Massive modes on brane

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Based on: [PRD 109 (2024) 024017]; [EPJC 83 (2023) 84];
[PRD 106 (2022) 044038]; [PRD 80 (2009) 065019]

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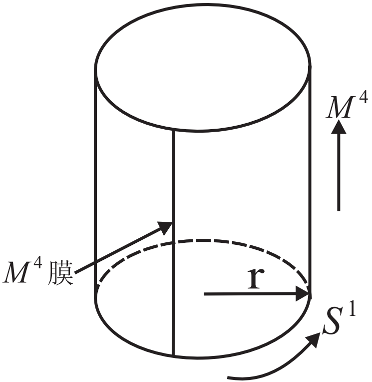
- 1. Introduction
- 2. Resonant KK modes on thick brane
- 3. Quasinormal modes on thick brane
- 4. Relation between them
- 5. Summary

Compact extra dimensions:

- **Kaluza-Klein theory** [Kaluza, 1921; Klein, 1926]
- **String theory** [Veneziano, 1968]
- **Large extra diemnsions** [Arkani-Hamed, Dimopoulos, and Dvali, 1998]
- **Warp extra dimension** [Randall and Sundrum, 1999]



Infinite extra dimensions (1):

Domain wall (DW) scenario [Akama, Rubakov, Shaposhnikov, 1983]

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dy^2$$

- **Our 4D world is a DW embedded in 5D flat space-time.**
- **It is generated by a scalar field:**

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - a(\phi^2 - v^2)^2, \qquad \phi(y) = v_0 \tanh(ky)$$

- **Fermions can be localized on the DW by**

$$\eta\phi\bar{\Psi}\Psi$$

- **Newtonian potential cannot be recovered**

$$U(r) \propto \frac{1}{r^2}$$

1. Introduction

Infinite extra dimensions (2):

Thin brane scenario

[Randall and Sundrum (RS), 1999]

$$ds^2 = e^{-2k|y|}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + dy^2$$

- Our 4D world is a brane embedded in a 5D space-time.
- Newtonian potential can be recovered on brane:

$$U(r) = G_N \frac{m_1 m_2}{r}\left(1 + \frac{1}{k^2 r^2}\right)$$

- The energy density: $\rho(y) \propto \sigma\delta(y)$

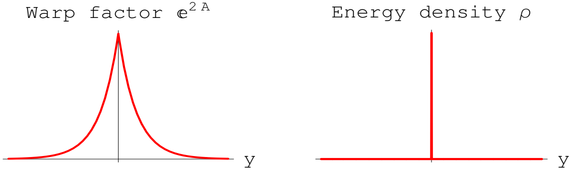


图: Thin brane

1. Introduction

Infinite extra dimensions (3):

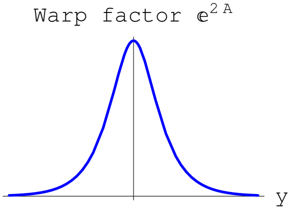
Thick brane scenario (Domain Wall)

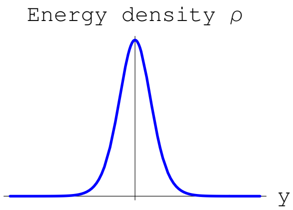
$$ds^2 = e^{2A(y)}\,\eta_{\mu\nu}(x)\,dx^\mu dx^\nu + dy^2$$

- **Infinite but warped extra dimension.**
- **The brane is generated by a scalar field, e.g.**

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - V(\phi), \quad \phi(y) = v_0 \tanh(ky)$$

- **The graviton zero mode is localized on the brane. Newtonian potential can be recovered.**

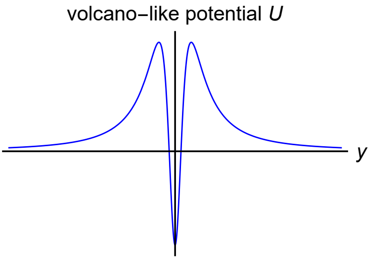




1. Introduction

KK modes of thick brane with volcano-like potential

- The zero mode is localized on the brane.
- Massive KK modes cannot be localized on the brane.
- Some KK modes could be quasi-localized on the brane (resonances).
- The thick brane is a dissipative system for massive KK modes, which means that the brane may possess quasinormal modes.



