Effective range expansion with the left-hand cut

Meng-Lin Du (UESTC)

University of Electronic Science and Technology of China (电子科技大学)

In collaboration with A. Fillin, V. Baru, X.-K. Dong, E. Epelbaum, F.-K. Guo, C. Hanhart, A. Nefediev, J. Nieves, Q. Wang, and B. Wu

Based on PRL 131,131903 (2023), and arXiv:2408.09375[hep-ph]

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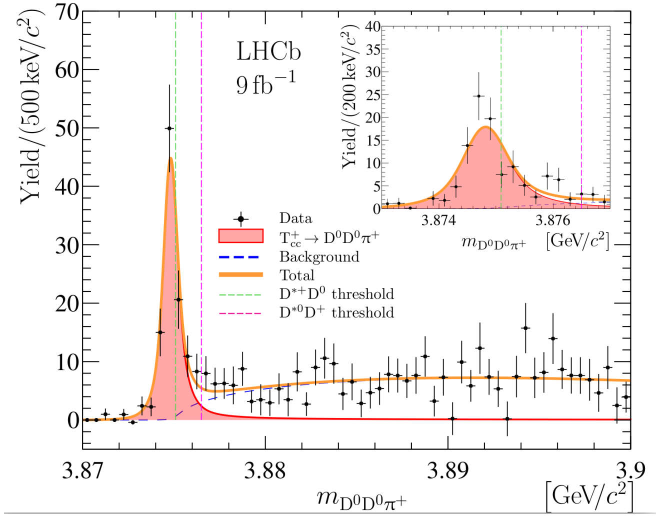
Doubly charmed tetraquark (Tcc)

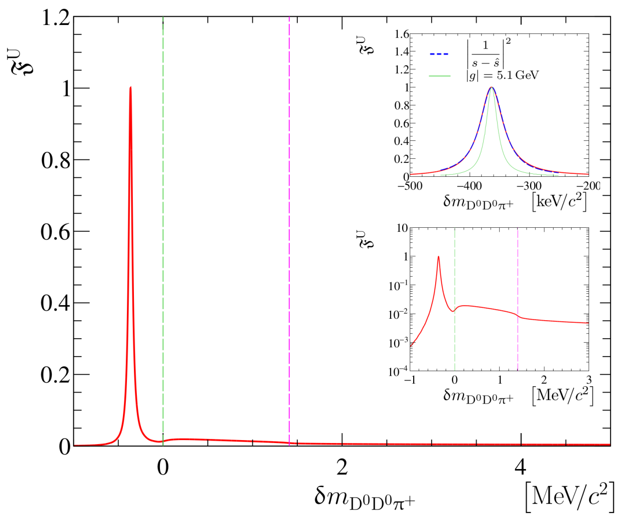


Breit-Wigner fit

LHCb, Nature Phys. 18, (2022) 751

Parameter	Value
N	117 ± 16
$\delta m_{\rm BW}$	-273 ± 61 keV
$\Gamma_{\rm BW}$	410 ± 165 keV

☞ $\mathfrak{R} \sim 400$ keV.



Unitarized and analytical

LHCb, Nature Commun. 13 (2022), 3351

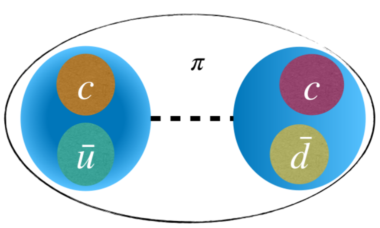
$\delta m = m_{T_{cc}^+} - m_{D^{*+}} - m_{D^0}$

$$\delta m_{\rm pole} = -360 \pm 40^{+4}_{-0} \text{ keV}$$

$$\Gamma_{\rm pole} = 48 \pm 2^{+0}_{-14} \text{ keV}$$

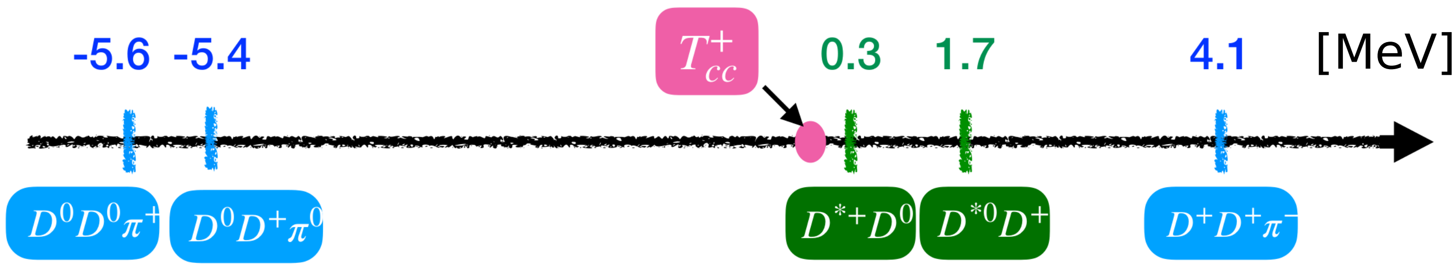
☞ $I = 0$: isoscalar

↪ $D^+D^0\pi^+$, D^+D^+ ✗



☞ T_{cc}^+ resides near D^*D thresholds LHCb, Nature Commun. 13 (2022)
↪ approximate 90% of $D^0D^0\pi^+$ events contain a D^{*+}.

