介子相对论波函数的求解和应用

Guo-Li Wang
Hebei University

In collaboration with

Chao-Hsi Chang, Tianhong Wang, Qiang Li, Wei Li, ...

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Outline

- 介子的描述
- Salpeter equation
- 波函数
- 应用举例
- Summary

Meson and its description

- Usually using $^{2S+1}L_J$ or $J^{P(C)}$
- nL and $^{2S+1}L_J$

	$S=0$	$S=1$	$S=1$	$S=1$
S	$^1S_0\ (0^{-+})$	$^3S_1\ (1^{--})$		
P	$^1P_1\ (1^{+-})$	$^3P_0\ (0^{++})$	$^3P_1\ (1^{++})$	$^3P_2\ (2^{++})$
D	$^1D_2\ (2^{-+})$	$^3D_1\ (1^{--})$	$^3D_2\ (2^{--})$	$^3D_3\ (3^{--})$
F	$^1F_3\ (3^{+-})$	$^3F_2\ (2^{++})$	$^3F_3\ (3^{++})$	$^3F_4\ (4^{++})$
G	$^1G_4\ (4^{-+})$	$^3G_3\ (3^{--})$	$^3G_4\ (4^{--})$	$^3G_5\ (5^{--})$

- S-D mixing
$$|\psi(3770)\rangle = |1^3D_1\rangle\cos\theta + |2^3S_1\rangle\sin\theta,$$
$$|\psi(3686)\rangle = -|1^3D_1\rangle\sin\theta + |2^3S_1\rangle\cos\theta$$

- ${}^1P_1 - {}^3P_1$ mixing
$$|D_1(2420)\rangle = |\frac{3}{2}\rangle = \cos\theta|^1P_1\rangle + \sin\theta|^3P_1\rangle,$$
$$|D_1'(2430)\rangle = |\frac{1}{2}\rangle = -\sin\theta|^1P_1\rangle + \cos\theta|^3P_1\rangle.$$

- 这两个例子说明，${}^{2S+1}L_J$ 描述粒子有时候有问题

- 而用 $J^{P(C)}$ 描述粒子不会出问题，始终是对的

Meson and its description

- $J^{P(C)}$

$J=0$	$0^{-(+)}\ ({}^1S_0)$	$0^{+(+)}\ ({}^3P_0)$		
$J=1$	$1^{-(-)}\ ({}^3S_1,\ {}^3S_1-{}^3D_1,\ {}^3D_1)$	$1^{++}\ ({}^3P_1)$	$1^{+-}\ ({}^1P_1)$	$1^{+}\ ({}^3P_1-{}^1P_1)$
$J=2$	$2^{+(+)}\ ({}^3P_2,\ {}^3P_2-{}^3F_2,\ {}^3F_2)$	$2^{--}\ ({}^3D_2)$	$2^{-+}\ ({}^1D_2)$	$2^{-}\ ({}^3D_2-{}^1D_2)$
$J=3$	$3^{-(-)}\ ({}^3D_3,\ {}^3D_3-{}^3G_3,\ {}^3G_3)$	$3^{++}\ ({}^3F_3)$	$3^{+-}\ ({}^1F_3)$	$3^{+}\ ({}^3F_3-{}^1F_3)$

- 3 categories

1. 0^- and 0^+
2. Natural parity 1^-, 2^+ and 3^-
3. Unnatural parity 1^+, 2^- and 3^+

- 两种描述的差别 ${}^{2S+1}L_J$ or $J^{P(C)}$
- 给定 ${}^{2S+1}L_J$ 对应确定的 $J^{P(C)}$
- 反过来不成立，给定 $J^{P(C)}$，不对应确定的 ${}^{2S+1}L_J$
- 可见，$J^{P(C)}$ 是更基本的物理量
- 因此构建波函数的表示时应该依据 $J^{P(C)}$，而不是 ${}^{2S+1}L_J$
  因此说粒子是S、P、D波，正如psi(3770)，不一定对。

依据J^P构建波函数表示

- 对介子进行P变换

$$P' = (P_0, -\vec{P}) \text{ and } q' = (q_0, -\vec{q})$$

$$\varphi_P(q) = \eta_P \gamma_0 \varphi_{P'}(q') \gamma_0,$$

- C变换

$$\varphi_P(q) = \eta_C C \varphi_P^T(-q) C^{-1},$$

$$C\gamma_5^T C^{-1} = \gamma_5 \text{ and } C\gamma_\mu^T C^{-1} = -\gamma_\mu$$

依据J^P构建波函数表示

- The wave function of a meson can be composed meson mass M, momentum P, internal relative momentum q between quark and antiquark, and Dirac matrix, possible polarization vector or tensor, etc.
- Universal wave function for a 0^- meson in Instantaneous approximation ($P \cdot q_\perp = 0$), $q_{\parallel}^{\mu} = (P \cdot q/M^2)P^{\mu}$, $q_{\perp}^{\mu} = q^{\mu} - q_{\parallel}^{\mu}$

$$\varphi_P^{0^-}(q_\perp) = \left(a_1 M + a_2 \not{P} + a_3 \not{q}_\perp + a_4 \frac{\not{q}_\perp \not{P}}{M} \right) \gamma^5$$

Four unknown unknown radial wave functions $a_i \equiv a_i(-q_\perp^2)$

介子结构波函数的时空对称性质*

北京大学理論物理研究室基本粒子理論組
中国科学院数学研究所理論物理研究室

对于 0^- 介子，$\eta_P=-1$，$\eta_C=1$，故波函数有一般形式

$$\chi_P(p)=\gamma_5 f_1+\frac{i\hat{P}}{m}\gamma_5 f_2+\frac{i\hat{p}}{M'}\frac{(Pp)}{mM'}\gamma_5 f_3+\frac{i}{mM'}P_\mu p_\nu\sigma_{\mu\nu}\gamma_5 f_4.$$

1^- 介子，$\eta_P=-1$，$\eta_C=-1$，故波函数有一般形式

$$\chi_P^\sigma(p)=\chi_{P\mu}(p)e_\mu^\sigma,$$

$$\chi_{P\mu}(p)=\gamma_\mu g_1+\frac{i\hat{P}}{m}\gamma_\mu g_2+\frac{ip_\mu}{M'}g_3+\frac{1}{M'^2}p_\mu\hat{p}g_4+\frac{1}{mM'^2}p_\mu p_\nu P_\lambda\sigma_{\nu\lambda}g_5+$$

$$+\frac{i}{mM'}\varepsilon_{\mu\nu\rho\sigma}p_\nu P_\rho\gamma_\sigma\gamma_5 g_6+\frac{1}{M'}\sigma_{\mu\nu}p_\nu\frac{(Pp)}{mM'}g_7+\frac{p_\mu}{mM'}\hat{p}\frac{(Pp)}{mM'}g_8$$

求解问题

- 薛定谔方程是非相对论的，只能求解一个未知量，例如

$$\varphi_P^{0^-}(q_\perp) = a\,(M + \not{P})\,\gamma^5$$

- 多未知量: relativistic Bethe-Salpeter equation

$$\chi_P(q) = iS(p_1)\int \frac{d^4k}{(2\pi)^4} V(P,k,q)\chi_P(k)S(-p_2)$$

- Instantaneous verson: Salpeter equation

$$\varphi(q_{P\perp}) \equiv i\int \frac{dq_P}{2\pi}\chi_P(q), \qquad \eta_P(q_{P\perp}) \equiv \int \frac{dk^3_{P\perp}}{(2\pi)^3} V(k_{P\perp}, q_{P\perp})\varphi(k_{P\perp})$$

$$\chi_P(q) = S(p_1)\eta_P(q_{P\perp})S(-p_2)$$

Salpeter euqation

- 正负能投影算符

$$\Lambda^{\pm}(p_{1_{P\perp}}) = \frac{1}{2\omega_1}\left[\frac{\not{P}}{M}\omega_1 \pm (m_1 + \not{p}_{1_{P\perp}})\right],$$

$$\Lambda^{\pm}(-p_{2_{P\perp}}) = \frac{1}{2\omega_2}\left[\frac{\not{P}}{M}\omega_2 \pm (-m_2 + \not{p}_{2_{P\perp}})\right]$$

- Salpeter equation

$$\varphi(q_{P\perp}) = \frac{\Lambda^{+}(p_{1_{P\perp}})\eta_P(q_{P\perp})\Lambda^{+}(-p_{2_{P\perp}})}{(M - \omega_1 - \omega_2)} - \frac{\Lambda^{-}(p_{1_{P\perp}})\eta_P(q_{P\perp})\Lambda^{-}(-p_{2_{P\perp}})}{(M + \omega_1 + \omega_2)}$$