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The braneworld model

- **The action of a thick brane model:**

$$S = \int d^5x\sqrt{-g}\left(\frac{1}{2\kappa_5^2}R - \frac{1}{2}g^{MN}\partial_M\varphi\partial_N\varphi - V(\varphi)\right). \quad (1)$$

- **The five-dimensional metric**

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \quad (2)$$

- **The dynamical field equations**

$$3A'' = -\phi'^2, \quad (3)$$

$$6A'^2 = \frac{1}{2}\phi'^2 - V, \quad (4)$$

$$\phi'' + 4A'\phi' = \frac{\partial V}{\partial \phi}. \quad (5)$$

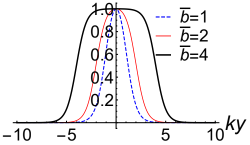
2. Resonant KK modes on thick brane

One solution is given by

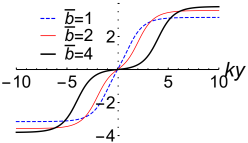
$$A(y) = \ln\Big[\tanh\big(k(y+b)\big) - \tanh\big(k(y-b)\big)\Big]. \tag{6}$$

$$\begin{aligned}\phi(y) = -i\sqrt{3}\,\mathrm{sech}(kb)\Big[&\cosh(2kb)\mathrm{F}\big(iky;\tanh^2(kb)+1\big)\\ &-2\sinh^2(kb)\Pi\big(\mathrm{sech}^2(kb);iky;\tanh^2(kb)+1\big)\Big],\end{aligned} \tag{7}$$

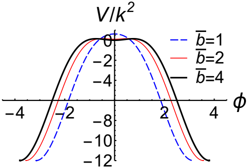
$$\begin{aligned}V(\phi(y)) = \frac{3}{4}k^2\Big[&-4\big(\tanh(k(y-b))+\tanh(k(b+y))\big)^2\\ &+\mathrm{sech}^2(k(y-b))+\mathrm{sech}^2(k(b+y))\Big].\end{aligned} \tag{8}$$



(a) The warp factor



(b) The scalar field



(c) The scalar potential

2. Resonant KK modes on thick brane

Based on this braneworld background, we consider a free massless test scalar field and study its evolution.

- **With $dz = e^{-A}dy$, the metric (2) becomes**

$$ds^2 = e^{2A(z)} \left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) . \tag{9}$$

- **The field equation $\Box^{(5)}\psi = 0$:**

$$\left[\partial_z^2 + 3(\partial_z A)\partial_z + \eta^{\mu\nu}\partial_\mu\partial_\nu\right]\psi = 0. \tag{10}$$

- **Then, we introduce the following decomposition**

$$\psi(x^M) = e^{-\frac{3}{2}A(z)}\,\Phi(t,z)\,\Xi(x^i). \tag{11}$$

- **Substituting Eq. (11) into Eq. (10) yields**

$$-\partial_t^2\Phi + \partial_z^2\Phi - U(z)\Phi - a^2\Phi = 0, \tag{12}$$

where the effective potential $U(z)$ is

$$U(z) = \frac{3}{2}\partial_z^2 A + \frac{9}{4}(\partial_z A)^2. \tag{13}$$

- **The function $\Phi(t,z)$ can be further decomposed into oscillating modes as $\Phi(t,z)=e^{i\omega t}u(z)$, which yields**

$$-\partial_z^2 u(z)+U(z)u(z)=m^2u(z), \tag{14}$$

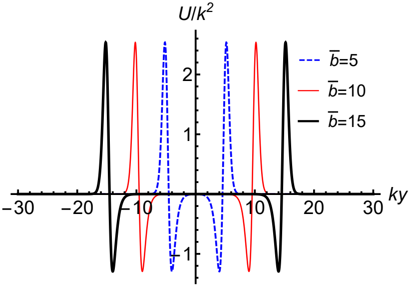
where $m=\sqrt{\omega^2-a^2}$ is the mass of the KK mode $u(z)$.