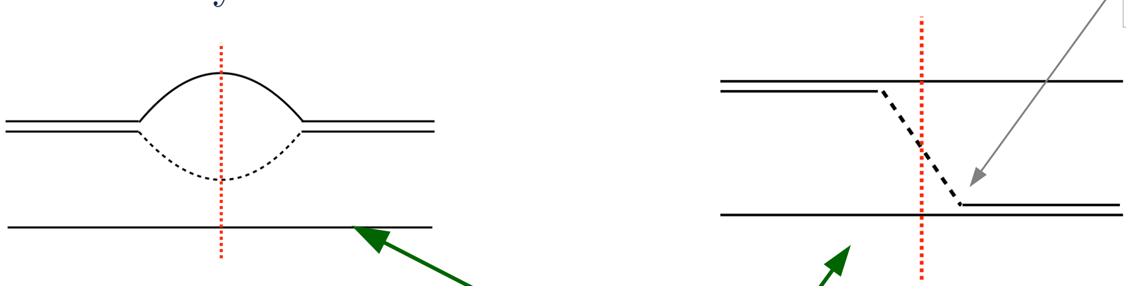
The three-body cut



☞ LO Chiral Lagrangian (g determined from $D^* \to D\pi$)

$$\mathcal{L} = \frac{1}{4} g \,\mathrm{Tr}\left(\vec{\sigma}\cdot\vec{u}_{ab} H_b H_a^\dagger\right)$$

☞ Three-body cuts



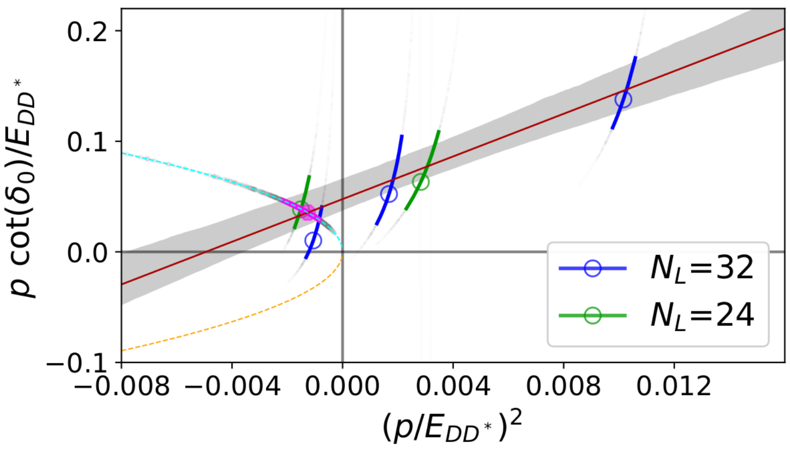
☞ $$U_\alpha(M,p) = P_\alpha - \sum_\beta \int \frac{d^3\vec{q}}{(2\pi)^3} V_{\alpha\beta}(M,p,q) G_\beta(M,q) U_\beta(M,q)$$

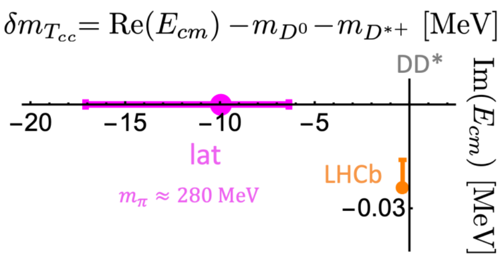
$$\hookrightarrow \quad G_\alpha(M,p) = \frac{1}{m_\alpha^* + m_\alpha + \frac{p^2}{2\mu_\alpha} - M - \frac{i}{2}\Gamma_\alpha(M,p)}$$

Doubly Charm Tetraquark on the Lattice

Padmanath *et al*, PRL129,032002(2022)

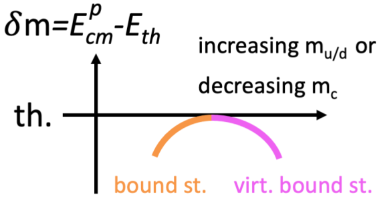
	m_D (MeV)	m_{D^*} (MeV)	$M_{\rm av}$ (MeV)	$a_{l=0}^{(J=1)}$ (fm)	$r_{l=0}^{(J=1)}$ (fm)	$\delta m_{T_{cc}}$ (MeV)	T_{cc}
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(h)}$)	1927(1)	2049(2)	3103(3)	1.04(29)	$0.96(^{+0.18}_{-0.20})$	$-9.9^{+3.6}_{-7.2}$	Virtual bound st.
Lattice ($m_\pi \simeq 280$ MeV, $m_c^{(l)}$)	1762(1)	1898(2)	2820(3)	0.86(0.22)	$0.92(^{+0.17}_{-0.19})$	$-15.0(^{+4.6}_{-9.3})$	Virtual bound st.
Experiment [2,41]	1864.85(5)	2010.26(5)	3068.6(1)	−7.15(51)	[−11.9(16.9), 0]	−0.36(4)	Bound st.



