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Improved analytic solution of Kerr (-Newman) black hole superradiance

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Abstract

The superradiant instabilities of Kerr(-Newman) black holes with charged or uncharged massive **spin-0 fields** are calculated analytically to the **next-to-leading order** in the limit of α ~ $r_g\mu \ll 1$ [1, 2, 3]. The next-to-leading order (NLO) result has a compact form and is in **good agreement with existing numerical calculations** [4, 5].

[1]. S. S. Bao, **QX** and H. Zhang, Phys. Rev. D 106 (2022) no.6, 064016
[2]. S. S. Bao, **QX** and H. Zhang, Phys. Rev. D 107 (2023) no.6, 064037
[3]. S. L. Detweiler, Phys. Rev. D 22 (1980), 2323-2326
[4]. H. Furuhashi and Y. Nambu, Prog. Theor. Phys. 112 (2004), 983-995
[5]. S. R. Dolan, Phys. Rev. D 76 (2007), 084001

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Black Hole Superradiance

Introduction

Boson condensate could form around a rotating black hole (BH) if the boson's Compton wavelength is comparable to the size of the BH horizon. If a light scalar boson exists with a proper value of mass, it could form **gravitational bound states** around spinning BHs (引力原子). The bound states can continuously **extract energy and angular momentum** from the host BHs until the angular momentum of the BH is below some critical value.

(Superradiance condition: $0 < \omega < \omega_c$)

Applied in many research frontiers

- Study the stability of spinning BHs.
- **The search of axion-like-particles.**
- The superradiant boson clouds could modify the **gravitational waveform** of two-BH-merger events.
- The gravitation wave signals generated by the spinning boson clouds around the host BHs.

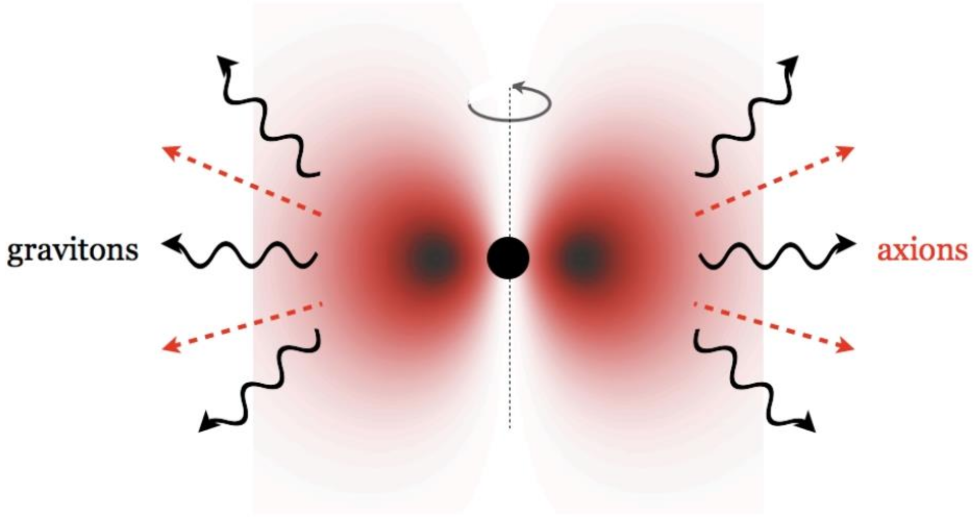

Figure 1: The spinning black hole "feeds" superradiant states
copy from Ref. [6]

[6] A. Arvanitaki, S. Dubovsky. Phys.Rev.D 83 (2011) 044026

Analytic Solution

The phenomenological study of BH superradiance depends on the accurate determination of the bound state's eigenfrequency ω.

Scalars in Kerr-Newman spacetime

$$ds^2 = -\left(1 - \frac{2r_g r - Q^2}{\Sigma^2}\right) dt^2 + \frac{\Sigma^2}{\Delta} dr^2 + \Sigma^2 d\theta^2 + \left[(r^2 + a^2) + \frac{(2r_g r - Q^2)a^2 \sin^2\theta}{\Sigma^2}\right] \sin^2\theta d\varphi^2 - \frac{2(2r_g r - Q^2)a \sin^2\theta}{\Sigma^2} dt d\varphi, \quad \text{Eq. (1)}$$

with

$$\Sigma^2 = r^2 + a^2 \cos^2\theta, \quad \text{Eq. (2)}$$

$$\Delta = r^2 - 2r_g r + a^2 + Q^2. \quad \text{Eq. (3)}$$

The equation $\Delta = 0$ gives two event horizons at $r_\pm = r_g \pm b$ with $b = \sqrt{r_g^2 - a^2 - Q^2}$.

Klein-Gordon equation

$$(\nabla^\alpha - iqA^\alpha)(\nabla_\alpha - iqA_\alpha)\phi - \mu^2\phi = 0, \quad \text{Eq. (4)}$$

where μ and q are the mass and electric charge of the scalar field, respectively.

