Exotic hadrons with heavy quarks
Part 2: Methods

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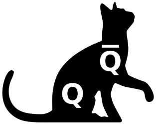
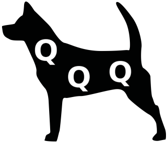
Ordinary hadrons



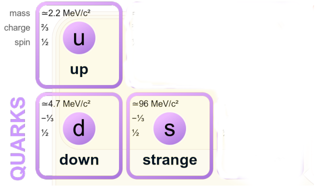


Quark model: The structure of hadrons

1964 — Quark model by Gell-Mann & Zweig $\Longrightarrow$ $SU(3)$ multiplets

"Ordinary" hadrons*:

- Meson consists of quark and antiquark
- Baryon consists of 3 quarks



* Compact "exotic" hadrons anticipated

All hadrons understood $\Longrightarrow$ No "exotic" states

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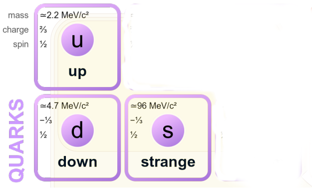

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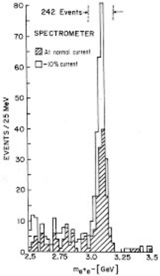
Prediction of the fourth quark:

- Glashow & Bjorken (1964)
- Glashow, Iliopoulos & Maiani (1970)

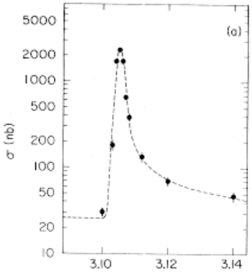
November revolution 1974: Discovery of charm

BNL $(p + Be \to e^+e^- X)$



$m_J = 3.1$ GeV
$\Gamma_J \approx 0$

SLAC $(e^+e^- \to \text{hadrons})$

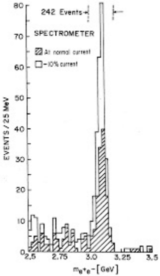


$m_\psi = 3.105 \pm 0.003$ GeV
$\Gamma_\psi \leqslant 1$ MeV

Narrow resonance J/ψ with mass around 3.1 GeV

November revolution 1974: Discovery of charm

BNL ($p + Be \to e^+e^- X$)



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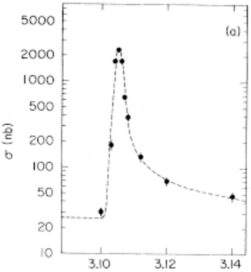


$m_\psi = 3.105 \pm 0.003$ GeV
$\Gamma_\psi \leqslant 1$ MeV

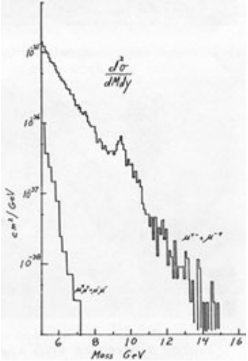
Narrow resonance J/ψ with mass around 3.1 GeV

5 years later $\Longrightarrow$ 10 charmonia states!

Bottomonia

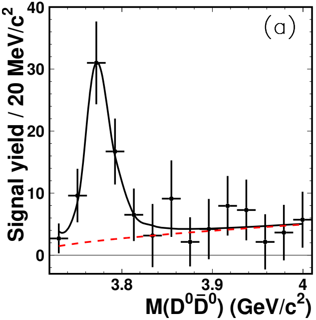
- 1977 — L.Lederman (Fermilab): discovery $\Upsilon(1S)$ with mass 9.54 GeV

$$p + (Cu, Pt) \to \mu^+ + \mu^- + \text{anything}$$



- 1978 — DESY (Germany): discovery of $\Upsilon(2S)$
- 1980 — CESR (USA): discovery of $\Upsilon(3S)$ and $\Upsilon(4S)$

Breit-Wigner parametrisation: Mass, Width, Poles



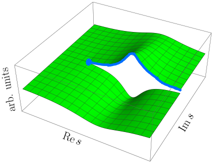
$$\mathcal{A} = \mathcal{A}_{\rm bg} + \mathcal{A}_{\rm BW}$$

$$\mathcal{A}_{\rm BW} \propto \frac{1}{M^2 - {M_0}^2 + iM\Gamma_0}$$

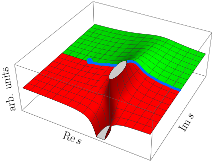
$$\Gamma_0 = \Gamma(R \to H_1 H_2)$$

$$M_0 > M_{H_1} + M_{H_2}$$

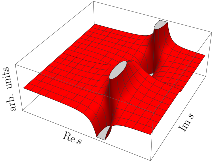
Pole positions: $\begin{cases} M_{\rm pole} \approx M_0 - \frac{i}{2}\Gamma_0 \\ M^*_{\rm pole} \approx M_0 + \frac{i}{2}\Gamma_0 \end{cases}$



first Riemann sheet

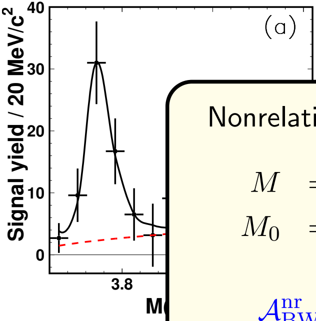


transition from first to second Riemann sheet



second Riemann sheet

Breit-Wigner parametrisation: Mass, Width, Poles



$$\mathcal{A} = \mathcal{A}_{\rm bg} + \mathcal{A}_{\rm BW}$$

$$\mathcal{A}_{\rm BW} \propto \frac{1}{\;\cdots\; M_0{}^2 + iM\Gamma_0}$$

$\to H_1 H_2)$

$+ M_{H_2}$

$\text{pole} \approx M_0 - \frac{i}{2}\Gamma_0$

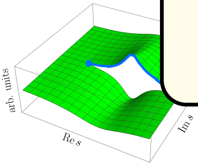
$^*\text{pole} \approx M_0 + \frac{i}{2}\Gamma_0$

Nonrelativistic expansion:

$$\begin{aligned} M &= M_{H_1} + M_{H_2} + E \\ M_0 &= M_{H_1} + M_{H_2} + E_0 \end{aligned}$$

$$\mathcal{A}_{\rm BW}^{\rm nr} \propto \frac{1}{E - E_0 + \frac{i}{2}\Gamma_0}$$

$$|\Psi|^2 \sim \left| e^{-iE_0 t - \frac{1}{2}\Gamma_0 t} \right|^2 \sim e^{-\Gamma_0 t}$$



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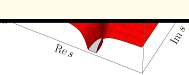