- **Solving Eq. (14) we could get a series of resonant modes, which can be treated as the initial data of the scalar field.**
- **The evolution is dominated by Eq. (12):**

$$-\partial_t^2\Phi+\partial_z^2\Phi-U(z)\Phi-a^2\Phi=0. \tag{12}$$

2. Resonant KK modes on thick brane

- **The effective potential in the coordinate y:**



- **The resonant modes can be studied by the relative probability method** [YXL et al, PRD 80 (2009) 065019]:

$$P(m^2) = \frac{\int_{-z_b}^{z_b} |u(z)|^2 dz}{\int_{-10z_b}^{10z_b} |u(z)|^2 dz}. \quad (15)$$

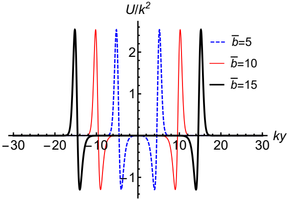
- **The wave functions can be even or odd since the potential is symmetric.**
- **Boundary conditions for Eq. (14):**

$$u_{\mathrm{even}}(0) = 1, \quad \partial_z u_{\mathrm{even}}(0) = 0; \tag{16}$$

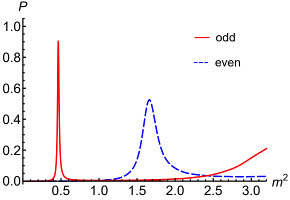
$$u_{\mathrm{odd}}(0) = 0, \quad \partial_z u_{\mathrm{odd}}(0) = 1. \tag{17}$$

- **Then we can get the relative probability $P(m^2)$ of scalar KK modes.**

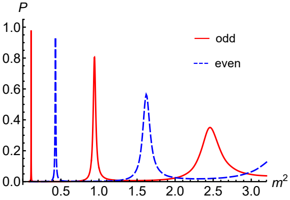
2. Resonant KK modes on thick brane



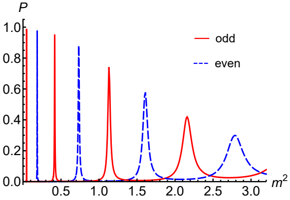
(a) The effective potential



(b) P with $\bar{b} = kb = 5$



(c) $kb = 10$



(d) $kb = 15$

2. Resonant KK modes on thick brane

- **Treating the scalar resonances as the initial data, we can evolve the scalar field under the evolution equation (12):**

$$-\partial_t^2\Phi + \partial_z^2\Phi - U(z)\Phi - a^2\Phi = 0. \qquad (12)$$

- **We impose the maximally dissipative boundary condition:**

$$\partial_n\Phi = \partial_t\Phi, \qquad (18)$$

  n **is the outward unit normal vector to the boundary.**
- **Equation (12) is solved numerically.**

- **To display the evolution of the scalar field, we define the energy** [V. Pavlidou et al, PRD 62 (2000) 084020].