Analytic Solution

Radial equation

The scalars can be written with the separation of variables

$$\phi(t,r,\theta,\varphi)=R_{lm}(r)S_{lm}(\theta)e^{im\varphi}e^{-i\omega t}. \qquad \text{Eq. (5)}$$

Inserting it into Eq. (4)

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR_{lm}}{dr}\right)+\left[\omega^2\left(r^2+a^2\right)^2-4ar_g rm\omega+a^2m^2-\left(\mu^2r^2+a^2\omega^2+\Lambda_{lm}\right)\Delta\right]R_{lm}=0, \qquad \text{Eq. (6)}$$

where Λ_{lm} is the separation constant which has the expanded form $\Lambda_{lm}=l(l+1)+\mathcal{O}(\alpha^4)$.

Analytic Solution

Asymptotic matching method

The basic idea of this method is to first obtain **two solutions** of the radial equation in large r and small r respectively. The two solutions have an **overlap region** in the small α limit. Then according to the same behavior for these two solution in the overlap region, we can obtain the eigenequation of ω, which can be solved numerically or perturbatively.

It is convenient to define

$$\omega = \omega_0 + \omega_1 \delta\lambda, \qquad \text{Eq. (7)}$$

where $\omega_1 \delta\lambda$ is the imaginary part of ω.

Analytic Solution

Large r limit

The radial function at the $r \gg r_g$ limit can be simplified as

$$\frac{d^2}{dr^2}(rR) + \left[(\omega^2 - \mu^2) + \frac{2(2r_g\omega^2 - r_g\mu^2 - qQ\omega)}{r} - \frac{l'(l'+1)}{r^2} + \mathcal{O}(r^{-3})\right] rR = 0, \quad \text{Eq. (8)}$$

and the solution can be written in terms of the confluent hypergeometric function up to an arbitrary normalization,

$$R(r) = e^{-\kappa r}(2\kappa r)^{l'+1} U(l' + 1 - \lambda, 2l' + 2; 2\kappa r), \quad \text{Eq. (9)}$$

where

$$\lambda = \frac{2r_g\omega^2 - r_g\mu^2 - qQ\omega}{\kappa}, \quad \kappa = \sqrt{\mu^2 - \omega^2}. \quad \text{Eq. (10)}$$

Here, $l' = l + \epsilon$, $\epsilon \sim \mathcal{O}(\alpha^2)$ plays the role of a regulator in LO.

Analytic Solution

Small r limit

It is more convenient to write the radial function in terms of $z = (r - r_+)/2b$ in the small r limit,

$$z(z+1)\frac{d}{dz}\left[z(z+1)\frac{dR}{dz}\right] + U(z)R = 0, \qquad \text{Eq. (11)}$$

where

$$\begin{aligned} U(z) = p^2 + z\left[\frac{4r_g r_+\omega}{b}\left(r_+\omega - \frac{am}{2r_+} - \frac{Q^2\omega}{2r_g}\right) - (\Lambda_{lm} + r_+^2\mu^2 + a^2\omega^2) + \frac{qQ}{b}(am + r_+ qQ - a^2\omega - 3r_+^2\omega)\right] \\ + z^2(a^2\omega^2 - \Lambda_{lm} + 2\mu^2 a^2 - 3\mu^2 r_+^2 + 6r_+^2\omega^2 + 2Q^2\mu^2 + q^2Q^2 - 6r_+ qQ\omega) \\ + 4z^3 b\left[r_g\mu^2 + 2r_+(\omega^2 - \mu^2) - qQ\omega\right] + 4z^4 b^2(\omega^2 - \mu^2), \end{aligned} \qquad \text{Eq. (12)}$$

in which,

$$p = \frac{(r_+^2 + a^2)}{2b}(\omega - \omega_c). \qquad \text{Eq. (13)}$$

Analytic Solution

Eq. (11) can be simplified as,

$$z(z+1)\frac{d}{dz}\left[z(z+1)\frac{dR}{dz}\right]+\left[p^2-l'(l'+1)z(1+z)\right]R=0. \qquad \text{Eq. (14)}$$

The solution is,

$$R(r)=\left(\frac{r-r_+}{r-r_-}\right)^{-ip}{}_2F_1\left(-l',l'+1;1-2ip;-\frac{r-r_+}{2b}\right), \qquad \text{Eq. (15)}$$

up to an arbitrary normalization.

Analytic Solution

Matching two solutions

Two solutions have an overlapped region in the limit $\alpha \ll 1$.

$$\text{Eq. (9)} \quad \xrightarrow{r\to 0} \quad \frac{(2\kappa)^{l'}\Gamma(-2l'-1)}{\Gamma(-l'-\lambda)} r^{l'} + \frac{(2\kappa)^{-l'-1}\Gamma(2l'+1)}{\Gamma(l'+1-\lambda)} r^{-l'-1}.$$

$$\text{Eq. (15)} \quad \xrightarrow{r\to +\infty} \quad \frac{(2b)^{-l'}\Gamma(2l'+1)}{\Gamma(l'+1)\Gamma(l'+1-2ip)} r^{l'} + \frac{(2b)^{l'+1}\Gamma(-2l'-1)}{\Gamma(-l'-2ip)\Gamma(-l')} r^{-l'-1}.$$