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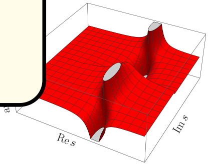
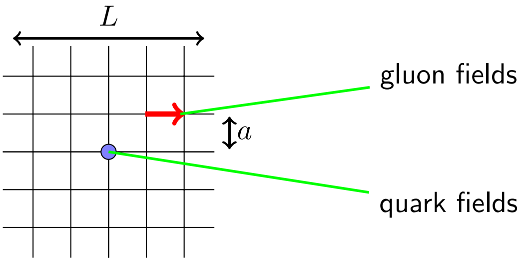


second Riemann sheet

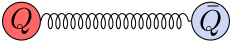
Lattice simulations



$$C_{ij}(t) = \langle 0|O_i(t)O_j(0)|0\rangle = \sum_n \frac{e^{-E_n t}}{2E_n}\langle 0|O_i(0)|n\rangle\langle n|O_j^\dagger(0)|0\rangle$$

- Continuum limit $\Longrightarrow a \to 0$
- Infinite box $\Longrightarrow L \to \infty$
- Unphysical light quark mass $\Longrightarrow$ Chiral extrapolation

Quark model: Adding dynamics



$$\boxed{\hat{H}_0\psi = E_0\psi}$$

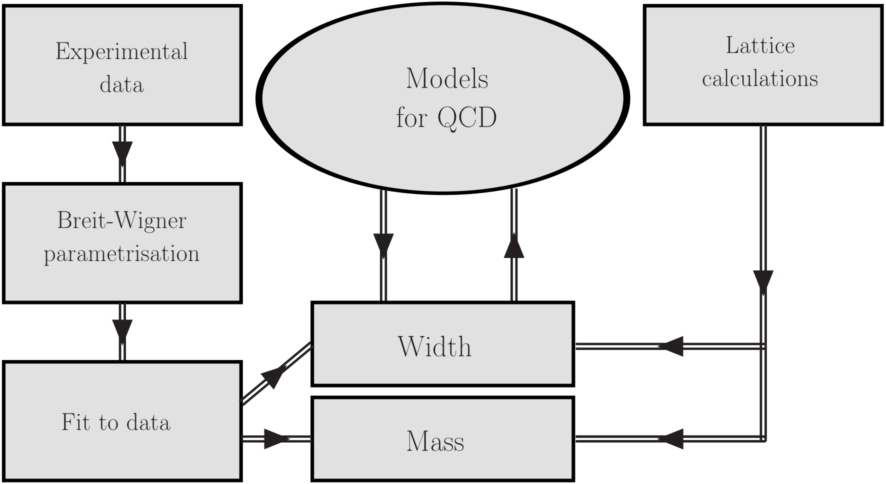
$$\hat{H}_0 = \frac{p^2}{m_Q} + V_0(r) + V_{\rm SD}(r)$$

$$V_0(r) = \sigma r - \frac{\frac{4}{3}\alpha_s}{r} + C_0$$ (Cornell potential)

$$V_{SD}(r) = \underbrace{V_{LS}(r)(\boldsymbol{L}\cdot(\boldsymbol{S}_Q + \boldsymbol{S}_{\bar{Q}}))}_{\text{fine structure}} + \underbrace{V_{SS}(r)(\boldsymbol{S}_Q\cdot\boldsymbol{S}_{\bar{Q}})}_{\text{hyperfine structure}}$$

$$+\underbrace{V_{ST}(r)\Big((\boldsymbol{S}_Q\cdot\boldsymbol{S}_{\bar{Q}}) - 3(\boldsymbol{S}_Q\cdot\boldsymbol{n})(\boldsymbol{S}_{\bar{Q}}\cdot\boldsymbol{n})\Big)}_{\text{spin}-\text{tensor force}} \quad \propto \quad \frac{1}{m_Q^2}$$

Approach to ordinary states



Hadronic physics: Consensus before 2003

- Quark model provides a decent description of low-lying hadrons
- Quark model works surprisingly well even for light flavours
- Heavy flavours (c and b) comply with nonrelativistic theory
- Relativistic corrections improve the description
- Experiment gradually fills “missing states”
- Lattice provides additional/alternative source of information

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General conclusion: Hadronic physics is well understood

Exotic states with heavy quarks

"Exotic animal is more unusual and rare than normal domesticated pets like cats or dogs"

Revolution of 2003: Enfant terrible $X(3872)$

- $I = 0$, $J^{PC} = 1^{++}$, contains $c\bar{c}$
- Too light compared with Quark Model prediction

$$M^{\rm QM}_{\chi_{c1}(2P)} - M^{\rm exp}_X \sim 100 \text{ MeV}$$

- Strongly attracted to $D\bar{D}^*$ threshold

$$M^{\rm exp}_X - (M_{D^0} + M_{\bar{D}^{*0}}) \sim 0$$

- Large ($\sim 40\%$) probability of the decay into $D\bar{D}^*$
- Strong isospin violation

$$Br(X \to \pi^+\pi^-\pi^0 J/\psi) \approx Br(X \to \pi^+\pi^- J/\psi)$$

Revolution of 2003: Enfant terrible $X(3872)$

- $I = 0$, $J^{PC} = 1^{++}$, contains $c\bar{c}$

- ~2500 citations (the most cited paper by Belle)
- $J^{PC} = 1^{++}$ unambiguously established by LHCb in 2013
- Nature of $X(3872)$ still under debate
- New name by PDG — $\chi_{c1}(3872)$

$$M_X - (M_{D^0} + M_{\bar{D}^{*0}}) \sim 0$$

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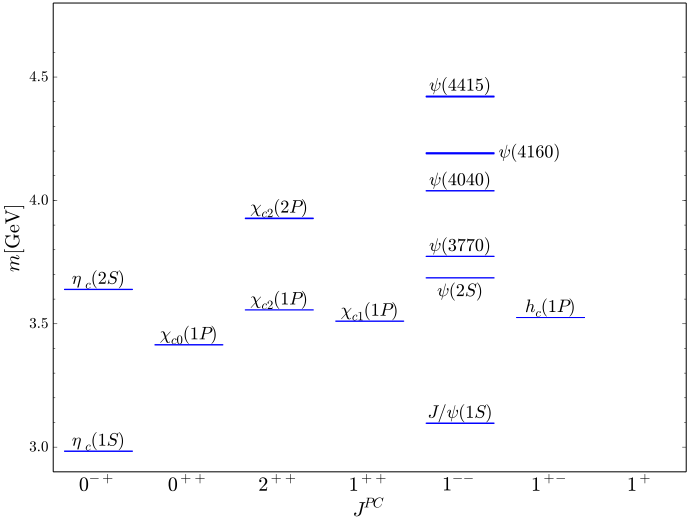
Spectrum of charmonium



