中国科学院大学
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协变轨道-自旋耦合方案

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JHEP 06 (2023) 039 景豪杰, 贲迪, 吴蜀明, 吴佳俊, 邹冰松.
Nucl.Phys.A 1040 (2023) 122761 李晓宇, 董相坤, 景豪杰.
arXiv 2405.06576 景豪杰, 吴蜀明, 吴佳俊.

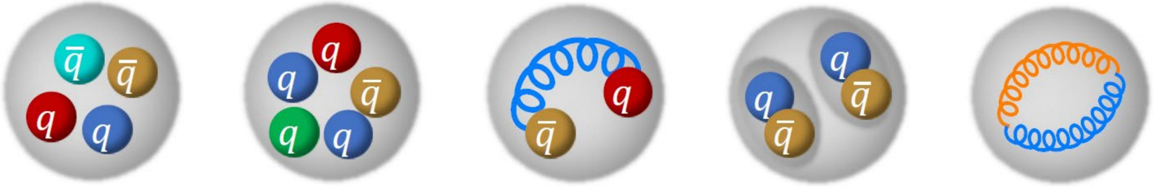
01 Introduction

- Motivation
- Review of the covariant L-S scheme

Motivation

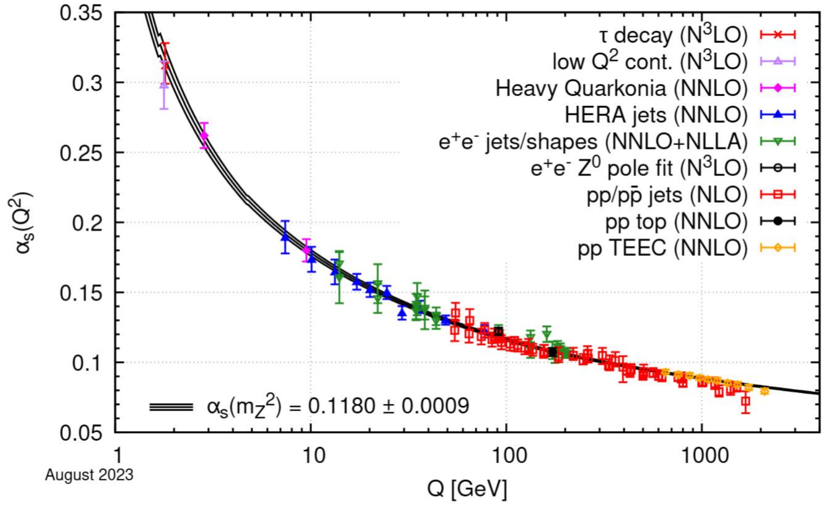
- QCD is the fundamental theory of strong interactions
  - Asymptotic freedom and color confinement
  - Non-perturbative in the low energy region
  - EFT: hadrons as the basic d.o.f.
- Hadron spectral physics: classifying hadrons
  - Traditional quark model: mesons and baryons

  - Exotic states:

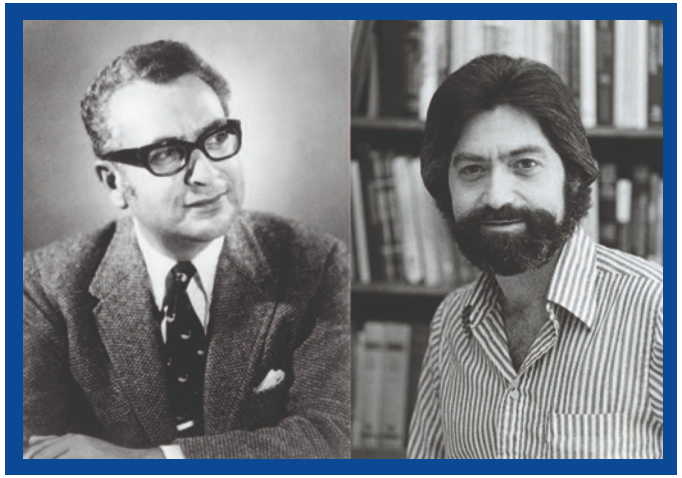


[https://itp.cas.cn/kxyj/kydt/202103/t20210331_5987996.html]

  - XYZ states, Pc states, Tcc states, etc.



PDG, Prog.Theor.Exp.Phys. 2022, 083C01 (2022)



M. Gell-Mann, Phys.Lett. 8, 214-215 (1964)
G. Zweig, CERN Report No.8182/TH.401 (1964)

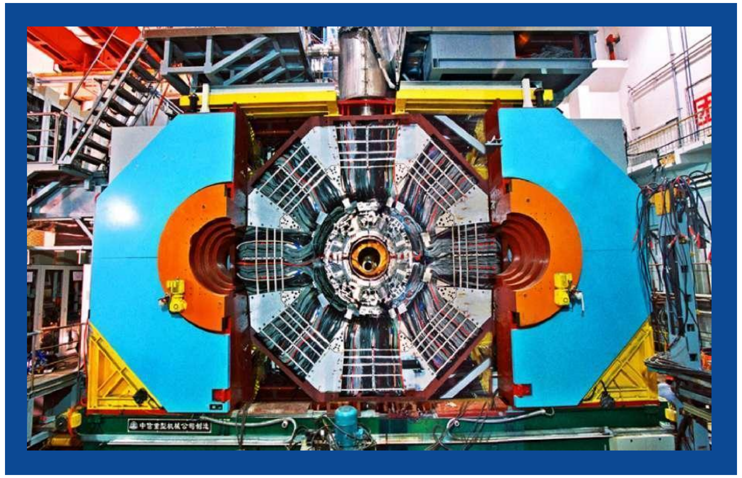
Motivation

- **Basic properties of hadrons**
  - **Intrinsic properties: mass, lifetime, spin, parity, charge, etc.**
  - **Observables: cross section, invariant mass distribution, angular distribution, etc.**

- **Beijing Electron Positron Collider (BEPC)** [http://bepclab.ihep.cas.cn/bepczz/zxfzt/]
  - **Main goals: tau-charm physics and synchrotron radiation**

The injector

The storage ring



Beijing Spectrometer III (BESIII)

Motivation

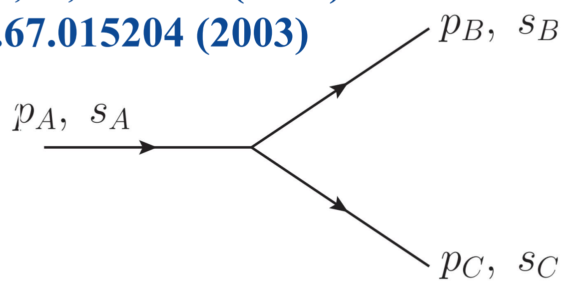
- Partial Wave Analysis (PWA)
  - a standard method for extracting spin and parity from angular distributions

- Commonly used PWA formalism
  - ➢ multipole analysis — Mesons and baryons: systematization and methods of analysis. (2008) A.V.Anisovich, V.V.Anisovich, etc. (see Appendix 5.C. Multipoles)
  - ➢ helicity scheme — K.C.Chou and M.I.Shirokov, J.Exptl.Theoret.Phys. (U.S.S.R.) 34,1230-1239 (1958); M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959); S.U.Chung, SPIN FORMALISMS [https://suchung.web.cern.ch/spinfm1.pdf] (2014)
  - ➢ covariant effective Lagrangian approach — M.Benmerrouche, etc. Phys.Rev.Lett. 77, 4716-4719 (1996); K.Nakayama, J.Speth, T.S.H.Lee, Phys.Rev.C 65, 045210 (2002); W.H.Liang, P.N.Shen, J.X.Wang and B.S.Zou, J.Phys.G 28, 333-343 (2002)
  - ➢ covariant L-S scheme — B.S.Zou and D.V.Bugg, Eur.Phys.J.A,16,537-547 (2003); B.S.Zou and F.Hussain, Phys.Rev.C.67.015204 (2003)

- Why covariant L-S scheme?
  - ✓ manifest Lorentz covariant form: convenient for multistep chain processes
  - ✓ with definite L-S quantum numbers: convenient for including L dependent form factors

Review of the covariant L-S scheme

B.S.Zou and D.V.Bugg, Eur.Phys.J.A,16,537-547 (2003)
B.S.Zou and F.Hussain, Phys.Rev.C.67.015204 (2003)



$$r = p_B - p_C$$
$$-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \equiv -\tilde{g}_{\mu\nu}(p)$$
$$\tilde{\gamma}_\mu = \tilde{g}_{\mu\nu}(p_A)\gamma^\nu$$

Building blocks:

- **Spin wave function for bosons**

$$\phi_{\mu_1\cdots\mu_S}$$

- **Pure spin wave function for fermion pairs**

$$\psi^{(n)}_{\mu_1\cdots\mu_n} = \bar{u}_{\mu_1\cdots\mu_n}(p_B, s_B)\gamma_5 v(p_C, s_C) \qquad \Psi^{(n+1)}_{\mu_1\cdots\mu_{n+1}} = \bar{u}_{\mu_1\cdots\mu_n}(p_B, s_B)\left(\gamma_{\mu_{n+1}} - \frac{r_{\mu_{n+1}}}{m_A + m_B + m_C}\right)v(p_C, S_C) + \cdots$$

$$\phi^{(n)}_{\mu_1\cdots\mu_n} = \bar{u}(p_B, S_B)u_{\mu_1\cdots\mu_n}(p_A, S_A) \qquad \Phi^{(n+1)}_{\mu_1\cdots\mu_{n+1}} = \bar{u}(p_B, s_B)\gamma_5\tilde{\gamma}_{\mu_{n+1}}u_{\mu_1\cdots\mu_n}(p_A, s_A) + \cdots$$

