Some phenomenological applications of Khuri-Treiman equations

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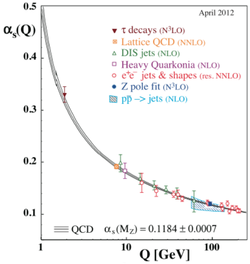
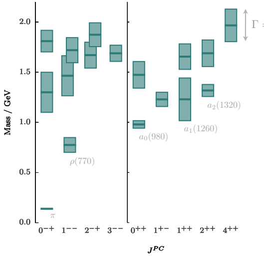
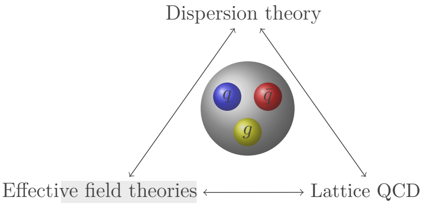
Introduction

- **Asymptotic freedom**:
  "like QED", but only at high energies
- **Confinement**:
  at low energies the gluons bind the quarks together to form the hadrons





- Quark models give gross structure of hadron spectrum
- Need non-pertubative **approaches** to describe these hadrons *rigorously*:



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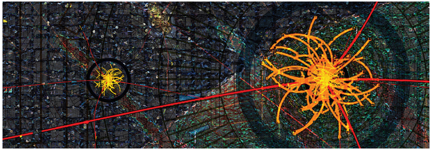
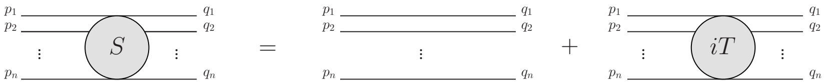
Experimental data



Dispersive framework

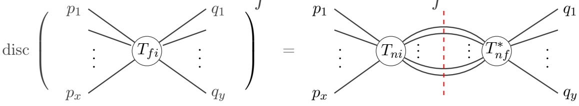
- Model independent and non-perturbative resummation of final-state interactions (FSI), based on:
  - **Unitarity** ($\sim$ probability conservation) gives rise to the *optical theorem*:



no interactions

interactions

$$\sum_f \text{Prob}_{i\to f} = \sum_f |\langle f|S|i\rangle|^2 = 1 \Rightarrow SS^\dagger = S^\dagger S = \mathbb{1} \quad (S = \mathbb{1} + iT)\,,$$

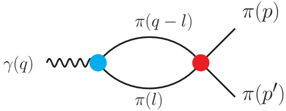


$$\text{disc}T_{fi} = T_{fi} - T^*_{if} = i\sum_n \int d\Pi_n \delta^4 (P_i - K_n)\, T^*_{nf} T_{ni}\,,$$

  - **Analyticity**: Dispersion relations reconstruct the whole amplitude with knowledge about discontinuity

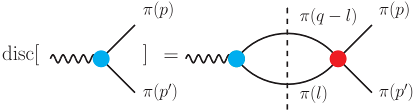
Example: the pion vector form factor

- **Full** loop calculation



$$\mathcal{M} = \frac{i}{2}\int \frac{d^4 l}{(2\pi)^4} \frac{T^{I*}_{\pi\pi}(s,z_l)(q-2l)_\mu F_\pi(s)}{[l^2-m_\pi^2+i\varepsilon][(q-l)^2-m_\pi^2+i\varepsilon]}\,,$$

- **Cutkosky** rule: propagators yield Im parts on-shell $\frac{1}{p^2-M^2+i\varepsilon} \longrightarrow -2\pi i\delta(p^2-M^2)$
- Calculation of the **discontinuity** (unitarity)



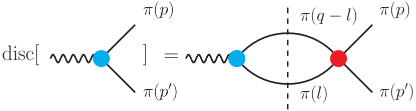
$$(p-p')_\mu \mathrm{disc} F_\pi(s) = \frac{i}{2}\int \frac{d^4 l}{(2\pi)^4}(2\pi)\delta(l^2-m_\pi^2)(2\pi)\delta((q-l)^2-m_\pi^2)T^{I*}_{\pi\pi}(s,z_l)(q-2l)_\mu F_\pi(s)\,,$$

- Use the p-w expansion of the $\pi\pi$ amplitude and the orthogonality condition

$$T^I_{\pi\pi}(s,z_l) = 32\pi\sum_{\ell=0}^{\infty}(2\ell+1)t^I_\ell(s)P_\ell(z_l)\,,\quad (2\ell+1)\int_{-1}^{1} dz_l P_\ell(z_l)P_{\ell'}(z_l) = 2\delta_{\ell\ell'}\,,$$

Example: the pion vector form factor

- **Unitarity**:



(only $\ell = 1$ is projected out)

$$\mathrm{disc}F_\pi(s) \quad = \quad 2i\mathrm{Im}F_\pi(s) = 2i\sigma_\pi(s)F_\pi(s)t_1^{1*}(s) = 2iF_\pi(s)\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\,,$$