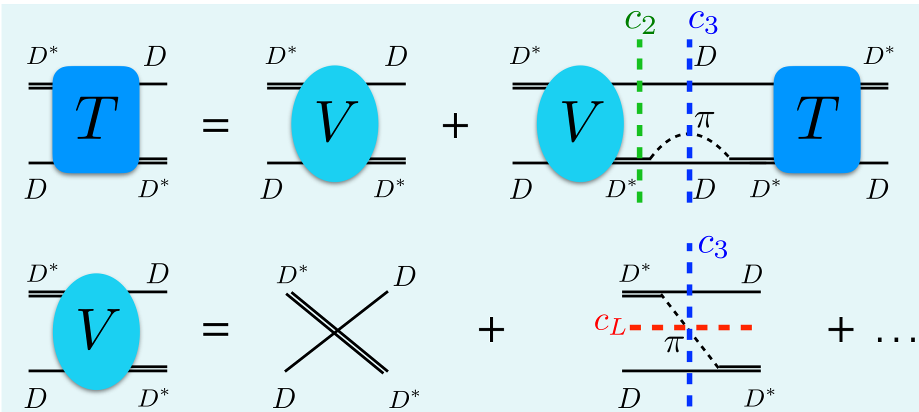


$$t = \frac{E_{\rm cm}}{2}\frac{1}{p\cot\delta - ip},$$

$$p\cot\delta = \frac{1}{a_0} + \frac{1}{2}r_0 p^2,$$

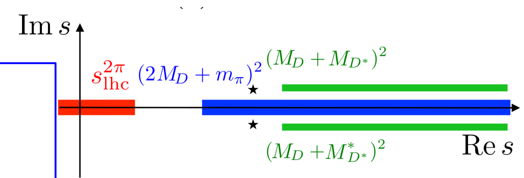


The three-body cut vs. left-hand cut



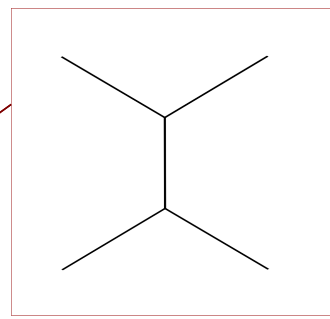
☞ three-body cut

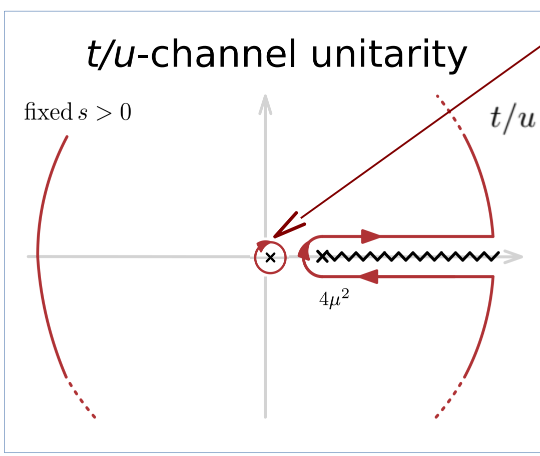
$E > M_D + M_D + M_\pi$



☞ left-hand cut

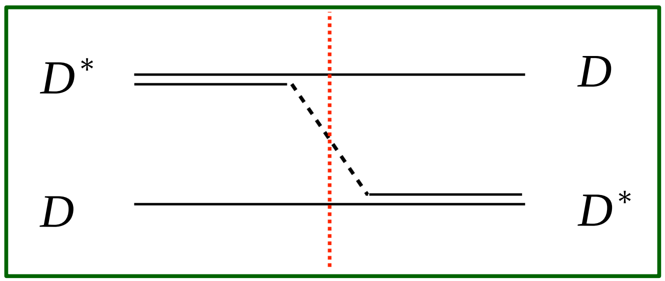
$$\int_{-1}^{1} d\cos\theta G_\pi(E,p,p)$$

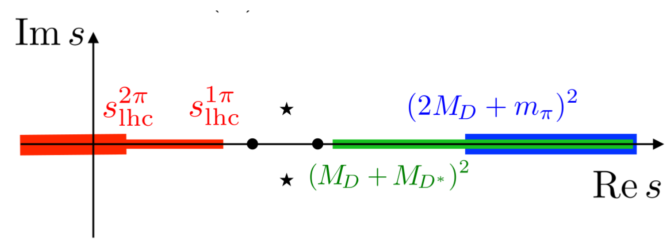


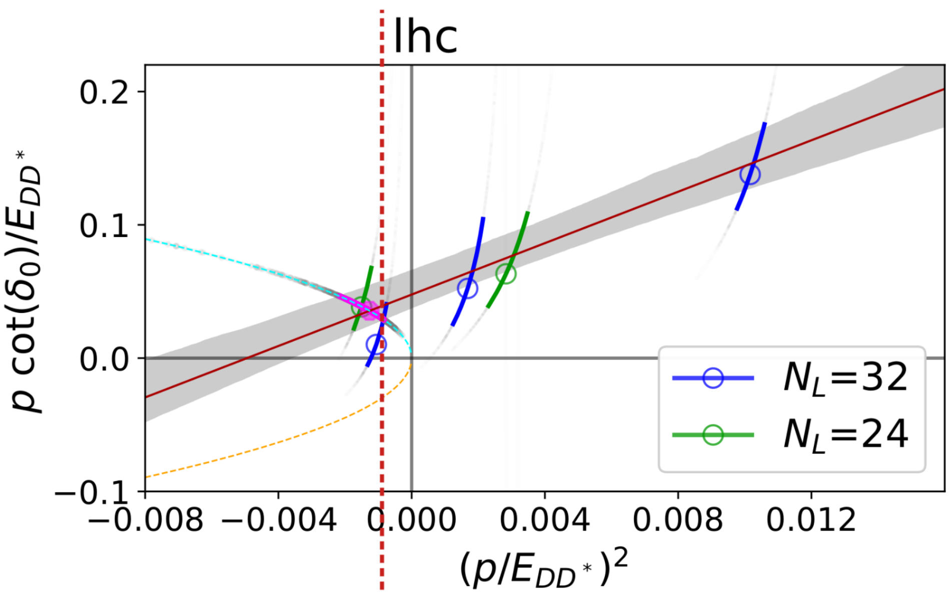


$$G_\pi^{-1}(E,\mathrm{k},\mathrm{k}') \xrightarrow[\text{on shell: } k=k'=p]{\cos\theta=\pm 1} E_{D*}(p^2) - E_D(p^2) - \omega_\pi(4p^2/0) = 0$$