- Positive and negative wave functions

$$\varphi^{\pm\pm} = \Lambda^{\pm}(p_{1_{P\perp}})\frac{\not{P}}{M}\varphi\frac{\not{P}}{M}\Lambda^{\pm}(-p_{2_{P\perp}})$$

$$\varphi(q_{P\perp}) = \varphi^{++}(q_{P\perp}) + \varphi^{+-}(q_{P\perp}) + \varphi^{-+}(q_{P\perp}) + \varphi^{--}(q_{P\perp})$$

Salpeter euqation

- Salpeter euqation 第二种表述

$$\varphi^{++}(q_{P_\perp}) = \frac{\Lambda^+(p_{1P_\perp})\eta(q_{P_\perp})\Lambda^+(-p_{2P_\perp})}{(M-\omega_1-\omega_2)},$$

$$\varphi^{--}(q_{P_\perp}) = -\frac{\Lambda^-(p_{1P_\perp})\eta(q_{P_\perp})\Lambda^-(-p_{2P_\perp})}{(M+\omega_1+\omega_2)},$$

$$\varphi^{+-}(q_{P_\perp}) = \varphi^{-+}(q_{P_\perp}) = 0\ ,$$

- 一般 $M+\omega_1+\omega_2 \gg M-\omega_1-\omega_2$

$$\varphi^{++}(q_{P_\perp}) \gg \varphi^{--}(q_{P_\perp}),\quad \varphi(q_{P_\perp}) \approx \varphi^{++}(q_{P_\perp})$$

- 只正能波函数及其方程：对也不对

对比

- 薛定谔方程只求解单参数方程

  Non-relativistic $\varphi_P^{0^-}(q_\perp) = a\,(M + \not{P})\,\gamma^5$

- Salpeter方程可求解4参数（8参数见后面）relativistic

$$\varphi_P^{0^-}(q_\perp) = \left(a_1 M + a_2 \not{P} + a_3 \not{q}_\perp + a_4 \frac{\not{q}_\perp \not{P}}{M}\right)\gamma^5$$

- 球坐标系，用球谐函数表示

$$\varphi_P^{0^-}(q_\perp) = \sqrt{4\pi}\left[M Y_{00}\left(a_1 + a_2\gamma^0\right) - \frac{|\vec{q}|}{\sqrt{3}}(Y_{1-1}\gamma^+ + Y_{11}\gamma^- - Y_{10}\gamma^3)(a_3 + a_4\gamma^0)\right]\gamma^5$$

a1 and a2 terms are S waves, non-relativistic

a3 and a4 terms are P waves, relativistic correction

波函数结果

利用Salpeter方程组后两个方程，2个参数2个方程

$$\varphi_P^{0^-}(q_\perp) = M\left(a_1 + a_2\frac{\not{P}}{M} - a_1 x_- \not{q}_\perp + a_2 x_+ \frac{\not{q}_\perp \not{P}}{M}\right)\gamma^5$$

$$x_+ = \frac{\omega_1 + \omega_2}{m_1\omega_2 + m_2\omega_1}, \quad x_- = \frac{\omega_1 - \omega_2}{m_1\omega_2 + m_2\omega_1} \qquad \omega_1 = \sqrt{m_1^2 - q_\perp^2}, \ \omega_2 = \sqrt{m_2^2 - q_\perp^2}$$

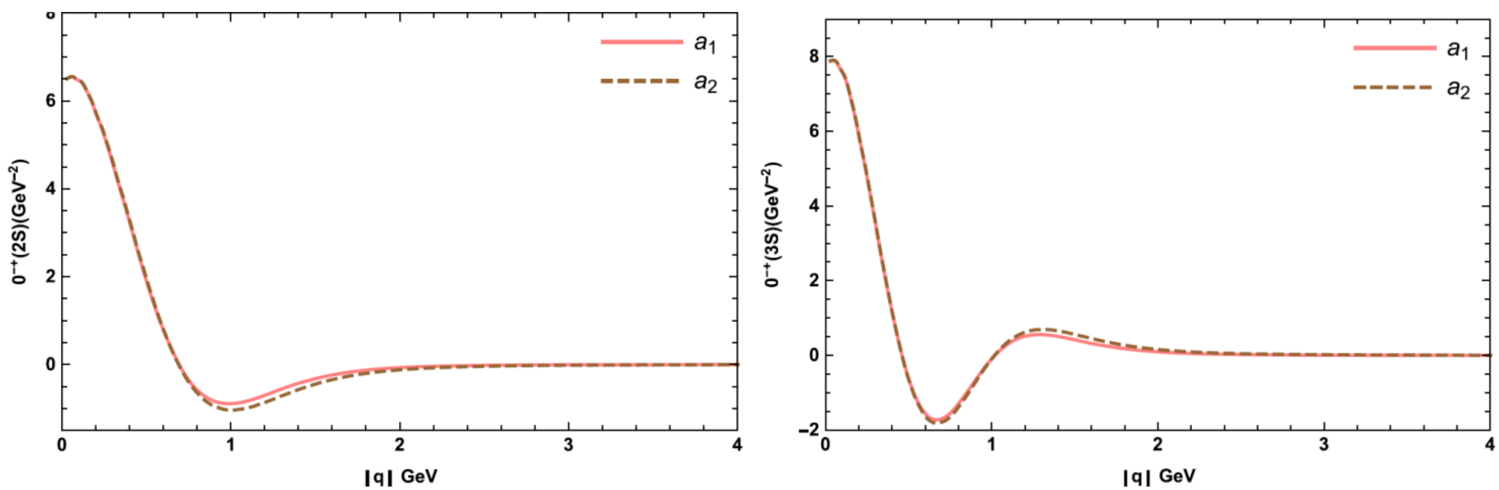
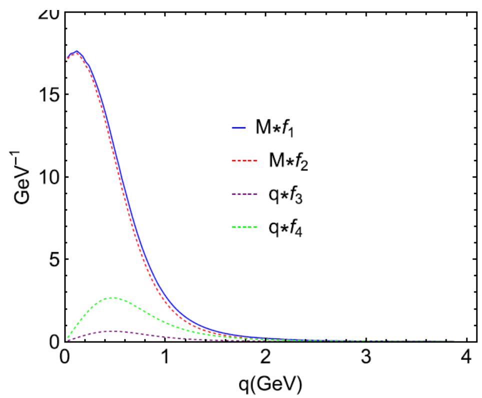


FIG. 3. The radial wave functions of the $\eta_c(2S)$ and $\eta_c(3S)$.

波函数结果

- B



wave functions of the 0^- mesons B^+ (left) and

Bc

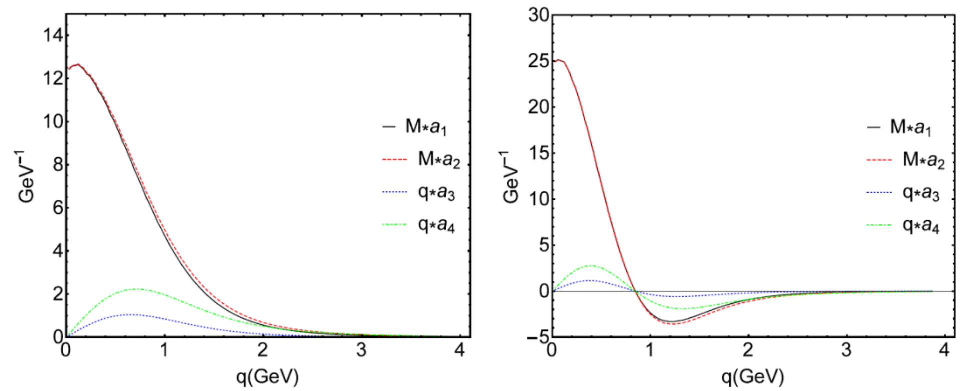
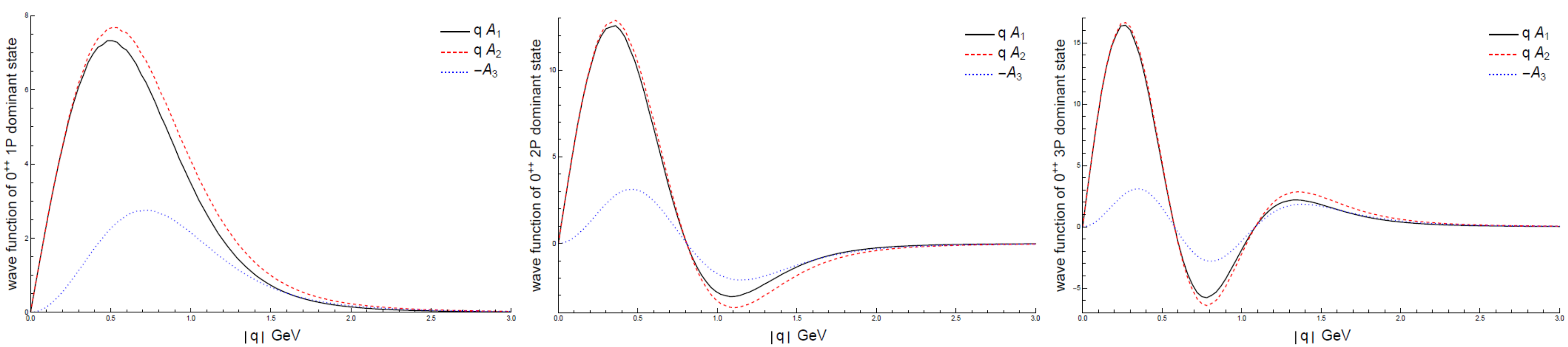


Figure 1. The 0^- wave functions of the ground state $B_c(1S)$ (left) and the first excited state $B_c(2S)$ (right). a_1 and a_2 terms are S waves; a_3 and a_4 terms are P waves.