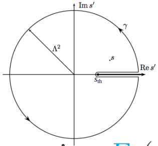
- **Watson's** theorem [Watson, Phys.Rev. 95, 228 (1954)]:

$$\mathrm{Im}F_\pi(s) = |F_\pi(s)|e^{i\delta_{F_\pi}(s)}\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\theta(s-4m_\pi^2)\,,$$

$$\Rightarrow \quad \delta_{F_\pi}(s) = \delta_1^1(s)\,,$$

Omnès equation

- Invoke Cauchy integral



$$F_\pi(s) = \frac{1}{2\pi i}\oint_\gamma \frac{F_\pi(s')}{s'-s}ds' ,$$

- Assuming $F_\pi(s) \to 0$ when $\Lambda^2 \to \infty$, only the integral over the cut remains

$$F_\pi(s) = \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{disc}F_\pi(s')}{s'-s}ds'$$

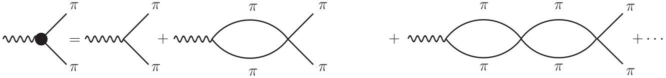
- Analytic solution, **Omnès** equation [Omnès, Nuovo Cimento 8, 316 (1958)]

$$F_\pi(s) = P(s)\Omega_1^1(s) , \quad \Omega_1^1(s) = \frac{f(s)}{f(0)} = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\} ,$$

- Polynomial ambiguity $P(s)$, not fixed by unitarity
  - — Matched to the results of the effective theory, or by experimental data

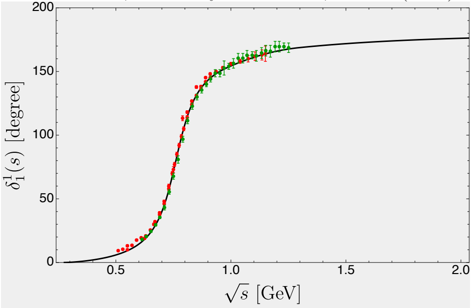
Omnès equation

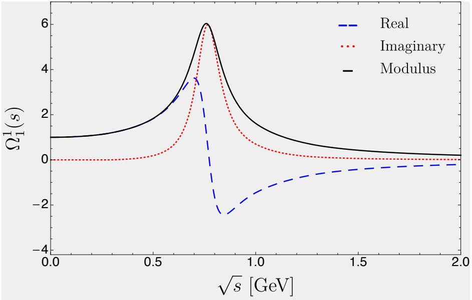
- Diagrammatic interpretation



- Solution: depends solely on the P-wave phase shift of $\pi\pi$

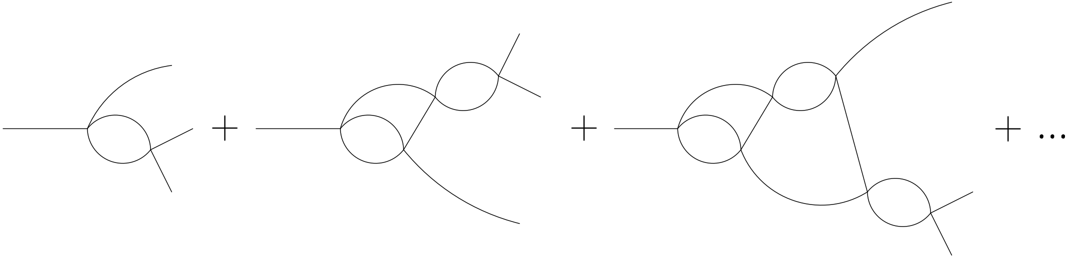
Garcia-Martin, *et.al.* Phys. Rev. D83, 074004 (2011)





Three-particle decay

- In many decay processes one wants to take into account unitarity/FSI in the three possible channels
- Khuri-Treiman equations:
  - Include full rescattering effects



Khuri-Treiman formalism: citation summary

- Developed for $K \to 3\pi$

Pion-Pion Scattering and K + /- --> 3pi Decay

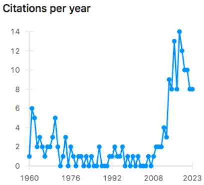
N.N. Khuri (Princeton, Inst. Advanced Study), S.B. Treiman (Princeton U.)

