respect to point x, but is a scalar for point x'. So, $\sigma_{\mu\nu} := \nabla_\nu\partial_\mu\sigma$ involves covariant derivatives, whereas $\sigma_{\mu\nu'} := \partial_{\nu'}\partial_\mu\sigma$ does not. We thus conclude

Metric $\Leftrightarrow$ Geodesic

Note: the geodesics should be arbitrary and the ends are free to move in the bulk.

Metric $\Leftrightarrow$ Geodesic

Note raising or lowering indices and taking the limit commute. Some identities for the Synge's world function

- $$\sigma_\mu \sigma^\mu = 2\sigma.$$
- $$g_{\mu\nu} = [\sigma_{\mu\nu}] = -[\sigma_{\mu\nu'}]$$
  .
- $$[\sigma_{\alpha\mu\beta\nu}] = \frac{1}{3}\Big(R_{\alpha\nu\beta\mu} + R_{\alpha\beta\nu\mu}\Big).$$
- $$\Sigma_{\mu\nu} := [\sigma^{\alpha\beta}\,\sigma_{\alpha\mu\beta\nu}] = g^{\alpha\beta}[\sigma_{\alpha\mu\beta\nu}] = \frac{1}{3}R_{\mu\nu}.$$
- $$\Sigma := [\sigma^{\mu\nu}\sigma^{\alpha\beta}\,\sigma_{\alpha\mu\beta\nu}] = g^{\mu\nu}\Sigma_{\mu\nu} = \frac{1}{3}R.$$

Geodesic version Einstein equation

With respect to metric, we have Einstein action and equation

$$S_E = \frac{1}{16\pi G} \int d^D x \sqrt{-g}(R - 2\Lambda),$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{1}$$

where $D = d + 1$ is the spacetime dimension. We only address vacuum with $\Lambda \neq 0$. Thus contracting $g^{\mu\nu}$ with eqn. (1), we find $R = \frac{2D}{D-2}\Lambda$ and the Einstein equation can also be put into

$$R_{\mu\nu} - \frac{1}{D} g_{\mu\nu} R = 0.$$

So, the geodesic version Einstein equation is

$$\Sigma_{\mu\nu} - \frac{1}{D} [\sigma_{\mu\nu}] \Sigma = 0,$$

taking the identical form as the metric Einstein equation.

Line of reasoning

We now entering the trickiest part, RT formula:

Geodesic = Entanglement entropy

Once this relations is built, we can use the geodesic version Einstein equation

$$\Sigma_{\mu\nu} - \frac{1}{D}[\sigma_{\mu\nu}]\,\Sigma = 0,$$

to get the dynamical equation of Entanglement Entropy.

Entanglement in QM

Entanglement in quantum mechanics, for example, the triplet and singlet:

$$
\begin{cases}
|11\rangle = & |\uparrow\uparrow\rangle, & \text{No entangling,} \\
|1-1\rangle = & |\downarrow\downarrow\rangle, & \text{No entangling,} \\
|10\rangle = & \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big) & \text{Entangling,} \\
|00\rangle = & \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big) & \text{Entangling}
\end{cases}
$$

Define density matrix

$$\rho_{\rm tot} = |\Psi\rangle\langle\Psi|, \quad \rho_1 = {\rm Tr}_2\, \rho_{\rm tot}, \quad \text{and} \quad S_1 = -{\rm Tr}\rho_1 \log \rho_1, \quad S_2 = S_1.$$

<u>No entangling state $|11\rangle$:</u>

$$\rho_{\rm tot} = |11\rangle\langle 11|, \quad \rho_1 = {\rm Tr}_2\, \rho_{\rm tot} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad S_1 = -{\rm Tr}\rho_1 \log \rho_1 = 0.$$

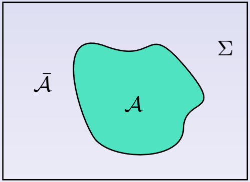
<u>Entangling singlet $|00\rangle$:</u>

$$\rho_{\rm tot} = |00\rangle\langle 00|, \quad \rho_1 = {\rm Tr}_2\, \rho_{\rm tot} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad S_1 = -{\rm Tr}\rho_1 \log \rho_1 = \log 2.$$