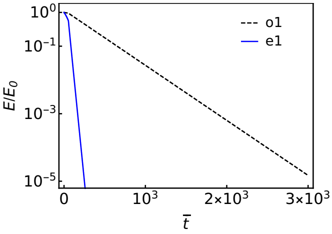
$$E(t) = \int_{-10z_b}^{10z_b} \frac{1}{2}\left((\partial_t\Phi)^2 + \left(\partial_z\Phi - \frac{3}{2}\partial_z A\Phi\right)^2\right) dz, \qquad (19)$$

- **The energy decay can be fitted as an exponential function:**

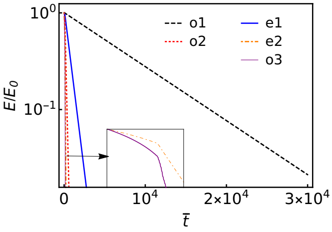
$$E(t) = E_0 \, \mathbf{exp}(-s\bar{t}), \qquad (20)$$

s **is the fitting parameter,** $\bar{t} = kt$.

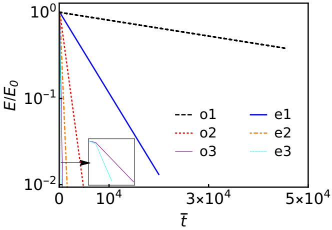
2. Resonant KK modes on thick brane



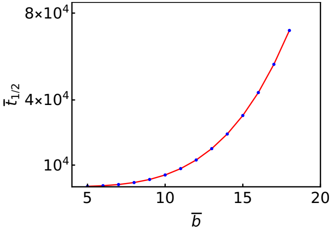
(a) $E(t)$ with $kb = 5$



(b) $E(t)$ with $kb = 10$

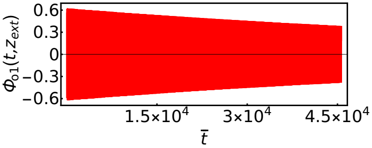


(c) $E(t)$ with $kb = 15$

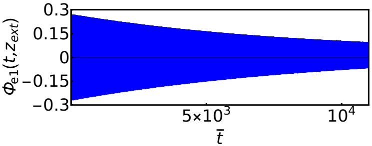


(d) $\bar{t}_{1/2} \sim kb$

In addition, we analyze the numerical evolution by extracting a time series for the resonance amplitude at a fixed point $z_{\mathbf{ext}}$. We set $kb = 15$.



first odd resonance at $kz_{\mathbf{ext}} = 3$.



first even resonance at $kz_{\mathbf{ext}} = 3$.

2. Resonant KK modes on thick brane

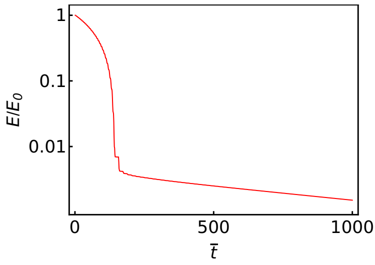
kb	$\bar{m}_1^2$	s	$t_{1/2}$ (if $k = 10^{-2}$eV)
6	0.3177	1.4624×10^{-3}	3.0714×10^{-11}seconds
8	0.1736	3.6373×10^{-4}	1.2349×10^{-10}seconds
10	0.1088	1.2867×10^{-4}	3.4908×10^{-10}seconds
12	0.0744	5.6502×10^{-5}	7.9494×10^{-10}seconds
14	0.0540	2.8609×10^{-5}	1.5700×10^{-9}seconds
16	0.0410	1.6000×10^{-5}	2.8072×10^{-9}seconds
18	0.0322	9.6335×10^{-6}	4.6625×10^{-9}seconds

表: The first resonant mass spectrum $\bar{m}_1^2$, fitting parameter s, and half-life $t_{1/2}$ for different values of the parameter kb.

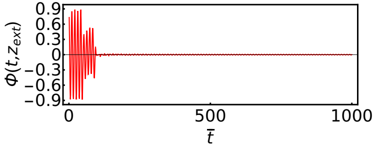
The lifetime of the first resonance increases with the brane width.