Aug 1, 1960

6 pages

Published in: *Phys.Rev.* 119 (1960) 1115-1121

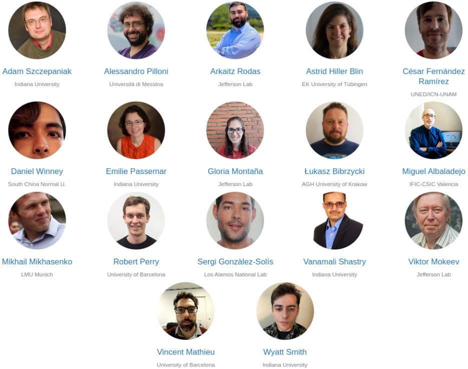
DOI: 10.1103/PhysRev.119.1115



- Most prominent applications:
  - $\eta \to 3\pi$ [Kambor *et.al.* 1995, Anisovich *et.al.* 1996, Descotes-Genon *et.al.* 2014, JPAC 2015&17, Colangelo 2016, Albaladejo *et.al.* 2017]
  - $\eta' \to \eta\pi\pi$ [Isken *et.al.* 2017]
  - $\omega/\phi \to 3\pi, \pi\gamma^*$ [Kubis *et.al.* 2012, JPAC 2014&20, Dax *et.al.* 2018]
  - $J/\psi \to 3\pi, \pi\gamma^*$ [JPAC 2023] $\to$ this talk
  - $e^+e^- \to 3\pi/\pi\gamma$ (a_μ) [Hoferichter *et.al.* 2018&19, Hoid 2020]
  - $D \to K\pi\pi$ [Niecknig *et.al.* 2015]
  - $J^{PC} \to 3\pi$ [JPAC 2019, Stamen *et.al.* 2022]

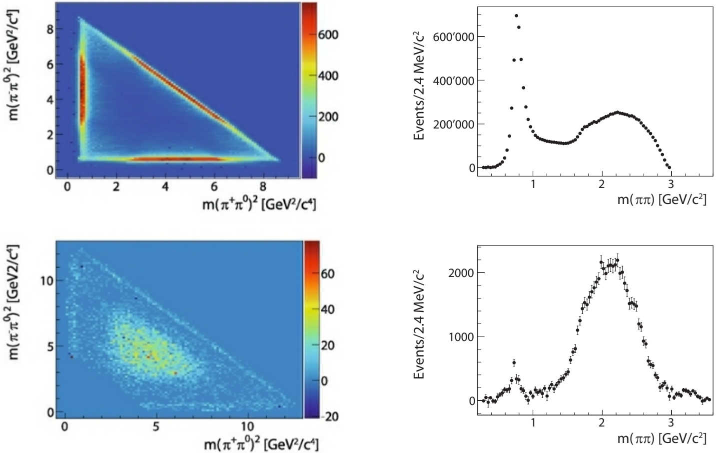
JPAC: Joint Physics Analysis Center

- Work in **theoretical/experimental/phenomenological** analysis
- Light/heavy meson **spectroscopy**
- Interaction with many **experimental collaborations** (BESIII, CLAS, GlueX, KLOE, LHCb, MAMI,...) and LQCD groups
- Website: https://www.jpac-physics.org



$J/\psi/\psi(2S) \to 3\pi$ decay

- BESIII data [Phys.Lett. B710, 594 (2012)] show $\rho\pi$ puzzle
  - $J/\psi \to \pi^0\pi^+\pi^-$: 1.9 milion events, $\rho\pi$ dominance
  - $\psi(2S) \to \pi^0\pi^+\pi^-$: 7872 events, $\rho\pi$ subleading



Definitions

- Decay amplitude for $V \to \pi^+\pi^-\pi^0$ $(V = \omega, \phi, J/\psi)$

$$\mathcal{M}(s,t,u) \quad = i\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu}p_+^{\nu}p_-^{\alpha}p_0^{\beta}\mathcal{F}(s,t,u)\,,$$

- Mandelstam variable, s-channel scattering angle and momenta

$$s = (p_+ + p_-)^2\,, \quad t = (p_0 + p_+)^2\,, \quad u = (p_0 + p_-)^2\,,$$

$$\cos\theta_s(s,t,u) = \frac{t-u}{4\,p(s)\,q(s)}\,, \quad \sin\theta_s(s,t,u) = \frac{\sqrt{\phi(s,t,u)}}{2\sqrt{s}\,p(s)\,q(s)}\,,$$

$$p(s) = \frac{\lambda^{\frac{1}{2}}(s, m_{\pi^+}^2, m_{\pi^-}^2)}{2\sqrt{s}}\,, \quad q(s) = \frac{\lambda^{\frac{1}{2}}(s, m_{J/\psi}^2, m_{\pi^0}^2)}{2\sqrt{s}}\,,$$

$$|\mathcal{M}(s,t,u)|^2 = \frac{1}{4}\left(2\sqrt{s}\sin\theta_s p(s)q(s)\right)^2|\mathcal{F}(s,t,u)|^2 = \mathcal{P}(s,t,u)|\mathcal{F}(s,t,u)|^2\,,$$