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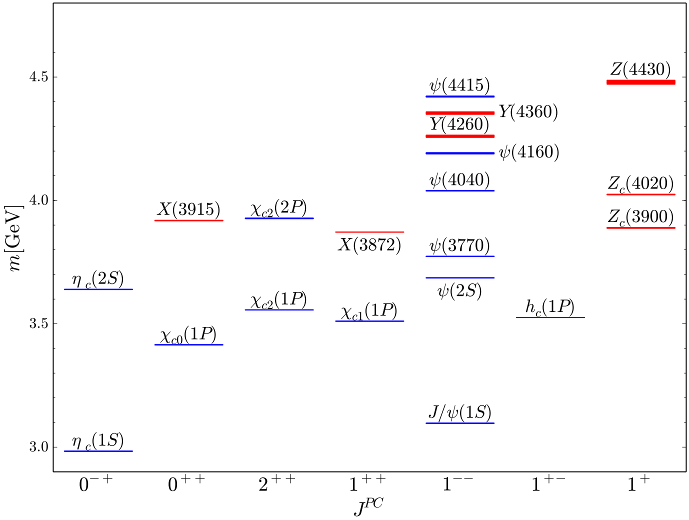

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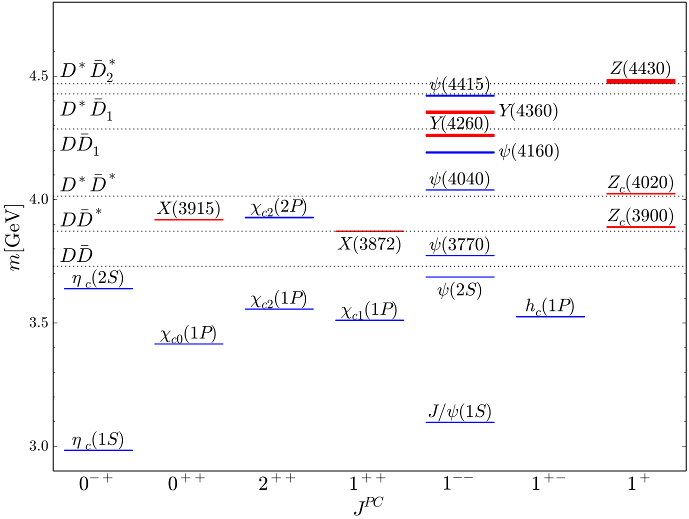

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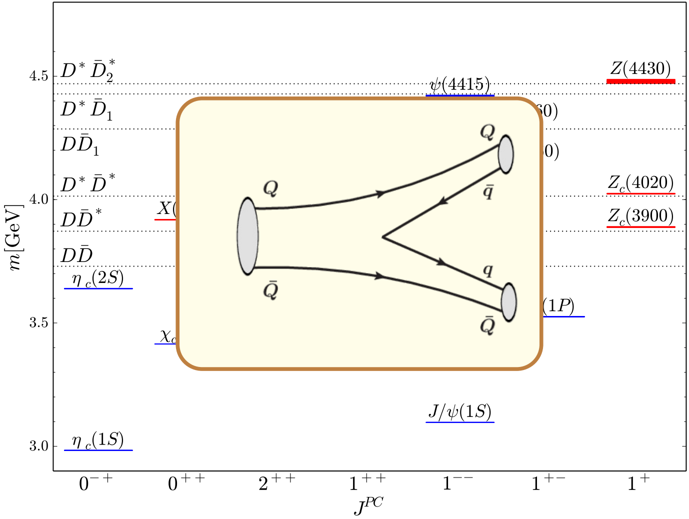
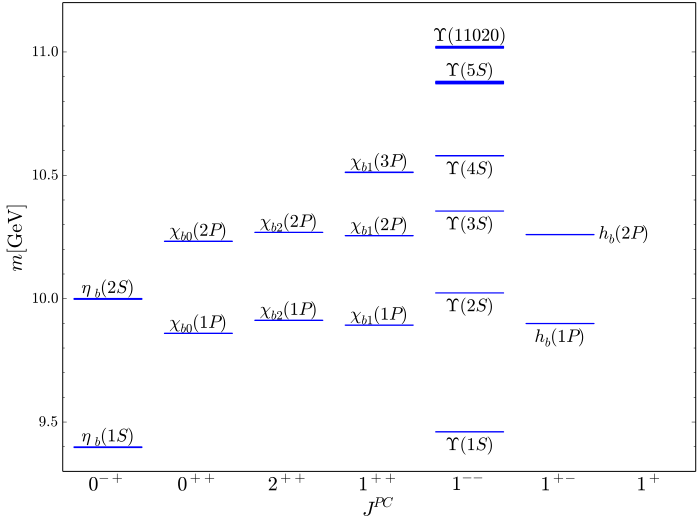


Spectrum of bottomonium



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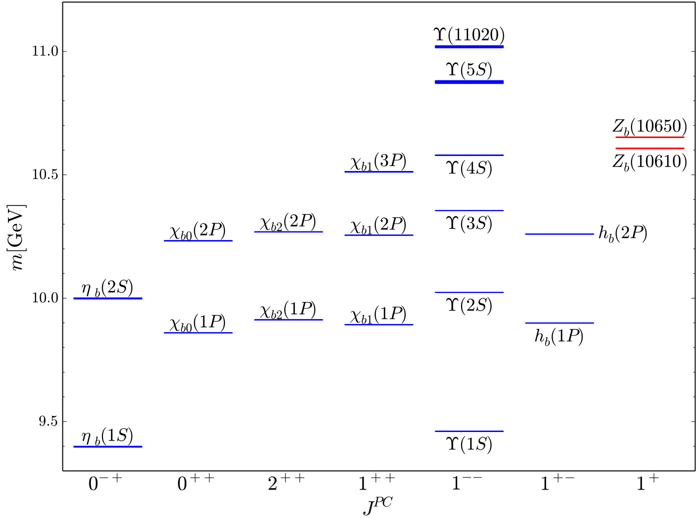

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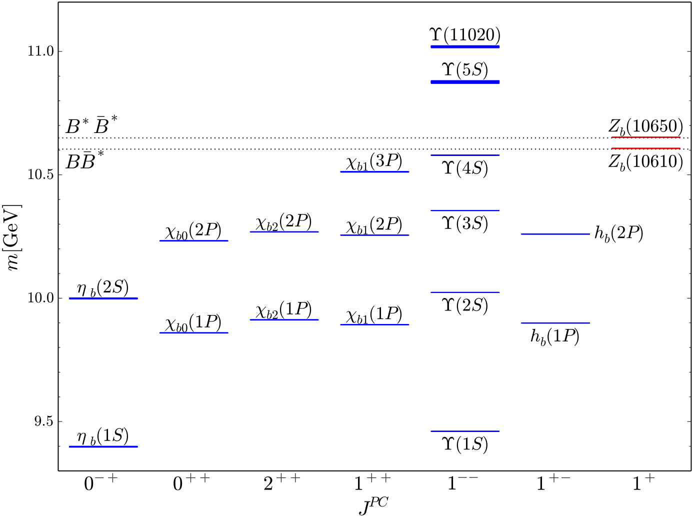