The ratio of the coefficients of the $r^{l'}$ and $r^{-l'-1}$ should be the same for the two solutions in the overlap region. After some algebra, one can arrive at,

$$\lambda = \frac{r_g(2\omega_0^2-\mu^2) - qQ\omega_0}{\sqrt{\mu^2-\omega_0^2}} + \frac{r_g\omega_0\omega_1(3\mu^2-2\omega_0^2) - qQ\mu^2\omega_1}{(\mu^2-\omega_0^2)^{3/2}}\delta\lambda^{(0)} + \mathcal{O}\left((\delta\lambda^{(0)})^2\right) \quad \text{Eq. (16)}$$

with

$$\delta\lambda^{(0)} = -ip\,(4\kappa b)^{2l+1}\frac{(n+2l+1)!(l!)^2}{n!\,[(2l)!(2l+1)!]^2}\prod_{j=1}^{l}(j^2+4p^2) \quad \text{Eq. (17)}$$

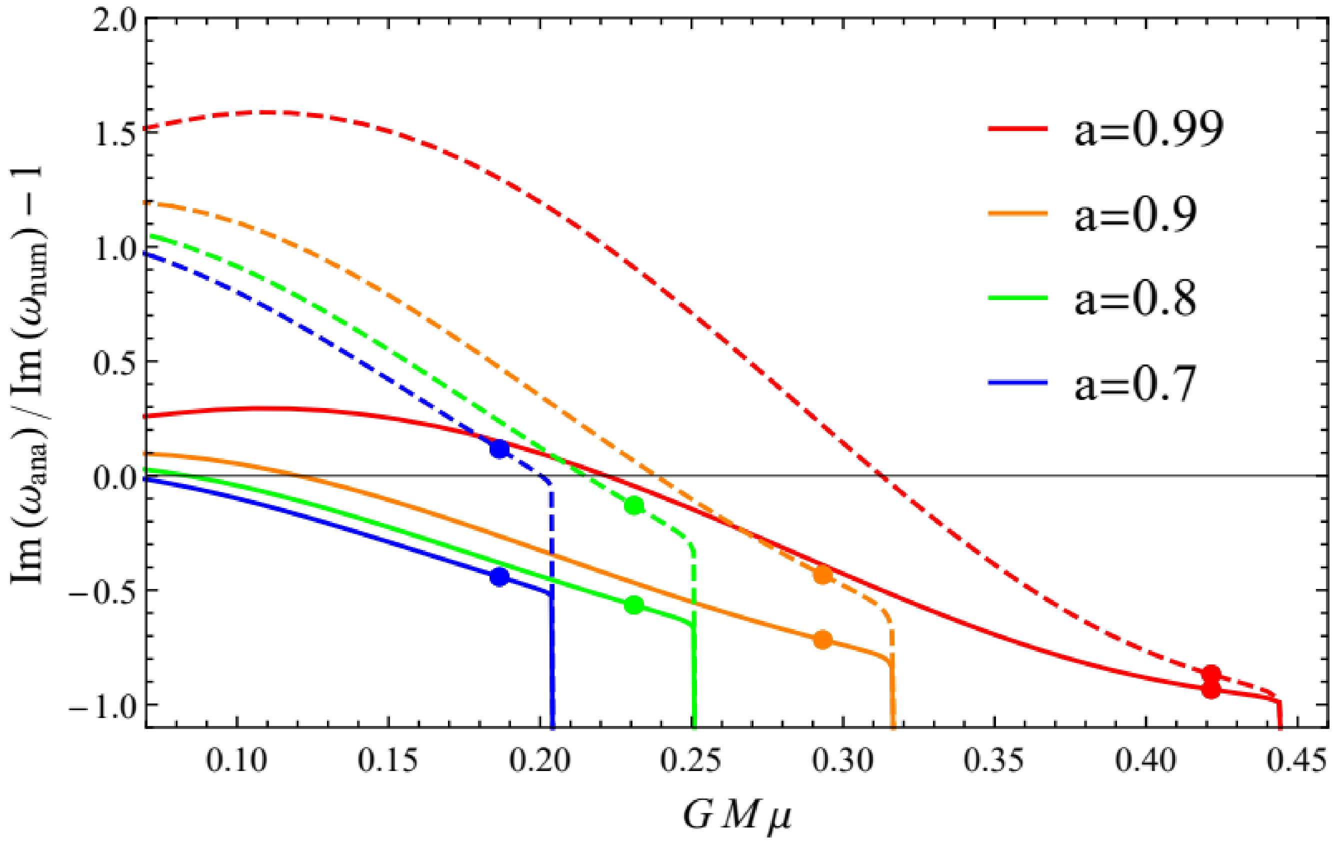

Figure 2: Comparison of the numerical result and the analytic approximation. The solid curves and the dashed curves are from Eq. (17) and Ref. [3], respectively. The numerical values from Ref. [4].