- **Orbital angular momentum tensor**

$$\tilde{t}^{(L)}_{\mu_1\cdots\mu_L}$$

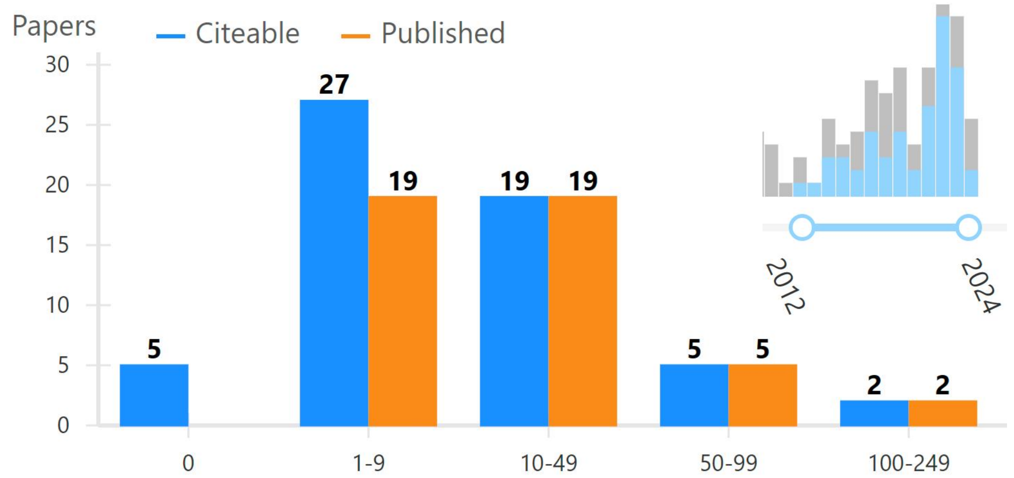
- **Lorentz structures**

$$(p_A)_\mu,\ g_{\mu\nu},\ \epsilon_{\mu\nu\rho\sigma}$$

For radiative decay process of baryons:

- **additional conditions due to gauge invariance**

S.Dulat and B.S.Zou, Eur.Phys.J.A, 26,125-134 (2005)
S.Dulat, J.J.Wu and B.S.Zou, PhysRevD.83.094032 (2011)



The work on the strong and radiative decay of vector mesons has currently been cited 58 times by BESIII!

02 Theoretical Part

- Relativistic spin wave function (SWF)
- Lorentz covariant coupling structure

Relativistic SWF

- Klein–Fock–Gordon equation (1926)
  - for spinless particles O.Klein, Z.Phys. 37, 895-906 (1926)
    W.Gordon, Z.Phys. 40, 1, 117-133 (1926)
    V.Fock, Z.Phys. 39, 226-232 (1926)

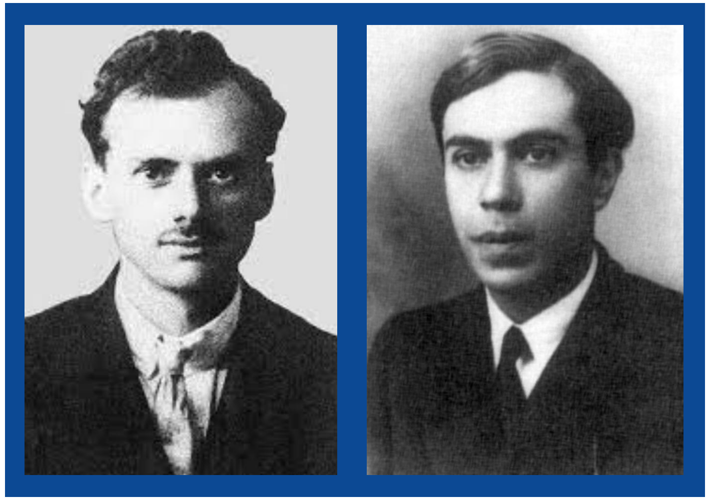
$$(\Box + m^2)\psi = 0$$

- Dirac equation (1928)
  - for spin-1/2 particles P.A.M.Dirac, Proc.Roy.Soc.Lond.A 117 (1928)

$$(i\gamma^\mu \partial_\mu + m)(i\gamma^\nu \partial_\nu - m)\psi = 0$$

- Majorana equation (1932)
  - for spin-1/2 particles E.Majorana, Nuovo.Cim. 9, 335-344 (1932)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$



Relativistic SWF

- The Dirac–Fierz–Pauli formalism (1936~1939)
  - for spin-(1/2+n) particles

$$p_{\gamma\dot{\alpha}} A^{\dot{\alpha}\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\epsilon_1\epsilon_2\cdots\epsilon_n} = m B^{\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\gamma\epsilon_1\epsilon_2\cdots\epsilon_n}$$
$$p^{\gamma\dot{\alpha}} B^{\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\gamma\epsilon_1\epsilon_2\cdots\epsilon_n} = m A^{\dot{\alpha}\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\epsilon_1\epsilon_2\cdots\epsilon_n}$$

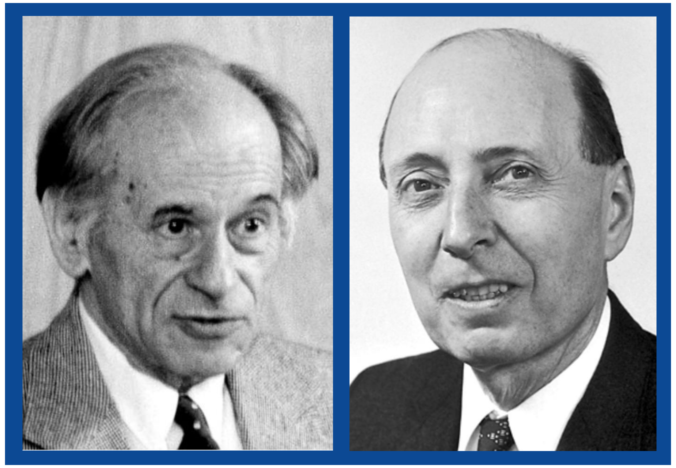
P. A. M. Dirac, Proc.Roy.Soc.Lond.A 155 (1936)
M. Fierz, W. Pauli, Proc.Roy.Soc.Lond.A 173 (1939)

- Rarita–Schwinger equations (1941)
  - for spin-(1/2+k) particles

$$(i\gamma^\alpha\partial_\alpha - m)\psi_{\mu_1\ldots\mu_k} = 0 \qquad \gamma^\alpha\psi_{\alpha\mu_2\ldots\mu_k} = 0$$
$$\partial^\alpha\psi_{\alpha\mu_2\ldots\mu_k} = 0 \qquad \psi^\alpha{}_{\alpha\mu_3\ldots\mu_k} = 0$$

W. Rarita, J. Schwinger, Phys.Rev. 60, 61 (1941)

Relativistic SWF



- **Group theoretical discussion by Bargmann and Wigner (1948)**
  - **Casimir operators of the Poincaré group**

$$C_1 = p^\mu p_\mu \qquad C_2 = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} p_\alpha p^\alpha - M_{\mu\alpha} M^{\nu\alpha} p^\mu p_\nu$$

  - **Classified by irreducible representation (IRREP)**

$[P_s]$ $\quad C_1 > 0$ & $C_2 \geq 0$: Particles of finite mass and spin s.

$[O_s]$ $\quad C_1 = 0$ & $C_2 = 0$: Particles of zero rest mass and discrete spin.

$[O(\Xi)]$ and $[O'(\Xi)]$ $\quad C_1 = 0$ & $C_2 = \Xi^2 > 0$: Particles of zero rest mass and continuous spin.

  - **Bargmann-Wigner equations for spin-(N/2) particles (N=1,2,3...)**

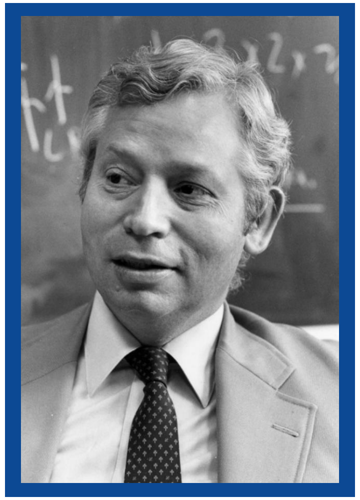
$$(i\gamma_k^\mu \partial_\mu - m)\psi = 0 \quad (k = 1, 2, \cdots, N)$$

V. Bargmann, E. P. Wigner, Proc.Nat.Acad.Sci. 34, 211 (1948)

Relativistic SWF

- **Weinberg's general causal fields (1964~1969)**
  - **Poincaré invariance + Causality + Cluster decomposition**
  - **The IRREP (s_L, s_R) $[L_p \simeq \mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{SU}(2)_\mathrm{R}]$ can describe particles with spin s $(|s_L - s_R| \leq s \leq s_L + s_R)$.**
  - **Equations of motion : eliminating excess d.o.f. in SWFs**

S.Weinberg, Phys.Rev. 133, B1318 (1964)
S.Weinberg, Phys.Rev. 134, B882 (1964)
S.Weinberg, Phys.Rev. 181, 1893 (1969)



- **Joos–Weinberg equation (1962~1964)**
  - **for spin-j particles (j=1/2, 1, 3/2, 2, ...)**

$$\left(i^{2j}\gamma^{\mu_1\mu_2\cdots\mu_{2j}}\,\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}} + m^{2j} \right)\psi = 0$$

H.Joos, Fortsch.Phys. 10. 65-146 (1962)
S.Weinberg, Phys.Rev. 133, B1318 (1964)

Rep.	Physical correspondence
$(0,0)$	Lorentz scalar
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	Dirac spinor for spin $1/2$
$(\frac{1}{2},\frac{1}{2})$	Lorentz four-vector
$(1,0)\oplus(0,1)$	Maxwell fields
$(\frac{3}{2},0)\oplus(0,\frac{3}{2})$	Joos-Weinberg spinor for spin $3/2$
$(1,1)$	Lorentz order-2 traceless symmetric tensor
$(1,\frac{1}{2})\oplus(\frac{1}{2},1)$	Rarita-Schwinger spinor for spin $3/2$
$(2,0)\oplus(0,2)$	Einstein fields
$\vdots$	$\vdots$