The left-hand cut







$m_\pi = 280$ MeV

☞ two-body branch point:

$E = M_D + M_{D^*}$

$\Longrightarrow p^2_{\rm rhc_2} = 0$

☞ three-body branch point:

$E = M_D + M_D + m_\pi$

$\Longrightarrow \left(\frac{p_{\rm rhc_3}}{E_{DD^*}}\right)^2 = +0.019$

☞ left-hand cut branch point:

$\Longrightarrow \left(\frac{p^{1\pi}_{\rm lhc}}{E_{DD^*}}\right)^2 = -0.001$

$\left(\frac{\tilde{p}^{1\pi}_{\rm lhc}}{E_{DD^*}}\right)^2 = -0.190$

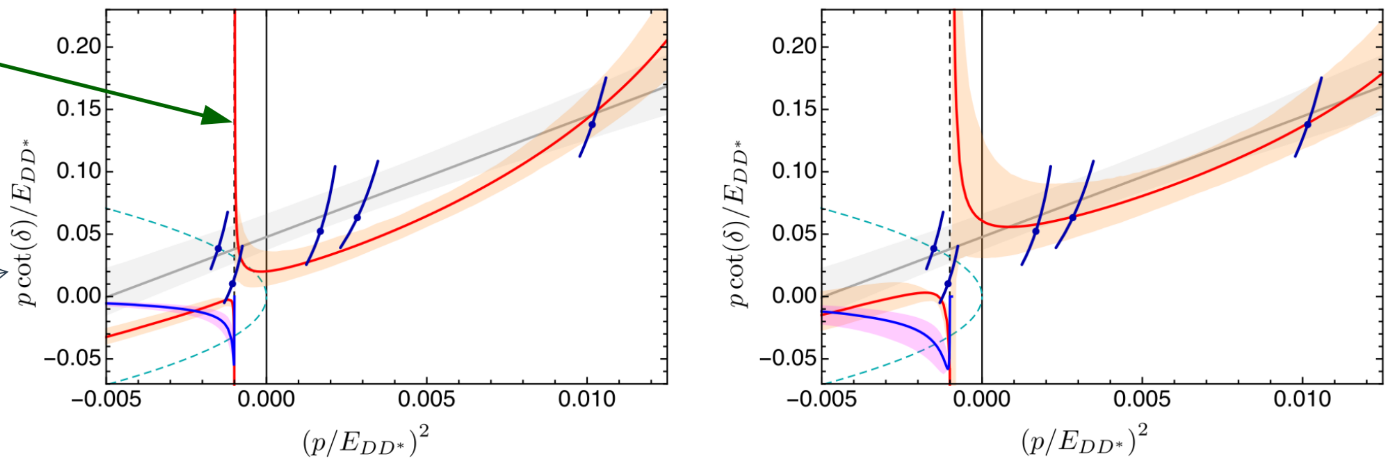
Phase shift with the left-hand cut: LSE

Du *et al.*, PRL 131,131903 (2023)

$M_D = 1927$ MeV, $M_{D^*} = 2049$ MeV, $m_\pi = 280$ MeV

Limit ERE

$$p\cot\delta = -\frac{2\pi}{\mu}\frac{1}{T} + ik$$



also happens in NN scattering… see in Liuming Liu's talk

Related recent works on FV w/ LHC…

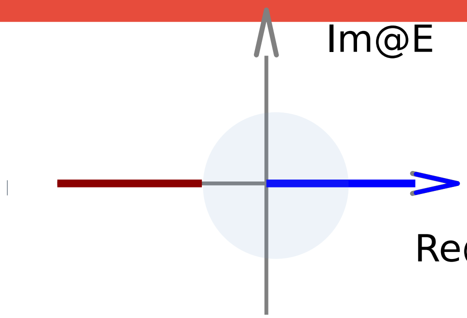
- Plane-wave basis to treat long-range interactions — Meng and Epelbaum, JHEP (2021); Meng et al., PRD (2024)
  - Project to irep. of the cubic to avoid the lhc associated to the partial wave projection — Mai and Döring, EPJA (2017), PRL (2019)
- Generalization of the Lüscher + *K*-matrix — Hansen and Raposo, JHEP (2024)
- Three-body framework (automatically includes lhc) — Dawid et al., PRD (2023); Hansen et al., PRD (2024)
- Modify the Lüscher formula via "modified effective range expansion" — Bubna et al., JHEP (2024)

The N/D method

ERE

$$f(k^2) = \frac{1}{k\cot\delta - ik}$$

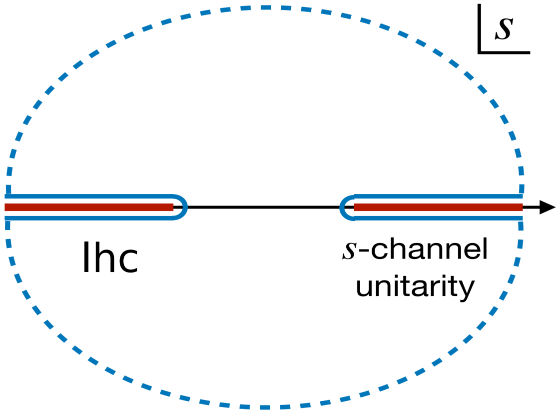
$$k\cot\delta = \frac{1}{a} + \frac{1}{2}rk^2 + \mathcal{O}(k^4)$$



$$T(s) = \frac{N(s)}{D(s)}$$

$$\mathrm{Im}D = \mathrm{Im}\frac{N}{T} = N\mathrm{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\mathrm{thr}} \\ 0, & s < s_{\mathrm{thr}} \end{cases}$$

$$\mathrm{Im}N = \begin{cases} \mathrm{Im}TD, & s < s_{\mathrm{lhc}} \\ 0, & s > s_{\mathrm{lhc}} \end{cases}$$