[3]. S. L. Detweiler, Phys. Rev. D 22 (1980), 2323-2326.
[5]. S. R. Dolan, Phys. Rev. D 76 (2007), 084001

Next-to-leading-order approximation

The first NLO correction appears as ϵ in the asymptotic radial wave function at large r, which is given in Eq. (9). It can be calculated from the definition of $l' = l + \epsilon$. Here, $\epsilon \sim \mathcal{O}(\alpha^2)$ plays the role of a regulator in LO.

$$\epsilon = \frac{-8r_g^2\mu^2 + Q^2\mu^2 + 8r_g qQ\mu - q^2Q^2}{2l+1} + \mathcal{O}(\alpha^4). \qquad \text{Eq. (18)}$$

The second NLO contribution is from the asymptotic radial wave function at small r. The potential $U(z)$ defined in Eq. (12) can be approximated by $p^2 - l'(l'+1)z(z+1) + zd$, where d is defined as,

$$d = (4r_g\mu - 2qQ)p - 2(4r_g - r_+)r_g\mu^2 + 2\mu qQ(4r_g - r_+) - q^2Q^2 + \mathcal{O}(\alpha^3). \qquad \text{Eq. (19)}$$

Here $l'(l'+1) = \Lambda_{lm} + 4r_g^2(\mu^2 - 3\omega^2) + a^2(\omega^2 - \mu^2) + Q^2(2\omega^2 - q^2 - \mu^2) + 8r_g qQ\omega.$

Thus, the corresponding radial function at NLO is,

$$R(r) = \frac{(r-r_-)^{\sqrt{d-p^2}}}{(r-r_+)^{ip}}\, {}_2F_1\Big(-l' - ip + \sqrt{d-p^2},\ l' + 1 - ip + \sqrt{d-p^2}; 1 - 2ip; -\frac{r-r_+}{2b}\Big). \qquad \text{Eq. (20)}$$

Next-to-leading-order approximation

Following similar matching steps, the NLO contribution of $\delta\lambda$ could be obtained after some algebra,

$$\delta\lambda^{(1)} = \left(\frac{d}{2\epsilon} - \frac{\epsilon}{2} - ip\right) \frac{(4\kappa b)^{2l'+1}\,\Gamma(n+2l'+2)\Gamma_{pd}}{n!\,[\Gamma(2l'+1)\Gamma(2l'+2)]^2}, \qquad \text{Eq. (21)}$$

where Γ_{pd} is defined as,

$$\Gamma_{pd} = \frac{\left|\Gamma(l'+1+ip+\sqrt{d-p^2})\Gamma(l'+1+ip-\sqrt{d-p^2})\right|^2 \Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-ip-\sqrt{d-p^2}-\epsilon)\Gamma(1+ip+\sqrt{d-p^2}+\epsilon)\Gamma(1-ip+\sqrt{d-p^2}-\epsilon)\Gamma(1+ip-\sqrt{d-p^2}+\epsilon)}. \qquad \text{Eq. (22)}$$