SWF for massive particle

- **A massive particle can be stationary** $p_\mu = \Lambda_\mu{}^\nu k_\nu \;\; [\Lambda = R\cdot B_z \cdot R^{-1},\; k_\mu = (m,0,0,0)_\mu]$

$$C_{L/R} = (J_1 \pm iK_1)^2 + (J_2 \pm iK_2)^2 + (J_3 \pm iK_3)^2$$

$$C_{\mathrm{U}(1)} = J_3$$

$$L_p \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{U}(1)$$

$$C_{\mathrm{SU}(2)} = J_1^2 + J_2^2 + J_3^2$$

Eigenfunction Method: J.Q.Chen, M.J.Gao, and G.Q.Ma, Rev. Mod. Phys. 57, 211(1985)

$$(C_{L/R})_\alpha{}^\beta \; u_\beta^\sigma(s) = s_{L/R}\,(s_{L/R}+1)\; u_\alpha^\sigma(s)$$
$$(C_{\mathrm{SU}(2)})_\alpha{}^\beta \; u_\beta^\sigma(s) = s(s+1)\; u_\alpha^\sigma(s)$$
$$(C_{\mathrm{U}(1)})_\alpha{}^\beta \; u_\beta^\sigma(s) = \sigma\; u_\alpha^\sigma(s)$$

→

$$D_\alpha{}^\beta(R)\; u_\beta^\sigma(s) = u_\beta^{\sigma'}(s)\; D_{\sigma'}^{(s)\sigma}(R)$$
$$D_\alpha{}^\beta(R)\; u_\beta^{\sigma'}(s)\; D_{\sigma'}^{(s)\sigma}\left(R^{-1}\right) = u_\alpha^\sigma(s)$$

→

$$u_\alpha^\sigma(\mathbf{p},s) \equiv D_\alpha{}^\beta(\Lambda)\; u_\beta^\sigma(s)$$

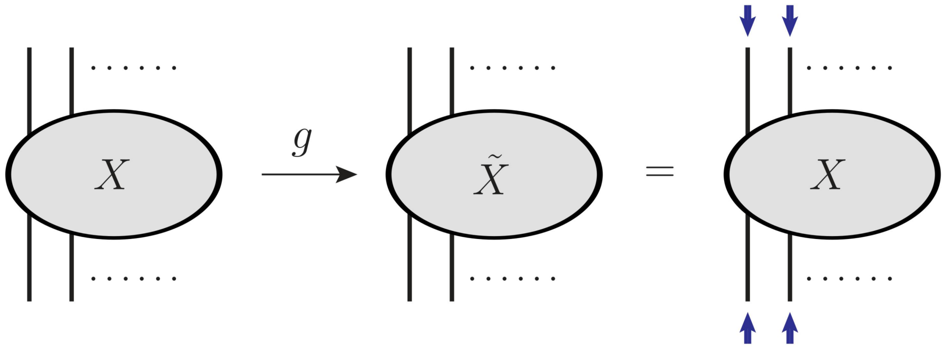
Lorentz covariant coupling structure

- **The general form of a three-particle amplitude**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\;\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{p}_1,s_1)\,u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2,s_2)\,u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3,s_3)$$

- **Covariant tensor (COVTEN)**

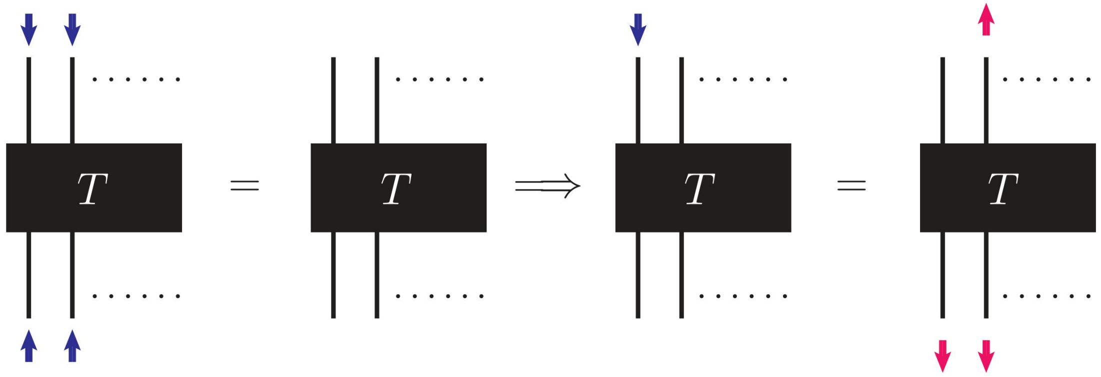
$$X^{a_1a_2\cdots}_{b_1b_2\cdots}\xrightarrow{g\in G}\tilde{X}^{a_1a_2\cdots}_{b_1b_2\cdots}=D_{b_1}{}^{b_1'}(g)\,D_{b_2}{}^{b_2'}(g)\,D^{a_1}{}_{a_1'}(g)\,D^{a_2}{}_{a_2'}(g)\cdots X^{a_1'a_2'\cdots}_{b_1'b_2'\cdots}$$



- **Invariant tensor (INVTEN)**

R.Penrose graphical notation: Applications of negative dimensional tensors (1971)

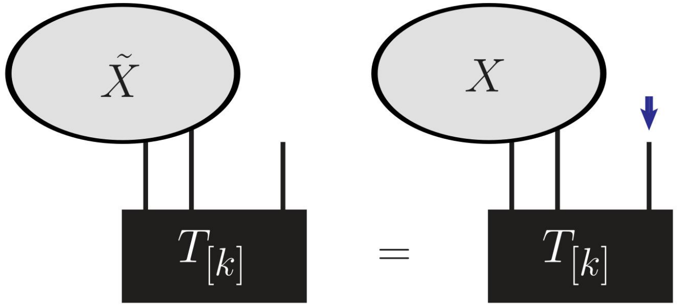
$$T^{a_1a_2\cdots}_{b_1b_2\cdots}\xrightarrow{g\in G}\tilde{T}^{a_1a_2\cdots}_{b_1b_2\cdots}=T^{a_1a_2\cdots}_{b_1b_2\cdots}$$



Lorentz covariant coupling structure

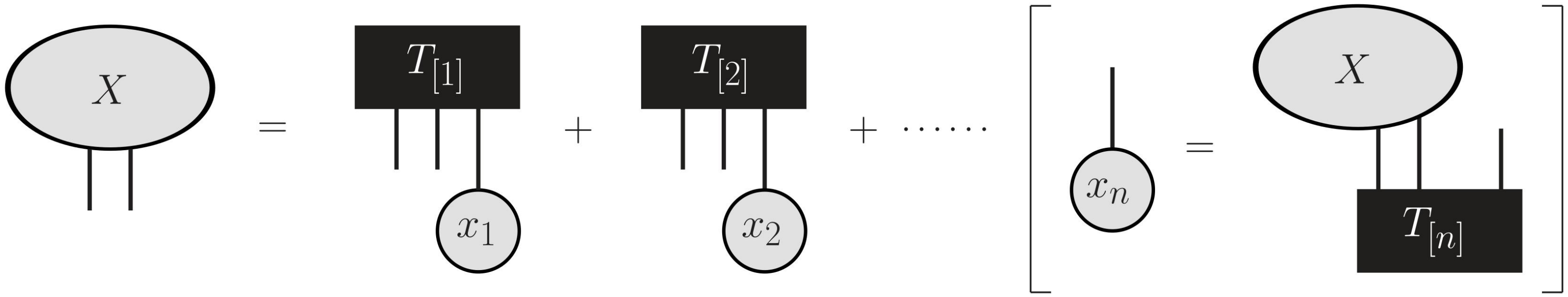
- **The projection properties of INVTEN** $[i] \otimes [j] = [k_1] \oplus [k_2] \oplus \cdots$

$$T^{ijk} X_{ij} \xrightarrow{g \in G} T^{ijk}\, \tilde{X}_{ij} = T^{ijk}\, D_i{}^{i'}(g)\, D_j{}^{j'}(g)\, X_{i'j'} = D^k{}_{k'}(g)\, T^{i'j'k'} X_{i'j'}$$



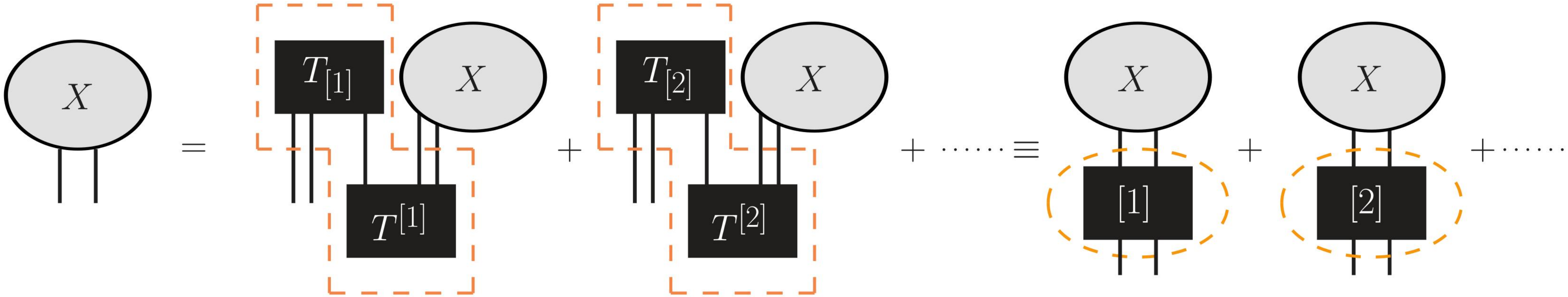
- **Reducibility of COVTEN**

$$X_{ij} = x^{k_1} T_{ijk_1} + x^{k_2} T_{ijk_2} + \cdots \quad \text{with} \quad x^{k_n} \xrightarrow{g \in G} \tilde{x}^{k_n} = D_{k_n'}{}^{k_n}(g) x^{k_n'}$$



Lorentz covariant coupling structure

- **COVTEN can be further decomposed into INVTEN**
  - ➢ **INVTEN cannot be further decomposed**
  - ➢ **INVTENs are also called irreducible tensors (IRTENs)**