$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s-s_0)^n}{\pi}\int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s-s_0)^{n-\ell}}{\pi}\int_{-\infty}^{s_{\mathrm{left}}} ds' \frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

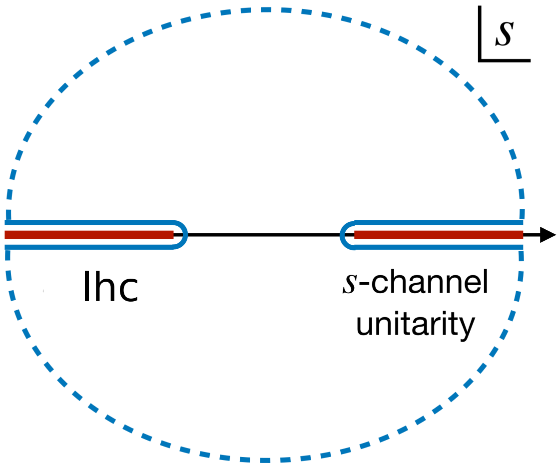
N=1

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + P(s) + G(s)$$

$$T(s) = \frac{1}{D(s)}$$

The N/D method

$$T(s) = \frac{N(s)}{D(s)}$$



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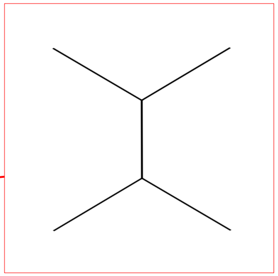
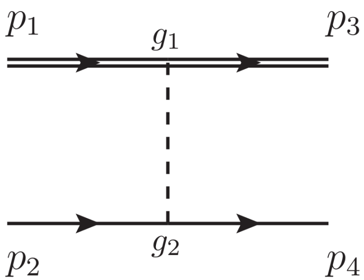
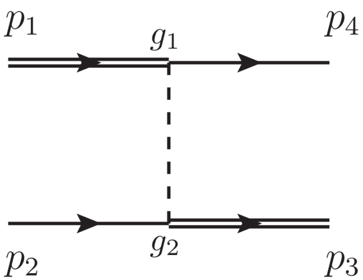


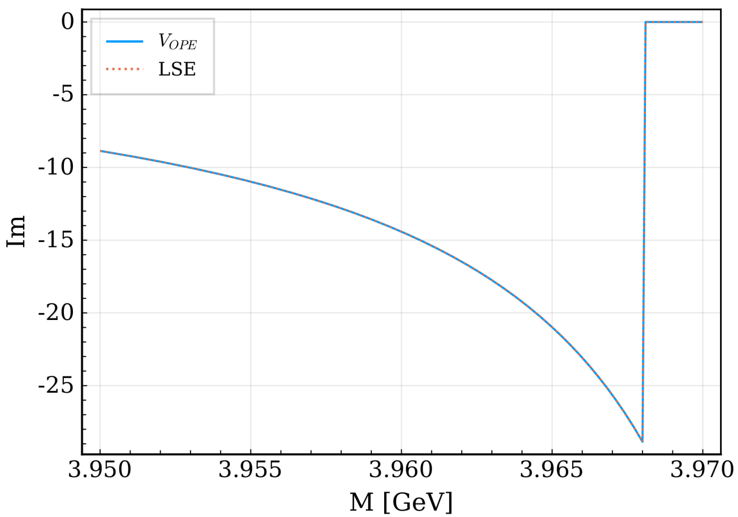
$$\frac{1}{T_\ell^{\mathrm{II}}} = \frac{1}{T_\ell} + 2i\rho \quad \Rightarrow \quad T^{\mathrm{II}} = \frac{1}{\frac{D}{N} + 2i\rho} = \frac{N}{D + 2i\rho N}$$

Along the lhc, $i\rho$ and D is real, N has imaginary part. $\Rightarrow$ $D + 2i\rho N \neq 0$

The left-hand cut







$$\mathrm{Im}\, f(k^2) = c\,\mathrm{Im}\, L(k^2) = -\frac{c}{4k^2}\pi, \quad \text{for } k^2 < k^2_{\mathrm{lhc}}$$

Solving LSE could be time-consuming.

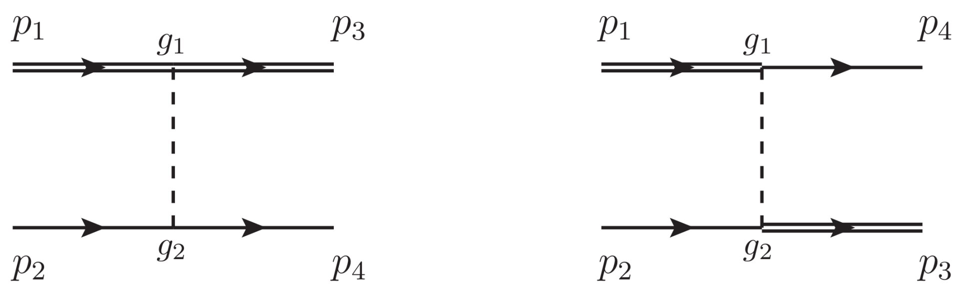
For a t-channel exchange at low-energies, an S-wave amplitude reads

$$L_t(s) = \frac{1}{2}\int \frac{1}{t - m_5^2}\mathrm{d}\cos\theta = -\frac{s}{\lambda(s, m_1^2, m_2^2)}\log\left(\frac{s - 2(m_1^2 + m_2^2) + m_5^2 + \frac{(m_1^2 - m_2^2)^2}{s}}{m_5^2}\right),$$

with m_5 the mass of changed particle. Likewise, the u-channel exchanged S-wave amplitude reads

$$L_u(s) = \frac{1}{2}\int \frac{1}{u - m_5^2}\mathrm{d}\cos\theta = -\frac{s}{\lambda(s, m_1^2, m_2^2)}\left(\log(s + m_5^2 - 2(m_1 + m_2)^2) - \log(m_5^2 - \frac{(m_1^2 - m_2^2)^2}{s})\right).$$