- If no dynamics: $|\mathcal{F}(s,t,u)|^2 = 1 \Rightarrow |\mathcal{M}(s,t,u)|^2$ follows P-wave distribution
  - In 1961, ω spin and parity from $\omega \to 3\pi$ was consistent with a P-wave
  - Current $\omega \to 3\pi$ precision Dalitz analyses show deviation from the P-wave

Khuri-Treiman representation of the amplitude

- s-channel partial-wave (pw) expansion of the amplitude

$$F(s,t,u) = \sum_{J=0}^{\infty} (p(s)\, q(s))^{J-1}\, P'_J(z_s)\, f_J(s)\,,$$

- Khuri-Treiman/isobar decomposition of the amplitude

$$F(s,t,u) = \sum_{J=0}^{J_{\max}} (p(s)\, q(s))^{J-1}\, P'_J(z_s)\, F_J(s) + (s \leftrightarrow t) + (s \leftrightarrow u)\,,$$

- Consider only $J = 1$ isobars

$$F(s,t,u) = F_1(s) + F_1(t) + F_1(u)\,,$$

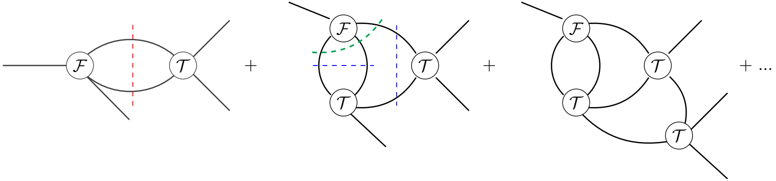
- pw projection of the KT decomposition

$$f_1(s) = F_1(s) + \hat{F}_1(s)\,, \quad \hat{F}_1(s) = 3\int_{-1}^{1} \frac{dz_s}{2}(1 - z_s^2) F_1(t(s,z_s))\,,$$

  — $F_1(s)$: right-hand cut (RHC)
  — $\hat{F}_1(s)$: left-hand cut (given by the RHC of the crossed channels $F_1(t), F_1(u)$)

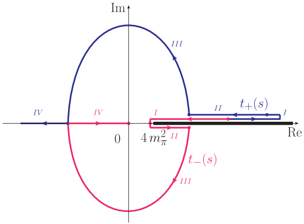
Unitarity and analyticity

- Unitarity relation for the isobar amplitude $F_1(s)$



$$\mathrm{disc}F_1(s) = 2i\left(F_1(s) + \hat{F}_1(s)\right)\sin\delta(s)e^{-i\delta(s)}\theta(s-4m_\pi^2)\,,$$

- Complications: integration contour for $\hat{F}_1(s)$
- Classic' strategy (Khuri-Treiman, 1960)
  — deform path of angular integral to avoid crossing branch cuts
- Alternative approach (Gasser and Rusetsky, 2018)
  — deform path of dispersion relation

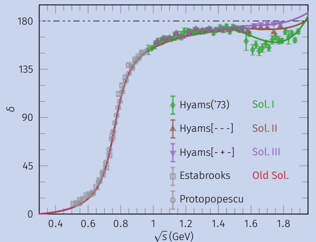


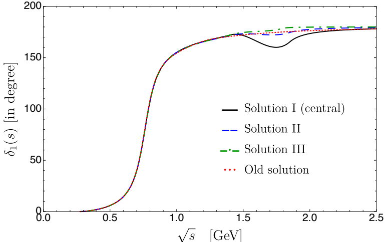
KT equations: DR, solutions, subtractions

- Unsubtracted dispersion relation:

$$F_1(s) = \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \,\frac{\text{disc}\, F_1(s')}{s'-s}\,, \quad F_1(s) = \Omega_1(s)\left(a + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'}\frac{\sin\delta_1(s')\,\hat{F}_1(s')}{|\Omega_1(s')|\,(s'-s)}\right),$$

- $\delta_1(s)$: $\pi\pi$ phase taken as input:
  - Old solution [Garcia-Martin, *et.al.* Phys. Rev. D83, 074004 (2011)]
  - New solutions [Pelaez, Rodas, Ruiz De Elvira, Eur. Phys. J. C79, 1008 (2019)]
  - **Central** analysis performed with solution I for $\delta_1(s)$
  - The spread in the other solutions: **theoretical uncertainty**





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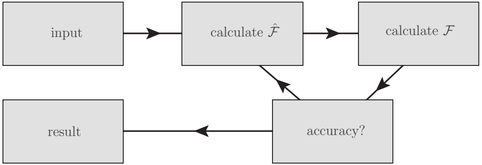
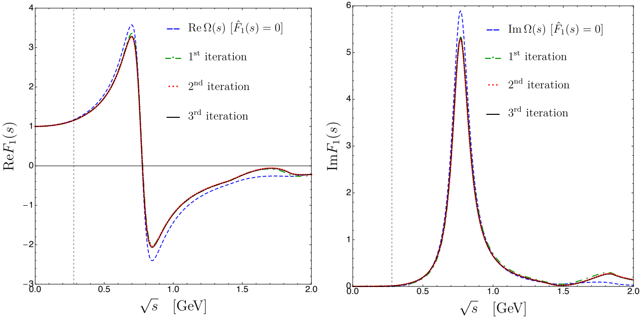
- Solution by numerical iteration
  - Initial input: $F_1 = \Omega_1(s)$