Results

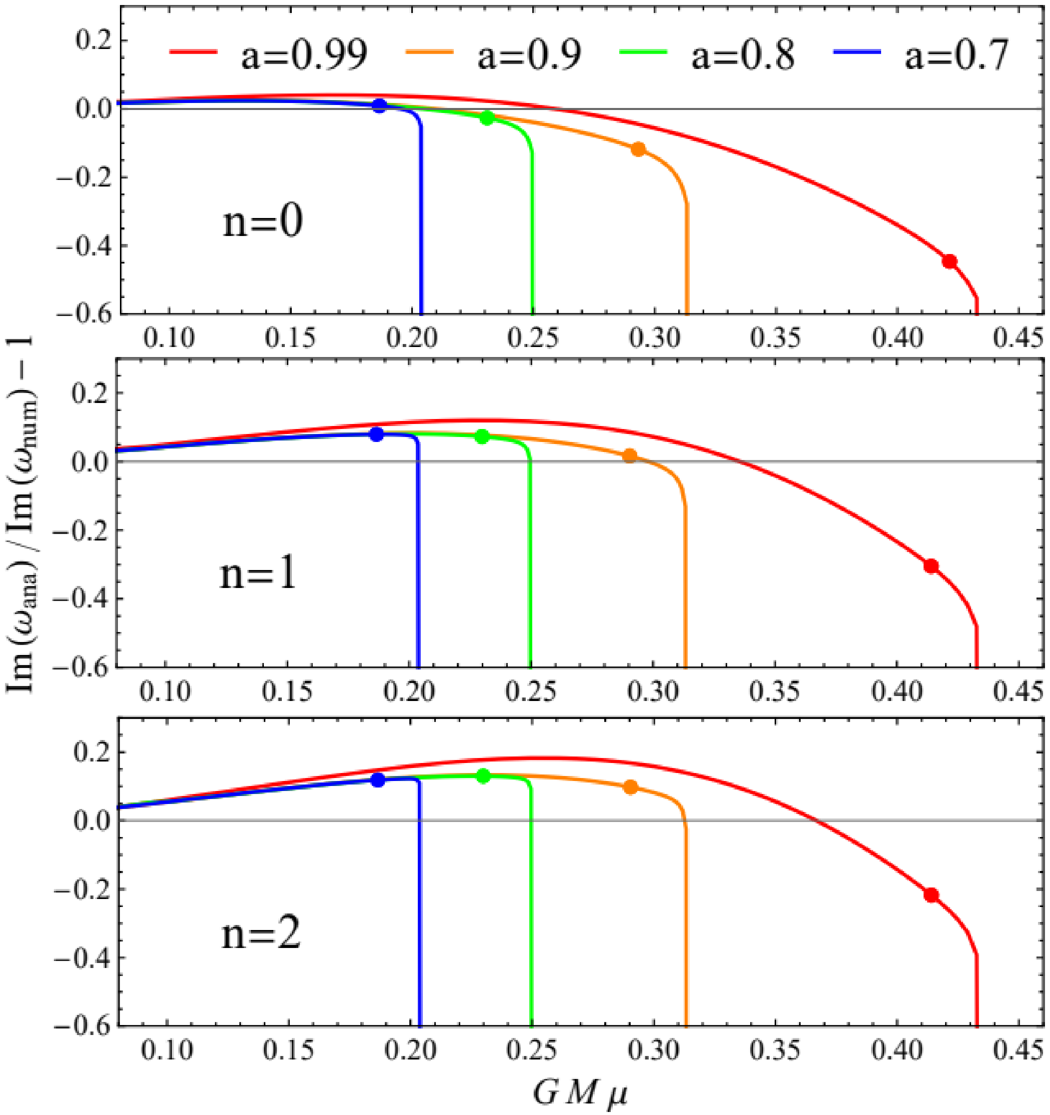

Figure 3: Comparison of the numerical result and the improved analytic approximations in Eq.(21) for $n = 0, 1{,}2$ and $l = m = 1$.

Results

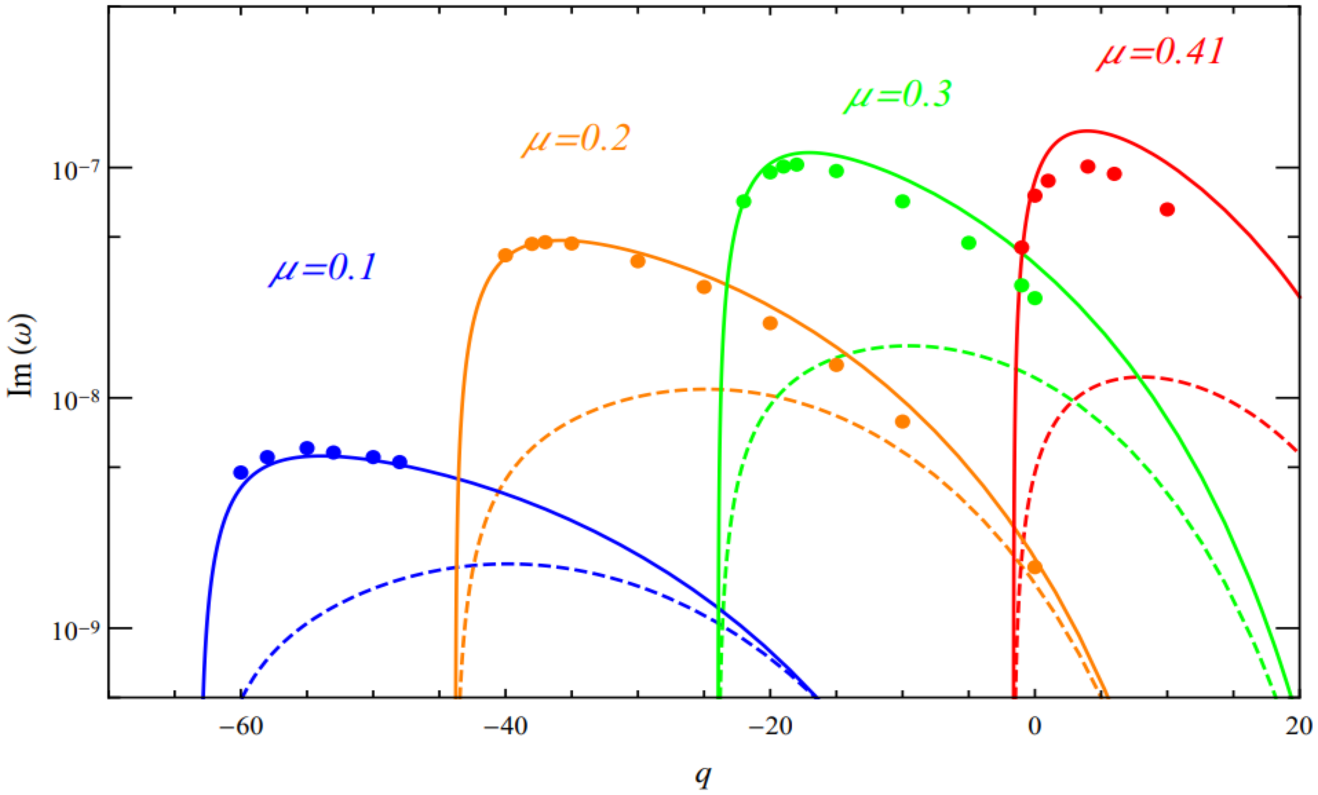

Figure 4: Comparison of the numerical result and the analytic approximations for $n = 0$, $l = m = 1$, $a = 0.98$, and $Q = 0.01$, with r_g chosen to be 1 for compacity. The imaginary part of ω is plotted as a function of the scalar field charge q. The dashed (solid) curves are the LO (NLO) approximations and the scattered dots are numerical results taken from Fig. 6 in Ref. [4]. The curves with different colors correspond to different values of μ, labeled above the corresponding curves with the same color.