- **Clebsch–Gordan coefficients (CGCs) are order-3 IRTENs**
  - **Example: CGCs of SU(2)**

$$\left[C^{j}_{j_1 j_2}\right]^{m}_{m_1 m_2} \xrightarrow{g\in \mathrm{SU}(2)} D^{(j)m}_{m'}\left(g^{-1}\right) D^{(j_1)m_1'}_{m_1}(g)\, D^{(j_2)m_2'}_{m_2}(g) \left[C^{j}_{j_1 j_2}\right]^{m'}_{m_1' m_2'} = \left[C^{j}_{j_1 j_2}\right]^{m}_{m_1 m_2}$$

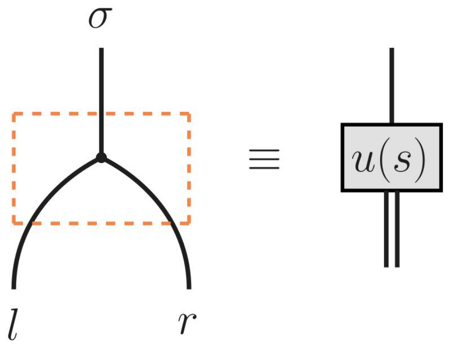
Lorentz covariant coupling structure

- **SWFs are IRTENs of SU(2)**

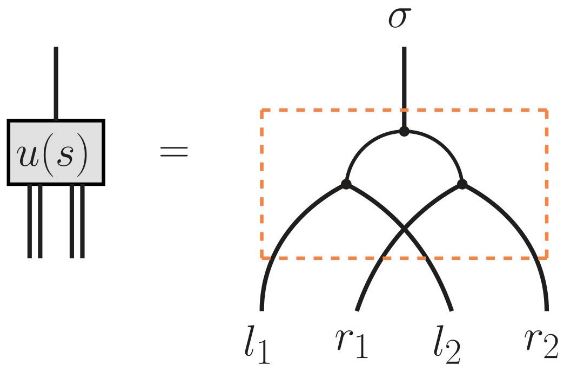
$$D_\alpha{}^{\beta}(R)\; u^\sigma_\beta(s) = u^{\sigma'}_\beta(s)\; D^{(s)\sigma}_{\sigma'}(R)$$

$$D_\alpha{}^{\beta}(R)\; u^{\sigma'}_\beta(s)\; D^{(s)\sigma}_{\sigma'}\left(R^{-1}\right) = u^\sigma_\alpha(s)$$

$$u^\sigma_\alpha(s) = u^\sigma_{lr}(s) \propto \left(C^s_{s_L s_R}\right)^\sigma_{lr}$$

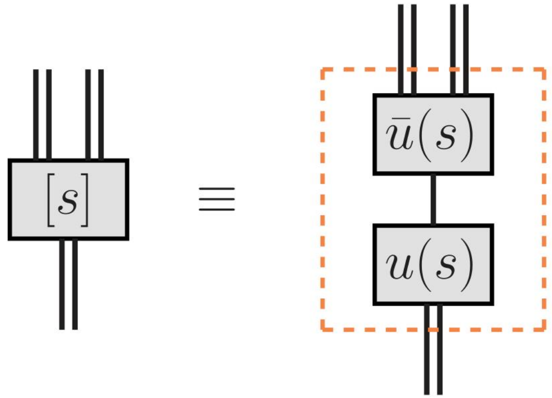


$$u^\sigma_{\alpha_1\alpha_2}(\chi, s) = u^\sigma_{l_1 r_1 l_2 r_2}((s_L, s_R), s)$$



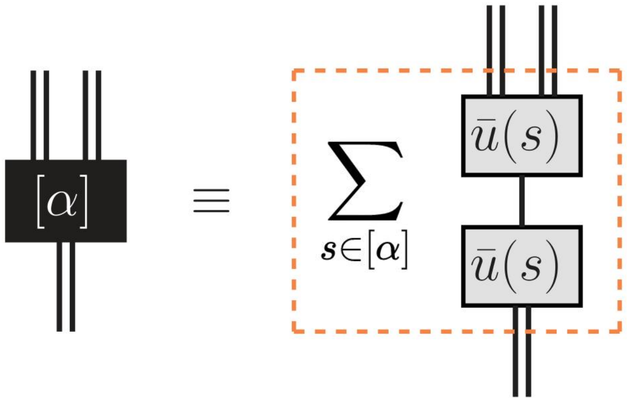
- **Lorentz covariant spin projection tensor (SPT)**

$$P^{\alpha_1\alpha_2}_\alpha(\mathbf{p}; \chi, s) = u^\sigma_\alpha(\mathbf{p}; s)\bar{u}^{\alpha_1\alpha_2}_\sigma(\mathbf{p}; \chi^*, s)$$



- **IRTENs of the Lorentz group**

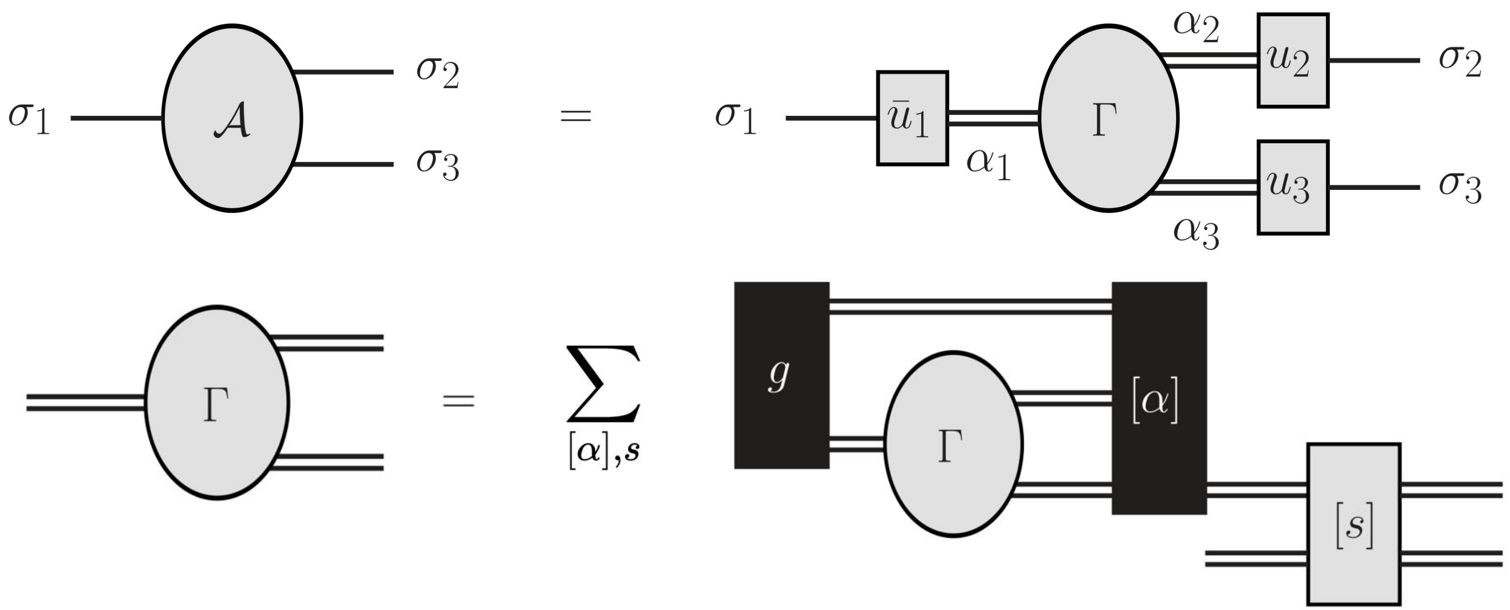
$$T^{\alpha_1\alpha_2}_\alpha = \sum_{s\in[\alpha]} u^\sigma_\alpha(s)\bar{u}^{\alpha_1\alpha_2}_\sigma([\alpha]^*, s)$$



Lorentz covariant coupling structure

- **The general form of a three-particle amplitude**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\right)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\right)\ \bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{p}_1,s_1)\,u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2,s_2)\,u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3,s_3)$$



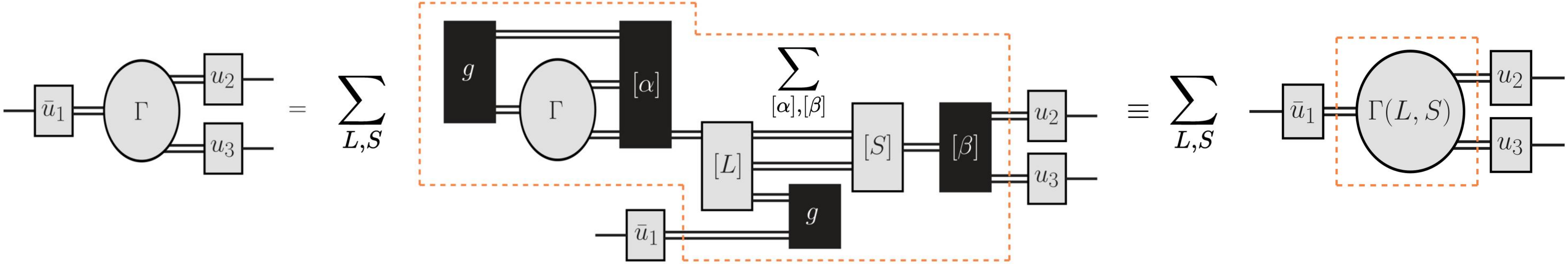
- **SPTs are building blocks for constructing Lorentz covariant coupling structure!**

03 Application in PWA

- Partial wave decomposition of amplitudes
- Construction of partial wave amplitudes

Partial wave decomposition of amplitudes

- **The pure-orbital (L) and pure-spin (S) component**
  - **C-scheme** $$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)}_{\text{pure-spin part}}$$
  - **H-scheme** $$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\, D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$



$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha_L\alpha_S}(\mathbf{k}_1;\chi_{LS},s_1)P_{\alpha_S}^{\alpha_2\alpha_3}(\mathbf{k}_1;\chi_{23},S)\,\tilde{t}_{\alpha_L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