KT equations: DR, solutions, subtractions

- Once-subtracted dispersion relations

$$F_1(s) = \Omega_1(s)\left(a + b'\,s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2}\frac{\sin\delta_1(s')\,\hat{F}_1(s')}{|\Omega_1(s')|\,(s'-s)}\right)$$

- The subtraction constant satisfies a sum rule:

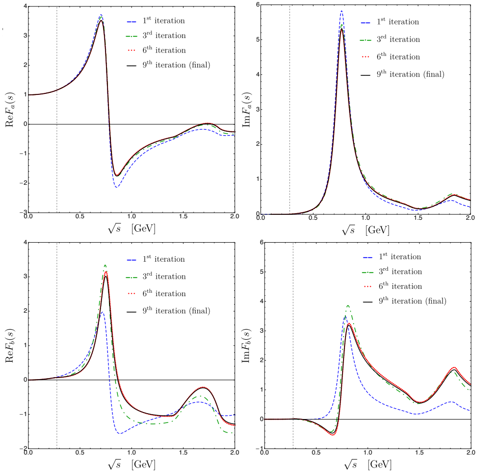
$$b \equiv b'/a = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2}\frac{\sin\delta_1(s')\,\hat{F}_1(s')/a}{|\Omega_1(s')|}\,.$$

- Solution

$$F_1(s) = a\left[F_a(s) + b\,F_b(s)\right]\,,$$

$$F_a(s) = \Omega_1(s)\left[1 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1(s')\,\hat{F}_a(s')}{|\Omega_1(s')|(s'-s)}\right]\,,$$

$$F_b(s) = \Omega_1(s)\left[s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1(s')\,\hat{F}_b(s')}{|\Omega_1(s')|(s'-s)}\right]\,.$$



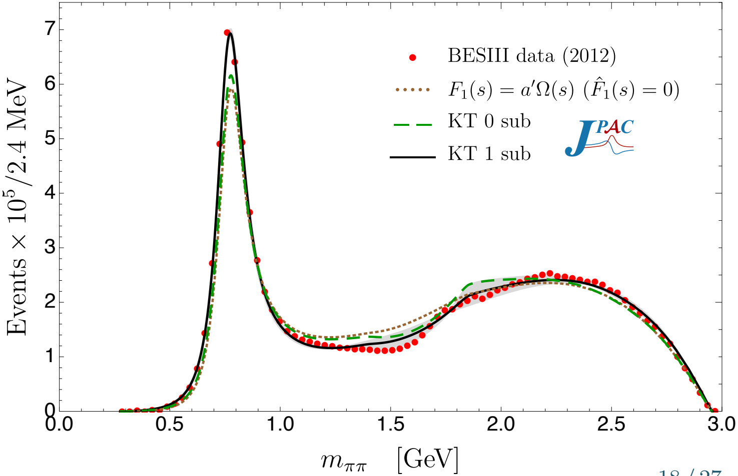
JPAC $J/\psi \to 3\pi$ BESIII data analysis

- Dominated by the ρ
- BESIII data [PLB 710 (2012)]
- JPAC analysis:
  - Two-body
  - KT unsub basic features
  - KT 1-sub improves the description

$$b_{\text{fit}} = 0.198(35)e^{i2.675(300)}\ \text{GeV}^{-2}$$

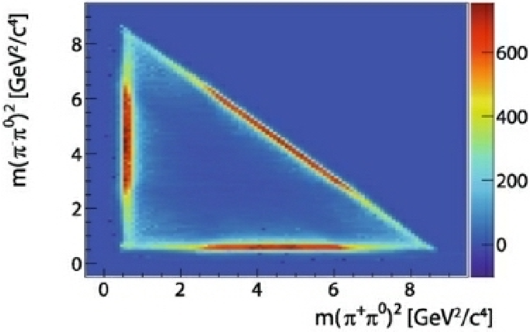
$$b_{\text{sr}} = 0.141e^{i2.32}\ \text{GeV}^{-2}$$

