Exercises for Lattice QCD, summer school 2019

Akaki Rusetsky

Exercise

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1 Bound state in the one-dimensional potential well

Consider a one-dimensional potential well

$$V(x) = \begin{cases} -V_0, & \text{if } |x| < b/2; \\ 0, & \text{if } |x| > b/2. \end{cases}$$

Solve the Schrödinger equation on the class of the symmetric wave functions $\Psi(x) = \Psi(-x)$.

a) Adjust the parameters so that one shallow bound state emerges (the size of the bound state is much larger than b) . Find the energy of the bound state in the infinite volume.

Solution

We have Schrödinger equation,

$$\left[-\frac{\nabla^2}{2m} + V(x)\right]\Psi(x) = E\Psi(x). \tag{1.1}$$

For bound state we have $-V_0 < E < 0$, so we write down $E = -\kappa^2/(2m)$ where κ is binding momentum.
Inside the well, wave function reads [a]

$$\Psi(x) = A\cos\omega x, \quad \omega = \sqrt{2mV_0 - \kappa^2}. \tag{1.2}$$

The right part out of the well reads [b],

$$\Psi(x) = A_B e^{-\kappa x}. \tag{1.3}$$

The boundary condition at the right edge of well gives that

$$\begin{aligned} A\cos\omega\frac{b}{2} &= A_B e^{-\kappa\frac{b}{2}}; \\ -A\omega\sin\omega\frac{b}{2} &= -A_B\kappa e^{-\kappa\frac{b}{2}}. \end{aligned} \tag{1.4}$$

Dividing the second equation by the first one, we find that

$$\omega\tan\omega\frac{b}{2} = \kappa. \tag{1.5}$$

Scaling everything by b,

$$\sqrt{2mV_0b^2-(\kappa b)^2}\tan\left(\frac{1}{2}\sqrt{2mV_0b^2-(\kappa b)^2}\right)=\kappa b. \tag{1.6}$$

The existing of shallow bound state means $\kappa b \ll 1$, therefore,

$$\sqrt{2mV_0b^2}\tan\left(\frac{1}{2}\sqrt{2mV_0b^2}\right)\simeq\kappa b\to 0 \tag{1.7}$$

yielding to the constrain of parameters,

$$\sqrt{2mV_0b^2}\to 2n\pi,\quad n\in\mathbb{Z}. \tag{1.8}$$

Then the binding momentum is obtained that

$$\kappa\simeq\sqrt{2mV_0}\tan\left(\frac{b}{2}\sqrt{2mV_0}\right) \tag{1.9}$$

and the binding energy in the infinite volume,

$$E=-\frac{\kappa^2}{2m}\simeq -V_0\tan^2\left(\frac{b}{2}\sqrt{2mV_0}\right). \tag{1.10}$$

[a]We pick up even parity solution, $\Psi(x)=\Psi(-x)$.

[b]The infinite volume boundary condition asks for $\Psi(+\infty)=0$

b) Find the asymptotic normalization coefficient.

Solution

The normalization condition reads [a],

$$|A|^2\left[\int_0^{b/2}dx\,\cos^2\omega x+\cos^2\left(\omega\frac{b}{2}\right)e^{\kappa b}\int_{b/2}^{\infty}dx\,e^{-2\kappa x}\right]=\frac{1}{2}. \tag{1.11}$$

Carrying out the integrals, we have

$$|A|^2\left[\frac{\omega b+\sin\omega b}{4\omega}+\frac{1+\cos\omega b}{2}\frac{1}{2\kappa}\right]=\frac{1}{2}. \tag{1.12}$$

So

$$|A|^2=\kappa\left[\kappa\frac{\omega b+\sin\omega b}{2\omega}+\frac{1+\cos\omega b}{2}\right]^{-1}\simeq\kappa\left[\frac{1+\cos\sqrt{2mV_0b^2}}{2}\right]^{-1}\to\kappa\left[\frac{1+\cos 2n\pi}{2}\right]^{-1}=\kappa \tag{1.13}$$

and

$$|A_B|^2 = \cos^2\left(\omega\frac{b}{2}\right)e^{\kappa b}|A|^2 \to \kappa\cos^2(n\pi) = \kappa. \tag{1.14}$$

The wave function has asymptotic behavior,

$$\Psi(x) \to \sqrt{\kappa}e^{-\kappa x}, \quad x \to \infty. \tag{1.15}$$

So asymptotic normalization coefficient $|A_B|^2 = \kappa$.

[a] Still, we pick up even parity solution, $\Psi(x) = \Psi(-x)$ and using equation $A\cos\omega\frac{b}{2} = A_B e^{-\kappa\frac{b}{2}}$.

c) Assume now that the system is put on a large torus $b/L \ll 1$. Impose the periodic boundary conditions of the wave fucntion:

$$\Psi(L/2) = \Psi(-L/2), \quad \Psi'(L/2) = \Psi'(-L/2).$$

Find the energy of the bound state on the torus. Does it agree with the expression which was derived in the lecture?

Solution

Now boundary condition $\Psi(+\infty) = 0$ breaks down. It means wave function out of the well becomes,

$$\Psi(x) = Be^{-\kappa_L|x|} + Ce^{\kappa_L|x|}. \tag{1.16}$$

New boundary condition

$$\Psi'(L/2) = \Psi'(-L/2) \tag{1.17}$$

yields to

$$-B\kappa_L e^{-\kappa_L\frac{L}{2}} + C\kappa_L e^{\kappa_L\frac{L}{2}} = B\kappa_L e^{-\kappa_L\frac{L}{2}} - C\kappa_L e^{\kappa_L\frac{L}{2}} \tag{1.18}$$

and then

$$e^{\kappa_L L} = B/C. \tag{1.19}$$

On the other hand, the boundary condition at the right edge of well now reads [a],

$$A\cos\omega\frac{b}{2} = Be^{-\kappa_L\frac{b}{2}} + Ce^{\kappa_L\frac{b}{2}};$$

$$-A\omega\sin\omega\frac{b}{2} = -B\kappa_L e^{-\kappa_L\frac{b}{2}} + C\kappa_L e^{\kappa_L\frac{b}{2}}. \tag{1.20}$$

Further, we derive that

$$\omega\tan\omega\frac{b}{2} = \kappa_L\frac{e^{\kappa_L(L-b)}-1}{e^{\kappa_L(L-b)}+1}. \tag{1.21}$$

Still we have $\kappa_L b \ll 1$,

$$\sqrt{2mV_0b^2}\tan\frac{1}{2}\sqrt{2mV_0b^2} = \kappa_L b\frac{e^{\kappa_L L}-1}{e^{\kappa_L L}+1}. \tag{1.22}$$