- 粲偶素 χ_{c0} $\quad \varphi_{0^{++}}(q_\perp) = A_1 \not{q}_\perp + A_2 \not{P} \not{q}_\perp / M + A_3 M$



矢量介子的普遍波函数S-P-D混合

- 1^- $$\varphi_{1^-}(q_\perp) = q_\perp.\epsilon\left(D_1 + \frac{\not P}{M}D_2 + \frac{\not q_\perp}{M}D_3 + \frac{\not P\not q_\perp}{M^2}D_4\right) + M\not\epsilon\left(D_5 + \frac{\not P}{M}D_6 + \frac{\not q_\perp}{M}D_8 + \frac{\not P\not q_\perp}{M^2}D_7\right)$$

D_5, D_6 terms are S waves, D_1, D_2, D_7 and D_8 terms are P waves.

D_3 and D_4 terms are $S - D$ mixture.

The D waves are

$$\left(\epsilon\cdot q_\perp\not q_\perp - \frac{1}{3}q_\perp^2\not\epsilon\right)\left(\frac{1}{M}D_3 - \frac{\not P}{M^2}D_4\right),$$

and the complete S waves are

$$M\not\epsilon\left(D_5 + \frac{\not P}{M}D_6\right) + \frac{1}{3}q_\perp^2\not\epsilon\left(\frac{1}{M}D_3 - \frac{\not P}{M^2}D_4\right).$$

矢量波函数的不同选择

- 1 $$\varphi_{1^-}(q_\perp) = M\not{\epsilon}\left(D_5 + \frac{\not{P}}{M}D_6\right)$$ 方程解：1S, 2S, 3S,… 仅S波
- 2 $$\varphi_{1^-}(q_\perp) = q_\perp.\epsilon\left(D_1 + \frac{\not{P}}{M}D_2\right) + M\not{\epsilon}\left(D_5 + \frac{\not{P}}{M}D_6 + \frac{\not{q}_\perp}{M}D_8 + \frac{\not{P}\not{q}_\perp}{M^2}D_7\right)$$

  方程解：1S, 2S, 3S,… S波为主，少量P波
- 3 $$\varphi_{1^-}(q_\perp) = q_\perp.\epsilon\left(\frac{\not{q}_\perp}{M}D_3 + \frac{\not{P}\not{q}_\perp}{M^2}D_4\right)$$

  方程解：1D, 2D, 3D,… D波为主，S波（约D波三分之一）
- 4 $$\varphi_{1^-}(q_\perp) = q_\perp.\epsilon\left(D_1 + \frac{\not{P}}{M}D_2 + \frac{\not{q}_\perp}{M}D_3 + \frac{\not{P}\not{q}_\perp}{M^2}D_4\right) + M\not{\epsilon}\left(\frac{\not{q}_\perp}{M}D_8 + \frac{\not{P}\not{q}_\perp}{M^2}D_7\right)$$

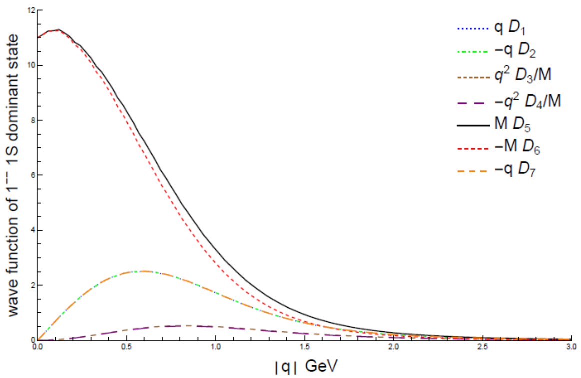
  方程解：1D, 2D, 3D,… D波为主，S波，少量P波

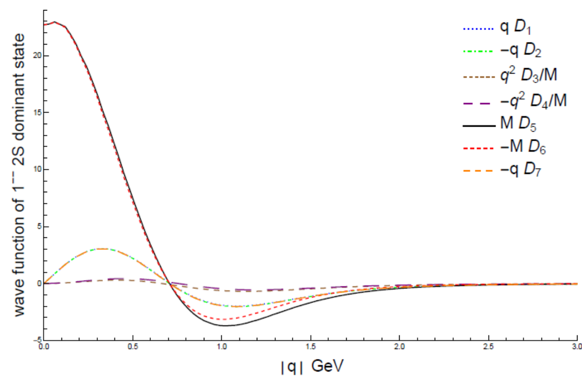
完整矢量波函数的解

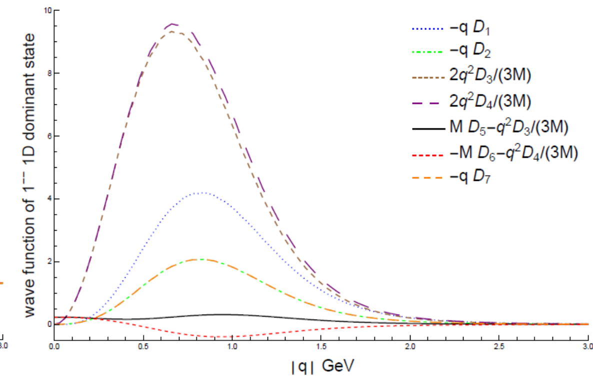
- 完整解：1S，2S，1D，3S，2D…

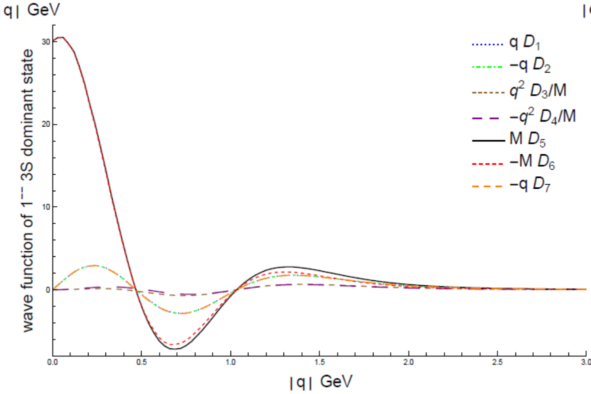
  粲偶素质量谱：3097, 3688, 3779, 4057, 4111 MeV
- 两类：（1）1S,2S,3S：S波为主，少量P波，D波可略