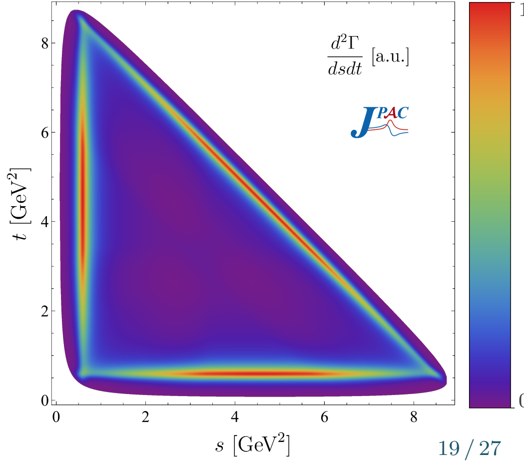
JPAC Coll, Phys.Rev. D108, 014035 (2023)



JPAC $J/\psi \to 3\pi$ BESIII data analysis

- Dominated by the ρ
- BESIII data [PLB 710 (2012)]
- JPAC analysis:
  - Dalitz plot distribution similar to experimental one





Contribution of the F-wave

- Isobar decomposition of the amplitude including F-wave:

$$F(s,t,u) = F_1(s) + F_1(t) + F_1(u)$$
$$+ (p(s)q(s))^2 P_3'(z_s) F_3(s) + (p(t)q(t))^2 P_3'(z_t) F_3(t) + (p(u)q(u))^2 P_3'(z_u) F_3(u)\,,$$

- $F_{1,3}(s)$ is the P, F-wave isobar amplitude
- Discontinuity of the F-wave:

$$\operatorname{disc} F_3(s) = 2i\left(F_3(s) + \hat{F}_3(s)\right)\,\sin\delta_3(s)\,e^{-i\delta_3(s)}\,\theta(s-4m_\pi^2)\,,$$

- We neglect $\hat{F}_3(s)$ (for simplicity)

$$F_3(s) = p_3(s)\Omega_3(s)\,,\quad \Omega_3(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_3(s')}{s'-s}\right]\,,$$

Model of the F-wave: $\rho_3(1690)$ exchange

- Exchange of a $\rho_3(1690)$ in the s-channel with an energy-dependent width:

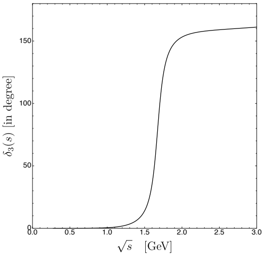
$$F_3(s)|_{\mathrm{BW}} = \frac{m_{\rho_3}^2}{m_{\rho_3}^2 - s - i m_{\rho_3}\Gamma_{\rho_3}^{\ell=3}(s)}\,, \qquad \Gamma_R^{\ell}(s) = \frac{\Gamma_R m_R}{\sqrt{s}}\left(\frac{p(s)}{p(m_R^2)}\right)^{2\ell+1}\left(F_R^{\ell}(s)\right)^2\,,$$

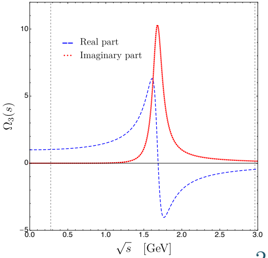
$$p(s) = \frac{\sqrt{s}}{2}\sigma_\pi(s)\,, \quad F_R^{\ell=3}(s) = \sqrt{\frac{z_0(z_0-15)^2 + 9(2z_0-5)^2}{z(z-15)^2+9(2z-5)^2}}\,, \quad z = r_R^2 p^2(s)\,, \quad z_0 = r_R^2 p^2(m_{\rho_3}^2)\,,$$

- Extraction of the F-wave phase and Omnès F-wave function:

$$\tan\delta_3(s) = \frac{\mathrm{Im}F_3(s)|_{\mathrm{BW}}}{\mathrm{Re}F_3(s)|_{\mathrm{BW}}}\,,$$

$$\Omega_3(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_3(s')}{s'-s}\right]\,,$$





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- JPAC analysis:
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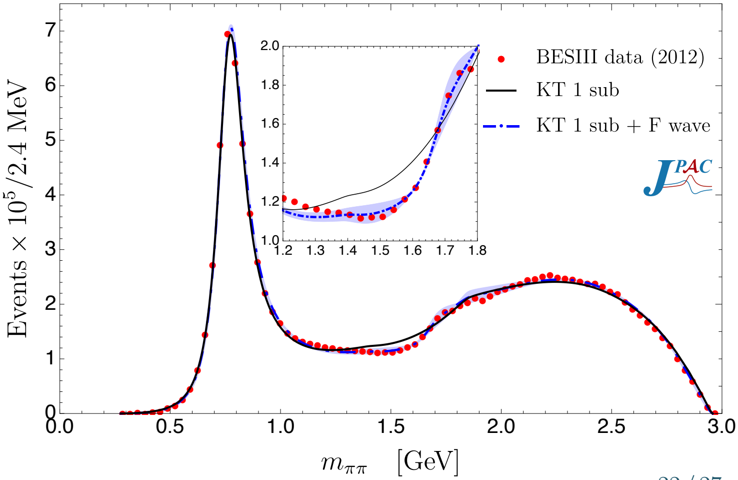
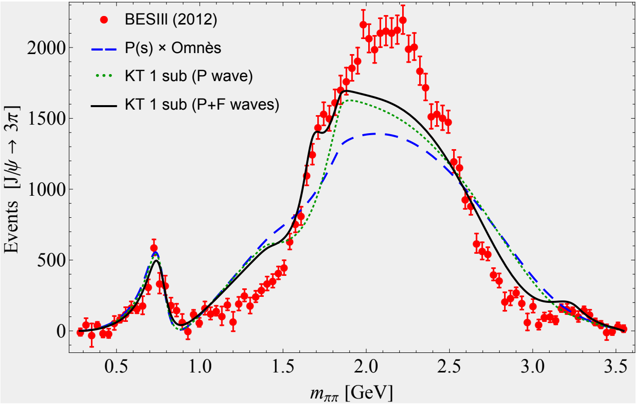
  - KT unsub basic features
  - KT 1-sub improves the description
  - KT 1-sub+F-wave describe better $m_{\pi\pi} \sim 1.5$ GeV.