[b]Notice that in the infinite volume, $\kappa = \sqrt{2mV_0b^2}\tan\frac{1}{2}\sqrt{2mV_0b^2}$, we then write down,

$$\kappa_L - \frac{2\kappa_L}{e^{\kappa_L L}+1} = \kappa. \tag{1.23}$$

Further [c]

$$\begin{aligned}\kappa_L =& \kappa\left[1-\frac{2}{e^{(\kappa+\delta\kappa)L}+1}\right]^{-1}\\ \to& \kappa\left[1-\frac{2}{e^{\kappa L}+1}\right]^{-1}\\ \simeq& \kappa\left[1+2e^{-\kappa L}\right].\end{aligned} \tag{1.24}$$

Then we have

$$E_L = -\frac{\kappa_L^2}{2m} \simeq -\frac{\kappa^2}{2m}\left[1+2e^{-\kappa L}\right]^2 \simeq -\frac{\kappa^2}{2m}\left[1+4e^{-\kappa L}\right] = E\left[1+4e^{-\kappa L}\right] \tag{1.25}$$

yielding to

$$\delta E = E_L - E = 4Ee^{-\kappa L}. \tag{1.26}$$

It is consistant with our result in the lecture [d],

$$\delta E = 4|A_B|^2\frac{E}{\kappa}e^{-\kappa L} = 4Ee^{-\kappa L} \tag{1.27}$$

since in this case, asymptotic normalization coefficient $|A_B|^2 = \kappa$.

[a]Note that now $\omega = \sqrt{2mV_0 - \kappa_L^2}$.

[b]We can check that,the above equation recovers to the infinite volume version as $L \to \infty$.

[c]We call $\kappa_L = \kappa + \delta\kappa$ and $\delta\kappa$ vanishes as $L \to \infty$,

[d]Note that for the definitions of E_L and E, we have a sign difference from the lecture,

2 Energy level shift of an atom

We consider a pair of charged particles bound in a hydrogen-like atom by a static Coulomb force $V_C(|\mathbf{x}|) = e^2/|\mathbf{x}|$. Add now the perturbation $\lambda V(\mathbf{x})$ to the purely Coulomb Hamiltonian, assuming that the range of this potential is much smaller that the characteristics distances in the atom (what is the characteristic distance in the atom?).

a) Calculate the energy shift of the ground-state level and show that this shift to the lowest order in the fine structure constant $\alpha = e^2/(4\pi)$ and *in all orders of* λ is proportional to the scattering length on the potential $\lambda V(\mathbf{x})$.

Solution

We have the wave function of the ground-state level,

$$\psi_0(\mathbf{x}) = \frac{2}{R^{3/2}} \exp\left(-\frac{|\mathbf{x}|}{R}\right), \tag{2.1}$$

where Bohr radius $R = 1/(\mu e^2)$ [a].
We have energy shift of the ground-state level,

$$\begin{aligned} \Delta E =& \langle \psi_0 | \lambda V(\mathbf{x}) | \psi_0 \rangle \\ =& \int d^3x \, |\psi_0(\mathbf{x})|^2 \lambda V(\mathbf{x}) \\ =& \frac{\lambda}{R^3} \times \left[4 \int d^3x \, V(\mathbf{x}) \exp\left(-\frac{2|\mathbf{x}|}{R}\right)\right]. \end{aligned} \tag{2.2}$$

The short range potential is [b],

$$\lambda V(\mathbf{x}) = \frac{2\pi\lambda}{\mu} \delta^{(3)}(\mathbf{x}). \tag{2.3}$$

So energy shift at the lowest order is

$$\Delta E = \frac{8\pi\lambda}{\mu R^3}. \tag{2.4}$$

[a] μ is the reduced mass. Bohr radius is the characteristic distance in the atom.

[b] λ has length dimension so the total potential has mass dimension. We can define λ as scattering length. To see this, let us do Fourier transformaiton,

$$\int d^3x e^{-ikx} C\delta^{(3)}(x) = C, \quad C = -\frac{2\pi\lambda}{\mu}.$$

Then LS equation gives,

$$\begin{aligned} T =& C + 2\mu C \int \frac{d^3k}{(2\pi)^3 i} \frac{1}{2\mu E - k^2 + i\epsilon} T \\ =& C + 2\mu C i \rho T \end{aligned}$$

where $\rho = k_*/(4\pi)$. It means

$$T = \frac{2\pi}{\mu} \frac{1}{\frac{2\pi}{\mu C} - ik_*}$$

and match equation, $\frac{2\pi}{\mu C} = -\frac{1}{a}$ yielding to $\lambda = a$.

b) Derive an analogy to the Lüscher formula.

Solution

We have normalized wavefunction in a box [a],

$$\psi_0(\mathbf{x}) = \frac{1}{L^{3/2}} e^{i\mathbf{k}_n \mathbf{x}}, \quad \mathbf{k}_n = \frac{2\pi \mathbf{n}}{L}, \quad \mathbf{n} \in \mathbb{Z}^3. \tag{2.5}$$

For the potential

$$\lambda V(\mathbf{x}) = \frac{2\pi\lambda}{\mu} \delta^{(3)}(\mathbf{x}), \tag{2.6}$$

we have

$$\begin{aligned}\Delta E =& \langle \psi_0 | \lambda V(\mathbf{x}) | \psi_0 \rangle \\ =& \int d^3x \, |\psi_0(\mathbf{x})|^2 \lambda V(\mathbf{x}) \\ =& \frac{2\pi\lambda}{\mu L^3}.\end{aligned} \tag{2.7}$$

We can also calculate energy shift in momentum space [b].

[a] Normalization in a box,

$$a^3 \sum_{\mathbf{x}} |\psi(\mathbf{x})|^2 = 1.$$

[b] We consider contact potential in momentum space,

$$V = -\frac{2\pi\lambda}{\mu}.$$

The T-matrix in a finite volume is calculated by unitaritization,

$$T_L = V + V I_L V + \cdots$$

where phase space integral is

$$\begin{aligned} I_L =& \int \frac{dk}{(2\pi)i} \frac{1}{L^3} \sum_{\mathbf{k}} \left(\frac{k^2}{2m} - k_0 - i\epsilon \right)^{-1} \left[\frac{k^2}{2M} - (E - k_0) - i\epsilon \right]^{-1} \\ =& \frac{1}{L^3} \sum_{\mathbf{k}} \left[\frac{k^2}{2\mu} - E \right]^{-1} \end{aligned}$$

$$
\begin{aligned}
&= 2\mu \mathrm{PV} \cancel{\int \frac{d^3k}{(2\pi)^3} \left[k^2 - k_*^2\right]^{-1}} + \frac{2\mu}{L^3} \sum\!\!\!\!\!\!\!\int_{\mathbf{k}} \left[k^2 - k_*^2\right]^{-1} \\
&\equiv \frac{\mu}{2\pi} \left\{S_L\right\}.
\end{aligned}
$$

We define that

$$
S_L = \frac{4\pi}{L^3} \sum\!\!\!\!\!\!\!\int_{\mathbf{k}} \left[k^2 - k_*^2\right]^{-1}.
$$