Partial wave decomposition of amplitudes

- **Example: amplitude with spin-1/2** $\mathcal{A}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*) = \epsilon_\mu(\mathbf{k}_1)\bar{u}(\mathbf{p}_2^*)\Gamma^\mu(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*)v(\mathbf{p}_3^*)$

$$\mathcal{A}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*) = \epsilon_\mu(\mathbf{k}_1)\langle \underbrace{\bar{U}_2\Gamma^\mu(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*)U_3}_{\text{纯轨道部分}} \underbrace{v(\mathbf{k}_3)\bar{u}(\mathbf{k}_2)}_{\text{纯自旋部分}} \rangle$$

$$\bar{u}(\mathbf{p}_2^*) = \bar{u}(\mathbf{k}_2)\,\bar{U}_2$$
$$v(\mathbf{p}_3^*) = U_3\,v(\mathbf{k}_3)$$

Partial wave decomposition of amplitudes

- **Example: amplitude with spin-1/2** $\mathcal{A}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*) = \epsilon_\mu(\mathbf{k}_1)\bar{u}(\mathbf{p}_2^*)\Gamma^\mu(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*)v(\mathbf{p}_3^*)$

$$\boxed{\mathcal{A}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*) = \epsilon_\mu(\mathbf{k}_1)\langle \underbrace{\bar{U}_2\Gamma^\mu(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*)U_3}_{\text{纯轨道部分}} \underbrace{v(\mathbf{k}_3)\bar{u}(\mathbf{k}_2)}_{\text{纯自旋部分}} \rangle}$$

$$\bar{u}(\mathbf{p}_2^*) = \bar{u}(\mathbf{k}_2)\,\bar{U}_2$$
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$$\left[\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)\right] \otimes \left[\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)\right] = (0,0)_L \oplus (0,0)_R \oplus (1,0) \oplus (0,1) \oplus \left(\frac{1}{2},\frac{1}{2}\right)_L \oplus \left(\frac{1}{2},\frac{1}{2}\right)_R$$

$$v(\mathbf{k}_3)\bar{u}(\mathbf{k}_2) = A + B\gamma_5 + C_\mu\gamma^\mu + D_\mu\gamma_5\gamma^\mu + E_{\mu\nu}\sigma^{\mu\nu}$$

Partial wave decomposition of amplitudes

- **Example: amplitude with spin-1/2** $\mathcal{A}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)=\epsilon_\mu(\mathbf{k}_1)\bar{u}(\mathbf{p}_2^*)\Gamma^\mu(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)v(\mathbf{p}_3^*)$

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$$\Gamma^\mu=\gamma^\mu$$

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Spin part: $C_\mu=\frac{1}{4}\bar{u}(\mathbf{k}_2)\gamma_\mu v(\mathbf{k}_3)$ $\qquad C_\mu=P_\mu{}^\nu(S=1)C_\nu$ $\qquad$ **S=1**

Partial wave decomposition of amplitudes

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$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right) = (0,0)\oplus(1,0)\oplus(0,1)\oplus(1,1)$$

$$\left[T_{(0,0)}\right]^{\mu\nu,\mu'\nu'} = \frac{1}{4}g^{\mu\nu}g^{\mu'\nu'}$$
$$\left[T_{[(1,0)\oplus(0,1)]}\right]^{\mu\nu,\mu'\nu'} = \frac{1}{2}\left(g^{\mu\mu'}g^{\nu\nu'} - g^{\mu\nu'}g^{\nu\mu'}\right)$$
$$\left[T_{(1,1)}\right]^{\mu\nu,\mu'\nu'} = \frac{1}{2}\left(g^{\mu\mu'}g^{\nu\nu'} + g^{\mu\nu'}g^{\nu\mu'}\right) - \frac{1}{4}g^{\mu\nu}g^{\mu'\nu'}$$

Partial wave decomposition of amplitudes

- **Example: amplitude with spin-1/2** $\mathcal{A}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*) = \epsilon_\mu(\mathbf{k}_1)\bar{u}(\mathbf{p}_2^*)\Gamma^\mu(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*)v(\mathbf{p}_3^*)$

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Orbital part: $\langle \bar{U}_2\gamma^\mu U_3\gamma^\nu\rangle = P^{\nu\mu}_{\alpha\beta}(L=0)C_0^{\alpha\beta} + P^{\nu\mu}_{\alpha\beta}(L=1)C_1^{\alpha\beta} + P^{\nu\mu}_{\alpha\beta}(L=2)C_2^{\alpha\beta}$ **L=0,1,2**

Partial wave decomposition of amplitudes

- **Example: amplitude with spin-1/2** $\mathcal{A}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)=\epsilon_\mu(\mathbf{k}_1)\bar{u}(\mathbf{p}_2^*)\Gamma^\mu(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)v(\mathbf{p}_3^*)$

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Total angular momentum: $\mathcal{A}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)=\epsilon_\mu(\mathbf{k}_1)C_\nu\langle\bar{U}_2\gamma^\mu U_3\gamma^\nu\rangle\equiv\epsilon_\mu(\mathbf{k}_1)\left(\mathcal{A}_0^\mu+\mathcal{A}_1^\mu+\mathcal{A}_2^\mu\right)$

$P^\mu{}_\nu(J=1)\mathcal{A}_0^\nu=\mathcal{A}_0^\mu\quad P^\mu{}_\nu(J=0)\mathcal{A}_1^\nu=\mathcal{A}_1^\mu\quad P^\mu{}_\nu(J=1)\mathcal{A}_2^\nu=\mathcal{A}_2^\mu$ **J=0,1** → ${}^3P_0\ \ {}^3S_1\ \ {}^3D_1$

Partial wave decomposition of amplitudes

- **Example: amplitude with spin-1/2**

$\bar{\psi}_2\gamma_\mu\psi_1$ $\quad {}^3P_0 \; {}^3S_1 \; {}^3D_1$

$\bar{\psi}_2\overleftrightarrow{\partial}_\mu\psi_1$ $\quad {}^3S_1 \; {}^3P_0 \; {}^3D_1$

- **Pure SWF for fermion pairs in $V \to B\bar{B}$**

$$[{}^1S_0] \quad \psi^{(0)} = \bar{u}(p_B, s_B)\gamma_5 v(p_C, s_C)$$

$$[{}^3S_1] \quad \Psi^{(1)}_{\mu_1} = \bar{u}(p_B, s_B)\left(\gamma_{\mu_1} - \frac{r_{\mu_1}}{m_A + m_B + m_C}\right)v(p_C, S_C)$$

B.S.Zou and F.Hussain, Phys.Rev.C.67.015204 (2003)

- **One can obtain partial wave amplitudes through the linear combination of various Lorentz structures.**

Lorentz structure	Partial wave components $({}^{2S+1}L_J)$
$\bar{\psi}_2\psi_1$	${}^3P_0$ ${}^1S_0$
$\bar{\psi}_2\gamma_5\psi_1$	${}^1S_0$ ${}^3P_0$
$\bar{\psi}_2\gamma_\mu\psi_1$	${}^3P_0 \; {}^3S_1 \; {}^3D_1$ ${}^1S_0 \; {}^1P_1 \; {}^3P_1$
$\bar{\psi}_2\gamma_5\gamma_\mu\psi_1$	${}^3P_1 \; {}^1S_0 \; {}^1P_1$ ${}^3P_0 \; {}^3S_1 \; {}^3D_1$
$\bar{\psi}_2\sigma_{\mu\nu}\psi_1$	${}^1P_1 \; {}^3S_1 \; {}^3P_1 \; {}^3D_1$ ${}^1P_1 \; {}^3S_1 \; {}^3P_1 \; {}^3D_1$
$\bar{\psi}_2 \overleftrightarrow{\partial}_\mu \psi_1$	${}^3S_1 \; {}^3P_0 \; {}^3D_1$ ${}^1S_0$
$\partial_\mu(\bar{\psi}_2\psi_1)$	${}^3P_0$ ${}^1S_0 \; {}^1P_1$
$\vdots$	$\vdots$

Partial wave decomposition of amplitudes

- **Example: amplitude with spin-3/2**

➢ **With the increase of spin, the amount of computation will increase dramatically.**

➢ **It is useful to consider how to construct the Lorentz covariant partial wave amplitude.**

Lorentz structure	Partial wave components $\left({}^{2S+1}L_J\right)$
$\bar{\psi}_2\psi_{1\mu}$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\bar{\psi}_2\gamma_5\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3D_1$ ${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$
$\bar{\psi}_2\gamma_\nu\psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$
$\bar{\psi}_2\gamma_5\gamma_\nu\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$ ${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\sigma_{\nu\rho}\psi_{1\mu}$	${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\ \overleftrightarrow{\partial}_\nu\ \psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5F_1\ {}^5S_2\ {}^3D_2\ {}^5D_2\ {}^5G_2$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\partial_\nu\left(\bar{\psi}_2\psi_{1\mu}\right)$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^3D_1\ {}^3P_2\ {}^3F_2$
$\vdots$	$\vdots$

Construction of partial wave amplitudes

- **Lorentz covatiant partial wave formulas (PWFs)**

  - **C-scheme**

$$\mathcal{C}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\,\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\,u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\,u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)$$

Consistent with the covariant tensor amplitude.

S.U.Chung, Phys.Rev.D.57.431-442(1998)
B.S.Zou and D.V.Bugg, Eur.Phys.J.A,16,537-547 (2003)

  - **H-scheme**

$$\mathcal{H}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\,\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\,u_{\alpha_2}^{\sigma_2}(s_2)\,u_{\alpha_3}^{\sigma_3}(s_3)$$

Consistent with the helicity amplitude.