[4]. H. Furuhashi and Y. Nambu, Prog. Theor. Phys. 112 (2004), 983-995

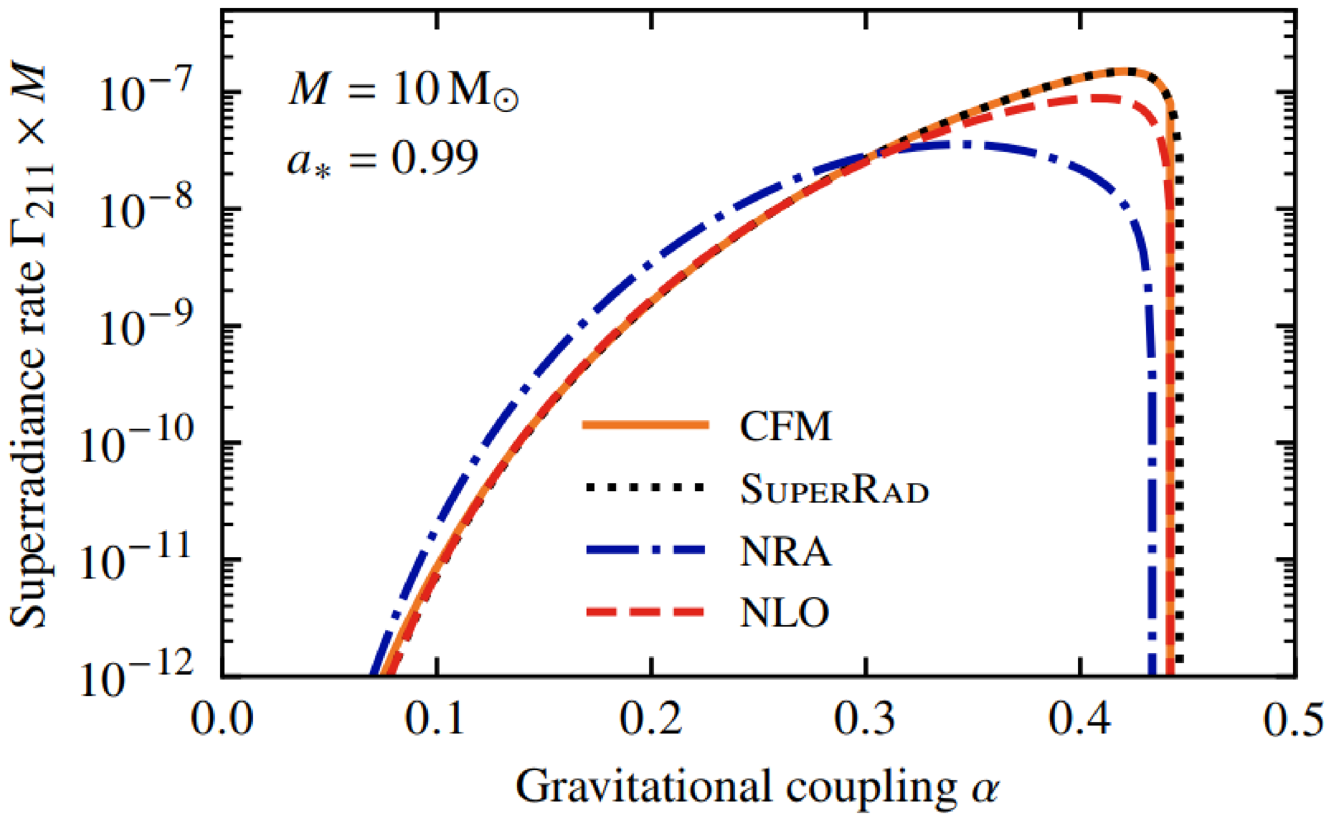

Figure 5: Copy from arXiv: 2406.10337. The |211⟩ superradiance rate (times the gravitational radius, M), computed using different methods. For benchmark values of M = 10 M⊙ and $a*$ = 0.99, they compare the continued fraction method (CFM; orange line), implementation of the next-to-leading-order corrections (NLO; dashed, red line).

Summary

- From the LO analytic approximation given by Ref. [3], the percentage error at small α increases with the BH spin parameter a and/or charge Q. We find there is a NLO term which is enhanced by a factor of $\frac{1}{b}$. For such **nearly extremal BHs**, this term can be as important as the LO contribution. So, for the case of Kerr BHs, by comparing the NLO solution to the numerical result, we find they agree to each other at small α with different values of a.

- For the case of KNBHs, the percentage error of the NLO increases with α, from **a few percent** for $\alpha \sim 0.1$ to about 50% for $\alpha \sim 0.4$.

[3]. S. L. Detweiler, Phys. Rev. D 22 (1980), 2323-2326.