Then we have

$$
\begin{aligned}
T_L &= \left(-\frac{2\pi\lambda}{\mu}\right) + \left(-\frac{2\pi\lambda}{\mu}\right) I_L \left(-\frac{2\pi\lambda}{\mu}\right) + \cdots \\
&= \left(-\frac{2\pi\lambda}{\mu}\right) \left(1 + \frac{2\pi\lambda}{\mu} I_L\right)^{-1} \\
&= \left(\frac{2\pi}{\mu}\right) \left(-\frac{1}{\lambda} - S_L\right)^{-1}.
\end{aligned}
$$

The S-wave Lüscher formula reads now,

$$
-\frac{1}{\lambda} = \frac{4\pi}{L^3} \sum_{\mathbf{k}} \left[k^2 - k_*^2\right]^{-1}.
$$

We conclude that $k_*^2 \propto L^{-3}$. Otherwise, the right-handed side of the above equation cannot be at $O(L^0)$ thus equation breaks down. Then

$$
-\frac{1}{\lambda} = \underbrace{-\frac{4\pi}{L^3} \left[k_*^{-2}\right]}_{O(1)} + \underbrace{\frac{4\pi}{L^3} \sum_{\mathbf{k}\neq 0} \left[k^2 - k_*^2\right]^{-1}}_{O(1/L)}. \tag{2.8}
$$

The leading order result is

$$
k_*^2 = \frac{4\pi\lambda}{L^3} + O(L^{-4}) \tag{2.9}
$$

yielding to

$$
\Delta E = \frac{2\pi\lambda}{\mu L^3}. \tag{2.10}
$$

3 Three particles

The action of the free neutral scalar field on the lattice is given by

$$
S = a^4 \sum_{x,\mu} \frac{1}{2} \partial_\mu \varphi(x) \partial_\mu \varphi(x) + a^4 \sum_x \frac{m^2}{2} \varphi^2(x),
$$

where a is the lattice constant and

$$
\partial_\mu \varphi(x) = \frac{1}{a} \left(\varphi(x + a\hat{\mu}) - \varphi(x)\right).
$$

Consider the operator

$$\Phi_{\mathbf{P}}(t) = \sum_x e^{-i\mathbf{P}\mathbf{x}} \varphi^3(\mathbf{x}, t), \quad \Phi^\dagger_{\mathbf{P}}(t) = \sum_x e^{i\mathbf{P}\mathbf{x}} \varphi^3(\mathbf{x}, t).$$

The two-point function of this operator has the form

$$D_{\mathbf{P}}(t) = \langle \Phi_{\mathbf{P}}(t) \Phi^\dagger_{\mathbf{P}}(0) \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \Phi_{\mathbf{P}}(t) \Phi^\dagger_{\mathbf{P}}(0) e^{-S}.$$

The spectrum is determined from the asymptotics at large t:

$$D_{\mathbf{P}}(t) \sim \sum_n a_n e^{-E_n(\mathbf{P})t}, \qquad \text{as } t \to \infty.$$

Consider the box $L^3 \times L_t$ with $L_t \gg L$. Take $\mathbf{P} = \frac{2\pi}{L}\mathbf{n}$, $\mathbf{n} = (0, 0, 1)$ (here, L is the size of the box). Write down first few energy levels $E_n(\mathbf{P})$. Consider the case $a \neq 0$.

a) Three-particle state's energy is the sum of three single particles' energies.

Solution

We calculate [a]

$$\begin{aligned}
D_{\mathbf{P}}(t) =& \langle \Phi_{\mathbf{P}}(t) \Phi^\dagger_{\mathbf{P}}(0) \rangle \\
=& \sum_{x,y} e^{-i\mathbf{P}(\mathbf{x}-\mathbf{y})} \langle \varphi^3(\mathbf{x}, t) \varphi^3(\mathbf{y}, 0) \rangle \\
=& 3! \sum_{x,y} e^{-i\mathbf{P}(\mathbf{x}-\mathbf{y})} G^3(\mathbf{x}-\mathbf{y}, t) \\
=& 3! \sum_{x,y} e^{-i\mathbf{P}(\mathbf{x}-\mathbf{y})} \left(\frac{1}{L^3}\right)^3 \not{\sum\!\!\!\!\int}_{p_1,p_2,p_3} e^{i(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(\mathbf{x}-\mathbf{y})} e^{-(E_1+E_2+E_3)t} G(p_1)G(p_2)G(p_3) \\
=& 3! \left(\frac{1}{L^3}\right)^3 \int \frac{dE_1}{2\pi i} \frac{dE_2}{2\pi i} \frac{dE_3}{2\pi i} e^{-(E_1+E_2+E_3)t} \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} \delta_{\mathbf{P},\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} G(p_1)G(p_2)G(p_3) \\
=& 3! \left(\frac{1}{L^3}\right)^3 \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} \delta_{\mathbf{P},\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} \left[\left(\frac{2}{a}\right)^3 \sinh \omega_1 a \sinh \omega_2 a \sinh \omega_3 a\right] e^{-(\omega_1+\omega_2+\omega_3)t} \qquad (3.1)
\end{aligned}$$

with the dispersion relation,

$$\omega = \frac{2}{a} \text{arcsinh} \sqrt{\frac{b-1}{2}} = \frac{1}{a} \log(b + \sqrt{b^2-1}), \quad b = 1 + \frac{1}{2}\left[4 \sum_{j=1,2,3} \sin^2 \frac{ap_j}{2} + m^2 a^2\right]. \qquad (3.2)$$

Then we can calculate the energy level, $E = \omega_1 + \omega_2 + \omega_3$.

[a] We consider $L_t \sim \infty$ which means Fourier transformation of energy is continuous.

b) The first few energy levels.

Solution

The ground state, e.g., $\mathbf{n}_1 = \mathbf{n}_2 = (0,0,0)$ and $\mathbf{n}_3 = (0,0,1)$,

$$E_0 = \frac{4}{a}\text{arcsinh}\left(\frac{1}{2}ma\right) + \frac{2}{a}\text{arcsinh}\sqrt{\sin^2\frac{\pi a}{L} + \frac{1}{4}m^2a^2}. \tag{3.3}$$

The 1st excited state, e.g., $\mathbf{n}_1 = (0,-1,0)$, $\mathbf{n}_2 = (0,1,0)$ and $\mathbf{n}_3 = (0,0,1)$,

$$E_1 = \frac{6}{a}\text{arcsinh}\sqrt{\sin^2\frac{\pi a}{L} + \frac{1}{4}m^2a^2}. \tag{3.4}$$

Another choice $\mathbf{n}_1 = (0,-1,1)$, $\mathbf{n}_2 = (0,1,0)$ and $\mathbf{n}_3 = (0,0,0)$,

$$E_1' = \frac{2}{a}\text{arcsinh}\sqrt{2\sin^2\frac{\pi a}{L} + \frac{1}{4}m^2a^2} + \frac{2}{a}\text{arcsinh}\sqrt{\sin^2\frac{\pi a}{L} + \frac{1}{4}m^2a^2} + \frac{2}{a}\text{arcsinh}\left(\frac{1}{2}ma\right). \tag{3.5}$$

The 2nd excited state, e.g., $\mathbf{n}_1 = (0,0,0)$, $\mathbf{n}_2 = (0,0,2)$ and $\mathbf{n}_3 = (0,0,-1)$ [a],

$$E_2 = \frac{2}{a}\text{arcsinh}\left(\frac{1}{2}ma\right) + \frac{2}{a}\text{arcsinh}\sqrt{\sin^2\frac{\pi a}{L} + \frac{1}{4}m^2a^2} + \frac{2}{a}\text{arcsinh}\sqrt{\sin^2\frac{2\pi a}{L} + \frac{1}{4}m^2a^2}. \tag{3.6}$$

[a] We can consider continue limit and non-relativistic limit to estimate the order of the states, i.e., $E \propto |\mathbf{n}_1|^2 + |\mathbf{n}_2|^2 + |\mathbf{n}_3|^2$.