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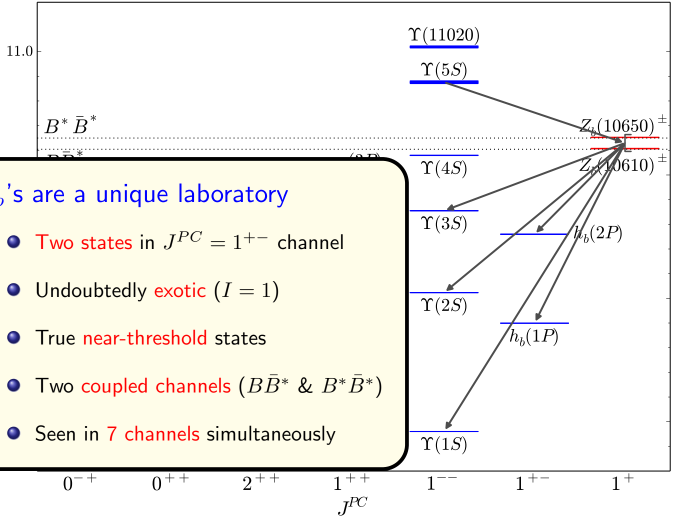
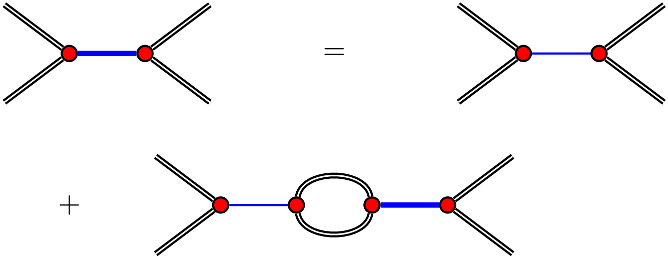


Z_b's are a unique laboratory

- Two states in $J^{PC} = 1^{+-}$ channel
- Undoubtedly exotic $(I = 1)$
- True near-threshold states
- Two coupled channels $(B\bar{B}^* \text{ \& } B^*\bar{B}^*)$
- Seen in 7 channels simultaneously

Effect of hadronic loops

$$|\Psi\rangle = \begin{pmatrix} \sqrt{Z}|\psi_0\rangle \\ \chi(\boldsymbol{k})|H_1 H_2\rangle_{L=0} \end{pmatrix}$$



$$\frac{1}{E - E_0 + \frac{i}{2}\Gamma_0} \quad \Longrightarrow \quad \frac{1}{E - E_f + \frac{i}{2}(gk + \Gamma_0)} \qquad k = \sqrt{2\mu E}$$

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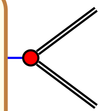
Flatté parametrisation:

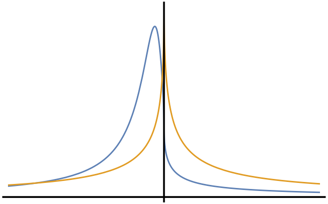


+ Simple and physically transparent
+ Accounts for threshold phenomena
– Difficult multichannel generalisation
– Obscure effect of particle exchanges
– Not systematically improvable

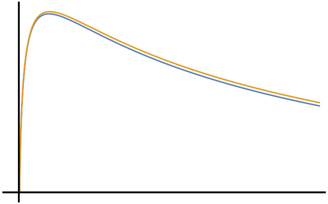
$$\frac{1}{E-E_0+\frac{i}{2}\Gamma_0} \Longrightarrow \frac{1}{E-E_f+\frac{i}{2}(gk+\Gamma_0)} \qquad k=\sqrt{2\mu E}$$

Examples of line shapes

$$\frac{\Gamma_0}{\left|E-E_f+\frac{i}{2}(gk+\Gamma_0)\right|^2}$$



$$\frac{gk}{\left|E-E_f+\frac{i}{2}(gk+\Gamma_0)\right|^2}$$

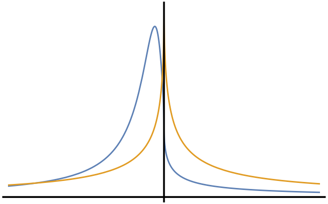


- Bound state $(E_f < 0)$ — blue curve
- Virtual state $(E_f > 0)$ — yellow curve

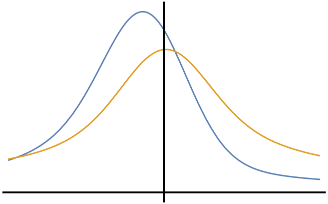
Pole resides on real axis below threshold on RS-I or RS-II

Effect of experimental resolution

$$\frac{\Gamma_0}{\left|E-E_f+\frac{i}{2}(gk+\Gamma_0)\right|^2}$$



$$\int \frac{\Gamma_0 f_{\rm res}(E'-E)dE'}{\left|E'-E_f+\frac{i}{2}(gk+\Gamma_0)\right|^2}$$

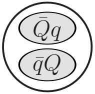


- Left plot — before convolution with resolution
- Right plot — after convolution with resolution

Sharp structures turn to broad humps

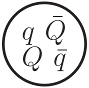
Models for exotic states

- **Hadronic Molecule**



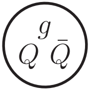
Extended object made of $(\bar{Q}q)$ and $(\bar{q}Q)$

- **Compact Tetraquark**



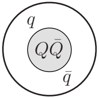
Compact object made of $(Q\bar{Q}q\bar{q})$

- **Hybrid**



Compact object made of $(Q\bar{Q})$ + gluon(s)

- **Hadro-Quarkonium**



$(Q\bar{Q})$ surrounded by light quarks

Models for exotic states

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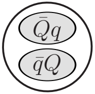


Extended object made of $(\bar{Q}q)$ and $(\bar{q}Q)$

- ${}^3S_1$ NN system with $I=0$:

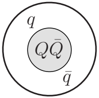
  Pole on RS-I with $E_B = 2.23$ MeV $\Longrightarrow$ deuteron

- ${}^1S_0$ NN system with $I=1$:

  Pole on RS-II with $E_B = 0.067$ MeV $\Longrightarrow$ virtual state

Compact object made of $(Q\bar{Q})$ + gluon(s)

- **Hadro-Quarkonium**



$(Q\bar{Q})$ surrounded by light quarks

Some generalities

Composite or elementary?