Entanglement in $D = d + 1$ QFT, taking a time-slice $\Sigma = \mathcal{A} \cup \bar{\mathcal{A}}$, subregion $\mathcal{A}$ entangles with $\bar{\mathcal{A}}$,

$d = 1$ system

$d = 2$ system



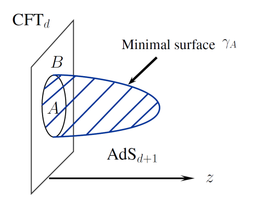
The entanglement entropy (EE) S_A is defined as

$$\rho_{\mathrm{tot}} = \frac{1}{Z} |\Psi\rangle\langle\Psi|, \qquad \rho_A = \mathrm{Tr}_{\bar{A}} \rho_{\mathrm{tot}}, \qquad S_A = -\mathrm{Tr} \rho_A \log \rho_A,$$

Holography

The holographic principle states that the data of gravity are stored in its lower dimensional boundary which looks like a holographic image.

Inspired by AdS/CFT and Bekenstein-Hawking entropy, Ryu and Takayanagi (2006,PRL) proposed an equivalence between the entanglement entropy (EE) of CFT$_D$ and the minimal surface area in AdS$_{D+1}$,



$$S_A = \frac{\text{Area}\,(\gamma_A)}{4G_N^{D+1}}.$$

This RT formula was for pure state in the original proposal.

General relativity: Gravitational force is an illusion of spacetime curvature.

It from qubit: spacetime structure emerges from quantum entanglements.

RT formula for pure state

As $D = 2$, we have $\mathrm{AdS}_3/\mathrm{CFT}_2$, the minimal surfaces are geodesics. The simplest configuration is the plain CFT_2/Poincare AdS_3,

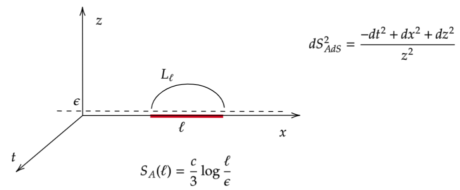


Figure: ϵ is the UV cutoff, $\ell >> \epsilon$ is the entangling segment.

On a time slice, for an interval $\Delta x = \ell >> \epsilon = z_1 = z_2$,

$$L_\ell = \frac{c}{6} \cosh^{-1} \frac{\Delta x^2 + z_1^2 + z_2^2}{2 z_1 z_2} \simeq \frac{c}{3} \log \frac{\ell}{\epsilon} = S_A(\ell)$$

Q: Can we apply "Geodesic = EE" to get the EOM of EE now? — Negative!
The reasons are:

- The geodesics are attached on the boundary — one special class.
- Both sides of the equality are divergent — too loose to make assertions.

Mixed states

- The geodesics are attached on the boundary — one special class.
- Both sides of the equality are divergent — too loose to make assertions.

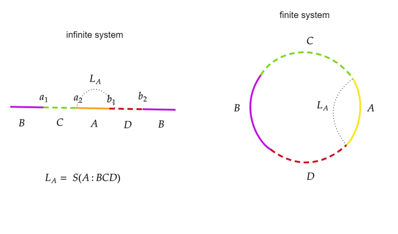
The two problems are really just one. Imagine, if the geodesics are not constrained to the boundary, i.e. the end points are free to move in the bulk, we certainly have arbitrary geodesics and the equality should be exact.

This is what we need!

To this end, let us turn to mixed states, which have the pure states as limits.

Mixed states: introduction

Pictorially, a bipartite mixed state is



Q: What are the entanglement between disconnected segments A and B, i.e. $S(A:B)$ and its holographic dual?