4 Electromagnetic form factor of a pion

The electromagnetic form factor of a pion is defined through

$$\langle \pi_+(\mathbf{p}_1)|J^\mu(0)|\pi_+(\mathbf{p}_2)\rangle = (p_1+p_2)^\mu F_\pi(t), \quad t = (p_1-p_2)^2,$$

where J_μ is the electromagnetic current

$$J^\mu(x) = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d.$$

It is convenient to choose the Breit Frame

$$P_1 = (E_\pi, \mathbf{p}), \quad p_2 = (E_\pi, -\mathbf{p}), \quad E_\pi = \sqrt{M_\pi^2 + \mathbf{p}^2}, \quad t = -Q^2, \quad \mathbf{Q} = 2\mathbf{p}.$$

Field operators for the pions, moving with a momentum **p**, on the lattice are given by

$$\phi_{\mathbf{p}}(t) = \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \bar{d} i\gamma_5 u, \quad \phi^\dagger_{\mathbf{p}}(t) = \sum_{\mathbf{x}} e^{i\mathbf{p}\mathbf{x}} \bar{u} i\gamma_5 d.$$

a) Define

$$\begin{aligned} V^\mu_{\mathbf{p}}(t,t') =& \langle 0|\phi_{\mathbf{p}}(t) J^\mu(0) \phi^\dagger_{-\mathbf{p}}(t')|0\rangle, \\ D_{\mathbf{p}}(t,t') =& \langle 0|\phi_{\mathbf{p}}(t)\phi^\dagger_{\mathbf{p}}(t')|0\rangle \end{aligned}$$

and show that

$$\langle \pi_+(\mathbf{p})|J^\mu(0)|\pi_+(-\mathbf{p})\rangle = \lim_{t\to\infty}\lim_{t'\to-\infty} \frac{V^\mu_{\mathbf{p}}(t,t')}{D_{\mathbf{p}}(t,t')}.$$

Solution

We insert unitarity,

$$\begin{aligned} V^\mu_{\mathbf{p}}(t,t') =& \langle 0|\phi_{\mathbf{p}}(t) J^\mu(0) \phi^\dagger_{-\mathbf{p}}(t')|0\rangle \\ =& \sum_{n,m} \int \frac{d^3q}{2E_n(\mathbf{q})} \frac{d^3k}{2E_m(\mathbf{k})} \langle 0|\phi_{\mathbf{p}}(t)|n,\mathbf{q}\rangle\langle n,\mathbf{q}|J^\mu(0)|m,\mathbf{k}\rangle\langle m,\mathbf{k}|\phi^\dagger_{-\mathbf{p}}(t')|0\rangle \\ =& \sum_{n,m} \frac{1}{2E_n(\mathbf{p})} \frac{1}{2E_m(-\mathbf{p})} \langle 0|\phi_{\mathbf{p}}(0)|n,\mathbf{p}\rangle e^{-E_n(\mathbf{p})t}\langle n,\mathbf{p}|J^\mu(0)|m,-\mathbf{p}\rangle e^{E_m(-\mathbf{p})t'}\langle m,-\mathbf{p}|\phi^\dagger_{-\mathbf{p}}(0)|0\rangle. \end{aligned} \tag{4.1}$$

In the adibatic limit, $t\to\infty$, $t'\to-\infty$, the lowest energy state [a], i.e., one π_+ state is projected out,

$$\begin{aligned} |n,\mathbf{p}\rangle \to& \sqrt{2E_\pi(\mathbf{p})}|\pi_+(\mathbf{p})\rangle; \\ |m,-\mathbf{p}\rangle \to& \sqrt{2E_\pi(-\mathbf{p})}|\pi_+(-\mathbf{p})\rangle. \end{aligned} \tag{4.2}$$

So we have

$$V^\mu_{\mathbf{p}}(t,t') \to Z(\mathbf{p})F^\mu(\mathbf{p})e^{-E_n(\mathbf{p})t+E_m(\mathbf{p})t'},$$

where

$$Z(\mathbf{p}) = \langle 0|\phi_{\mathbf{p}}(0)|\pi_+(\mathbf{p})\rangle\langle\pi_+(-\mathbf{p})|\phi^\dagger_{-\mathbf{p}}(0)|0\rangle$$

$$F^\mu(\mathbf{p}) = \langle\pi_+(\mathbf{p})|J^\mu(0)|\pi_+(-\mathbf{p})\rangle.$$

By the same token, we have

$$D_{\mathbf{p}}(t,t^{'}) = \langle 0|\phi_{\mathbf{p}}(t)\phi^\dagger_{\mathbf{p}}(t^{'})|0\rangle \to Z^{'}(\mathbf{p})e^{-E_n(\mathbf{p})t+E_m(\mathbf{p})t^{'}} \tag{4.3}$$

where

$$Z^{'}(\mathbf{p}) = \langle 0|\phi_{\mathbf{p}}(0)|\pi_+(\mathbf{p})\rangle\langle\pi_+(\mathbf{p})|\phi^\dagger_{\mathbf{p}}(0)|0\rangle. \tag{4.4}$$

Consider π_+ is pseudo-scalar, we have

$$\begin{aligned}\langle\pi_+(\mathbf{p})|\phi^\dagger_{\mathbf{p}}(0)|0\rangle =& \langle\pi_+(\mathbf{p})|P\left(P\phi^\dagger_{\mathbf{p}}(0)P\right)P|0\rangle\\ =&(-)^2\langle\pi_+(-\mathbf{p})|\phi^\dagger_{-\mathbf{p}}(0)|0\rangle.\end{aligned} \tag{4.5}$$

So it turns out that $Z = Z^{'}$.
At last, we arrive at

$$\lim_{t\to\infty}\lim_{t^{'}\to-\infty}\frac{V^\mu_{\mathbf{p}}(t,t^{'})}{D_{\mathbf{p}}(t,t^{'})} = F^\mu(\mathbf{p}) = \langle\pi_+(\mathbf{p})|J^\mu(0)|\pi_+(-\mathbf{p})\rangle. \tag{4.6}$$

[a] The state has the same quantum number as operator ϕ.

b) What changes, if one wishes to extract the timelike form factor, i.e., $\langle\mathbf{p},-\mathbf{p}|j^\mu(0)|0\rangle$?