Effective range expansion: $-a^{-1}+\frac{1}{2}rk^2-ik$

$$a=\frac{2(1-Z)}{(2-Z)}\frac{1}{\sqrt{2\mu E_B}}+O\left(\frac{1}{\beta}\right) \qquad r=-\frac{Z}{(1-Z)}\frac{1}{\sqrt{2\mu E_B}}+O\left(\frac{1}{\beta}\right)$$

β ($\gg k$) — (inverse) range of force

(Weinberg'1960s)

Elementary (confined) state

- Two near-threshold poles

Composite (molecular) state

- One near-threshold pole

$\Longrightarrow$ pole counting rules (Morgan'1992)

Effective range of a molecule

- Smorodinsky: $r > 0$ for finite-range negative potential
  (Smorodinsky'1948,Esposito et al.'2021)
- Wigner: causality bounds r from above ($r < 0$ for zero-range potentials)
  (Wigner'1955)
- Molecule: r is defined by range corrections (Weinberg'1960s)

$$r = \underbrace{-\frac{Z}{(1-Z)}\frac{1}{\sqrt{2\mu E_B}}}_{\text{small for } Z\to 0} + \Delta r(\beta)$$

$$\Delta r(\beta \sim m_\pi) = -2\frac{\partial}{\partial k^2}\left[\frac{2}{\pi}\int^{m_\pi}\frac{q^2 dq}{q^2-k^2-i0} - ik\right]_{|k^2=0} \sim \frac{1}{m_\pi} \sim 1\ \text{fm} > 0$$

Weinberg(like) analysis in physics of heavy flavours

- Resonances reside near S-wave two-body threshold (Yes)
- Bound states (Not always)

  Solution: $\bar{X} = 1 - Z \to 1/\sqrt{1+2|r/a|}$ (Matuschek et al.'2021)
- Stable constituents (Almost never)

  Solution: $k_{\rm eff} = \sqrt{2\mu(E + i\frac{\Gamma}{2})} \Longrightarrow$ ERE at complex point (Braaten et al.'2010)
- No additional thresholds near by (Rarely)

  Solution: Expand contributions from additional channels at $k_1 \to 0$ (!!!)
- No additional singularities (Matter of luck)
  - CDD (Castillejo-Dalitz-Dyson) poles
  - Left-hand cuts
  - ...

  Solution: No general solution...

CDD pole

Let direct interaction between hadrons H_1 and H_2 produce a near-threshold pole

$$t_V(E) \approx \frac{1}{-\gamma_V - ik}$$

Then the amplitude reads

$$f(E) = \frac{1}{E - E_f + \frac{i}{2}gk - \frac{(E-E_f)^2}{E-E_C}} \quad \text{with} \quad E_C = E_f - \frac{1}{2}g\gamma_V$$

- $|E_C| \gg |E_f|$

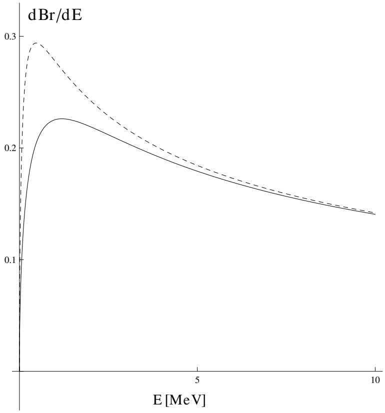
$$f(E) \approx \frac{1}{E - E_f + \frac{i}{2}gk}$$

- $|E_C| \sim |E_f|$

$$f(E) \propto (E - E_C)$$

CDD pole at work

$|E_C| \gg |E_f|$



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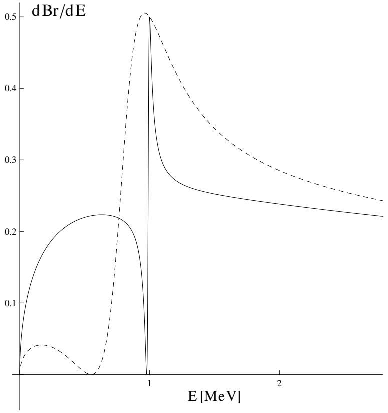

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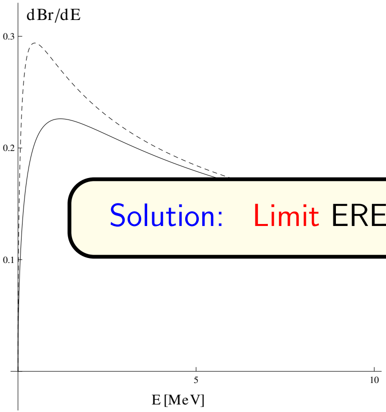

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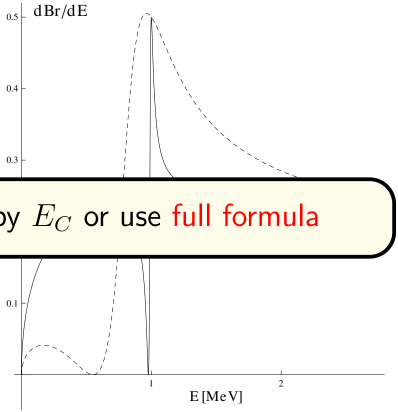
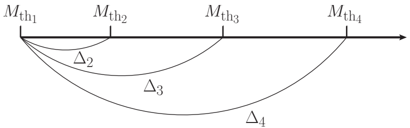


Solution: Limit ERE by E_C or use full formula

Generalisation to multiple hadronic channels



$$|\Psi\rangle = \begin{pmatrix} \sqrt{Z}|\psi_0\rangle \\ \chi_1(\boldsymbol{p})|H_{11}H_{12}\rangle \\ \chi_2(\boldsymbol{p})|H_{21}H_{22}\rangle \\ \cdots \end{pmatrix} \qquad H = \begin{pmatrix} E_0 & f_1 & f_2 & \cdots \\ f_1 & H_{h_1} & V_{12} & \cdots \\ f_2 & V_{21} & H_{h_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$H_{h_i}(\boldsymbol{p},\boldsymbol{p}') = \left(\Delta_i + \frac{p^2}{2\mu_i}\right)\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') + V_{ii}(\boldsymbol{p},\boldsymbol{p}')$$

For two channels V_{ij} $(i,j=1,2)$ is parametrised through the singlet and triplet inversed scattering lengths γ_s and γ_t:

- γ_s governs the position of the zero E_C
- γ_t governs the relevance of the term k_1k_2

Solution of the Lippmann-Schwinger equation

$$t_s = \frac{1}{2}(t_{11}+t_{22})+t_{12} = \frac{(E-E_C)\,(2\gamma_t+i(k_1+k_2))}{4\pi^2\mu\ D(E)}$$

$$t_t = \frac{1}{2}(t_{11}+t_{22})-t_{12} = \frac{2\gamma_s(E-E_f)+i(k_1+k_2)(E-E_C)}{4\pi^2\mu\ D(E)}$$

$$t_{st} = \frac{1}{2}(t_{11}-t_{22}) = \frac{i(k_2-k_1)(E-E_C)}{4\pi^2\mu\ D(E)}$$