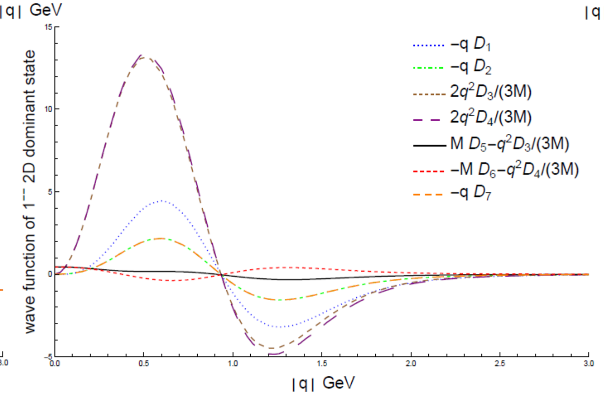
  （2）1D,2D,3D：D波为主，少量P波，S波可略

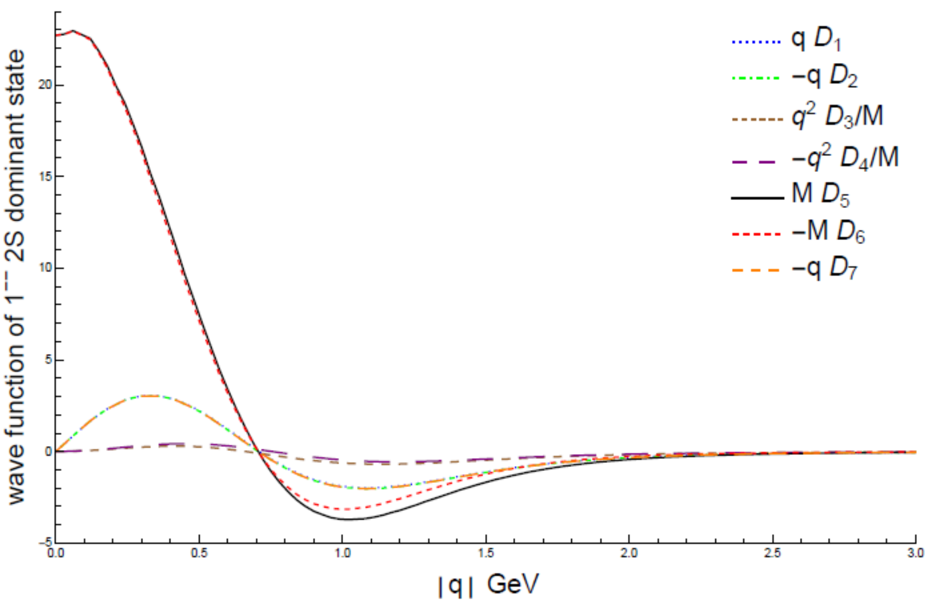




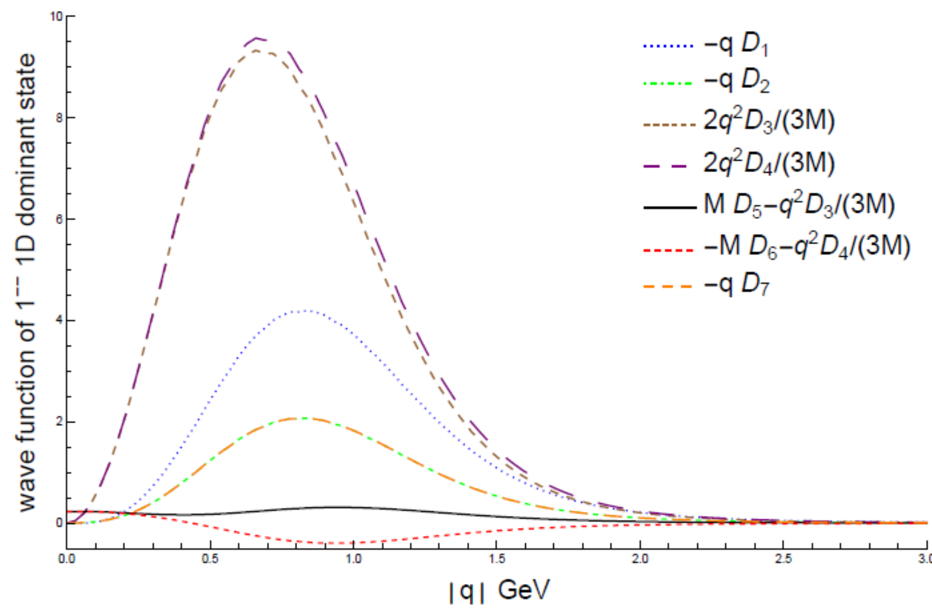








wave function of 1⁻⁻ 2S dominant state
|q| GeV
q D1
−q D2
q2 D3/M
−q2 D4/M
M D5
−M D6
−q D7



wave function of 1⁻⁻ 1D dominant state
|q| GeV
−q D1
−q D2
2q2D3/(3M)
2q2D4/(3M)
M D5−q2D3/(3M)
−M D6−q2D4/(3M)
−q D7

Wave function of a 1^+ state and its partial waves

$$\varphi_P^{1^+}(q_\perp) = \epsilon\cdot q_\perp\left(f_1 + f_2\frac{\not P}{M} + f_3\frac{\not q_\perp}{M} + f_4\frac{\not q_\perp \not P}{M^2}\right)\gamma^5$$
$$+\frac{i\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu q_\perp^\rho \epsilon^\sigma}{M}\left(g_1 + g_2\frac{\not P}{M} + g_3\frac{\not q_\perp}{M} + g_4\frac{\not q_\perp \not P}{M^2}\right),$$

f_i terms are 1^{+-} ($^1P_1$), g_i are 1^{++} ($^3P_1$), so it is $^1P_1 -^3 P_1$ mixing state.

$$\varphi_P^{1^+}(q_\perp) = \epsilon\cdot q_\perp\left(f_1 + f_2\frac{\not P}{M} - f_1 x_- \not q_\perp + f_2 x_+\frac{\not q_\perp \not P}{M}\right)\gamma^5$$
$$+\frac{i\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu q_\perp^\rho \epsilon^\sigma}{M}\left(g_1 + g_2\frac{\not P}{M} - g_1 x_- \not q_\perp + g_2 x_+\frac{\not q_\perp \not P}{M}\right).$$

Wave function of a 1^+ *state and its partial waves*

- Normalization

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{8\omega_1\omega_2\vec{q}^2}{3M(m_1\omega_2 + m_2\omega_1)}(f_1 f_2 - 2g_1 g_2) \equiv \cos^2\theta + \sin^2\theta = 1.$$

where the mixing angle is defined by wave function

- There is also P and D partial waves, f1,f2,g1,g2 terms are pure P waves, others are D waves.

Wave function of a 1^+ *state and its partial waves*

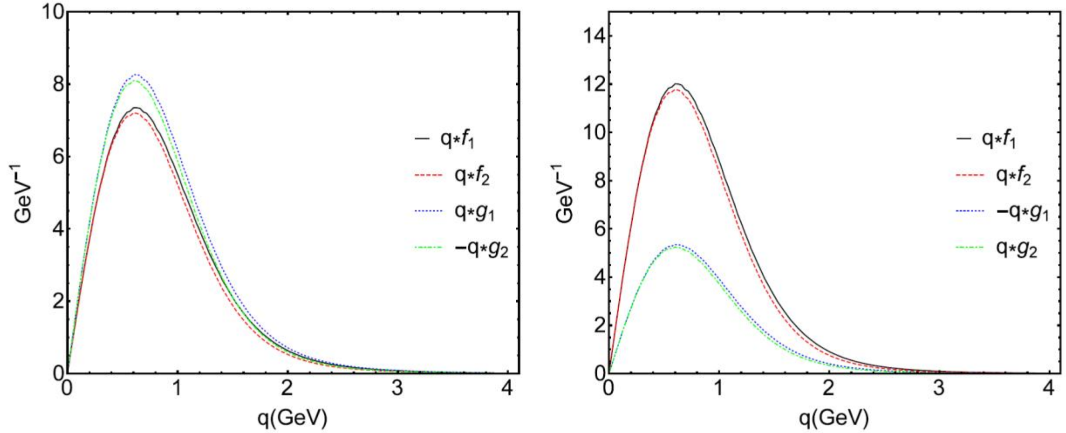
- Solutions appear in pairs, first and second are 1P, third and fourth are 2P, etc
- First two $1^1P_1 : 1^3P_1 = 0.284 : 0.716$ and $0.716 : 0.284$ with mixing angle $\theta_{1P} = -57.8^\circ$ or 32.2°
- 3, 4 $2^1P_1 : 2^3P_1 = 0.263 : 0.737$ and $0.737 : 0.263$,

$$\theta_{2P} = -59.1^\circ \text{ or } 30.9^\circ.$$

- Pure P wave: $\varphi_{nP} = \theta_{nP}$
- $P : D = 1 : 0.0971$ for two 1P Bc(1P)

  $P : D = 1 : 0.0936$ for two 2P Bc(2P)

1^+ 态Bc波函数和质量谱



$1\ P_1$	1^+	6739 (input)
$1\ P_1'$	1^+	6748
θ_{1P}		$32.2°(-57.8°)$
$2\ P_1$	1^+	7144
$2\ P_1'$	1^+	7149
θ_{2P}		$30.9°(-59.1°)$

Figure 6. The 1^+ wave functions of the $1^1P_1 - 1^3P_1$ mixing states $B_{c1}(1P)$ (left) and $B'_{c1}(1F$ (right). f_1 and f_2 terms are ${}^1P_1$ waves; g_1 and g_2 terms are ${}^3P_1$ waves.

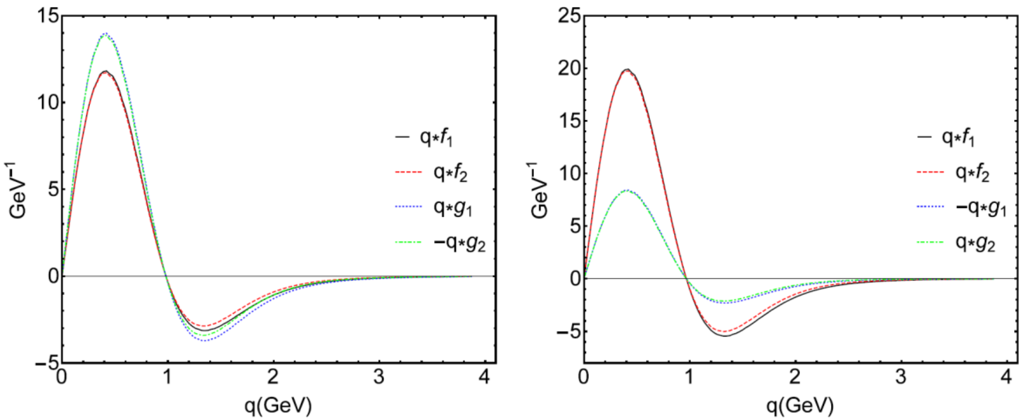


Figure 7. The 1^+ wave functions of the $2^1P_1 - 2^3P_1$ mixing states $B_{c1}(2P)$ (left) and $B'_{c1}(2P)$ (right). f_1 and f_2 terms are ${}^1P_1$ waves; g_1 and g_2 terms are ${}^3P_1$ waves.