Solution

Now let us consider [a],

$$\begin{aligned}V^\mu_{\mathbf{p}}(t) =& \langle 0|\phi_{\mathbf{p}}(t)\phi^\dagger_{-\mathbf{p}}(t)J^\mu(0)|0\rangle\\ =&\sum_n\int\frac{d^3q}{2E_n(\mathbf{q})}\langle 0|\phi_{\mathbf{p}}(t)\phi^\dagger_{-\mathbf{p}}(t)|n,\mathbf{q}\rangle\langle n,\mathbf{q}|J^\mu(0)|0\rangle\\ =&\sum_n\frac{1}{2E_n(0)}\langle 0|\phi_{\mathbf{p}}(0)\phi^\dagger_{-\mathbf{p}}(0)|n,0\rangle e^{-E_n(0)t}\langle n,0|J^\mu(0)|0\rangle.\end{aligned} \tag{4.7}$$

The adiabatic limit, i.e., $t\to\infty$ only projects the threshold state thus is useless for the exacting timelike form factor.

$$V^\mu_{\mathbf{p}}(t) \to \langle 0|\phi_{\mathbf{p}}(0)\phi^\dagger_{-\mathbf{p}}(0)|\mathrm{th}\rangle\langle\mathrm{th}|J^\mu(0)|0\rangle e^{-E_{\mathrm{th}}t}. \tag{4.8}$$

The threshold state is the linear combination of asymptotic states [b],

$$|\mathrm{th}\rangle = \sum_{\mathbf{p}}\psi_{\mathrm{th}}(\mathbf{p})|\mathbf{p},-\mathbf{p}\rangle. \tag{4.9}$$

We have to consider new operators $\phi_{\mathbf{q}}(t)\phi^{\dagger}_{-\mathbf{q}}(t)$ and the excited states $|n\rangle$, i.e., $n = 1, 2, 3, \cdots$ besides threshold state.

[a] The insersion is only allowed when the time-ordering is clear.
[b] In a box, $\mathbf{p}$ can only take integer value.

5 The finite-volume shift of the nucleon mass

The lowest-order pion-pion and pion-nucleon Lagrangians in the $SU(2)$ ChPT is given by

$$\begin{aligned}\mathcal{L}^{(2)}_{\pi\pi} =& \frac{F^2}{4}\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger + \chi^\dagger U + U^\dagger \chi),\\ \mathcal{L}^{(1)}_{\pi N} =& \bar{\Psi}(i\not{D} - m_0 + \frac{g_A}{2}\not{u}\gamma_5)\Psi.\end{aligned}$$

Here, Ψ is the two-component nucleon field (proton and neutron), and

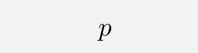
$$D_\mu = \partial_\mu + \Gamma_\mu, \quad \Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u],$$
$$U = u^2, \quad u_\mu = iu^\dagger \partial_\mu U u^\dagger, \quad \chi = 2B\hat{m}.$$

The higher-order terms in the Lagrangian are collected, e.g., in [hep-ph/0206068]. In order to solve this problem, you will not need them, however.

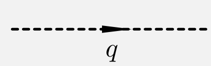
a) The problem: calculate the finite-volume shift of the nucleon mass at $O(p^3)$.

Solution

We have propagators for pion and nucleon.

$G(p) =$ 

$= \dfrac{1}{m - \not{p}}$

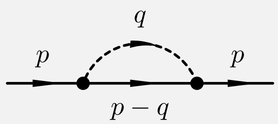
$D(q) =$ 

$= \dfrac{1}{M_\pi^2 - q^2}$

The interaction term betwenn πN reads,

$$\mathcal{L}^{\mathrm{int}}_{\pi N} = \frac{g_A}{2F}\bar{\Psi}\gamma_5 \not{\partial}\phi^a \tau^a \Psi + \cdots$$

We calculate the self-energy of nucleon up to $\mathcal{O}(p^3)$.

$\Sigma(p) \ =$ 

$$= \int \frac{d^Dq}{(2\pi)^D i}\left(\frac{g_A}{2F}\right)^2 \gamma_5 \not{q} \tau^a \frac{1}{m - \not{p} + \not{q}} \gamma_5 \not{q} \tau^a \frac{1}{M_\pi^2 - q^2}$$

Then,

$$\Sigma(p) = \left(\frac{3g_A^2}{4F^2}\right) \int \frac{d^Dq}{(2\pi)^D i} \frac{\gamma_5 \not{q}\left[\not{p} - \not{q} + m\right]\gamma_5 \not{q}}{\left[(p-q)^2 - m^2\right]\left[q^2 - M_\pi^2\right]}. \tag{5.1}$$

The numerator with Dirac structure is dealt with as

$$\begin{aligned} N =& \not{q}\left[\not{p} - \not{q} - m\right]\not{q} \\ =& \not{q}\not{p}\not{q} - q^2\not{q} - mq^2 \\ =& 2pq\not{q} - q^2\not{p} - q^2\not{q} - mq^2. \end{aligned} \tag{5.2}$$

Put on on-shell condition, $\not{p} \to m$.

$$\begin{aligned} \frac{N}{D} \to& \frac{(2pq - q^2)\not{q} - 2mq^2}{\left[(p-q)^2 - m^2\right]\left[q^2 - M_\pi^2\right]} \\ =& \frac{(m^2 - p^2 + 2pq - q^2)\not{q} - 2m(q^2 - M_\pi^2) - 2mM_\pi^2}{\left[(p-q)^2 - m^2\right]\left[q^2 - M_\pi^2\right]} \\ =& -\frac{\not{q}}{q^2 - M_\pi^2} - \frac{2m}{(p-q)^2 - m^2} - \frac{2mM_\pi^2}{\left[(p-q)^2 - m^2\right]\left[q^2 - M_\pi^2\right]} \\ \equiv& -X_1 - X_2 - X_3. \end{aligned} \tag{5.3}$$

So we have in the infinite volume [a],

$$\Sigma(p) = \left(-\frac{3g_A^2}{4F^2}\right) \int \frac{d^Dq}{(2\pi)^D i}(\cancel{X_1} + X_2 + X_3). \tag{5.4}$$