2. Resonant KK modes on thick brane

- **We also consider the evolution of the non-resonances. The energy and amplitude of non-resonances decay very fast at early stage, but later they decay like those of resonances.**



(a) $E(t)$ with $kb = 15$

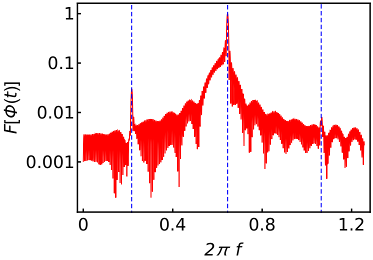


(b) $kz_{\text{ext}} = 30$

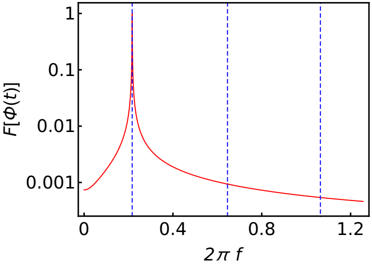
- **To have a better understanding of the above results, we calculate the discrete Fourier transform in time of the scalar field**

$$F[\Phi(t)](f) := |A\sum_{p}\Phi(t_p, z_j)\mathbf{exp}(-2\pi i f t_p)|, \qquad (21)$$

t_p are the discrete time values.



(a) non-resonance with $m^2 = 0.36$ for $kb = 15$



(b) first resonance with $m^2 = 0.047$ for $kb = 15$

- The results show that, non-resonances can evolve into combinations of resonances.
- From this point of view, resonances seem to play a similar role in the braneworld as the quasinormal modes in black holes physics.

Summary 1:

- We investigated the evolution of a free massless scalar field in a thick brane model.
- The resonances decay very slowly compared to the non-resonances and can exist on the brane for a very long time.
- Such resonances might be a candidate for dark matter.
- Nonresonances can evolve into combinations of resonances. This arouses our interest in quasinormal modes in thick brane models.

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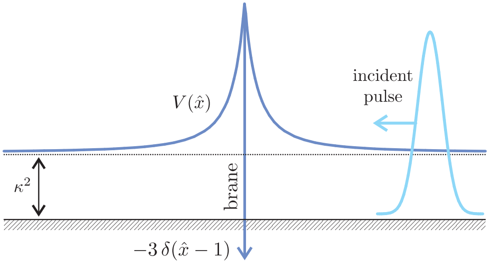
3. Quasinormal modes on thick brane

- **In 2005, Seahra studied the scattering of KK gravitons in the Randall-Sundrum-II model and found that the brane possesses a series of discrete quasinormal modes (QNMs) [S.S. Seahra, PRD 72 (2005) 066002].**



- **As a smooth extension of the Randall-Sundrum-II model, a thick brane should also have QNMs.**

The braneworld model

- **The action and the metric:**

$$S = \int d^5x\sqrt{-g}\left(\frac{1}{2\kappa_5^2}R - \frac{1}{2}g^{MN}\partial_M\varphi\partial_N\varphi - V(\varphi)\right),$$
$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2.$$

- **To investigate the QNMs of a thick brane, we consider the following thick brane solution** [1]

$$A(y) = -b\ln\left(\cosh(ky)\right),$$
$$\varphi(y) = \sqrt{6b}\arctan\left(\tanh\left(\frac{ky}{2}\right)\right),$$
$$V(\varphi) = \frac{3bk^2}{8}\left(1 - 4b - (1+4b)\cos\left(\sqrt{\frac{8}{3b}}\varphi\right)\right).$$

[1] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Phys. Rev. D 62, 046008 (2000).