应用: **Electromagnetic decays of $X(3823)$ as the $\psi_2(1^3D_2)$ state**

- 非相对论近似下，因反冲仍有含部分相对论修正

$$\varphi_{2^{--}}^{++}(q_\perp) = i\epsilon_{\mu\nu\alpha\beta}\frac{P^\nu}{M}q_\perp^\alpha q_{\perp\delta}\epsilon^{\beta\delta}\gamma^\mu\left(1-\frac{\not{P}}{M}\right)F_1,$$

$$\varphi_{0^{-+}}^{++}(q_{f_\perp}) = \left(1+\frac{\not{P}_{f_\perp}}{M_f}\right)\gamma^5 A_{f_1},$$

$$\varphi_{0^{++}}^{++}(q_{f_\perp}) = \left(\frac{\not{q}_{f_\perp}}{M_f}+\frac{\not{P}_f\not{q}_{f_\perp}}{M_f^2}\right)B_{f_2},$$

$$\varphi_{1^{++}}^{++}(q_{f_\perp}) = i\epsilon_{\mu\nu\alpha\beta}\frac{P_f^\nu}{M_f}q_{f_\perp}^\alpha\epsilon_f^\beta\gamma^\mu\left(1-\frac{\not{P}_f}{M_f}\right)C_{f_1},$$

$$\varphi_{2^{++}}^{++}(q_{f_\perp}) = M_f\epsilon_{f,\mu\nu}\gamma^\mu q_{f_\perp}^\nu\left(1-\frac{\not{P}_f}{M_f}\right)D_{f_5}.$$

$X(3823) \to \eta_c(^1S_0)\gamma$

$$\Gamma_1 = \frac{2\alpha E_\gamma^3}{9MM_f}\left[\int \frac{q^2 dq d\cos\theta}{(2\pi)^2}\left(\frac{2q^2 F_1}{\sqrt{5M}}\right)\cdot 1 \cdot \left(\frac{(A_{f_1} + A'_{f_1})}{\sqrt{M_f}}\right)(3\cos^2\theta - 1)\right]^2_{M_1},$$

$$A_{f_1} + A'_{f_1} = 2A_{f_1} + \frac{r^2}{M^2}\cos^2\theta \frac{\partial^2 A_{f_1}}{\partial(\frac{r}{M}\cos\theta)^2},$$

$$A_{f_1} - A'_{f_1} = 2\frac{r}{M}\cos\theta \frac{\partial A_{f_1}}{\partial(\frac{r}{M}\cos\theta)} + \frac{1}{3}\frac{r^3}{M^3}\cos^3\theta \frac{\partial^3 A_{f_1}}{\partial(\frac{r}{M}\cos\theta)^3}$$

$$\Gamma_1 = \frac{32\alpha r^7}{10125\pi^4 M^6 M_f^2}\left(\int dq q^4 F_1 \frac{\partial^2 A_{f_1}}{\partial(\frac{r}{M}\cos\theta)^2}\right)^2_{M_1}$$

- 过程 $X(3823) \to \chi_{c1}({}^3P_0)\gamma$
- E1为0，因为其完整的形状因子：

$$t_1 = \int \frac{q^2 dq d\cos\theta}{(2\pi)^2} 4\left\{ \left[\frac{F_3 q^2}{M^2 M_f}(B_{f_1} + B'_{f_1}) + \frac{F_2}{M}(B_{f_1} + B'_{f_1}) \right] \frac{q^2}{2|\vec{P}_f|^2}(3\cos^2\theta - 1) + \frac{1}{MM_f}\left[\frac{F_1}{M}(B_{f_2} - B'_{f_2}) - \frac{F_2}{M_f}(B_{f_3} - B'_{f_3}) \right] \frac{q^3}{2|\vec{P}_f|}(\cos^3\theta - \cos\theta) - \frac{1}{MM_f}\left[\frac{F_3 \alpha_f}{M}(B_{f_2} - B'_{f_2}) \right] \frac{q^3}{2|\vec{P}_f|}(3\cos^3\theta - \cos\theta) \right\}.$$

$$\Gamma_2 = \frac{2\alpha E_\gamma^3}{9MM_f}\left[-\int \frac{q^2 dq d\cos\theta}{(2\pi)^2} \left(\frac{2q^2 F_1}{\sqrt{5}M} \right) \cdot 1 \cdot \left(\frac{q(B_{f_2} - B'_{f_2})}{\sqrt{M_f^3}} \right) (\cos^3\theta - \cos\theta) \right]_{M_2}^2$$

$$\Gamma_2 = \frac{32\alpha r^5}{10125\pi^4 M^4 M_f^4}\left(\int dq q^5 F_1 \frac{\partial B_{f_2}}{\partial(\frac{r}{M}\cos\theta)} \right)_{M_2}^2$$

$$X(3823) \to \chi_{c1}(^3P_1)\gamma$$

$$\Gamma_3 = \frac{7\alpha E_\gamma^3 E_f^2 (M+M_f)^2}{36 M M_f^3}\left[\langle 1\rangle_{E_1}^2 + \frac{16M(E_f - E_\gamma)}{7M_f E_f (M+M_f)}\langle 1\rangle_{E_1}\langle 2\rangle_{M_2} + \frac{4M_f}{E_f(M+M_f)}\cdot\langle 1\rangle_{E_1}\langle 3\rangle_{M_2}\right]$$

$$\langle 1\rangle_{E_1} = \int \frac{d^3q}{(2\pi)^3}\left(\frac{2q^2F_1}{\sqrt{5M}}\right)\cdot\frac{1}{q}\cdot\left(\frac{\sqrt{2}q(C_{f_1}+C'_{f_1})}{\sqrt{3M_f}}\right)(3\cos^2\theta - 1),$$

$$\langle 2\rangle_{M_2} = \int \frac{d^3q}{(2\pi)^3}\left(\frac{2q^2F_1}{\sqrt{5M}}\right)\cdot 1\cdot\left(\frac{\sqrt{2}q(C_{f_1}-C'_{f_1})}{\sqrt{3M_f}}\right)(3\cos^3\theta - \cos\theta),$$

$$\langle 3\rangle_{M_2} = \int \frac{d^3q}{(2\pi)^3}\left(\frac{2q^2F_1}{\sqrt{5M}}\right)\cdot 1\cdot\left(\frac{\sqrt{2}q(C_{f_1}-C'_{f_1})}{\sqrt{3M_f}}\right)\left[\left(\frac{M+E_f}{E_\gamma}+\frac{3M}{2M_f}-1\right)\cos^3\theta + \left(1-\frac{M}{M_f}\right)\cos\theta\right].$$

$$\Gamma_3 = \frac{56\alpha r^5 (M+M_f)^2}{10125\pi^4 M^4 M_f^2}\left[\frac{r^2}{3M^2}\left(\int dq q^4 F_1 \frac{\partial^2 C_{f_1}}{\partial(\frac{r}{M}\cos\theta)^2}\right)_{E_1}^2 \right.$$
$$+ 2\left(\int dq q^4 F_1 \frac{\partial^2 C_{f_1}}{\partial(\frac{r}{M}\cos\theta)^2}\right)_{E_1}$$
$$\left.\times\left(\int dq q^5 F_1 \frac{\partial C_{f_1}}{\partial(\frac{r}{M}\cos\theta)}\right)_{M_2}\right], \qquad (40)$$

$$X(3823) \to \chi_{c2}(^3P_2)\gamma$$

$$\Gamma_4 = \frac{7\alpha E_\gamma^3 E_f^2 (M+M_f)^2}{36 M M_f^3}\left[\left(1+\frac{4E_\gamma^2}{7(M+M_f)^2}\right)\langle 4\rangle_{E_1}^2 + \frac{4}{E_\gamma}\left(1+\frac{4E_\gamma^2}{7(M+M_f)^2}\right)\langle 4\rangle_{E_1}\langle 5\rangle_{M_2}\right.$$
$$\left. + \frac{4}{7E_f(M+M_f)^2}(-8E_f E_\gamma + 2M_f E_\gamma - 7M^2 + 3MM_f + 10M_f^2)\langle 4\rangle_{E_1}\langle 6\rangle_{M_2}\right],$$