JPAC Coll, Phys.Rev. D108, 014035 (2023)



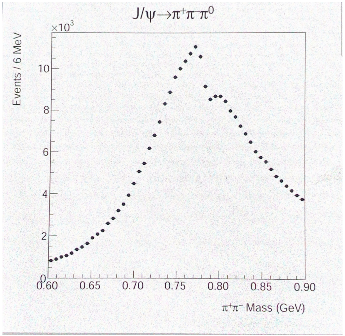
$\psi(2S) \to \pi^+\pi^-\pi^0$ decays

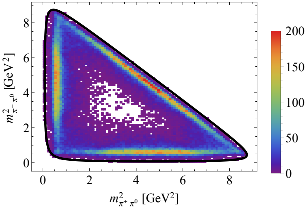
- Completely analogous formalism
- Larger phase space,
  $m_{\psi(2S)} = 3.68610$ GeV
- BESIII data [PLB 710 (2012)]
- Why does the $\psi(2S) \to \pi^+\pi^-\pi^0$ Dalitz plot differ so dramatically from $J/\psi \to \pi^+\pi^-\pi^0$?

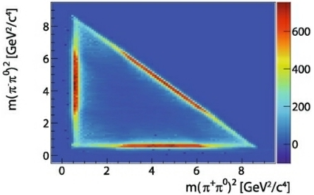


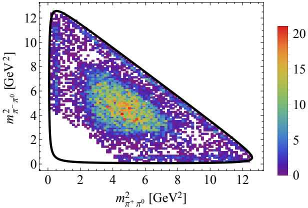
$J/\psi \to 3\pi$: future analyses

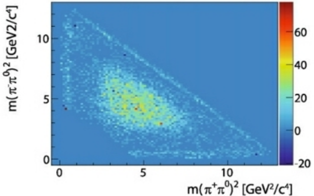
- In 2022, BESIII collected 10 billion J/ψ events
  - Expect to see roughly 200 million $J/\psi \to 3\pi$ events (vs ∼1.9 million in 2012)
  - High-precision ρ-ω mixing extraction (JPAC, ongoing)





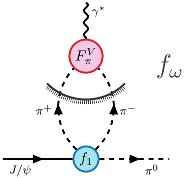






$J/\psi \to \pi^0\gamma^*$ transition form factor

- Dispersive representation (once-subtracted)

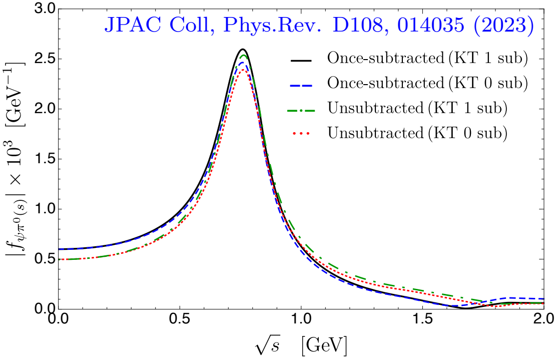


$$f_{\omega\pi^0}(s) = |f_{J/\psi\pi^0}(0)|\, e^{i\phi_{J/\psi\pi^0}(0)} + \frac{s}{12\pi^2}\int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^{3/2}} \frac{p^3(s')\, F_\pi^{V\,*}(s')\, f_1^{J/\psi\to 3\pi}(s')}{(s'-s)},$$

- $|f_{J/\psi\pi^0}(0)|$ from data:

$$\Gamma(J/\psi \to \pi^0\gamma) = \frac{e^2(m^2_{J/\psi} - m^2_{\pi^0})^3}{96\pi m^3_{J/\psi}} |f_{J/\psi\pi^0}(0)|^2\,,$$