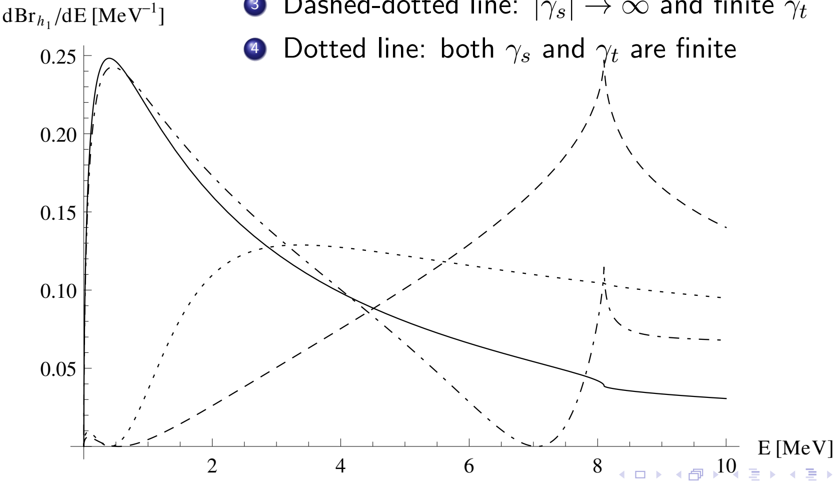
$$D(E) = \gamma_s\Big(2\gamma_t+i(k_1+k_2)\Big)(E-E_f)-\Big(2k_1k_2-i\gamma_t(k_1+k_2)\Big)(E-E_C)$$

$$E_C = E_f - \frac{1}{2}g\gamma_s$$

(Artoisenet et al.'2010, Hanhart et al.'2011)

Examples of the line shapes

1. Solid line: $|\gamma_s| \to \infty$ and $|\gamma_t| \to \infty$
2. Dashed line: finite γ_s and $|\gamma_t| \to \infty$
3. Dashed-dotted line: $|\gamma_s| \to \infty$ and finite γ_t
4. Dotted line: both γ_s and γ_t are finite



Contribution of second channel

Assume $|\gamma_s| \to \infty$ (no CDD pole) and $|\gamma_t| \to \infty$ (no channels entanglement)

- Naive expansion

$$E - E_f + \frac{i}{2} g(k_1 + k_2) = \frac{k_1^2}{2\mu} - E_f + \frac{i}{2} g\Big(k_1 + \underbrace{\sqrt{2\mu\Delta - k_1^2}}_{\text{expand for } k_1 \to 0}\Big)$$

$$r = r_0 + \delta r \qquad r_0 = -\frac{2}{\mu g} \qquad \delta r = -\frac{1}{\sqrt{2\mu\Delta}} \underset{\Delta \to 0}{\longrightarrow} \infty \ (!!!)$$

- Educated expansion: Use exact two-channel expression

$$Z = \left(1 - \frac{1}{r_0}\left(\frac{1}{\sqrt{2\mu E_B}} + \frac{1}{\sqrt{2\mu(E_B + \Delta)}}\right)\right)^{-1}$$

in Weinberg formula for r

$$r = r_0 \frac{\sqrt{E_B + \Delta}}{\sqrt{E_B} + \sqrt{E_B + \Delta}} \underset{\Delta \gg E_B}{\longrightarrow} r_0$$

Can we do without ERE?

Probalility to observe resonance in the α-th channel $(\alpha = 1, 2)$

(Hyodo et al,'2012,Aceti & Oset'2012)

$$X_\alpha = g_\alpha^2 \left[\frac{d}{dM^2} \int \frac{d^3p}{(2\pi)^3} G_\alpha(M,p) \right]_{|M=M_{\rm pole}}$$

with the couplings defined as residues

$$g_\alpha g_\beta = \lim_{M \to M_{\rm pole}} (M^2 - M_{\rm pole}^2) T_{\alpha\beta}(M)$$

In neglect of constituents widths

$$X_1 = \frac{\sqrt{E_B + \Delta}}{\sqrt{E_B} + \sqrt{E_B + \Delta}} \qquad X_2 = \frac{\sqrt{E_B}}{\sqrt{E_B} + \sqrt{E_B + \Delta}}$$

Generalisation to compact component

Single hadronic channel

$$Z \propto \sqrt{E_B} \qquad X = 1 - Z$$

Two hadronic channels ($\mu_1 = \mu_2 = \mu$)

$$Z = \frac{R_0}{R_0 + R_1 + R_2} \qquad X_1 = \frac{R_1}{R_0 + R_1 + R_2} \qquad X_2 = \frac{R_2}{R_0 + R_1 + R_2}$$

where

$$R_0 = \frac{2}{\mu g} = |r_0| \qquad R_1 = \frac{1}{\sqrt{2\mu E_B}} \qquad R_2 = \frac{1}{\sqrt{2\mu(E_B + \Delta)}}$$

$$\Delta = M_2^{\rm th} - M_1^{\rm th}$$

Spectral density

Hint: Extract information from continuum w.f. $(E = k^2/(2\mu))$

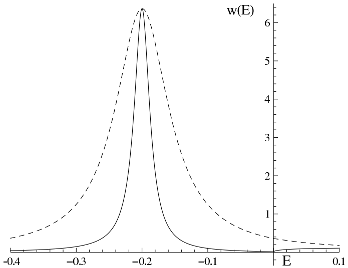
$$|\Psi\rangle = C_k|\psi_0\rangle + \chi_k(p)|H_1 H_2\rangle$$

$$w(E) = 4\pi\mu k|C_k|^2\Theta(E - E_{\rm th}^{\rm min}) = \frac{1}{2\pi i}\left[\frac{1}{E - E_0 + \Sigma^*(E)} - \text{c.c.}\right]$$

(Bogdanova et al.'1991, Baru *et al*'2004)

$$\text{“}Z\text{”} = W = \int_{E_{\rm th}-\delta}^{E_{\rm th}+\delta} w(E)dE$$

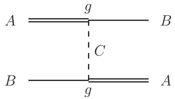
(δ is not well defined)



$$W_{\rm solid} = \int_{-0.6\,\rm MeV}^{0.2\,\rm MeV} w_{\rm solid}(E)dE \approx 0.3$$

$$W_{\rm dashed} = \int_{-0.6\,\rm MeV}^{0.2\,\rm MeV} w_{\rm dashed}(E)dE \approx 0.9$$

Unitarisation effects

Particles A and B interact exchanging particle C: 

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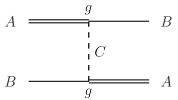

- Naive expectations ($\Gamma(A \to BC) \propto g^2$):

$$V_{AB} \propto g^2 \quad \underset{g\to\infty}{\Longrightarrow} \quad \text{deeply bound states}$$

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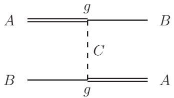

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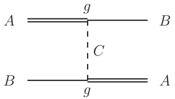

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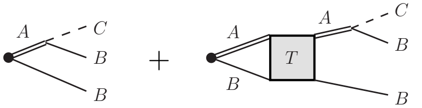
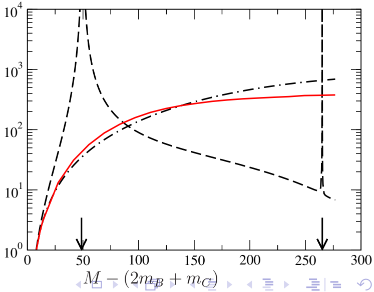
- Actuality: as g grows, re-scatterings become important
- Way to proceed: solve Lippmann-Schwinger equation $T = V - VGT$