Now in a box, we have [b],

$$\begin{aligned} \Sigma_L(p) =& \left(-\frac{3g_A^2}{4F^2}\right) \frac{1}{L^d} \sum_{\mathbf{q}} \int \frac{dq^0}{(2\pi)i}(\cancel{X_1} + X_2 + X_3) \\ =& \left(-\frac{3g_A^2}{4F^2}\right) \int \frac{dq^0}{(2\pi)i} \int \frac{d^dq}{(2\pi)^d} \sum_{\mathbf{n}} \left(\frac{2\pi}{L}\right)^d \delta^{(d)}(\mathbf{q} - \frac{2\pi\mathbf{n}}{L})(X_2 + X_3) \\ =& \left(-\frac{3g_A^2}{4F^2}\right) \int \frac{d^Dq}{(2\pi)^D i} \sum_{\mathbf{n}} \delta^{(d)}(\mathbf{n} - \frac{\mathbf{q}L}{2\pi})(X_2 + X_3) \\ =& \left(-\frac{3g_A^2}{4F^2}\right) \int \frac{d^Dq}{(2\pi)^D i} \sum_{\mathbf{n}} e^{i\mathbf{n}\mathbf{q}L}(X_2 + X_3). \end{aligned} \tag{5.5}$$

The nucleon mass is related to self-energy by [c]

$$
\begin{aligned}
G =& \frac{1}{m - \not{p}} + \frac{1}{m - \not{p}} \Sigma(p) \frac{1}{m - \not{p}} + \cdots \\
=& \frac{1}{m - \not{p} - \Sigma(p)}.
\end{aligned}
\tag{5.6}
$$

It means

$$
\Sigma(p = m) = 0. \tag{5.7}
$$

In the finite volume, we still use m as parameter of Lagrangian and we have

$$
m - m_L - \Sigma_L(p = m_L) = 0 \tag{5.8}
$$

yielding to

$$
\begin{aligned}
\Delta m \equiv m_L - m =& - \Sigma_L(p = m_L) \\
=& \Sigma(p = m) - \Sigma_L(p = m_L) \\
=& \Sigma(p = m) - \Sigma_L(p = m) + \text{high-order}.
\end{aligned}
\tag{5.9}
$$

Now let us calculate mass shift.

$$
\begin{aligned}
\Sigma(p = m) - \Sigma_L(p = m) =& \left(\frac{3g_A^2}{4F^2}\right) \sum_{\mathbf{n}\neq 0} \int \frac{d^4q}{(2\pi)^4 i} e^{i\mathbf{n}\mathbf{q}L} (X_2 + X_3) \\
\equiv& I_2 + I_3
\end{aligned}
\tag{5.10}
$$

where

$$
\begin{aligned}
I_2 =& \left(\frac{3g_A^2}{4F^2}\right) \sum_{\mathbf{n}\neq 0} \int \frac{d^4q}{(2\pi)^4 i} e^{i\mathbf{n}\mathbf{q}L} \frac{2m}{(p-q)^2 - m^2} \\
=& \left(\frac{3mg_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int \frac{d^4k}{(2\pi)^4 i} e^{i\mathbf{n}(\mathbf{k}+\not{\mathbf{p}})L} \frac{1}{k^2 - m^2} \\
=& \left(-\frac{3mg_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int \frac{d^4k_E}{(2\pi)^4} e^{in\cdot k_E L} \frac{1}{k_E^2 + m^2}, \quad (k_E = (0, \mathbf{k})) \\
=& \left(-\frac{3mg_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int \frac{d^4k_E}{(2\pi)^4} e^{in\cdot k_E L} \int_0^\infty ds\, e^{-(k_E^2 + m^2)s} \\
=& \left(-\frac{3mg_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int_0^\infty ds\, e^{-m^2 s - \frac{n^2 L^2}{4s}} \int \frac{d^4k_E}{(2\pi)^4} e^{-(k_E - \frac{inL}{2s})^2 s} \\
=& \left(-\frac{3mg_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int_0^\infty ds\, e^{-m^2 s - \frac{n^2 L^2}{4s}} \int \frac{2\pi^2 k^3 dk}{(2\pi)^4} e^{-k^2 s}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\frac{3mg_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int_0^\infty ds\, e^{-m^2 s - \frac{n^2L^2}{4s}} \left(\frac{1}{16\pi^2 s^2}\right) \\
&= \left(-\frac{3m^2 g_A^2}{8\pi^2 F^2}\right) \frac{1}{L} \sum_{\mathbf{n}\neq 0} \frac{1}{n} K_1(nmL) \\
&\sim \left(-\frac{3m^{3/2} g_A^2}{8\sqrt{2}\pi^{3/2} F^2}\right) \frac{1}{L^{3/2}} \sum_{\mathbf{n}\neq 0} \frac{1}{n^{3/2}} e^{-nmL}
\end{aligned} \tag{5.11}
$$

and

$$
\begin{aligned}
I_3 &= \left(\frac{3g_A^2}{4F^2}\right) \sum_{\mathbf{n}\neq 0} \int \frac{d^4q}{(2\pi)^4 i} e^{i\mathbf{nq}L} \frac{2mM_\pi^2}{[(p-q)^2 - m^2]\,[q^2 - M_\pi^2]} \\
&= \left(\frac{3mM_\pi^2 g_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int \frac{d^4q}{(2\pi)^4 i} e^{i\mathbf{nq}L} \int dx \frac{1}{\left[(q-xp)^2 - \Delta\right]^2}, \quad \Delta = x^2 m^2 + (1-x) M_\pi^2 \\
&= \left(\frac{3mM_\pi^2 g_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int dx \int \frac{d^4 l_E}{(2\pi)^4} e^{i\mathbf{n}(\mathbf{l}+\cancel{x\mathbf{p}})L} \frac{1}{\left[l_E^2 + \Delta\right]^2} \\
&= \left(\frac{3mM_\pi^2 g_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int dx \int_0^\infty ds\, s e^{-sx^2m^2 - s(1-x)M_\pi^2} \left(\int \frac{dl_0}{(2\pi)} e^{-sl_0^2}\right) \left(\int \frac{d^3 l}{(2\pi)^3} e^{i\mathbf{nl}L} e^{-sl^2}\right) \\
&= \left(\frac{3mM_\pi^2 g_A^2}{2F^2}\right) \sum_{\mathbf{n}\neq 0} \int dx \int_0^\infty ds\, s e^{-sx^2m^2 - s(1-x)M_\pi^2 - \frac{n^2L^2}{4s}} \left(\frac{1}{2\sqrt{\pi s}}\right) \left(\frac{1}{8\pi^{3/2} s^{3/2}}\right) \\
&= \left(\frac{3mM_\pi^2 g_A^2}{32\pi^2 F^2}\right) \sum_{\mathbf{n}\neq 0} \int dx \int_0^\infty \frac{ds}{s}\, e^{-sx^2m^2 - s(1-x)M_\pi^2 - \frac{n^2L^2}{4s}} \\
&= \left(\frac{3mM_\pi^2 g_A^2}{16\pi^2 F^2}\right) \sum_{\mathbf{n}\neq 0} \int dx\, K_0\!\left(nL\sqrt{x^2m^2 + (1-x)M_\pi^2}\right) \\
&\sim \left(\frac{3mM_\pi^2 g_A^2}{16\sqrt{2}\pi^{3/2} F^2}\right) \frac{1}{\sqrt{mL}} \sum_{\mathbf{n}\neq 0} \frac{1}{\sqrt{n}} \int dx\, \frac{1}{\sqrt[4]{x^2 + (1-x)M_\pi^2/m^2}} e^{-nmL\sqrt{x^2+(1-x)M_\pi^2/m^2}}.
\end{aligned} \tag{5.12}
$$

$I_2 \sim e^{-mL}/(mL)^{3/2}$ and $I_3 \sim e^{-\gamma mL}/(mL)^{\delta}$ [d].