The left-hand cut: nonrelativistic



$$\eta = |m_1 - m_2|/(m_1+m_2)$$
$$\mu_{\rm ex}^2 = m_{\rm ex}^2 - (m_1-m_2)^2$$
$$\mu_+^2 = 4\mu\mu_{\rm ex}^2/(m_1+m_2)$$

Exchanged-particle: relativistic

$$L_t(k^2) \equiv \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{t-m_{\rm ex}^2} = -\frac{1}{4k^2}\log\frac{m_{\rm ex}^2/4+k^2}{m_{\rm ex}^2/4},$$
$$L_u(k^2) \equiv \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{u-m_{\rm ex}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4+k^2}{\mu_+^2/4+\eta^2k^2},$$

$$\frac{1}{2}\int_{-1}^{+1}\frac{(\mathbf{p}_1-\mathbf{p}_3)^2}{t-m_{\rm ex}^2}\mathrm{d}\cos\theta = -\frac{m_{\rm ex}^2}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{t-m_{\rm ex}^2} - 1$$
$$\frac{1}{2}\int_{-1}^{+1}\frac{(\mathbf{p}_1-\mathbf{p}_4)^2}{u-m_{\rm ex}^2}\mathrm{d}\cos\theta \approx -\frac{\mu_{\rm ex}^2}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{u-m_{\rm ex}^2} - 1$$

$\mathcal{F}_\ell/2$

$$f(k^2) = \frac{n(k^2)}{d(k^2)} \qquad \begin{aligned} \mathrm{Im}\, d(k^2) &= -k\, n(k^2), && \text{for } k^2>0,\\ \mathrm{Im}\, n(k^2) &= d(k^2)\ \mathrm{Im}\, f(k^2), && \text{for } k^2<k_{\rm lhc}^2. \end{aligned}$$

The N/D method: nonrelativistic

$\propto \mathrm{Im} L$

$$n(k^2) = n_m(k^2) + \frac{(k^2)^m}{\pi} \int_{-\infty}^{k^2_{\rm lhc}} \frac{d(k'^2)\,\mathrm{Im}\, f(k'^2)}{(k'^2 - k^2)(k'^2)^m} \mathrm{d}k'^2$$

No singularity along lhc

$$n(k^2) = n'_m(k^2) + \frac{P(k^2)}{\pi} \int_{-\infty}^{k^2_{\rm lhc}} \frac{\mathrm{Im} f(k'^2)}{k'^2 - k^2} \mathrm{d}k'^2 = n'_m(k^2) + P(k^2)\tilde{g}L(k^2)$$

$$\begin{aligned} n(k^2) &= n_0 + n_1 k^2 + \frac{k^2}{\pi} \int_{-\infty}^{k^2_{\rm lhc}} \frac{(d_0 + d_1 k'^2)\mathrm{Im} f(k'^2)}{(k'^2 - k^2)k'^2} \mathrm{d}k'^2 \\ &= n_0 + n_1 k^2 - cL_0 + (d_0 + d_1 k^2)\frac{c}{\pi} \int_{-\infty}^{k^2_{\rm lhc}} \frac{\mathrm{Im} f(k'^2)}{k'^2 - k^2} \mathrm{d}k'^2 \\ &= n'_0 + n_1 k^2 + (d_0 + d_1 k^2) c L(k^2) \end{aligned}$$

$$n(k^2) = \tilde{n}(k^2) + \tilde{g}(L(k^2) - L_0)$$

$$L_0 = L(k^2 = 0) = -1/\mu^2_{\rm ex}$$

The N/D method: nonrelativistic

$$d(k^2) = d_n(k^2) - \frac{(k^2 - k_0^2)^n}{\pi} \int_0^\infty \frac{k' n(k'^2)\,\mathrm{d}k'^2}{(k'^2 - k^2)(k'^2 - k_0^2)^n}$$

$$d(k^2) = \tilde{d}(k^2) - ik(\tilde{n}(k^2) - \tilde{g}L_0) - \frac{\tilde{g}}{\pi} \int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2} \mathrm{d}k'^2$$

$$= \tilde{d}(k^2) - ik\, n(k^2) - \tilde{g} d^{\mathrm{R}}(k^2)$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k} \left(\log \frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log \frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k} \right)$$

It is worth stressing that $d(k^2)$ is free of lhc, as the lhc associated with $n(k^2)$ below the threshold is counterbalanced by $d^R(k^2)$, which is crucial to ensure that $f(k^2)$ exhibits the correct lhc behavior. Along the rhc, both $n(k^2)$ and $d^{\mathrm{R}}(k^2)$ are real such that $\mathrm{Im}d(k^2) = -k\, n(k^2)$.