K.C.Chou and M.I.Shirokov, J.Exptl.Theoret.Phys. (U.S.S.R.) 34,1230-1239 (1958)
M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)

➢ **C-scheme and H-scheme are equivalent with each other in the non-relativistic limit.**

04 About massless particle

- ◎ SWF for massless particles and gauge invariance (GI)
- ◎ Weight function

SWF for massless particles and GI

- **A massless particle can not be stationary** $p_\mu = \tilde{\Lambda}_\mu{}^\nu \, k_\nu \quad \left[\tilde{\Lambda} = R \cdot B_z, \; k_\mu = (\kappa, 0, 0, \kappa)_\mu\right]$

$$C_{L/R} = (J_1 \pm iK_1)^2 + (J_2 \pm iK_2)^2 + (J_3 \pm iK_3)^2$$

$$C_{\mathrm{U}(1)} = J_3$$

$$L_p \longrightarrow \mathrm{E}(2) \longrightarrow \mathrm{U}(1)$$

$$C_{\mathrm{SU}(2)} = J_1^2 + J_2^2 + J_3^2$$

$$C_{\mathrm{E}(2)} = (J_1 - K_2)^2 + (J_2 + K_1)^2$$

$$\left(C_{L/R}\right)_\alpha{}^\beta \; h_\beta^\sigma(\tilde{s}) = s_{L/R}\left(s_{L/R}+1\right) \; h_\alpha^\sigma(\tilde{s})$$
$$\left(C_{\mathrm{E}(2)}\right)_\alpha{}^\beta \; h_\beta^\sigma(\tilde{s}) = \tilde{s} \; h_\alpha^\sigma(\tilde{s})$$
$$\left(C_{\mathrm{U}(1)}\right)_\alpha{}^\beta \; h_\beta^\sigma(\tilde{s}) = \sigma \; h_\alpha^\sigma(\tilde{s})$$

$$\left(C_{L/R}\right)_\alpha{}^\beta \; h_\beta^\sigma(s,\tilde{s}) = s_{L/R}\left(s_{L/R}+1\right) \; h_\alpha^\sigma(s,\tilde{s})$$
$$\left(C_{\mathrm{E}(2)}\right)_\alpha{}^\beta \; h_\beta^\sigma(s,\tilde{s}) = \tilde{s} \; h_\alpha^\sigma(s,\tilde{s})$$
$$\left(C_{\mathrm{U}(1)}\right)_\alpha{}^\beta \; h_\beta^\sigma(s,\tilde{s}) = \sigma \; h_\alpha^\sigma(s,\tilde{s})$$
$$\left(C_{\mathrm{SU}(2)}\right)_\alpha{}^\beta \; h_\beta^\sigma(s,\tilde{s}) = s(s+1) \; h_\alpha^\sigma(s,\tilde{s})$$

$$X \equiv \left[C_{\mathrm{SU}(2)}, C_{\mathrm{E}(2)}\right]$$
$$X_\alpha{}^\beta \; h_\beta^\sigma(s,\tilde{s}) = 0$$
$$h_\beta^\sigma(s,\tilde{s}) \in \ker\left(X_\alpha{}^\beta\right)$$

$$h_\alpha^\sigma(\mathbf{p}, s, \tilde{s}) = D(\tilde{\Lambda})_\alpha{}^\beta \; h_\alpha^\sigma(s,\tilde{s})$$

$\ker\left(X_\alpha{}^\beta\right) \subsetneqq [\alpha]$ **Gauge transformation is needed**

$\ker\left(X_\alpha{}^\beta\right) = [\alpha]$ **Automatically meet GI**

SWF for massless particles and GI

- **Example: spin-1 massless particle**

$$[\alpha] = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \text{Potential } A_\mu$$

$$X = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \end{pmatrix}$$

$$\ker X = \{\varepsilon_\mu^\sigma \mid \sigma = \pm 1\} \subsetneq [\alpha]$$

$$D_\mu{}^\nu(\Lambda)\varepsilon_\nu^\sigma(\mathbf{p}^*) = e^{i\theta(\Lambda)}\varepsilon_\mu^\sigma(\mathbf{p}) + \lambda(\Lambda,\sigma)p_\mu$$

Gauge transformation is needed

$$[\alpha] = (1,0) \oplus (0,1) \qquad \text{Field strength } F_{\mu\nu}$$

$$X = 0 \qquad \ker X = [\alpha]$$

Automatically meet GI

$$D_\alpha{}^\beta(\Lambda)\, h_\beta^\sigma(\mathbf{p}^*, s, \tilde{s}) = e^{i\theta(\Lambda)}\, h_\alpha^\sigma(\mathbf{p}, s, \tilde{s}) + (\text{Non physical d.o.f.})$$

SWF for massless particles and GI

- **The pure-orbital (L) and pure-spin (S) component**

$$h^{\sigma}_{\alpha}(\mathbf{p}, s, \tilde{s}) = \sum_{\sigma'=-s}^{+s} \underbrace{D_{\alpha}{}^{\beta}(\Lambda)}_{\text{pure-orbital part}} \times \underbrace{u^{\sigma'}_{\beta}(s)\, D^{(s)\sigma}_{\sigma'}(R)}_{\text{pure-spin part}} \quad (\sigma = \pm s)$$

$$\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1, \mathbf{p}^*_2, \mathbf{p}^*_3) = \underbrace{\Gamma^{\beta_2\beta_3}_{\alpha_1}(\mathbf{k}_1, \mathbf{p}^*_2, \mathbf{p}^*_3)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\, D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma'_2}_{\alpha_2}(s_2)\, u^{\sigma'_3}_{\alpha_3}(s_3)\, D^{(s_2)\sigma_2}_{\sigma'_2}(R_{2^*})\, D^{(s_3)\sigma_3}_{\sigma'_3}(R_{3^*})}_{\text{pure-spin part}}$$

- **PWF including massless particles**

$$\boxed{\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1, \mathbf{p}^*_2, \mathbf{p}^*_3; L, S) = \Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1, \mathbf{p}^*_2, \mathbf{p}^*_3; L, S)\, \bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma'_2}_{\alpha_2}(s_2)\, u^{\sigma'_3}_{\alpha_3}(s_3)\, D^{(s_2)\sigma_2}_{\sigma'_2}(R_{2^*})\, D^{(s_3)\sigma_3}_{\sigma'_3}(R_{3^*})}$$

SWF for massless particles and GI

- **Example:** $J/\psi \to \gamma f_2$
  - **the possible (L,S) combinations are (0,1), (2,1), (2,2), (2,3), (4,3)**
  - **according to the PWF in the previous slide, one has (in helicity basis)**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(1,0) \propto \left[\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(1,2) \propto \left[\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(2,2) \propto \left[\begin{pmatrix} 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(3,2) \propto \left[\begin{pmatrix} 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(3,4) \propto \left[\begin{pmatrix} 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -2\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

➢ **only 3 of the 5 amplitudes are linearly independent !**

Weight function

- **The number of linear independent (L,S) bases**

<table>
<tr><th>Range</th><th>$N_0(s_1; s_2, s_3)$</th><th>$N_1(s_1; s_2, s_3)$</th><th>$N_2(s_1; s_2, s_3)$</th><th>$N_3(s_1; s_2, s_3)$</th></tr>
<tr><td>$s_1 < s_2 - s_3$</td><td rowspan="2">$(2s_1+1)(2s_3+1)$</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>$s_1 = s_2 - s_3$</td><td rowspan="2">$2(s_1 - s_2 + s_3 + 1)$</td><td rowspan="3">2</td><td>2</td></tr>
<tr><td>$|s_2 - s_3| < s_1 < s_2 + s_3$</td><td>$n(s_1; s_2, s_3)$</td><td>$0$</td></tr>
<tr><td>$s_1 = s_3 - s_2$</td><td rowspan="2">$(2s_1+1)(2s_2+1)$</td><td rowspan="2">$2(2s_1+1)$</td><td>2</td></tr>
<tr><td>$s_1 < s_3 - s_2$</td><td>0</td><td>0</td></tr>
<tr><td>$s_1 = s_2 + s_3$</td><td rowspan="2">$(2s_2+1)(2s_3+1)$</td><td rowspan="2">$2(2s_3+1)$</td><td rowspan="2">4</td><td>2</td></tr>
<tr><td>$s_1 > s_2 + s_3$</td><td>0</td></tr>
</table>

$$n(s_1; s_2, s_3) = -(s_1^2 + s_2^2 + s_3^2) + 2(s_1 s_2 + s_2 s_3 + s_1 s_3) + s_1 + s_2 + s_3 + 1$$

- **Weight function for choosing (L,S) bases**

One massless particle: $W(s_1, s_2, s_3, L, S) = F_S(s_1, s_2, s_3, S)$

Two or three massless particles: $W(s_1, s_2, s_3, L, S) = F_S(s_1, s_2, s_3, S) + F_L(s_1, s_2, s_3, L, S) + F_\sigma(s_1, s_2, s_3, L, S)$

$$\begin{aligned} F_S(s_1, s_2, s_3, S) &= -(s_2 + s_3 + 1)\,|S - s_1| + S \\ F_L(s_1, s_2, s_3, L, S) &= -2(s_2 + s_3 + 1)^2 \left| L - |S - s_1| - \frac{1}{2} \right| \end{aligned}$$

$$F_\sigma(s_1, s_2, s_3, L, S) = \begin{cases} -2(s_2 + s_3 + 1)^2 (s_1 + s_2 + s_3) & \text{for } \left(C_{s_1}^{LS}\right)_{s_2 \pm s_3}^{0\, s_2 \pm s_3} = 0 \\ 0 & \text{for others} \end{cases}$$

SWF for massless particles and GI

- **Example:** $J/\psi \to \gamma f_2$
  - **the possible (L,S) combinations are (0,1), (2,1), (2,2), (2,3), (4,3)**
  - **according to the PWF in the previous slide, one has (in helicity basis)**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(1,0) \propto \left[\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(1,2) \propto \left[\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(2,2) \propto \left[\begin{pmatrix} 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(3,2) \propto \left[\begin{pmatrix} 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(3,4) \propto \left[\begin{pmatrix} 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -2\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

```
s1 = 1; s2 = 1; s3 = 2;
WFunc1[s1_, s2_, s3_, S_, L_] :=
  - (s2 + s3 + 1) Abs[S - s1] + S;

WFunc1[s1, s2, s3, 1, 0]
WFunc1[s1, s2, s3, 1, 2]
WFunc1[s1, s2, s3, 2, 2]
WFunc1[s1, s2, s3, 3, 2]
WFunc1[s1, s2, s3, 3, 4]

1

1

-2

-5

-5
```