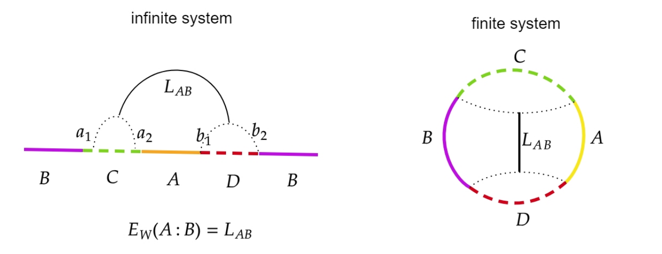
1. As $C, D \to 0$, we recover the pure state.
2. Since A and B are not complementary, ρ_{AB} is not pure, but mixed.
3. In order to calculate $S(A:B)$, we need to make A and B complementary somehow, and then construct a pure state ψ_{AB}.
4. Though only bipartite is concerned, the following derivations are generic since general multipartites are composed of bipartites.

Mixed states: EWCS, the holographic dual

Takayanagi and Umemoto (nature phys 2017) proposed:

$$S_{vN}(A:B) = E_W(A:B),$$

where $E_W(A:B)$ is the Entanglement wedge cross section (EWCS) in the bulk,

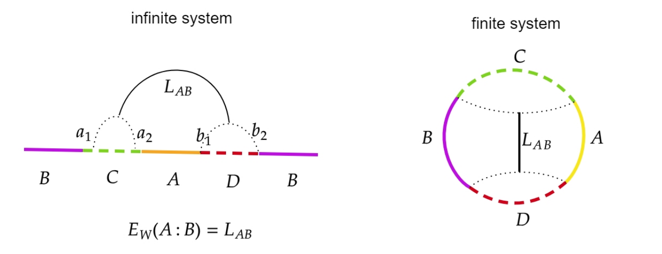


EWCS $E_W(A:B)$ for $A:B$ is obtained by minimizing the length of L_{AB}, turning out to be:

$$E_W(A:B) = \frac{c}{6}\log\left[1+\frac{2}{z}+2\sqrt{\frac{1}{z}(\frac{1}{z}+1)}\right], \quad z = \frac{(a_2-a_1)(b_2-b_1)}{(b_1-a_2)(b_2-a_1)}.$$

This geodesic with finite length, and free moving ends, is precisely what we need!

Mixed states: two approaches to calculate $S(A:B)$



There are two approaches to make A and B complementary and then calculate $S_{vN}(A:B)$:

1. Purification (Takayanagi and Umemoto:2017 Nature Phys): Replace unknown C and D by arbitrary states $\bar{A}$ and $\bar{B}$, define $S(A:B) := \min_{\bar{A}\bar{B}} S(A\bar{A}:B\bar{B})$ optimizing over $\bar{A}$ and $\bar{B}$. — very hard to do in practice.
   Reflected entropy S_R (Dutta and Faulkner:2019JHEP) was introduced by choosing $\bar{A}, \bar{B}$ as CPT of A, B respectively. They proved $S_R = 2E_W$.
2. Subtraction (Jiang, Wu, Wang, Yang:2406.09033): Simply subtract unknown C and D from the system in a covariant way. Yes, we confirmed $S_{vN}(A:B) = E_W(A:B)$, it is an exact equality, both sides are finite.

Our subtraction approach: calculation

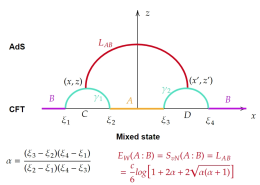


Figure: Bipartite mixed state in CFT_2 on a time slice. Geodesic γ_1 is completely fixed by ξ_1 and ξ_2, geodesic γ_2 is completely fixed by ξ_3 and ξ_4. From the Ultra-parallel theorem in hyperbolic geometry, as the shortest curve connecting γ_1 and γ_2, L_{AB} is a geodesic and unique. (x, z) and (x', z') are completely fixed by all four ξ_i's. The EWCS of A, B is defined as L_{AB}, which equals the CFT entanglement entropy between segments A and B, i.e. $S_{\mathrm{vN}}(A : B) = E_W(A : B) = L_{AB}$.