$$\langle 4\rangle_{E_1} = \int \frac{d^3q}{(2\pi)^3}\left(\frac{2q^2F_1}{\sqrt{5M}}\right)\cdot\frac{1}{q}\cdot\left(\frac{\sqrt{M_f}\,q(D_{f_5}+D'_{f_5})}{\sqrt{3}}\right)(3\cos^2\theta - 1),$$

$$\langle 5\rangle_{M_2} = \int \frac{d^3q}{(2\pi)^3}\left(\frac{2q^2F_1}{\sqrt{5M}}\right)\cdot 1\cdot\left(\frac{\sqrt{M_f}\,q(D_{f_5}-D'_{f_5})}{\sqrt{3}}\right)(5\cos^3\theta - 3\cos\theta),$$

$$\langle 6\rangle_{M_2} = \int \frac{d^3q}{(2\pi)^3}\left(\frac{2q^2F_1}{\sqrt{5M}}\right)\cdot 1\cdot\left(\frac{\sqrt{M_f}\,q(D_{f_5}-D'_{f_5})}{\sqrt{3}}\right)(\cos^3\theta - \cos\theta).$$

$$\Gamma_4 = \frac{28\alpha r^5 (M+M_f)^2}{10125\pi^4 M^4}\left[\frac{r^2}{3M^2}\left(\int dq q^4 F_1 \frac{\partial^2 D_{f_5}}{\partial(\frac{r}{M}\cos\theta)^2}\right)_{E_1}^2\right.$$
$$\left. - \frac{2r}{7M}\left(\int dq q^4 F_1 \frac{\partial^2 D_{f_5}}{\partial(\frac{r}{M}\cos\theta)^2}\right)_{E_1}\times\left(\int dq q^5 F_1 \frac{\partial D_{f_5}}{\partial(\frac{r}{M}\cos\theta)}\right)_{M_2}\right]$$

我们的结果：M1+E2+M3+…， 或者E1+M2+E3+…

TABLE I. The decay widths (keV) of the radiative transition $X(3823) \to \chi_{cJ}(1P)\gamma$ ($J = 0, 1, 2$), $X(3823) \to \eta_c(1S, 2S)\gamma$, and the ratio of $\frac{\Gamma(\psi_2(1D)\to\chi_{c2}(1P)\gamma)}{\Gamma(\psi_2(1D)\to\chi_{c1}(1P)\gamma)}$.

	[20]	[25]	[26]				[27]		Ours	
	RE	*RE*	*NR*	*RV*	*RS*	*RVS*	*NR*	*GI*	*RE*	EX [31]
$\Gamma(\psi_2(1D) \to \chi_{c1}(1P)\gamma)$	250	260	297	215	215	215	307	268	265	
$\Gamma(\psi_2(1D) \to \chi_{c2}(1P)\gamma)$	60	56	62	55	51	59	64	66	57	
$\frac{\Gamma(\psi_2(1D)\to\chi_{c2}(1P)\gamma)}{\Gamma(\psi_2(1D)\to\chi_{c1}(1P)\gamma)}$ %	24	22	21	26	24	27	21	25	22	$28^{+14}_{-11} \pm 2$
$\Gamma(\psi_2(1D) \to \chi_{c0}(1P)\gamma)$									1.2	
$\Gamma(\psi_2(1D) \to \eta_c(1S)\gamma)$									1.3	
$\Gamma(\psi_2(1D) \to \eta_c(2S)\gamma)$									0.069(0.067)	

	[27]		[47]			[29]		Ours
	NR	*GI*	NR_1	NR_2	NR_3	NR_1	NR_2	*RE*
$\Gamma(\psi_2(2D) \to \chi_{c0}(1P)\gamma)$								0.16
$\Gamma(\psi_2(2D) \to \chi_{c1}(1P)\gamma)$	26	23	17	26	10	68	68	33
$\Gamma(\psi_2(2D) \to \chi_{c2}(1P)\gamma)$	7.2	0.62	6.7	10	3.8	20	20	7.3
$\Gamma(\psi_2(2D) \to \chi_{c2}(1F)\gamma)$								6.2
$\Gamma(\psi_2(2D) \to \chi_{c0}(2P)\gamma)$								1.13
$\Gamma(\psi_2(2D) \to \chi_{c1}(2P)\gamma)$	298	225	140	178	92	223	188	237 (230)
$\Gamma(\psi_2(2D) \to \chi_{c2}(2P)\gamma)$	52	65	39	64	19	115	64	58
$\frac{\Gamma(\psi_2(2D)\to\chi_{c1}(1P)\gamma)}{\Gamma(\psi_2(2D)\to\chi_{c1}(2P)\gamma)}$ (%)	8.7	10	12	15	11	30	36	14
$\frac{\Gamma(\psi_2(2D)\to\chi_{c2}(2P)\gamma)}{\Gamma(\psi_2(2D)\to\chi_{c1}(2P)\gamma)}$ (%)	17	29	28	36	21	52	34	25
$\Gamma(\psi_2(2D) \to \eta_c(1S)\gamma)$								2.1
$\Gamma(\psi_2(2D) \to \eta_c(2S)\gamma)$								0.33 (0.32)
$\Gamma(\psi_2(2D) \to \eta_c(3S)\gamma)$								0.092

$M_{3D_2(2D)} = 4.154$ GeV

TABLE VII. The EM decay width (keV) of different partial waves for $\psi_2(1D) \to \chi_{c2}(1P)\gamma$.

$2^{--} \backslash 2^{++}$	$Complete$	$P\,wave(D_{f_5}, D_{f_6})$	$D\,wave(D_{f_1}, D_{f_2}, D_{f_7})$	$F\,wave(D_{f_3}, D_{f_4})$
$Complete$	57	18	1.5	0.23
$D\,wave(F_1, F_2)$	75	44	4.9	0.70
$F\,wave(F_3)$	1.7	6.1	1.4	0.0057

TABLE VIII. The EM decay width (keV) of different partial waves for $\psi_2(2D) \to \chi_{c2}(1P)\gamma$.

$2^{--} \backslash 2^{++}$	$Complete$	$P\,wave(D_{f_5}, D_{f_6})$	$D\,wave(D_{f_1}, D_{f_2}, D_{f_7})$	$F\,wave(D_{f_3}, D_{f_4})$
$Complete$	7.3	3.4	0.39	0.046
$D\,wave(F_1, F_2)$	9.2	4.9	0.56	0.066
$F\,wave(F_3)$	0.38	0.24	0.037	0.00028

TABLE IX. The EM decay width (keV) of different partial waves for $\psi_2(2D) \to \chi_{c2}(1F)\gamma$.

$2^{--} \backslash 2^{++}$	$Complete$	$P\,wave$	$D\,wave(D_{f_1}, D_{f_2}, D_{f_7})$	$F\,wave$
$Complete$	6.2	0.65	3.6	5.6
$D\,wave(F_1, F_2)$	4.8	0.4	2.9	4.3
$F\,wave(F_3)$	0.55	0.055	0.12	0.46

Summary

- J^P是更基本的量子数
- 依据J^P 给出介子普遍的波函数：相对论的；三大类；不是纯波，都含有其他分波
- The mixing of different waves naturally appears in relativistic method, not manmade, only one wave function is needed
- In a relativistic method, the mixing angle can be calculated by wave function, not potential
- 辐射电磁衰变中：M1+E2+M3+…，或者E1+M2+E3+…

$$\Gamma[X(3823)] \approx \Gamma(\eta_c\gamma) + \sum \Gamma(\chi_{cJ}\gamma) + \Gamma(J/\psi\pi\pi) + \Gamma(ggg) + \Gamma(gg\gamma) \approx 379 \text{ keV}.$$

$$\varphi_{2^{--}}^{++}(q_\perp) = i\epsilon_{\mu\nu\alpha\beta}\frac{P^\nu}{M}q_\perp^\alpha q_{\perp\delta}\epsilon^{\beta\delta}\gamma^\mu\left[F_1 + \frac{\not{P}}{M}F_2 + \frac{\not{P}\not{q}_\perp}{M^2}F_3\right]$$

$$\varphi_{1^{++}}^{++}(q_{f_\perp}) = i\epsilon_{\mu\nu\alpha\beta}\frac{P_f^\nu}{M_f}q_{f_\perp}^\alpha \epsilon_f^\beta\gamma^\mu\left[C_{f_1} + \frac{\not{P}_f}{M_f}C_{f_2} + \frac{\not{P}_f\not{q}_{f_\perp}}{M_f^2}C_{f_3}\right]$$