➢ **only 3 of the 5 amplitudes are linearly independent !**

05 About numerical calculation

About numerical calculation

- Currently commonly used PWA package (PKG) on BESIII
  - FDC-PWA： 协变有效拉氏量方法下协变振幅的自动化计算
    (王建雄研究员、平荣刚研究员@IHEP) [https://www1.ihep.ac.cn/wjx/pwa] (2000)
  - GPUPWA： 协变 L-S 方案下分波振幅的自动化计算 （刘北江研究员@IHEP）
    - 支持矢量介子的强衰变和辐射衰变过程 [https://sourceforge.net/projects/gpupwa/] (2011)
  - TF-PWA： 螺旋度方案下分波振幅的自动化计算 [https://tf-pwa.readthedocs.io] (2020)
    （蒋艺、刘寅睿、钱文斌教授、吕晓睿教授、郑阳恒教授@UCAS）
- Automatic calculation of PWF under the covariant L-S scheme
  - PKG for calculating PWF under C/H-scheme based on C++
    (与吴蜀明博士@UCAS合作) [https://github.com/Wu-ShuMing/PWFs] (2024)
  - Crosscheck our PKG with the TF-PWA
    (与蒋艺@UCAS, 马润秋@IHEP 和王石@LZU合作)

About numerical calculation

- **Building blocks for numerical calculation**
  - **PWF in the c.m. frame**
    - **H-scheme:** $$[\mathcal{H}^{*}_{L,S}]^{\sigma_2\sigma_3}_{\sigma_1} = \frac{|\mathbf{p}^{*}_2|^{L}}{\sqrt{2s_1+1}} (C^{SL}_{s_1})^{\sigma_S\sigma_L}_{\sigma_1} (C^{s_2 s_3}_{S})^{\sigma_2\sigma_3}_{\sigma_S}\, \mathcal{Y}_{L,\sigma_L}(\hat{\mathbf{p}}^{*}_2)$$
    - **C-scheme:** $$[\mathcal{C}^{*}_{L,S}]^{\sigma_2\sigma_3}_{\sigma_1} = \sum_{\sigma'_2,\sigma'_3} [\mathcal{H}^{*}_{L,S}]^{\sigma'_2\sigma'_3}_{\sigma_1}\, \mathcal{I}^{(s_2)\sigma_2}_{\sigma'_2}(\mathbf{p}^{*}_2)\, \mathcal{I}^{(s_3)\sigma_3}_{\sigma'_3}(\mathbf{p}^{*}_3)$$
  - **PWF in any frame** $$[\mathcal{A}_{L,S}]^{\sigma_2\sigma_3}_{\sigma_1} = \sum_{\sigma'_2,\sigma'_3} [\mathcal{A}^{*}_{L,S}]^{\sigma'_2\sigma'_3}_{\sigma_1} D^{(s_2)\sigma_2}_{\sigma'_2}(R_2) D^{(s_3)\sigma_3}_{\sigma'_3}(R_3)$$

Perform scaling transformation on polarization indices.

Thomas-Wigner rotation.

for massive particle-i :

$$R_i = R_{\hat{\mathbf{n}}_i} \cdot R_z(\psi_i) \cdot R^{-1}_{\hat{\mathbf{n}}_i}$$

$$\frac{\psi_i}{2} = \tan^{-1}\left[\sqrt{\frac{(E_1+m_1)(E_i+m_i)}{(E_1-m_1)(E_i-m_i)}} + \cos\theta_{1i}, \sin\theta_{1i}\right],$$

$$\hat{\mathbf{n}}_i = \frac{\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_i}{\sin\theta_{1i}}, \qquad \cos\theta_{1i} = \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_i \,;$$

for massless particle-i :

$$R_i = R_z(\psi^{+}_i + \psi^{-}_i), \qquad \hat{z} \equiv (0,0,1),$$

$$\psi^{\pm}_i = \tan^{-1}\left[\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_i \mp \hat{\mathbf{p}}_1 \cdot \hat{z} + \sqrt{\frac{(E_1+m_1)}{(E_1-m_1)}}(1 \mp \hat{\mathbf{p}}_i \cdot \hat{z}), \pm(\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_i)\cdot \hat{z}\right].$$

06

Summary and outlook

Summary and outlook

- The covariant L-S scheme is one of the commonly used PWA schemes in BESIII.
- PWFs can be systematically constructed by using the IRTENs.
- Both helicity scheme and covariant L-S scheme can be constructed within this framework.

- Constructing a PKG for calculating PWFs under the covariant L-S scheme.
- Try to incorporate contributions from loop diagrams within the existing framework.
- Other research related to PWA.

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Back Up

The scaling transformation: $\mathcal{I}_{\sigma'}^{(s)\sigma}(\mathbf{p}) = \sum_{\sigma''} D_{\sigma'}^{(s)\sigma''}(R_{\hat{\mathbf{p}}}) I(|\mathbf{p}|, s, \sigma'') D_{\sigma''}^{(s)\sigma}\left(R_{\hat{\mathbf{p}}}^{-1}\right),$

where $D_{\sigma'}^{(s)\sigma}(R)$ is the Wigner- D matrix; $R_{\hat{\mathbf{p}}}$ is a rotation that rotates the z-axis in $\hat{\mathbf{p}}$;

$I(|\mathbf{p}|, s, \sigma'')$ is the scale of the dilation.

The explicit form of the dilation scale is $I(|\mathbf{p}|, s, \sigma) = 2^{-(2s+1)}(2s+1)(s-\sigma)!(s+\sigma)!\, V(|\mathbf{p}|, s, \sigma).$

The value of $V(|\mathbf{p}|, s, \sigma)$ is shown in Eq. (A2) for integer s and shown in Eq. (A3) for half-integer s:

$$\sum_{k=\max\left(\sigma-\frac{s}{2},-\frac{s}{2}\right)}^{\min\left(\sigma+\frac{s}{2},\frac{s}{2}\right)} \frac{\sqrt{\pi}\, s!\, \vartheta^{2k-\sigma}}{\left(s+\frac{1}{2}\right)!\left(\frac{s}{2}-k\right)!\left(\frac{s}{2}+k\right)!\left(\frac{s}{2}+k-\sigma\right)!\left(\frac{s}{2}-k+\sigma\right)!}. \tag{A2}$$

$$\sum_{k=\max\left(\sigma-\frac{2s-1}{4},-\frac{2s+1}{4}\right)}^{\min\left(\sigma+\frac{2s-1}{4},\frac{2s+1}{4}\right)} \frac{\sqrt{\pi}\left(s-\frac{1}{2}\right)!\ \left(\vartheta^{2k-\sigma}-\vartheta^{\sigma-2k}\right)}{s!\left(\frac{s}{2}-k+\frac{1}{4}\right)!\left(\frac{s}{2}+k+\frac{1}{4}\right)!\left(\frac{s}{2}+k-\sigma-\frac{1}{4}\right)!\left(\frac{s}{2}-k+\sigma-\frac{1}{4}\right)!}. \tag{A3}$$

$$\vartheta_i = \begin{cases} (|\mathbf{p}_i| + E_i)/m_i & \text{for massive particle} \\ |\mathbf{p}_i|/|\mathbf{k}_i| & \text{for massless particle} \end{cases}.$$

Example 2 : $\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right)=(0,0)\oplus(1,0)\oplus(0,1)\oplus(1,1)\,.$

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right)\ \mapsto\ (0,0)\ :\ T^{\mu\nu}=\left(C_0^{\frac{1}{2}\,\frac{1}{2}}\right)_0^{l_1l_2}\left(C_0^{\frac{1}{2}\,\frac{1}{2}}\right)_0^{r_1r_2}T^{\mu}_{l_1r_1}T^{\nu}_{l_2r_2},$$

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right)\ \mapsto\ (1,0)\ :\ T^{\mu\nu}_l=\left(C_1^{\frac{1}{2}\,\frac{1}{2}}\right)_l^{l_1l_2}\left(C_0^{\frac{1}{2}\,\frac{1}{2}}\right)_0^{r_1r_2}T^{\mu}_{l_1r_1}T^{\nu}_{l_2r_2},$$

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right)\ \mapsto\ (0,1)\ :\ T^{\mu\nu}_r=\left(C_0^{\frac{1}{2}\,\frac{1}{2}}\right)_0^{l_1l_2}\left(C_1^{\frac{1}{2}\,\frac{1}{2}}\right)_r^{r_1r_2}T^{\mu}_{l_1r_1}T^{\nu}_{l_2r_2},$$

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right)\ \mapsto\ (1,1)\ :\ T^{\mu\nu}_{\mu^2}=U^{lr}_{\mu^2}\left(C_1^{\frac{1}{2}\,\frac{1}{2}}\right)_l^{l_1l_2}\left(C_1^{\frac{1}{2}\,\frac{1}{2}}\right)_r^{r_1r_2}T^{\mu}_{l_1r_1}T^{\nu}_{l_2r_2},$$

The above ir.tens can be expressed in a familiar way (only contains the Lorentz four-vector indices) as follows,

$$\left(T_{(0,0)}\right)^{\mu\nu,\mu'\nu'}=T^{\mu\nu}T^{\mu'\nu'}=\frac{1}{4}\,g^{\mu\nu}g^{\mu'\nu'},$$

$$\left(T_{[(1,0)\oplus(0,1)]^+}\right)^{\mu\nu,\mu'\nu'}\equiv T^{\mu\nu}_lT^{l\mu'\nu'}+T^{\mu\nu}_rT^{r\mu'\nu'}=\frac{1}{2}\,\left(g^{\mu\mu'}g^{\nu\nu'}-g^{\mu\nu'}g^{\nu\mu'}\right),$$

$$\left(T_{[(1,0)\oplus(0,1)]^-}\right)^{\mu\nu,\mu'\nu'}\equiv T^{\mu\nu}_lT^{l\mu'\nu'}-T^{\mu\nu}_rT^{r\mu'\nu'}=\frac{i}{2}\,\epsilon^{\mu\nu\mu'\nu'},$$

$$\left(T_{(1,1)}\right)^{\mu\nu,\mu'\nu'}=T^{\mu\nu}_{\mu^2}T^{\mu^2\mu'\nu'}=\frac{1}{2}\left(g^{\mu\mu'}g^{\nu\nu'}+g^{\mu\nu'}g^{\nu\mu'}\right)-\frac{1}{4}\,g^{\mu\nu}g^{\mu'\nu'}.$$