Thus we now have an exact equality with both sides finite

$$S_{\mathrm{vN}}(A:B) = \frac{c}{6}\log\left[1 + \frac{2}{z} + 2\sqrt{\frac{1}{z}(\frac{1}{z}+1)}\right] = E_W(A:B),$$

In the limit $\ell_A = \xi_3 - \xi_2 = \ell$ and $\ell_C = \ell_D = \epsilon \to 0$, this entanglement entropy simplifies to the pure state infinite one

$$S_{\mathrm{vN}}(A:B) = \frac{c}{3}\log\frac{\ell}{\epsilon}.$$

So, we now have

$$S_{\mathsf{vN}}(A:B) = E_W(A:B) = L_{AB}(x(\xi_i), z(\xi_i); x'(\xi_i), z'(\xi_i)),$$

and mixed states are the general configurations to quantify entanglement. Defining

$$\chi := \frac{1}{2} S_{\rm vN}^2, \quad g_{ij} := [\chi_{ij}], \quad R_{ij} := \left[\chi^{k\ell}\chi_{ki\ell j}\right], \quad R := [\chi^{ij} R_{ij}], \tag{2}$$

where the derivatives $i, j \cdots$ are with respect to four ξ_i's on the boundary, we are ready to obtain the dynamic equation of CFT$_2$ entanglement entropy

$$R_{ij} - \frac{1}{D} g_{ij} R = 0, \tag{3}$$

which is precisely the Einstein equation! Since all the quantities and parameters in this equation belong to CFT, it is indeed a CFT dynamical equation for the entanglement entropy.

Dynamics of entanglement entropy: Einstein equation

We can make the cosmological constant explicit in the equation,

$$R_{ij} - \frac{1}{2} g_{ij} R + \frac{\Lambda}{3} g_{ij} = 0,$$

from which a relation between the cosmological constant and CFT$_2$ entropy is found

$$\Lambda = \frac{3(D-2)}{2D} R \xrightarrow{D=3} \Lambda = \frac{1}{2} R = \frac{1}{2} \left[\chi^{ij} \chi^{k\ell} \chi_{ki\ell j} \right].$$

So, the cosmological constant is not really a free parameter, but determined by the CFT entanglement entropy.

Dynamics of entanglement entropy: Einstein equation

Now, given an EE, extracting the metric only becomes a trivial procedure:

$$g_{ij} = -[\chi_{ij'}] = -\left[\left(\frac{1}{2}S_{\mathrm{vN}}^2\right)_{ij'}\right],$$

where i, j' are ordinary derivatives since they act on different ends. This had been a very hard job from the pure state EE previously.

We see the metric is a derived but not a fundamental quantity. Perhaps this is why gravity is non-renormalizable and we should not try to quantize the metric itself.

Moreover, since the ordinary QFT and the induced gravity are linked through entanglement entropy, it is now possible to unify the black hole thermodynamics and Einstein equation.

RG equation as a geometric identity

For a classically scale-invariant theory living on a D-dimensional manifold $\mathcal{M}$ with the metric $ds^2 = \gamma_{ab} dx^a dx^b$, the Callan-Symanzik RG equation is:

$$\left[\ell \frac{\partial}{\partial \ell} - 2 \int_{\mathcal{M}} \gamma^{ab} \frac{\delta}{\delta \gamma^{ab}}\right] \log Z_{\mathsf{CFT}} = 0,$$

where ℓ is the length scale. Using the replica trick and Rényi entropy

$$S_{\mathsf{vN}}^{(n)} = \frac{1}{1-n} \log \left[\frac{Z_{\mathcal{M}_n}}{(Z_{\mathcal{M}})^n}\right],$$

where $\mathcal{M}_n$ is the replicated manifold, one can show the RG equation for the entanglement entropy is

$$\ell \frac{\partial}{\partial \ell} S_{\mathsf{vN}} = \frac{c}{6}.$$

The constant $\frac{c}{6}$ on the right hand side is for $D = 3$.