$$\varphi_{2^{++}}^{++}(q_{f_\perp}) = \epsilon_{f,\mu\nu}q_{f_\perp}^\mu q_{f_\perp}^\nu \times\left[D_{f_1} + \frac{\not{P}_f}{M_f}D_{f_2} + \frac{\not{q}_{f_\perp}}{M_f}D_{f_3} + \frac{\not{P}_f\not{q}_{f_\perp}}{M_f^2}D_{f_4}\right] + M_f\epsilon_{f,\mu\nu}\gamma^\mu q_{f_\perp}^\nu\left[D_{f_5} + \frac{\not{P}_f}{M_f}D_{f_6} + \frac{\not{P}_f\not{q}_{f_\perp}}{M_f^2}D_{f_7}\right]$$

$$1^{--}\quad (M-2\omega_1)\left\{\left(\psi_3(\vec{q})\frac{\vec{q}^2}{M^2}-\psi_5(\vec{q})\right)+\left(\psi_4(\vec{q})\frac{\vec{q}^2}{M^2}+\psi_6(\vec{q})\right)\frac{m_1}{\omega_1}\right\}=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2}{\omega_1^2}\left\{(V_s+V_v)\left(\psi_3(\vec{k})\frac{\vec{k}^2}{M^2}-\psi_5(\vec{k})\right)(\vec{k}\cdot\vec{q})\right.$$

$$\left.-(V_s-V_v)\left[m_1^2\left(\psi_3(\vec{k})\frac{(\vec{k}\cdot\vec{q})^2}{M^2\vec{q}^2}-\psi_5(\vec{k})\right)+m_1\omega_1\left(\psi_4(\vec{k})\frac{(\vec{k}\cdot\vec{q})^2}{M^2\vec{q}^2}+\psi_6(\vec{k})\right)\right]\right\},$$

$$(M+2\omega_1)\left\{\left(\psi_3(\vec{q})\frac{\vec{q}^2}{M^2}-\psi_5(\vec{q})\right)-\left(\psi_4(\vec{q})\frac{\vec{q}^2}{M^2}+\psi_6(\vec{q})\right)\frac{m_1}{\omega_1}\right\}=-\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2}{\omega_1^2}\left\{(V_s+V_v)\left[\left(\psi_3(\vec{k})\frac{\vec{k}^2}{M^2}-\psi_5(\vec{k})\right)\right](\vec{k}\cdot\vec{q})\right.$$

$$\left.-(V_s-V_v)\left[m_1^2\left(\psi_3(\vec{k})\frac{(\vec{k}\cdot\vec{q})^2}{M^2\vec{q}^2}-\psi_5(\vec{k})\right)-m_1\omega_1\left(\psi_4(\vec{k})\frac{(\vec{k}\cdot\vec{q})^2}{M^2\vec{q}^2}+\psi_6(\vec{k})\right)\right]\right\},$$

$$(M-2\omega_1)\left\{\left(\psi_3(\vec{q})+\psi_4(\vec{q})\frac{m_1}{\omega_1}\right)\frac{\vec{q}^2}{M^2}-3\left(\psi_5(\vec{q})-\psi_6(\vec{q})\frac{\omega_1}{m_1}\right)-\psi_6(\vec{q})\frac{\vec{q}^2}{m_1\omega_1}\right\}=$$

$$-\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}\left\{(V_s+V_v)\left[-\frac{2\omega_1}{m_1}\psi_6(\vec{k})-\psi_3(\vec{k})\frac{\vec{k}^2}{M^2}+\psi_5(\vec{k})\right](\vec{k}\cdot\vec{q})\right.$$

$$\left.+(V_s-V_v)\left[\omega_1^2\left(\psi_3(\vec{k})\frac{\vec{k}^2}{M^2}-3\psi_5(\vec{k})\right)+m_1\omega_1\left(\psi_4(\vec{k})\frac{\vec{k}^2}{M^2}+3\psi_6(\vec{k})\right)-\left(\psi_3(\vec{k})\frac{(\vec{k}\cdot\vec{q})^2}{M^2}-\psi_5(\vec{k})\vec{q}^2\right)\right]\right\},$$

$$(M+2\omega_1)\left\{\left[\psi_3(\vec{q})-\psi_4(\vec{q})\frac{m_1}{\omega_1}\right]\frac{\vec{q}^2}{M^2}-3\left(\psi_5(\vec{q})+\psi_6(\vec{q})\frac{\omega_1}{m_1}\right)+\psi_6(\vec{q})\frac{\vec{q}^2}{m_1\omega_1}\right\}=$$

$$\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}\left\{(V_s+V_v)\left[\frac{2\omega_1}{m_1}\psi_6(\vec{k})-\psi_3(\vec{k})\frac{\vec{k}^2}{M^2}+\psi_5(\vec{k})\right](\vec{k}\cdot\vec{q})\right.$$

$$\left.+(V_s-V_v)\left[\omega_1^2\left(\psi_3(\vec{k})\frac{\vec{k}^2}{M^2}-3\psi_5(\vec{k})\right)-m_1\omega_1\left(\psi_4(\vec{k})\frac{\vec{k}^2}{M^2}+3\psi_6(\vec{k})\right)-\left(\psi_3(\vec{k})\frac{(\vec{k}\cdot\vec{q})^2}{M^2}-\psi_5(\vec{k})\vec{q}^2\right)\right]\right\},$$

2+

$$MF_1(q_\perp) = (\omega_1+\omega_2)F_1(q_\perp) + \int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{24\omega_1\omega_2}\Big\{4(V_S-V_V)(e_1m_2+e_2m_1)\Big[-(F_3(k_\perp)-F_4(k_\perp)) - \frac{m_1-m_2}{e_1+e_2}(F_1(k_\perp)+F_2(k_\perp)) - \left(\frac{e_1-e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)) + (F_3(k_\perp)+F_4(k_\perp))\right)\frac{\omega_1+\omega_2}{e_1+e_2}\Big]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} - 9(V_S+V_V)\Big[(F_1(k_\perp)+F_2(k_\perp))(q_\perp^2+m_1m_2-\omega_1\omega_2) + (F_1(k_\perp)-F_2(k_\perp))(\omega_1m_2-\omega_2m_1)\frac{e_1-e_2}{m_1+m_2}\Big] \times\left(\frac{(\vec{k}\cdot\vec{q})^2}{k_\perp^4}-\frac{q_\perp^2}{3k_\perp^2}\right) + 3(V_S-V_V)(e_1m_2+e_2m_1)\Big[(F_1(k_\perp)+F_2(k_\perp))\frac{5m_1+m_2}{e_1+e_2} + 2(F_3(k_\perp)-F_4(k_\perp)) + \left(\frac{5e_1+e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)) + 2(F_3(k_\perp)+F_4(k_\perp))\right)\frac{\omega_1+\omega_2}{e_1+e_2}\Big]\left(\frac{(\vec{k}\cdot\vec{q})^3}{k_\perp^6}-\frac{q_\perp^2\vec{k}\cdot\vec{q}}{3k_\perp^4}\right)\Big\}; \tag{A1}$$

$$MF_2(q_\perp) = -(\omega_1+\omega_2)F_2(q_\perp) - \int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{24\omega_1\omega_2}\Big\{4(V_S-V_V)(e_1m_2+e_2m_1)\Big[-(F_3(k_\perp)-F_4(k_\perp)) - \frac{m_1-m_2}{e_1+e_2}(F_1(k_\perp)+F_2(k_\perp)) + \left(\frac{e_1-e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)+(F_3(k_\perp)+F_4(k_\perp)))\right)\frac{\omega_1+\omega_2}{e_1+e_2}\Big]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} - 9(V_S+V_V)\Big[(F_1(k_\perp)+F_2(k_\perp))(q_\perp^2+m_1m_2-\omega_1\omega_2) - (F_1(k_\perp)-F_2(k_\perp))(\omega_1m_2-\omega_2m_1)\frac{e_1-e_2}{m_1+m_2}\Big] \times\left(\frac{(\vec{k}\cdot\vec{q})^2}{k_\perp^4}-\frac{q_\perp^2}{3k_\perp^2}\right) + 3(V_S-V_V)(e_1m_2+3e_2m_1)\Big[(F_1(k_\perp)+F_2(k_\perp))\frac{5m_1+m_2}{e_1+e_2} + 2(F_3(k_\perp)-F_4(k_\perp)) - \left(\frac{5e_1+e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)) + 2(F_3(k_\perp)+F_4(k_\perp))\right)\frac{\omega_1+\omega_2}{e_1+e_2}\Big]\left(\frac{(\vec{k}\cdot\vec{q})^3}{k_\perp^6}-\frac{q_\perp^2\vec{k}\cdot\vec{q}}{3k_\perp^4}\right)\Big\}; \tag{A2}$$