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Reference

[1]. S. S. Bao, **QX** and H. Zhang, *Phys. Rev. D 106 (2022) no.6, 064016*

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[4]. H. Furuhashi and Y. Nambu, *Prog. Theor. Phys. 112 (2004), 983-995*

[5]. S. R. Dolan, *Phys. Rev. D 76 (2007), 084001*

[6] A. Arvanitaki, S. Dubovsky. *Phys.Rev.D 83 (2011) 044026*

Appendix

The wave version of the Penrose process

Kerr: has Killing vector $\hat{t} = \partial_t$

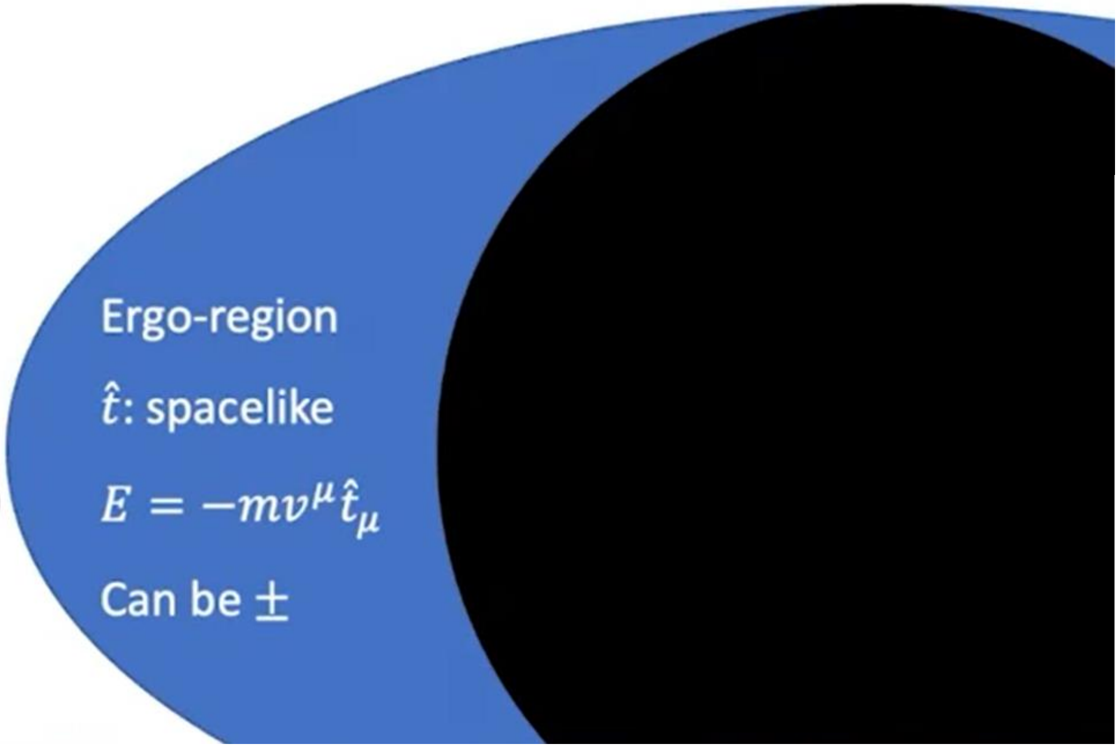
Killing energy $E = -p^{\mu}\hat{t}_{\mu} = -m\frac{dx^{\mu}}{d\tau}\hat{t}_{\mu} = -mv^{\mu}\hat{t}_{\mu}$

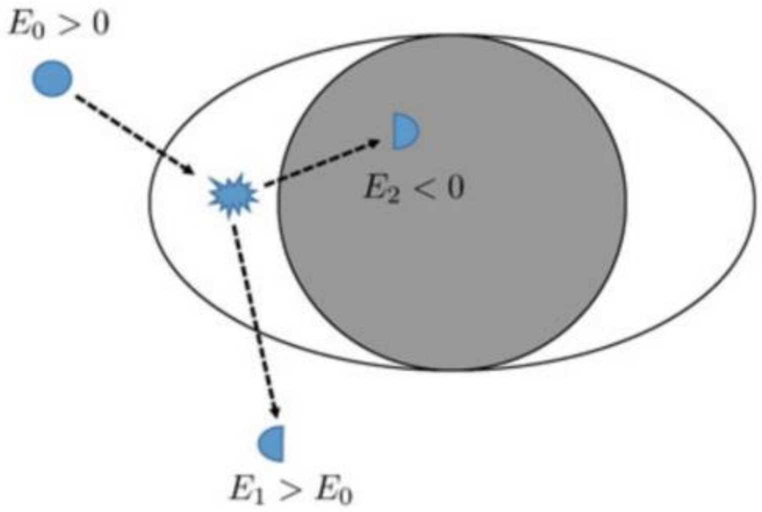
is conserved along geodesic

Exterior region

$\hat{t}$: timelike

$E = -mv^{\mu}\hat{t}_{\mu} > 0$





Appendix

Schrodinger-like equation

To obtain a constraint on the parameters that allow superradiance, we change to the tortoise coordinates,