[a] X_1 is odd in q thus the corresponding integral or sum vanishes.

[b] Since p is on-shell, X_2 and X_3 are both regular after integrating out q^0. Poisson formula can be applied therefore.

[c] We use physical mass in the Lagrangian.

[d] I roughly get $\gamma \simeq 1/4$ and $\delta \simeq 1/2$.

6 Calculation of sums

Calculate $\sum_{\mathbf{n}\neq 0} \frac{1}{n^2}$. This sum diverges and needs regularization. Use dimensional regularization and arrive to the result, shown in the text.

a) Regularization in the infrared part.

Solution

We introduce infrared cut-off μ.

$$
\begin{aligned}
I =& \sum_{\mathbf{n}\neq 0} \frac{1}{n^2} \\
=& \sum_{\mathbf{n}\neq 0} \left(\frac{1}{n^2} - \frac{1}{n^2+\mu^2} + \frac{1}{n^2+\mu^2} \right) \\
=& \mu^2 \sum_{\mathbf{n}\neq 0} \frac{1}{n^2(n^2+\mu^2)} + \sum_{\mathbf{n}\neq 0} \frac{1}{n^2+\mu^2} \\
\equiv& J_1 + I_1 .
\end{aligned} \tag{6.1}
$$

Here

$$
J_1 = \mu^2 \sum_{\mathbf{n}\neq 0} \frac{1}{n^2(n^2+\mu^2)} \tag{6.2}
$$

is convergent and calculated numerically. On the other hand,

$$
\begin{aligned}
I_1 =& \sum_{\mathbf{n}\neq 0} \frac{1}{n^2+\mu^2} \\
=& \sum_{\mathbf{n}} \frac{1}{n^2+\mu^2} - \frac{1}{\mu^2} \\
\equiv& I_2 - \frac{1}{\mu^2} .
\end{aligned} \tag{6.3}
$$

I_2 is well-defined in the infrared part thus Poisson formular can be applied.

b) Regularization in the ultra-violet part.

Solution

We use Poisson formular.

$$
\begin{aligned}
I_2 =& \sum_{\mathbf{n}} \frac{1}{n^2+\mu^2} \\
=& \int d^d x \sum_{\mathbf{n}} \delta^{(d)}(x-\mathbf{n}) \frac{1}{x^2+\mu^2} \\
=& \int d^d x \sum_{\mathbf{n}} e^{i2\pi \mathbf{n}\mathbf{x}} \frac{1}{x^2+\mu^2} \\
=& \int d^d x \frac{1}{x^2+\mu^2} + \sum_{\mathbf{n}\neq 0} \int d^d x e^{i2\pi \mathbf{n}\mathbf{x}} \frac{1}{x^2+\mu^2}
\end{aligned}
$$

$$=L_0 + \sum_{\mathbf{n}\neq 0} L_{\mathbf{n}}. \tag{6.4}$$

There is ultra-violet divergence in L_0 and we use dim.-reg. [a] .

$$\begin{aligned} L_0 =& \int d^d x \frac{1}{x^2+\mu^2} \\ =& \pi^{d/2}\Gamma(1-\frac{d}{2})(\frac{1}{\mu^2})^{1-\frac{d}{2}} \\ =& -2\pi^2\mu. \end{aligned} \tag{6.5}$$

The $L_{\mathbf{n}}$ is regular thus d can be recovered to 3 naturally.

$$\begin{aligned} L_{\mathbf{n}} =& \int d^3 x e^{i2\pi \mathbf{n}\mathbf{x}} \frac{1}{x^2+\mu^2} \\ =& 2\pi \int_0^\infty x^2 dx \int_{-1}^{1} dz e^{i2\pi n x z} \frac{1}{x^2+\mu^2} \\ =& 2\pi \int_0^\infty x^2 dx \frac{e^{i2\pi n x} - e^{-i2\pi n x}}{2\pi n x} \frac{1}{x^2+\mu^2} \\ =& \frac{1}{n}\int_{-\infty}^{\infty} x dx \frac{e^{i2\pi n x}}{x^2+\mu^2} \\ =& \frac{\pi}{n}\frac{e^{-2\pi n \mu}}{\mu}. \end{aligned} \tag{6.6}$$

[a] A latent couter-term has been introduced in dim.-reg. of 3-d integral.

7 Field Theory in $1+1$ dimensions

Consider NREFT with a single scalar field in $1+1$ dimensions.

a) In the infinite volume, derive the Lippmann-Schwinger equation and unitarity relation. Write down the representation of the scattering amplitude in the CM frame in terms of the scattering phase shift.

Solution

We can firstly calcualte phase space factor in the CM frame [a].

$$\begin{aligned} \rho(E) =& \frac{1}{2}\int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}(2\pi)^D \delta^{(D)}(P-q_1-q_2) \\ =& \frac{1}{2}\int dq_1 \delta(E-E_1-E_2) \\ =& \frac{1}{2}\int dq \delta(E-2m-\frac{q^2}{2m}-\frac{q^2}{2m}) \end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}\int dq \frac{\delta(q-\sqrt{m(E-2m)})}{\frac{2q}{m}}\\
=&\frac{m}{4p_*}
\end{aligned}
\tag{7.1}
$$

where we call $p_* = \sqrt{m(E-2m)}$.
We have LS equation,

$$
T(p,q) = V(p,q) + \int \frac{d^D k}{(2\pi)^D i} V(p,k) G(k) T(k,q). \tag{7.2}
$$

We can further simplify it as [b],

$$
T(E) = V(E) + V(E) G_0(E) T(E) \tag{7.3}
$$

where [c]

$$
T(E) = T(p_*, p_*), \quad V(E) = V(p_*, p_*) \tag{7.4}
$$

and

$$
\begin{aligned}
G_0 =& \int \frac{d^D k}{(2\pi)^D i} G(k)\\
=&\frac{1}{2}\int \frac{d^D k}{(2\pi)^D i} \frac{1}{\left[m+\frac{\mathbf{k}^2}{2m}-k_0-i\epsilon\right]\left[m+\frac{(-\mathbf{k})^2}{2m}-E+k_0-i\epsilon\right]}\\
=&\frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \frac{1}{2m-E+\frac{k^2}{m}-i\epsilon}\\
=&\frac{m}{2}\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2-p_*^2-i\epsilon}\\
=&\frac{m}{2(4\pi)^{d/2}}\Gamma(1-\frac{d}{2})\left(\frac{1}{-p_*^2-i\epsilon}\right)^{1-\frac{d}{2}} = \frac{im}{4p_*}.
\end{aligned}
\tag{7.5}
$$