$$
\begin{aligned}
MF_3(q_\perp) = {} & (\omega_1+\omega_2)F_3(q_\perp) + \int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{24\omega_1\omega_2}\Bigg\{-10(V_S+V_V)\Bigg[\Bigg(\frac{m_1-m_2}{m_1+m_2}\frac{e_1-e_2}{e_1+e_2} \\
& \times (F_1(k_\perp)+F_2(k_\perp)) + \frac{e_1-e_2}{m_1+m_2}(F_3(k_\perp)-F_4(k_\perp))\Bigg)(m_2\omega_1-m_1\omega_2) \\
& + \Bigg(\frac{e_1-e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)) + (F_3(k_\perp)+F_4(k_\perp))\Bigg)(q_\perp^2+m_1m_2-\omega_1\omega_2)\Bigg]\frac{(\vec{k}\cdot\vec{q})^2}{k_\perp^4} \\
& - 8(V_S-V_V)\frac{e_2m_1+e_1m_2}{(e_1+e_2)(m_1+m_2)}[m_2((e_1-e_2)(F_1(k_\perp)-F_2(k_\perp)) + (m_1+m_2)(F_3(k_\perp)+F_4(k_\perp))) \\
& + \omega_2((m_1-m_2)(F_1(k_\perp)+F_2(k_\perp)) + (e_1+e_2)(F_3(k_\perp)-F_4(k_\perp)))]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} \\
& + 10(V_S-V_V)(e_2m_1+e_1m_2)\Bigg[\frac{e_1-e_2}{e_1+e_2}(F_1(k_\perp)-F_2(k_\perp)) + \frac{m_1+m_2}{e_1+e_2}(F_3(k_\perp)+F_4(k_\perp)) \\
& + \frac{\omega_1+\omega_2}{m_1+m_2}\Bigg(\frac{m_1-m_2}{e_1+e_2}(F_1(k_\perp)+F_2(k_\perp)) + (F_3(k_\perp)-F_4(k_\perp))\Bigg)\Bigg]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} \\
& - 2(V_S-V_V)(e_2m_1+e_1m_2)\Bigg[\frac{5e_1+e_2}{e_1+e_2}(F_1(k_\perp)-F_2(k_\perp)) + 2\frac{m_1+m_2}{e_1+e_2}(F_3(k_\perp)+F_4(k_\perp)) \\
& + \frac{\omega_1+\omega_2}{m_1+m_2}\Bigg(\frac{5m_1+m_2}{e_1+e_2}(F_1(k_\perp)+F_2(k_\perp)) + 2(F_3(k_\perp)-F_4(k_\perp))\Bigg)\Bigg]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} \\
& + 3(V_S+V_V)\Bigg[(F_1(k_\perp)+F_2(k_\perp))(m_1\omega_2-m_2\omega_1) - \frac{e_1-e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)) \\
& \times (-q_\perp^2+m_1m_2-\omega_1\omega_2)\Bigg]\Bigg(\frac{q_\perp^2}{k_\perp^2} - \frac{3(\vec{k}\cdot\vec{q})^2}{k_\perp^4}\Bigg) + (V_S+V_V)\frac{1}{m_1+m_2}\Bigg[\frac{e_1-e_2}{e_1+e_2}((m_1-m_2) \\
& \times (F_1(k_\perp)+F_2(k_\perp)) + (e_1+e_2)(F_3(k_\perp)-F_4(k_\perp)))(m_2\omega_1-m_1\omega_2) + ((e_1-e_2) \\
& \times (F_1(k_\perp)-F_2(k_\perp)) + (m_1+m_2)(F_3(k_\perp)+F_4(k_\perp)))(q_\perp^2+m_1m_2-\omega_1\omega_2)\Bigg] \\
& \times \Bigg(\frac{3q_\perp^2}{k_\perp^2} + \frac{(\vec{k}\cdot\vec{q})^2}{k_\perp^4}\Bigg) + 2(V_S-V_V)\frac{e_2m_1+e_1m_2}{(e_1+e_2)(m_1+m_2)}[m_2((5e_1+e_2)(F_1(k_\perp)-F_2(k_\perp)) \\
& + 2(m_1+m_2)(F_3(k_\perp)+F_4(k_\perp))) + \omega_2((5m_1+m_2)(F_1(k_\perp)+F_2(k_\perp)) \\
& + 2(e_1+e_2)(F_3(k_\perp)-F_4(k_\perp)))]\Bigg(\frac{3(\vec{k}\cdot\vec{q})^3}{k_\perp^6} - \frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4}\Bigg)\Bigg\}; \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
MF_4(q_\perp) = &-(\omega_1+\omega_2)F_4(q_\perp) - \int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{24\omega_1\omega_2}\Bigg\{-10(V_S+V_V)\Bigg[\Bigg(-\frac{m_1-m_2}{m_1+m_2}\frac{(e_1-e_2)}{(e_1+e_2)} \\
&\times(F_1(k_\perp)+F_2(k_\perp)) + \frac{e_1-e_2}{m_1+m_2}(F_3(k_\perp)-F_4(k_\perp))\Bigg)(m_2\omega_1-m_1\omega_2) \\
&+\Bigg(\frac{e_1-e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)) + (F_3(k_\perp)+F_4(k_\perp))\Bigg)(q_\perp^2+m_1m_2-\omega_1\omega_2)\Bigg]\frac{(\vec{k}\cdot\vec{q})^2}{k_\perp^4} \\
&-8(V_S-V_V)\frac{(e_2m_1+e_1m_2)}{(e_1+e_2)(m_1+m_2)}[m_2((e_1-e_2)(F_1(k_\perp)-F_2(k_\perp)) + (m_1+m_2)(F_3(k_\perp)+F_4(k_\perp))) \\
&-\omega_2((m_1-m_2)(F_1(k_\perp)+F_2(k_\perp)) + (e_1+e_2)(F_3(k_\perp)-F_4(k_\perp)))]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} \\
&+10(V_S-V_V)(e_2m_1+e_1m_2)\Bigg[\frac{e_1-e_2}{e_1+e_2}(F_1(k_\perp)-F_2(k_\perp)) + \frac{m_1+m_2}{e_1+e_2}(F_3(k_\perp)+F_4(k_\perp)) \\
&-\frac{\omega_1+\omega_2}{m_1+m_2}\Bigg(\frac{m_1-m_2}{e_1+e_2}(F_1(k_\perp)+F_2(k_\perp)) + (F_3(k_\perp)-F_4(k_\perp))\Bigg)\Bigg]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} \\
&-2(V_S-V_V)(e_2m_1+e_1m_2)\Bigg[\frac{5e_1+e_2}{e_1+e_2}(F_1(k_\perp)-F_2(k_\perp)) + 2\frac{m_1+m_2}{e_1+e_2}(F_3(k_\perp)+F_4(k_\perp)) \\
&-\frac{\omega_1+\omega_2}{m_1+m_2}\Bigg(\frac{5m_1+m_2}{e_1+e_2}(F_1(k_\perp)+F_2(k_\perp)) + 2(F_3(k_\perp)-F_4(k_\perp))\Bigg)\Bigg]\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4} \\
&+3(V_S+V_V)\Bigg[-(F_1(k_\perp)+F_2(k_\perp))(m_1\omega_2-m_2\omega_1) - \frac{e_1-e_2}{m_1+m_2}(F_1(k_\perp)-F_2(k_\perp)) \\
&\times(-q_\perp^2+m_1m_2-\omega_1\omega_2)\Bigg]\Bigg(\frac{q_\perp^2}{k_\perp^2}-\frac{3(\vec{k}\cdot\vec{q})^2}{k_\perp^4}\Bigg) + (V_S+V_V)\frac{1}{m_1+m_2}\Bigg[-\frac{e_1-e_2}{e_1+e_2}((m_1-m_2) \\
&\times(F_1(k_\perp)+F_2(k_\perp)) + (e_1+e_2)(F_3(k_\perp)-F_4(k_\perp)))(m_2\omega_1-m_1\omega_2) + ((e_1-e_2) \\
&\times(F_1(k_\perp)-F_2(k_\perp)) + (m_1+m_2)(F_3(k_\perp)+F_4(k_\perp)))(q_\perp^2+m_1m_2-\omega_1\omega_2)\Bigg] \\
&\times\Bigg(\frac{3q_\perp^2}{k_\perp^2}+\frac{(\vec{k}\cdot\vec{q})^2}{k_\perp^4}\Bigg) + 2(V_S-V_V)\frac{e_2m_1+e_1m_2}{(e_1+e_2)(m_1+m_2)}[m_2((5e_1+e_2)(F_1(k_\perp)-F_2(k_\perp)) \\
&+2(m_1+m_2)(F_3(k_\perp)+F_4(k_\perp))) - \omega_2((5m_1+m_2)(F_1(k_\perp)+F_2(k_\perp)) \\
&+2(e_1+e_2)(F_3(k_\perp)-F_4(k_\perp)))]\Bigg(\frac{3(\vec{k}\cdot\vec{q})^3}{k_\perp^6}-\frac{q_\perp^2\vec{k}\cdot\vec{q}}{k_\perp^4}\Bigg)\Bigg\}. \qquad \text{(A4)}
\end{aligned}
$$