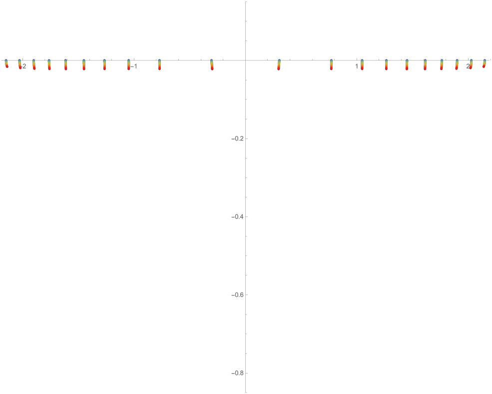




- Dashed line: neglected A width
- Dot-dashed line: constant A width
- Solid line: dynamical A width

Unitarisation effects

Resonances $R_n = (\bar{Q}Q)_n$ interact via $(\bar{q}Q)$ field φ ($\mathcal{L}_{\rm int} = g \sum_n R_n \bar{\varphi}\varphi$)



$g \sim 0$

Unitarisation effects

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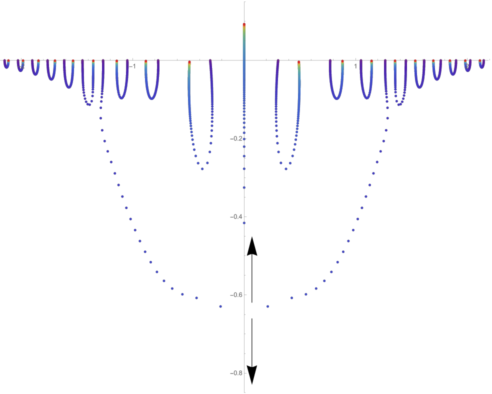
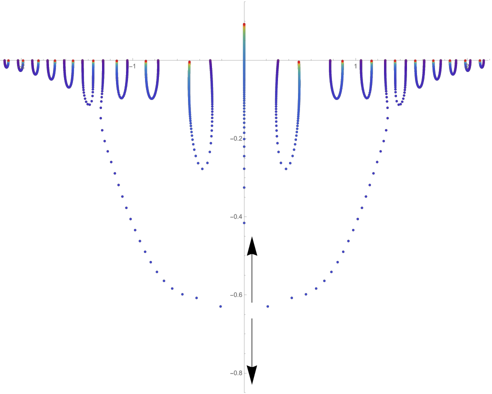


$g \rightarrow \infty$

Unitarisation effects

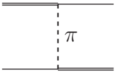
Resonances $R_n = (\bar{Q}Q)_n$ interact via $(\bar{q}Q)$ field φ ($\mathcal{L}_{\rm int} = g\sum_n R_n\bar{\varphi}\varphi$)

$g \to \infty$



Conclusion: For $g \to \infty$ dressed resonances decouple from each other and a molecule is formed

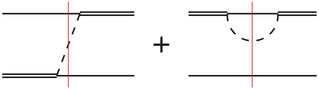
Pion exchange

$V_{\text{OPE}} =$ 

$\sim \frac{q_i q_j}{q^2 - m_\pi^2} \underset{\text{S-wave, recoil}}{\Longrightarrow} \frac{1}{3}\delta_{ij}\left(-1 + \overbrace{\frac{\mu_\pi^2}{\boldsymbol{q}^2 + \underbrace{[m_\pi^2 - (M_* - M)^2]}_{\text{Effective mass } \mu_\pi^2}}}^{\text{Long-range OPE}}\right)$

(S-wave, ~~recoil~~)

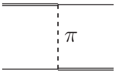
- Deuteron ($m_\pi \gg M_n - M_p \Longrightarrow \mu_\pi = m_\pi$) $\Longrightarrow V_{\text{OPE}}^{\text{long-range}} \sim \frac{1}{r} e^{-m_\pi r}$
- Charmonium system ($m_\pi < M_{D^*} - M_D \Longrightarrow \mu_\pi^2 < 0$ & $|\mu_\pi| \ll m_\pi$):

3-body unitarity: 

- Bottomonium system ($m_\pi > M_{B^*} - M_B \Longrightarrow \mu_\pi^2 > 0$ & $\mu_\pi < m_\pi$):

$\int d\Omega_{\hat{k}\hat{k}'} V_{\text{OPE}}(\boldsymbol{k} - \boldsymbol{k}') \sim \log \frac{\mu_\pi^2 + (k + k')^2}{\mu_\pi^2 + (k - k')^2} \underset{k' = k}{\Longrightarrow}$ left-hand cut at $k^2 < -\frac{1}{4}\mu_\pi^2$

Pion exchange

$V_{\text{OPE}} =$ 

$\sim \dfrac{q_i q_j}{q^2 - m_\pi^2}$ $\overset{\text{S-wave, }\not{\text{recoil}}}{\Longrightarrow}$ $\frac{1}{3}\delta_{ij}\left(-1 + \overbrace{\frac{\mu_\pi^2}{\boldsymbol{q}^2 + \underbrace{[m_\pi^2 - (M_* - M)^2]}_{\text{Effective mass } \mu_\pi^2}}}^{\text{Long-range OPE}}\right)$

If $m_\pi^{\text{lat}} > m_\pi^{\text{phys}} \Longrightarrow$ Interpretation of lattice results may be nontrivial

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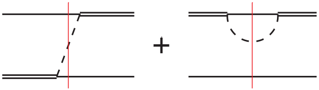
3-body unitarity: 

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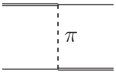
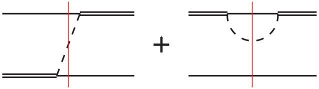
$V_{\text{OPE}} =$  $\sim \dfrac{q_i q_j}{q^2 - m_\pi^2} \underset{\text{S-wave, }\cancel{\text{recoil}}}{\Longrightarrow} \dfrac{1}{3}\delta_{ij}\left(-1 + \overbrace{\dfrac{\mu_\pi^2}{\boldsymbol{q}^2 + \underbrace{[m_\pi^2 - (M_* - M)^2]}_{\text{Effective mass } \mu_\pi^2}}}^{\text{Long-range OPE}}\right)$

If $m_\pi^{\text{lat}} > m_\pi^{\text{phys}} \Longrightarrow$ Interpretation of lattice results may be nontrivial

- For data in a broad energy range, D waves from OPE are important

3-body unitarity: 