- $\phi_{\omega\pi^0}(0)$ only free parameter
- No data, $\phi_{\omega\pi^0}(0) = 0$



Combined analysis of $\omega \to 3\pi$ and $\omega \to \pi^0\gamma^*$

- $\omega \to 3\pi$ Dalitz parameters: $X = \frac{t-u}{\sqrt{3}R_\omega} = \sqrt{Z}\cos\phi\,, Y = \frac{s_c - s}{R_\omega} = \sqrt{Z}\sin\phi\,,$

$$|\mathcal{F}(Z,\phi)|^2 = |N|^2\left[1 + 2\alpha Z + 2\beta Z^{3/2}\sin 3\phi + 2\gamma Z^2 + \mathcal{O}(Z^{5/2})\right],$$

Reference	$\alpha \times 10^3$	$\beta \times 10^3$	$\gamma \times 10^3$
BESIII [PRD98, 112007 (2018)]	111(18)	25(10)	22(29)
low	112(15)	23(6)	29(6)
high	109(14)	26(6)	19(5)

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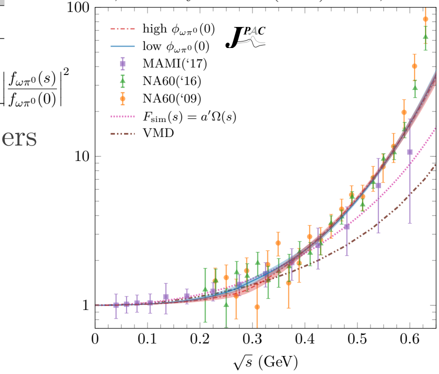

- $\omega \to 3\pi$ subtraction constant from Dalitz parameters

$$b_{\rm Fit} \simeq 3.15(22)e^{2.03(14)i}\ \mathrm{GeV}^{-2}$$

vs

$$b_{\rm sr} = 0.55e^{0.15i}\ \mathrm{GeV}^{-2}$$

- Only 1 sub KT describes both Dalitz-parameters and form factor

Outlook

- **Khuri-Treiman equations**:
    - Dispersive representation for **3-particle Final-State Interactions**
    - Based on fundamental principles: **analyticity, unitarity and crossing symmetry**
    - Input: $\pi\pi$ scattering **phase shifts**
    - Resonance shape affected by **left-hand cuts / 3-body effects**
    - **Predictive power** (subtraction constants)
    - Experimental **data** well described

Khuri-Treiman representation

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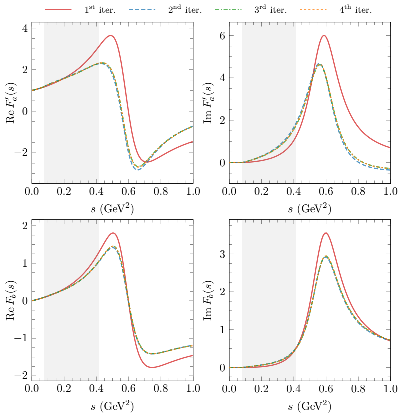
- $\mathcal{F}(s) = a\,[F_a(s) + b\,F_b(s)]$

$$F_a(s) = \Omega(s)\left[1 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{F}_a(s')}{|\Omega(s')|(s'-s)}\right],$$

$$F_b(s) = \Omega(s)\left[s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{F}_b(s')}{|\Omega(s')|(s'-s)}\right].$$



- Subtraction constant from Dalitz parameters

$$b_{\text{Fit}} \simeq 2.9e^{1.90(1)i}\,\text{GeV}^{-2} \quad \text{vs} \quad b_{\text{sr}} = 0.55e^{0.15i}\,\text{GeV}^{-2}$$

$\omega \to \pi^0\gamma^*$ form factor

- Dispersive representation

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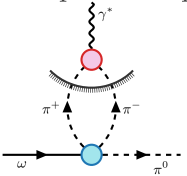

$$f_{\omega\pi^0}(s) = |f_{\omega\pi^0}(0)|\, e^{i\phi_{\omega\pi^0}(0)} + \frac{s}{12\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^{3/2}}\frac{p^3(s')\, F_\pi^{V*}(s')\, f_1^{\omega\to3\pi}(s')}{(s'-s)}\,,$$

- $|f_{\omega\pi^0}(0)|$ from data:

$$\Gamma(\omega\to\pi^0\gamma) = \frac{e^2(m_\omega^2 - m_{\pi^0}^2)^3}{96\pi m_\omega^3}\,|f_{\omega\pi^0}(0)|^2\,,$$

- Data: $F_{\omega\pi}(s) = \frac{f_{\omega\pi}(s)}{f_{\omega\pi}(0)}$ [NA60('09,'16), A2('17)]
- $\phi_{\omega\pi^0}(0)$ only free parameter
- Two minima are found (low and high)