RG equation as a geometric identity

In any geometry, not limited to gravity, there is an identity for the world function,

$$g^{\mu\nu}\sigma_\mu\sigma_\nu = 2\sigma, \quad \sigma(x,x') = \frac{1}{2}L^2(x,x'). \tag{4}$$

In AdS$_3$, except the vertical ones, any other geodesic is a segment of a semi-circle,

$$(x-x_0)^2 + z^2 = r^2.$$

We can map a semi-circle to a vertical line by the isometries of AdS$_3$.

$$(x,z) \to (\tilde{x},\tilde{z}) := \Big(\frac{x-x_0+r}{(x-x_0+r)^2+z^2} - \frac{1}{2r}, \frac{z}{(x-x_0+r)^2+z^2}\Big).$$

In terms of $(\tilde{x},\tilde{z})$, the semi-circle is a vertical line $\sigma = \sigma(0,\tilde{z}_1;0,\tilde{z}_2)$. Now, apply the geometric identity (4),

$$z^2\left[(\partial_x\sigma)^2 + (\partial_z\sigma)^2\right] = \tilde{z}^2(\partial_{\tilde{z}}\sigma)^2 = 2\sigma.$$

Obviously, $\tilde{z}$ is the renormalization scale ℓ. Including the coupling $c = 3R_{\mathsf{AdS}}/2G^{(3)}$, substituting $\sigma = \frac{1}{2}L^2 = \frac{1}{2}S^2_{\mathsf{vN}}$, we thus have in both bulk geometry and boundary CFT

$$\ell\frac{\partial}{\partial\ell}S_{\mathsf{vN}} = \frac{c}{6},$$

precisely the RG equation.

We now give a mathematical realization of ER=EPR (JWWY arXiv:2411.18485).

Briefly, we derived the Einstein-Rosen bridge from the quantum entanglement in the thermofield double CFT.

Consider the thermofield double state, an entangled pure state of two copies of thermal CFT_2,

$$|\mathrm{TFD}\rangle = \sum_n e^{-\frac{\beta}{2}E_n} |n_L\rangle \otimes |n_R\rangle.$$

Split the systems as in the figure,

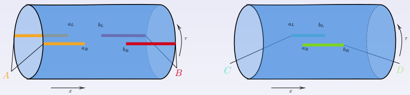


Figure: The density matrices ρ_{AB} and ρ_{CD}, indicate two different configurations for disjoint subsystems in the entangled TFD state, respectively.

Both ρ_{AB} and ρ_{CD} are mixed states.

Realization of ER=EPR

Using the subtraction approach (JWWY arXiv:2406.09033), we obtain

$$S_{\mathsf{vN}}(A:B)=\frac{c}{6}\text{arccosh}\left(\frac{\left(u^2+1\right)\left(u'^2+1\right)\cosh\left(\phi'-\phi\right)-4uu'}{\left(u^2-1\right)\left(u'^2-1\right)}\right).$$

So, the induced metric is

$$ds^2=\left[-\lim_{x'\to x}\partial_{x'^i}\partial_{x^j}\left(\frac{1}{2}S_{\mathrm{vN}}^2\right)\right]dx^i dx^j=\frac{4}{(1-u^2)^2}du^2+\left(\frac{1+u^2}{1-u^2}\right)^2 d\phi^2,$$

precisely revealing a geometry in which two spatial subregions ($u<0$ and $u>0$) are connected by a wormhole throat located at $u=0$. This metric is the $T=0$ $(u=-v)$ slice of the eternal black hole in Kruskal coordinates:

$$ds^2=-\frac{4}{(1+uv)^2}dudv+\left(\frac{1-uv}{1+uv}\right)^2 d\phi^2.$$

The RT surfaces of $S_{\mathsf{vN}}(A:B)$ and $S_{\mathsf{vN}}(C:D)$ are:

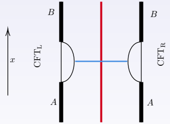


Figure: The red line is the RT surface of $S_{\mathsf{vN}}(C:D)$, the horizon. The blue line indicates the RT surface of $S_{\mathsf{vN}}(A:B)$.