- **Next, we consider the linear transverse-traceless tensor perturbation of the metric:**

$$g_{MN} = \begin{pmatrix} e^{2A(y)}(\eta_{\mu\nu} + h_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix}, \tag{22}$$

$$\partial_\mu h^{\mu\nu} = 0 = \eta^{\mu\nu} h_{\mu\nu}. \tag{23}$$

- **The linear equation of the tensor fluctuation is**

$$\left(e^{-2A} \Box^{(4)} h_{\mu\nu} + h''_{\mu\nu} + 4A' h'_{\mu\nu} \right) = 0, \tag{24}$$

where $\Box^{(4)} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$**.**

3. Quasinormal modes on thick brane

- **Introducing** $dz = e^{-A}dy$ **and decomposing** $h_{\mu\nu}$ **as**

$$h_{\mu\nu} = e^{-\frac{3}{2}A(z)}\Phi(t,z)e^{-ia_j x^j}\epsilon_{\mu\nu}, \quad \epsilon_{\mu\nu} = \textbf{constant}, \tag{25}$$

we obtain the wave equation for $\Phi(t,z)$

$$-\partial_t^2\Phi + \partial_z^2\Phi - U\Phi - a^2\Phi = 0. \tag{26}$$

- **Further decomposing** $\Phi(t,z) = e^{-i\omega t}\phi(z)$, **we have**

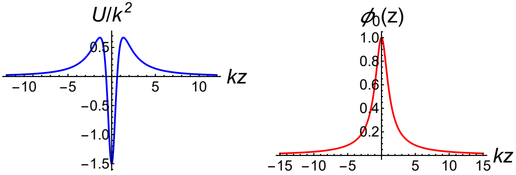
$$\left[-\partial_z^2 + U\right]\phi(z) = m^2\phi(z), \quad U = \frac{3}{2}\partial_z^2 A + \frac{9}{4}(\partial_z A)^2, \tag{27}$$

where $m^2 = \omega^2 - a^2$.

- **The effective potential** $U(z)$ **is given by** [O. DeWolfe, D.Z. Freedman, S.S. Gubser, and A. Karch, PRD 62 (2000) 046008]

$$U(z) = \frac{3k^2\left(5k^2z^2 - 2\right)}{4\left(k^2z^2+1\right)^2}, \tag{28}$$

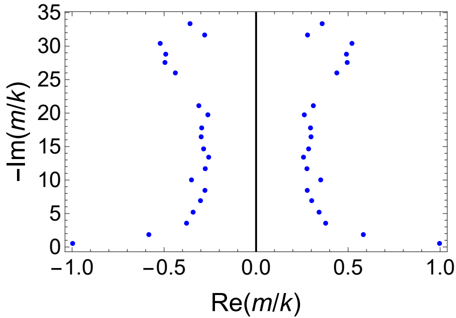
3. Quasinormal modes of thick brane



(a) Effective potential (b) Zero mode

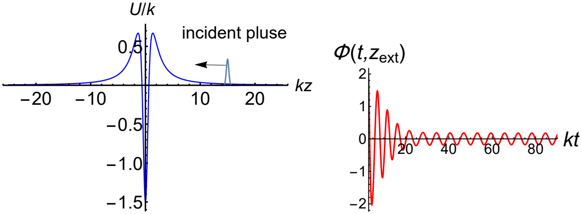
- **The boundary conditions for the QNMs are**

$$\phi(z) \propto \begin{cases} e^{+imz}, & z \to \infty, \\ e^{-imz}, & z \to -\infty. \end{cases} \tag{29}$$



n	Asymptotic iteration method		WKB method	
	$\mathrm{Re}(m/k)$	$\mathrm{Im}(m/k)$	$\mathrm{Re}(m/k)$	$\mathrm{Im}(m/k)$
1	0.997018	-0.526366	1.04357	-0.459859
2	0.582855	-1.85056	0.536087	-1.71224
3	0.377996	-3.55174	0.279715	-3.70181

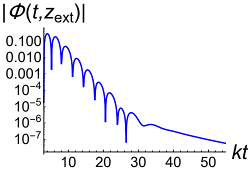
- **Numeric evolution of an initial wave packet (Gauss pulse) at a fixed location.**



(a) (b) $kz_{\text{ext}} = 3$

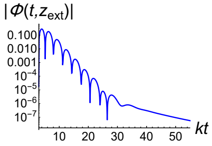
3. Quasinormal modes of thick brane

- To investigate the character of the odd QNMs, we give **an odd initial wave packet**.
- We choose $a/k = 0$ and $a/k = 1$ to show the effect of the parameter a.