Effective range expansion with the left-hand cut

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k}\left(\log\frac{\mu_+/2+ik}{\mu_+/2-ik} - \log\frac{\mu_+/2+i\eta k}{\mu_+/2-i\eta k}\right)$$

$$L(k^2) = -\frac{1}{4k^2}\log\frac{\mu_+^2/4+k^2}{\mu_+^2/4+\eta^2k^2}$$

$$\frac{1}{f(k^2)} = \frac{\tilde{d}(k^2)-\tilde{g}d^{\mathrm{R}}(k^2)}{\tilde{n}(k^2)+\tilde{g}\left(L(k^2)-L_0\right)} - ik$$

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n}\tilde{d}_i k^{2i} - \tilde{g}d^{\mathrm{R}}(k^2)}{1+\sum_{j=1}^{m}\tilde{n}_j k^{2j} + \tilde{g}(L(k^2)-L_0)} - ik$$

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0+\tilde{d}_1k^2-\tilde{g}d^{\mathrm{R}}(k^2)}{1+\tilde{g}(L(k^2)-L_0)} - ik\right]^{-1}$$

$\tilde{g}\to 0$

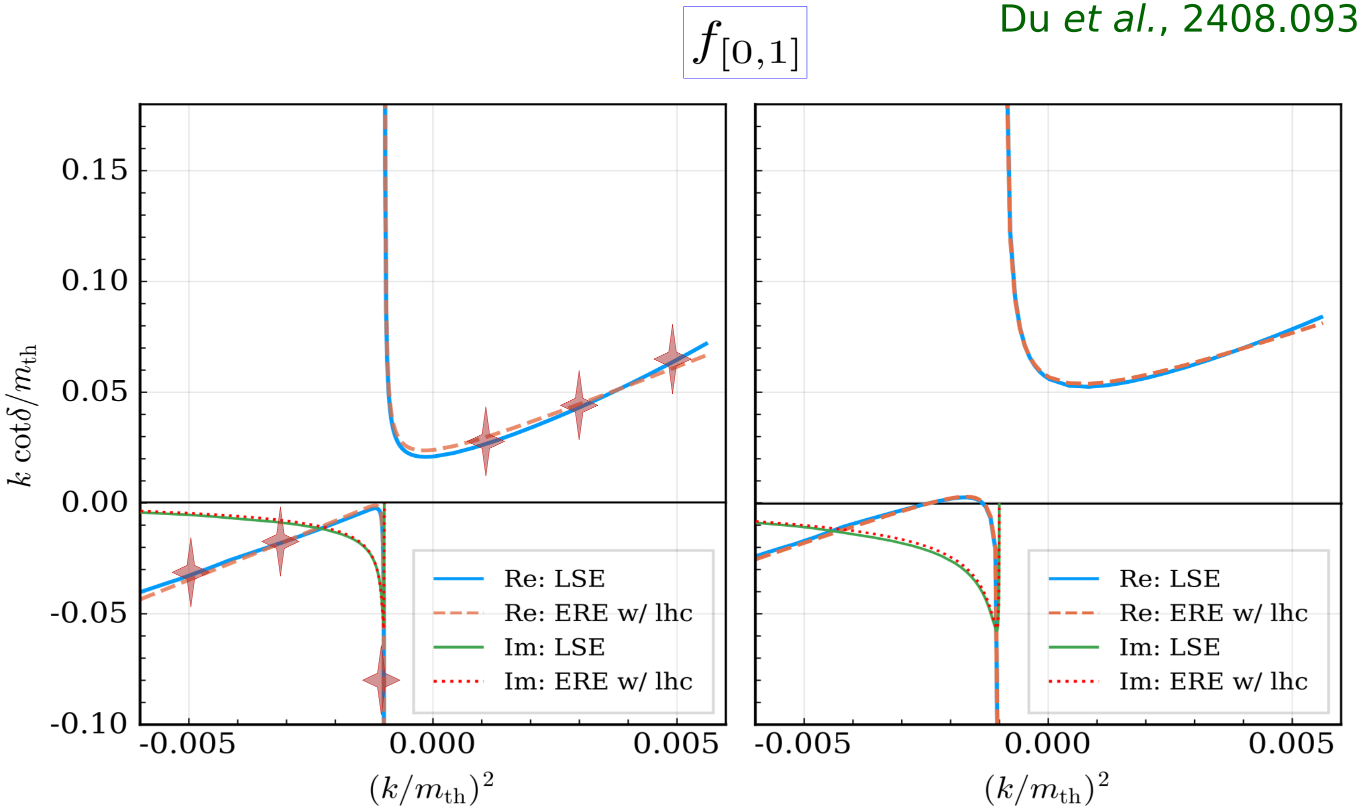
$$\frac{1}{f(k^2)} = \frac{1}{a} + \frac{1}{2}rk^2 - ik$$

Scattering length $$a = f(k^2=0) = \left[\tilde{d}_0 + \frac{\tilde{g}}{\mu_+}(1-\eta)\right]^{-1}$$

Effective range $$r = \left.\frac{\mathrm{d}^2(1/f+ik)}{\mathrm{d}k^2}\right|_{k=0} = 2\tilde{d}_1 - \frac{8\tilde{g}}{3\mu_+^3}\left(1-\eta^3\right) - \frac{4\tilde{g}}{\mu_+^4 a_u}\left(1-\eta^4\right)$$