$$dr_* = \frac{r^2 + a^2}{\Delta} dr,$$

with which the interesting region $r \in (r_+, +\infty)$ corresponds to $r_* \in (-\infty, +\infty)$. We also define,

$$R_*(r_*) = \sqrt{r^2 + a^2} R(r).$$

Then Eq. (6) can be rewritten into a Schrödinger-like equation,

$$\frac{d^2 R_*(r_*)}{dr_*^2} - V(r) R_*(r_*) = 0,$$

where the effective potential is,

$$\begin{aligned} V(r) = &- \left(\omega - \frac{am + qQr}{a^2 + r^2} \right)^2 + \frac{\Delta \mu^2}{a^2 + r^2} \\ &- \frac{\Delta}{(a^2 + r^2)^2} \left[2am\omega - \Lambda_{lm} + a^2(\mu^2 - \omega^2) \right] \\ &+ \frac{\Delta[\Delta + 2r(r - r_g)]}{(a^2 + r^2)^3} - \frac{3\Delta^2 r^2}{(a^2 + r^2)^4}. \end{aligned}$$

Critical frequency

In the region close to the outer horizon r_+, the potential has the asymptotic form,

$$\lim_{r \to r_+} V(r) = -(\omega - \omega_c)^2 + \mathcal{O}(r - r_+),$$

where the critical frequency is defined as

$$\omega_c = \frac{ma + qQr_+}{r_+^2 + a^2} = \frac{ma + qQr_+}{2r_g r_+ - Q^2}.$$

Appendix

Eigenfrequency

$$l' + 1 - \lambda = -n - \delta\lambda, \tag{35}$$

where $|\delta\lambda| \ll 1$ and n is zero or a positive integer. Following the convention in literature, we also define $\bar{n} = n + l + 1$. Then the above relation is re-expressed as $\lambda = \bar{n} + \epsilon + \delta\lambda$. At LO of α, it reduces to $\lambda = \bar{n} + \delta\lambda$.

$$\lambda = \frac{r_g(2\omega_0^2 - \mu^2) - qQ\omega_0}{\sqrt{\mu^2 - \omega_0^2}} + \frac{r_g\omega_0\omega_1(3\mu^2 - 2\omega_0^2) - qQ\mu^2\omega_1}{(\mu^2 - \omega_0^2)^{3/2}}\delta\lambda^{(0)} + \mathcal{O}\left((\delta\lambda^{(0)})^2\right). \tag{37}$$

On the other hand, we have $\lambda = \bar{n} + \delta\lambda^{(0)}$ from Eq. (35). Then it is straightforward to get,

$$\frac{r_g(2\omega_0^2 - \mu^2) - qQ\omega_0}{\sqrt{\mu^2 - \omega_0^2}} = \bar{n}, \tag{38a}$$

$$\frac{r_g\omega_0\omega_1(3\mu^2 - 2\omega_0^2) - qQ\mu^2\omega_1}{(\mu^2 - \omega_0^2)^{3/2}} = 1. \tag{38b}$$

$$\frac{\omega_0^{(0)}}{\mu} = 1 - \frac{1}{2}\left(\frac{r_g\mu - qQ}{\bar{n}}\right)^2 + \mathcal{O}(\alpha^4).$$

$$\frac{\omega_1^{(0)}}{\mu} = \frac{(r_g\mu - qQ)^2}{\bar{n}^3} + \mathcal{O}(\alpha^4).$$

Appendix

Missing factor

The calculation with the regulator is straightforward since both Γ functions are well defined. One could safely use $\Gamma(1+z) = z\Gamma(z)$ repeatedly and get,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\Gamma(-2l-1-2\epsilon)}{\Gamma(-l-\epsilon)} \\
&= \lim_{\epsilon\to 0}\frac{(-l-\epsilon)\ldots(-\epsilon)\Gamma(1-2\epsilon)}{(-2l-1-2\epsilon)\ldots(-2\epsilon)\Gamma(1-\epsilon)} \\
&= \frac{(-1)^{l+1}l!}{2(2l+1)!}.
\end{aligned} \tag{A1}$$

This result could also be obtained without the regulator. The following steps are provided by an anonymous referee and we list them here to show the readers a different way of doing the calculation. The key formula is

$$\Gamma\left(-\frac{n}{2}\right) = \frac{(-1)^{\frac{n+1}{2}}2^n\sqrt{\pi}}{n!}\left(\frac{n-1}{2}\right)!, \tag{A2}$$

which is valid when n is a positive odd integer. We will also use $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$. Then

$$\begin{aligned}
\frac{\Gamma(-2l-1)}{\Gamma(-l)} &= \frac{1}{(-2l-1)}\frac{\Gamma(-2l)}{\Gamma(-l)} = \frac{2^{-2l-1}}{(-2l-1)\sqrt{\pi}}\Gamma\left(\frac{-2l+1}{2}\right) \\
&= \frac{2^{-2l-1}}{(-2l-1)\sqrt{\pi}}\frac{(-1)^l 2^{2l-1}\sqrt{\pi}(l-1)!}{(2l-1)!} \\
&= \frac{(-1)^{l+1}l!}{2(2l+1)!}.
\end{aligned} \tag{A3}$$