(a) $kz_{ext} = 3$

图: **The case of** $a/k = 0$. $\omega/k = 1.01079 - 0.501256i$.

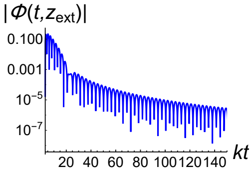


(a) $kz_{ext} = 3$

图: **The case of $a/k = 0$.**

- **For $a/k = 0$, there are two stages:**
- **(1) The exponentially decay stage. The frequency and damping time of these oscillations in this stage depend only on the characteristic structure of the thick brane.**
- **(2) The power-law damping stage. This situation is similar to the case of a massless field around a Schwarzschild black hole.**

3. Quasinormal modes of thick brane



(a) $kz_{ext} = 3$

图: **The case of $a/k = 1$.**

- **For $a/k = 1$, the quasinormal ringing governs the decay of the perturbation all the time.**
- **This is similar to the case of a massive field around a Schwarzschild black hole.**

Summary 2:

- Normal and quasinormal modes in a thick brane model.
    - A normal mode (the zero mode).
    - A series of discrete quasinormal modes.
- This provides a new way to investigate gravitational perturbations in thick brane models.

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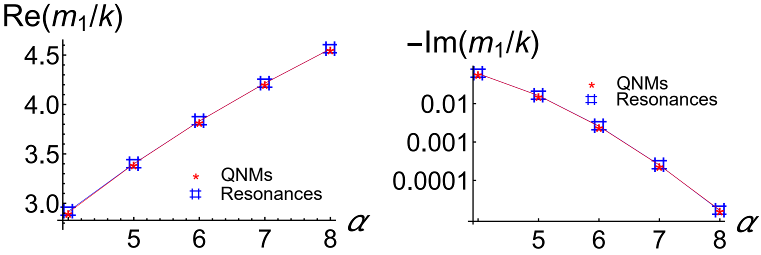
4. Relation between resonances and quasinormal modes

$$A(z) = -\frac{\alpha}{2} \ln\left(k^2 z^2 + 1\right). \tag{30}$$

Relation between resonances and quasinormal modes:

- **The oscillations of the resonances are equal to the real part of the quasinormal modes.**
- **While the decay rates of the resonances are equal to the imaginary part of the quasinormal modes.**



(a) $\mathrm{Re}(m_1/k) \sim \alpha$ (b) $-\mathrm{Im}(m_1/k) \sim \alpha$

4. Relation between resonances and quasinormal modes

- **The half-life time of the first quasinormal mode.**

α	$t_{1/2}$ (if $k = 10^{-2}$eV)
4	1.5588×10^{-12}seconds
5	1.9237×10^{-12}seconds
6	8.5406×10^{-12}seconds
7	8.8909×10^{-11}seconds
8	1.3724×10^{-9}seconds
9	2.6010×10^{-8}seconds
10	8.6655×10^{-7}seconds

- **For the Randall-Sundrum-II brane, the half-life time of the first quasinormal mode is about 10^{-14}seconds with $k = 10^{-2}$eV.**

Content

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5. Summary

- KK modes of a thick brane
  - A normal bound mode (the zero mode)
  - Resonant KK modes (resonances)
  - Nonresonant KK modes (non-resonances)
  - Quasinormal modes (QNMs)
- Resonances $\leftrightarrow$ non-resonances
  - Resonances decay slowly compared to non-resonances
  - Nonresonances can evolve into combinations of resonances
- Resonances $\leftrightarrow$ QNMs
  - Oscillation of resonances $\updownarrow$ Real part of QNMs
  - Decay rate of resonances $\updownarrow$ Imaginary part of QNMs
- These modes reflect structure of extra dimensions.

Thank you!