Example 3 : $[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]\otimes[(\frac{1}{2},0)\oplus(0,\frac{1}{2})] =$
$(0,0)_L \oplus (0,0)_R \oplus (1,0)\oplus(0,1)\oplus(\frac{1}{2},\frac{1}{2})_L \oplus (\frac{1}{2},\frac{1}{2})_R \,.$

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right] \;\mapsto\; (0,0)_L \;:\; (T_L)^{ab}\,,$$

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right] \;\mapsto\; (0,0)_R \;:\; (T_R)^{ab}\,,$$

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right] \;\mapsto\; (1,0) \;:\; T_l^{ab},$$

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right] \;\mapsto\; (0,1) \;:\; T_r^{ab},$$

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right] \;\mapsto\; \left(\frac{1}{2},\frac{1}{2}\right)_L \;:\; (T_L)_\mu^{ab}\,,$$

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right] \;\mapsto\; \left(\frac{1}{2},\frac{1}{2}\right)_R \;:\; (T_R)_\mu^{ab}\,,$$

Example 3 : $\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right] =$
$$(0,0)_L \oplus (0,0)_R \oplus (1,0) \oplus (0,1) \oplus \left(\tfrac{1}{2},\tfrac{1}{2}\right)_L \oplus \left(\tfrac{1}{2},\tfrac{1}{2}\right)_R .$$

$$
\begin{aligned}
(T_L)^{ab} &= \left(C_0^{\frac{1}{2}\frac{1}{2}}\right)_0^{l_1 l_2} \left(C_0^{00}\right)_0^{r_1 r_2} \ (U_L)^a_{l_1 r_1} \ (U_L)^b_{l_2 r_2} ,\\
(T_R)^{ab} &= \left(C_0^{00}\right)_0^{l_1 l_2} \left(C_0^{\frac{1}{2}\frac{1}{2}}\right)_0^{r_1 r_2} \ (U_R)^a_{l_1 r_1} \ (U_R)^b_{l_2 r_2} ,\\
T_l^{ab} &= \left(C_1^{\frac{1}{2}\frac{1}{2}}\right)_l^{l_1 l_2} \left(C_0^{00}\right)_0^{r_1 r_2} \ (U_L)^a_{l_1 r_1} \ (U_L)^b_{l_2 r_2} ,\\
T_r^{ab} &= \left(C_0^{00}\right)_0^{l_1 l_2} \left(C_1^{\frac{1}{2}\frac{1}{2}}\right)_r^{r_1 r_2} \ (U_R)^a_{l_1 r_1} \ (U_R)^b_{l_2 r_2} ,\\
(T_L)^{ab}_\mu &= \left(C_{\frac{1}{2}}^{\frac{1}{2}0}\right)_l^{l_1 l_2} \left(C_{\frac{1}{2}}^{0\frac{1}{2}}\right)_r^{r_1 r_2} T^{lr}_\mu \ (U_L)^a_{l_1 r_1} \ (U_R)^b_{l_2 r_2} ,\\
(T_R)^{ab}_\mu &= \left(C_{\frac{1}{2}}^{0\frac{1}{2}}\right)_l^{l_1 l_2} \left(C_{\frac{1}{2}}^{\frac{1}{2}0}\right)_r^{r_1 r_2} T^{lr}_\mu \ (U_R)^a_{l_1 r_1} \ (U_L)^b_{l_2 r_2} ,
\end{aligned}
$$

Example 3 : $\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]=$
$(0,0)_L\oplus(0,0)_R\oplus(1,0)\oplus(0,1)\oplus\left(\frac{1}{2},\frac{1}{2}\right)_L\oplus\left(\frac{1}{2},\frac{1}{2}\right)_R.$

Similarly, the above ir.tens can be expressed in a familiar way (only contains the Lorentz four-vector indices and Dirac spinor indices) as follows,

$$\begin{aligned}
\left(T_{(0,0)+}\right)^{ab} &= (T_L)^{ab}+(T_R)^{ab} \equiv g^{ab},\\
\left(T_{(0,0)-}\right)^{ab} &= (T_L)^{ab}-(T_R)^{ab} = g^{ac}\left(\gamma_5\right)_c{}^{b},\\
\left(T_{[1,0]+}\right)^{ab,\mu\nu} &= T_l^{ab}\,T^{l\mu\nu}+T_r^{ab}\,T^{r\mu\nu} = \frac{-i}{\sqrt{2}}g^{ac}\left(\sigma^{\mu\nu}\right)_c{}^{b},\\
\left(T_{[1,0]-}\right)^{ab,\mu\nu} &= T_l^{ab}\,T^{l\mu\nu}-T_r^{ab}\,T^{r\mu\nu} = \frac{-i}{\sqrt{2}}g^{ac}\left(\gamma_5\sigma^{\mu\nu}\right)_c{}^{b},\\
\left(T_{\left(\frac{1}{2},\frac{1}{2}\right)+}\right)^{ab}_{\mu} &= (T_L)^{ab}_{\mu}+(T_R)^{ab}_{\mu} = \frac{1}{\sqrt{2}}g^{ac}\left(\gamma_5\gamma_\mu\right)_c{}^{b},\\
\left(T_{\left(\frac{1}{2},\frac{1}{2}\right)-}\right)^{ab}_{\mu} &= (T_L)^{ab}_{\mu}-(T_R)^{ab}_{\mu} = \frac{1}{\sqrt{2}}g^{ac}\left(\gamma_\mu\right)_c{}^{b},
\end{aligned}$$

where g^{ab} is the *metric* of Dirac spinor space, the explicit form is $g^{ab}=\left[\left(-i\sigma^2\right)\oplus\left(-i\sigma^2\right)\right]^{ab}$.

Example 4 : Consider the spin projection tensors $P_{\beta}^{\alpha_1 \alpha_2}(\mathbf{p}; \chi_1, \chi_2, s)$ with $[\beta] = [\alpha_1] \equiv [\alpha]$ and $[\alpha_2] = (0, 0)$.

$$P_{\alpha}{}^{\alpha'}(\mathbf{p}; \chi_1, \chi_2, s) \;=\; \sum_{\sigma=-s}^{s} u_{\alpha}^{\sigma}(\mathbf{p}; \chi_1, s) \bar{u}_{\sigma}^{\alpha'}(\mathbf{p}; \chi_2, s).$$

By employing the orthogonal-normalization relation

$$\bar{u}_{\sigma}^{\alpha}(\mathbf{p}; \chi, s)\, u_{\alpha}^{\sigma'}(\mathbf{p}; \chi'^{*}, s) = \delta_{\sigma}{}^{\sigma'}\, \delta_{\chi\chi'},$$

one will get

$$P_{\alpha}{}^{\alpha'}(\mathbf{p}; \chi_1, \chi_2^{*}, s)\, u_{\alpha'}^{\sigma}(\mathbf{p}; \chi_1, s') \;=\; \delta_{\chi_1 \chi_2}\, \delta_{ss'}\, u_{\alpha}^{\sigma}(\mathbf{p}; \chi_1, s),$$

such equations are so-called relativistic motion equations of spin-s particles under rep. $[\alpha]$.

- ✍ $[\alpha] = (0, 0)$ & $s = 0$: Klein-Gordon equation.
- ✍ $[\alpha] = \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right]$ & $s = \frac{1}{2}$: Dirac equation.
- ✍ $[\alpha] = \left(\frac{1}{2}, \frac{1}{2}\right)$ & $s = 1$: Proca equation.
- ✍ $[\alpha] = \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right] \otimes \left(\frac{1}{2}, \frac{1}{2}\right)$ & $s = \frac{3}{2}$: Rarita-Schwinger equation for spin-$\frac{3}{2}$.

  $\cdots$ and so on.

Example 5 : The pure-orbital wave function $\tilde{t}^{(L)}_{\mu_1\cdots\mu_L}$ between two particles ($q_\mu = p_{1\mu} - p_{2\mu}$).

Consider the spin projection tensor of **Example 4** with $[\alpha] = \left(\frac{L}{2}, \frac{L}{2}\right) \equiv [\mu^L]$,

$$P_{\mu^L}{}^{\nu^L}\left(\mathbf{p}; [\mu^L], [\mu^L], s\right) = \sum_{\sigma=-s}^{s} u^{\sigma}_{\mu^L}\left(\mathbf{p}; [\mu^L], s\right) \bar{u}^{\nu^L}_{\sigma}\left(\mathbf{p}; [\mu^L], s\right).$$

Because of the following direct product decomposition,

$$\underbrace{\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right) \otimes \cdots \otimes \left(\frac{1}{2}, \frac{1}{2}\right)}_{L} = \left(\frac{L}{2}, \frac{L}{2}\right) \oplus \cdots \quad \mapsto \quad T^{\mu^L}_{\mu_1\cdots\mu_L},$$

the indices μ^L and ν^L can be replaced by Lorentz indices as follows,

$$P^{\nu_1\cdots\nu_L}_{\mu_1\cdots\mu_L}\left(\mathbf{p}; [\mu^L], [\mu^L], s\right) = T^{\mu^L}_{\mu_1\cdots\mu_L} T^{\nu_1\cdots\nu_L}_{\nu^L} P_{\mu^L}{}^{\nu^L}\left(\mathbf{p}; [\mu^L], [\mu^L], s\right).$$

The triangle relation under ir.rep $[\mu^L]$ is $0 \leq s \leq L$, take $s = L$ and one will get

$$\tilde{t}^{(L)}_{\mu_1\cdots\mu_L} = P^{\nu_1\cdots\nu_L}_{\mu_1\cdots\mu_L}\left(\mathbf{p}; [\mu^L], [\mu^L], L\right) q_{\nu_1} \cdots q_{\nu_L},$$

where the Lorentz indices $\mu_1 \cdots \mu_L$ are symmetrical and traceless.