Realization of ER=EPR: Bekenstein-Hawking entropy of the wormhole

As for $S_{\mathrm{vN}}(C:D)$, setting $a_L = a_R = a$ and $b_L = b_R = b$, we find

$$S_{\mathrm{vN}}(C:D) = \frac{2\pi r_+}{4G^{(3)}} = \frac{\mathsf{A}}{4G_N},$$

precisely the Bekenstein-Hawking entropy of the wormhole, with $A = 2\pi r_+$ and the parameter relations in AdS/CFT $\frac{|a-b|}{\beta} = \frac{r_+}{\ell}$.

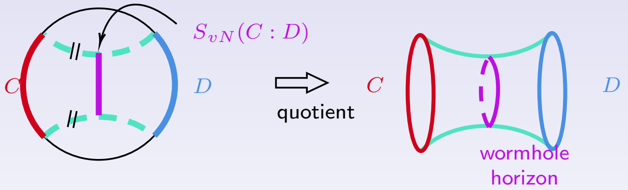


Figure: Left panel: An entanglement wedge is bounded by two cyan geodesics. The purple line is the EWCS. Right panel: A wormhole can be prepared by identifying two cyan geodesics of the entanglement wedge.

Furthermore, we find as $1/T = \beta \to \infty$,

$$S_{\mathrm{vN}}\left(C:D\right) = \frac{\pi c}{3\beta}\left|a-b\right| \to 0, \quad \text{and} \quad S_{\mathrm{vN}}\left(A:B\right) \sim \frac{c}{3}\log\beta \to \infty,$$

which provides a quantitative verification of Raamsdonk's conjecture that classically connected spacetime emerges from quantum entanglement.

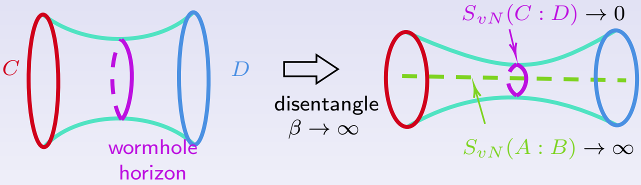


Figure: Disentangle the degrees of freedom in C and D by decreasing the temperature. The proper length between the corresponding spacetime regions increases to infinity, while the horizon area decreases to zero.

Summary and discussion

- We derived Einstein equation in CFT_2, which is the dynamics of EE.
- We proved that the RG equation is a geometrical identity. This explains why all QFTs share the same RG equation.
- So, the logic is:
    - Given a CFT_2, its EE generates a $D = 3$ gravity.
    - In this $3D$ curved spacetime, a $3D$ QFT can live and its EE(?) generates a $4D$ geometry.
    - Same pattern carries on to high dimensions.

Summary and discussion

- We have focused on AdS. Nevertheless, as long as RT formula holds, such as in dS, the derivations apply with the same fashion.
- Our derivations are for CFT_2. For higher dimensions, there could be two possibilities.
  1. Still EE satisfies Einstein equation. However, since the entangling surfaces are no longer points but co-dimension two surfaces, it is not obvious how to impose derivatives and taking the coincidence limit.
  2. More promising approach is to identify the dual CFT quantity of the bulk geodesics. This dual quantity automatically satisfies Einstein equation.
- Our results make "gravity as entropic force" straightforward. We explicitly realized ER=EPR.
- We showed that the metric $g_{ij} = -[\frac{1}{2}(S^2_{\mathrm{vN}})_{ij'}]$ is a derived but not a fundamental quantity. Perhaps this is why gravity is non-renormalizable and we should not try to quantize the metric itself.
- In physics, energy dominates and links everything. But now, we tend to believe the fundamental object in physics is entropy.

Thank you!

AUTHOR: Haitang Yang

ADDRESS: Center for Theoretical Physics
College of physics, Sichuan University
Sichuan, 610064, China

EMAIL: hyanga@scu.edu.cn