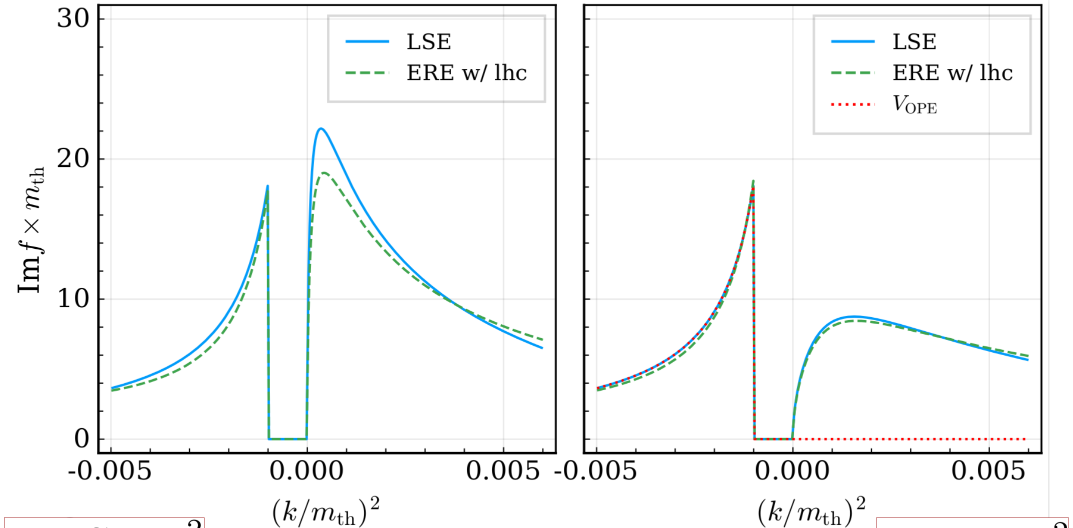
Example: Tcc on the Lattice [3 parameters]

Du *et al.*, 2408.09375 [hep-ph]



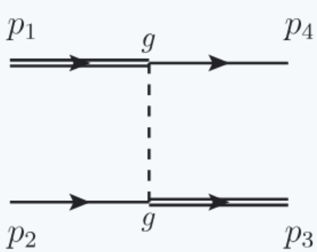
$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\rm R}(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik\right]^{-1}$$

Couplings to the exchanged-particle



$9.0\ \mathrm{GeV}^{-2}$ 	$9.4\ \mathrm{GeV}^{-2}$ ← $g_P = -\dfrac{2\pi\tilde{g}}{\mu d^{0,\rm lhc}\mathcal{F}_\ell}$

$$g^2_{D^*D\pi}/(4F^2) = 9.2\ \mathrm{GeV}^{-2}$$



$$-\frac{2\pi}{\mu}\mathrm{Im}f = \mathrm{Im}T = \mathrm{Im}V_{\rm OPE}(k^2) = g_P\frac{-\pi}{4k^2}\mathcal{F}_\ell, \quad \text{for } k^2 < k^2_{\rm lhc}$$

$$\mathrm{Im}\,n(k^2) = -\tilde{g}\frac{\pi}{4k^2}, \quad \text{for } k^2 < k^2_{\rm lhc}$$

$$d_u^{0,\rm lhc} = \tilde{d}_0 - \frac{\tilde{d}_1\mu_+^2}{4} + \frac{\mu_+}{2}\left(1 + \frac{\tilde{g}}{\mu_{\rm ex}^2}\right) + \frac{\tilde{g}\log[2/(1+\eta)]}{\mu_+}$$

The amplitude zero

At leading order, i.e., $\tilde{n}(k^2) = 1$,

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik\right]^{-1}$$

For a general u-channel exchange,

$$1 + \tilde{g}\left[L_u\left(k^2_{u,\mathrm{zero}}\right) + \frac{1}{\mu^2_{\mathrm{ex}}}\right] = 0,$$

for the case $|\Delta| \ll m_{\mathrm{th}}$ such that $\eta \ll 1$, ⇒ the t-channel exchange

$$k^2_{t,\mathrm{zero}} = -\frac{m^2_{\mathrm{ex}}}{4}\left[1 + \frac{1}{y} W(-e^{-y} y)\right]$$

where $y \equiv 1 + m^2_{\mathrm{ex}}/\tilde{g}$ and W is the Lambert W function.

$$y = 1 + \frac{1 + \frac{4}{3} a_t m_{\mathrm{ex}}(1 - \log 4) - \frac{4\pi a_t m^2_{\mathrm{ex}}}{\mu g_P \mathcal{F}_\ell}}{2 + a_t m_{\mathrm{ex}}\left(1 - m_{\mathrm{ex}} r_t/4\right)}.$$

Summary

★ Unphysical pion masses on the Lattice

$M_D = 1927$ MeV, $M_{D^*} = 2049$ MeV, $m_\pi = 280$ MeV

$\hookrightarrow$ the three-body cut above the two-body cut ($\sqrt{s_{\rm lhc}} = 3968$ MeV)

$\hookrightarrow$ The traditional ERE valid only in a very limited range

$\hookrightarrow$ An accurate extraction of the pole requires the OPE implemented

★ The ERE with the left-hand cut

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^R(k^2)}{1 + \tilde{g}(L(k^2) - L_0)} - ik\right]^{-1}$$

$\hookrightarrow$ correct behavior of the left-hand cut

$\hookrightarrow$ can be used to extract the couplings of the exchanged particle to the scattering particles

$\hookrightarrow$ amplitude zeros caused by the interplay between the short- and long-range interactions

Thank you very much for your attention!

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Without $d^R(k^2)$

$$d(k^2) = \tilde{d}(k^2) - ik(\tilde{n}(k^2) - \tilde{g}L_0) - \frac{\tilde{g}}{\pi}\int_0^\infty \frac{k' L(k'^2)}{k'^2 - k^2} \mathrm{d}k'^2$$
$$= \tilde{d}(k^2) - ik\, n(k^2) - \tilde{g} d^{\mathrm{R}}(k^2)$$

$$d^{\mathrm{R}}_u(k^2) = \frac{i}{4k}\left(\log\frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log\frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k}\right)$$

lhc

$$f^{-1} = \frac{\frac{1}{a} + \frac{1}{2}rk^2}{1 + \tilde{g}(L(k^2) - L_0)} - ik$$

$$d(k^2) = \frac{1}{a} + \frac{1}{2}rk^2 - ik - ik\tilde{g}(L(k^2) - L_0)$$