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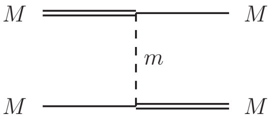
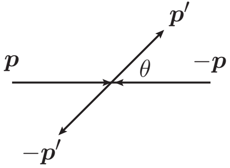
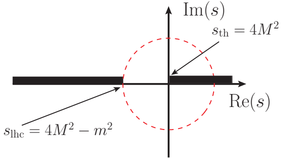
Left-hand cut







$$\mathcal{A} = \frac{1}{u - m^2} = -\frac{1}{m^2 + 2p^2(1 - \cos\theta)}$$

$$s = (p_1 + p_2)^2 = 4(p^2 + M^2) \quad \Longrightarrow \quad s_{\text{th}} = 4M^2$$

$$\mathcal{A}_S = \int \frac{d\Omega}{4\pi}\mathcal{A} = \frac{1}{4p^2}\log\frac{m^2 + 4p^2}{m^2} \quad \Longrightarrow \quad s_{\text{lhc}} = 4M^2 - m^2$$

Heavy-quark spin symmetry

- Exotic states contain heavy quarks (HQ)
- In the limit $m_Q \to \infty$ $(m_Q \gg \Lambda_{\rm QCD})$ spin of HQ decouples

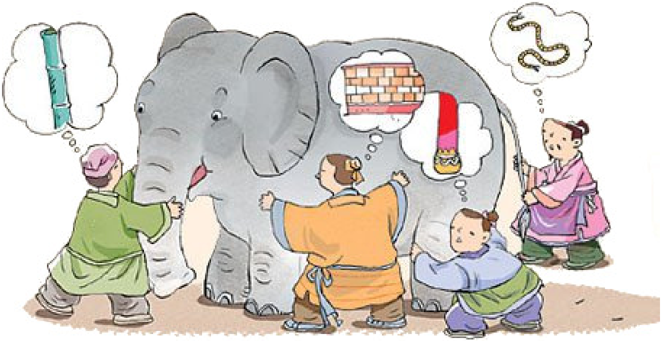
  $\Longrightarrow$ Heavy Quark Spin Symmetry (HQSS)

- For realistic m_Q's HQSS is approximate but accurate symmetry of QCD
- HQSS $=$ tool to relate properties of states with different HQ spin orientation

  $\Longrightarrow$ Spin partners

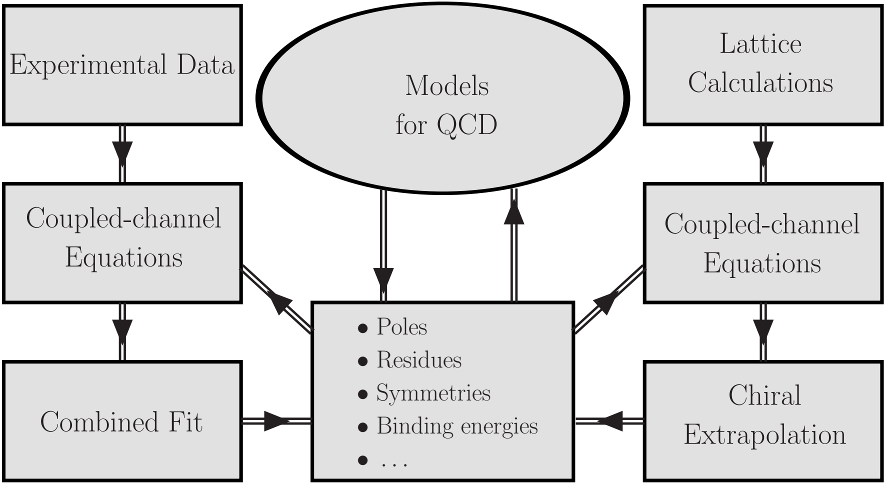
Combined analysis

Considering different channels separately is like blind study of elephant



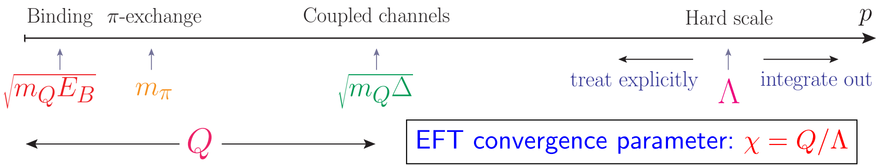
Combined analysis of all data sets is necessary!

Approach to exotic states



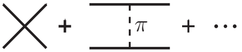
Effective Field Theory for Hadronic Molecules

Effective field theory for hadronic molecules



Interaction potential between heavy hadrons:

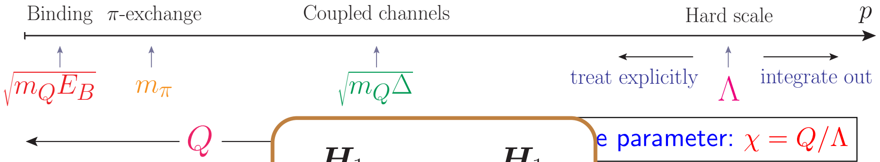
- Includes all relevant interactions



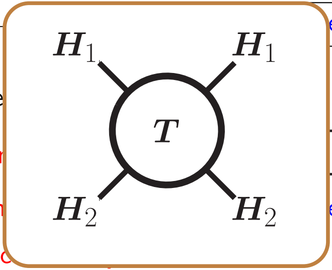
- Complies with relevant symmetries (chiral, HQSS, etc)
- Incorporates coupled-channel dynamics
- Expanded in powers of p^2/Λ^2 and truncated at necessary order (LO, NLO...)
- Iterated to all orders via (multichannel) Lippmann-Schwinger equation

$$T = V - VGT$$

Effective field theory for hadronic molecules



e parameter: $\chi = Q/\Lambda$



Interaction potential betwe

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$$T = V - VGT$$

Effective field theory for hadronic molecules

Free parameters:

- Low-energy constants
- (Bare) couplings to hadronic channels

Input (combined analysis):

- Line shapes (Dalitz plots)
- Partial branchings

Output:

- Pole position M_0 ("mass" $=$ Re(M_0), "width" $= 2 \times$ Im(M_0))
- Residues at the poles (dressed couplings)

Predictions:

- New properties of state: line shapes, partial widths,...
- Spin partners: poles, line shapes, partial widths,...
- Chiral extrapolations

Conclusions

- Collider experiments at energies above open-flavour thresholds started new era in hadronic physics
- Threshold phenomena, coupled channels, pion exchange are important
- Multibody unitarity and analyticity of amplitude need to be preserved
- Line shapes of non-Breit-Wigner form is current reality
- From "mass" and "width" to pole position and residues (couplings)
- EFT can be employed to a success as model-independent, systematically improvable analysis and prediction tool
- Results of EFT analysis to be used as input for QCD-inspired models
- Lattice simulations are important to fill the gap in experimental data and provide numerical experiment in "alternative Universe"

Backup

OPE sign

	$I=0$	$I=1$
PV	3	1
$(P\bar{V})_{C=\pm}$	$3C$	$-C$

$X(3872)$ $(I=0,\ C=+)$	T_{cc} $(I=0)$	Z_b $(I=1,\ C=-)$	W_{bJ} $(I=1,\ C=+)$
$+3$	$+3$	$+1$	-1