To simplify notation, $p_* \to p$. Now we can solve out T-matrix,

$$
T(E) = \frac{V(E)}{1-\frac{im}{4p}V(E)}. \tag{7.6}
$$

We can check the unitarity relation.

$$
\Im T = \frac{V}{1-\left(\frac{m}{4p}V\right)^2}\frac{m}{4p}V, \quad |T|^2 = \frac{V^2}{1-\left(\frac{m}{4p}V\right)^2} \tag{7.7}
$$

so we have

$$\Im T = \frac{m}{4p}|T|^2. \tag{7.8}$$

We write down T-matrix as

$$T = \frac{C}{\cot\delta - i} \tag{7.9}$$

yielding to

$$\Im T = \frac{C}{1+\cot^2\delta}, \quad |T|^2 = \frac{C^2}{1+\cot^2\delta}. \tag{7.10}$$

Therefore,

$$C = \frac{4p}{m}. \tag{7.11}$$

Also, we have match equation,

$$\frac{4p}{m}\frac{1}{\cot\delta - i} = \frac{4p}{m}\frac{1}{p\cot\delta - ip} = T = \frac{V}{1 - i\frac{m}{4p}V} = \frac{4p^2}{m}\frac{1}{\frac{4p^2}{m}V^{-1} - ip}. \tag{7.12}$$

It means [d]

$$p\cot\delta = \frac{4p^2}{m}V^{-1} = p^2[a_0 + a_1 p^2 + \cdots]. \tag{7.13}$$

---

[a] We have symmetry factor 1/2 in phase space factor because of identical particle.

[b] Call $G = N/D$, we can factorize V, T out of the integral,

$$\begin{aligned}
\int \frac{d^D k}{(2\pi)^D i} V(p,k)G(k)T(k,q) &= \int \frac{d^D k}{(2\pi)^D i}\frac{V(p,k)}{D(k)}N(k)T(k,q) \\
&= \int \frac{d^D k}{(2\pi)^D i}\frac{[V(p,k) - V(p,p_*) + V(p,p_*)]\,[T(k,q) - T(p_*,q) + T(p_*,q)]}{D(k)}N(k) \\
&\to V(p,p_*)\left[\int \frac{d^D k}{(2\pi)^D i}\frac{N(k)}{D(k)}\right]T(p_*,q) \\
&= V(p,p_*)\left[\int \frac{d^D k}{(2\pi)^D i}G(k)\right]T(p_*,q).
\end{aligned}$$

Here we have used the fact that regular function disappears in dim.-reg. integral. And functions

$$\frac{[V(p,k) - V(p,p_*)]}{D(k)}, \quad \frac{[T(k,q) - T(p_*,q)]}{D(k)}$$

are both regular.

[c] Notice the symmetry factor 1/2 because of identical particle.

[d] We have potential $V(p,q) = c_0 + c_1(p^2+q^2) + \cdots$ and on-shell potential $V(E) = c_0 + c_1' p_*^2 + \cdots$.

b) Write down the expression of the scattering amplitude in a finite volume and derive the Lüscher equation.

Solution

We have LS equation in a finite volume [a],

$$T_L = V + V G_0^L T_L \tag{7.14}$$

where

$$\begin{aligned} G_0^L =& \frac{1}{2}\int \frac{dk^0}{(2\pi)i}\frac{1}{L^d}\sum_{\mathbf{k}} \frac{1}{\left[m+\frac{\mathbf{k}^2}{2m}-k_0-i\epsilon\right]\left[m+\frac{(-\mathbf{k})^2}{2m}-E+k_0-i\epsilon\right]} \\ =& \frac{1}{2}\frac{1}{L^d}\sum_{\mathbf{k}}\frac{1}{\left[2m-E+\frac{k^2}{m}-i\epsilon\right]} \\ =& \frac{m}{2}\frac{1}{L}\sum_{k=-\infty}^{\infty}\frac{1}{[k^2-p_*^2]}. \end{aligned} \tag{7.15}$$

We can solve out T_L,

$$T_L = \frac{1}{V^{-1}-G_0^L}. \tag{7.16}$$

Lüsche equation is quantization condition which determines the poles of T-matrix.

$$V^{-1} = G_0^L. \tag{7.17}$$

From the match equation in the infinite volume, we have seen that

$$V^{-1} = \frac{m}{4p}\cot\delta. \tag{7.18}$$

Now we write down that

$$\frac{m}{4p}\cot\delta = \frac{m}{2}\frac{1}{L}\sum_{k=-\infty}^{\infty}\frac{1}{[k^2-p^2]} \tag{7.19}$$

that is

$$\cot\delta = 2p\mathcal{Z}(p,L). \tag{7.20}$$

The definition of Lüsche zeta funtion in $1+1$ dimensions is

$$\mathcal{Z}(p,L) = \frac{1}{L}\sum_{k=-\infty}^{\infty}\frac{1}{[k^2-p^2]}. \tag{7.21}$$

---

[a] We can factorize V, T out as well,

$$\int \frac{dk^0}{(2\pi)i}\frac{1}{L^d}\sum_{\mathbf{k}} V(p,k)G(k)T(k,q)$$

$$
\begin{aligned}
&= \int \frac{dk^0}{(2\pi)i} \frac{1}{L^d} \sum_{\mathbf{k}} \frac{V(p,k)}{D(k)} N(k) T(k,q) \\
&= \int \frac{dk^0}{(2\pi)i} \frac{1}{L^d} \sum_{\mathbf{k}} \frac{[V(p,k) - V(p,p_*) + V(p,p_*)]\,[T(k,q) - T(p_*,q) + T(p_*,q)]}{D(k)} N(k) \\
&\to V(p,p_*) \left[\int \frac{dk^0}{(2\pi)i} \frac{1}{L^d} \sum_{\mathbf{k}} \frac{N(k)}{D(k)} \right] T(p_*,q) \\
&= V(p,p_*) \left[\int \frac{dk^0}{(2\pi)i} \frac{1}{L^d} \sum_{\mathbf{k}} G(k) \right] T(p_*,q).
\end{aligned}
$$

Now we state that regular function disappears in sum.

$$
\int \frac{dk^0}{(2\pi)i} \frac{1}{L^d} \sum_{\mathbf{k}} f(k) = \underbrace{\int \frac{d^D k}{(2\pi)^D i} \cancel{f(k)}}_{\text{dim.-reg.}} + \underbrace{\left[\int \frac{dk^0}{(2\pi)i} \frac{1}{L^d} \sum_{\mathbf{k}} - \int \frac{d^D k}{(2\pi)^D i} \right] \cancel{f(k)}.}_{\text{exponentially suppressed}}
$$

c) Evaluate the Lüsche zeta function in $1+1$ dimensions explicitly and prove that the the Lüsche equation reproduces the result already known n 1-dimensional quantum mechanics.

Solution

The analytic continuation of Lüsche zeta function in $1+1$ dimensions is well-defined,

$$
\begin{aligned}
\mathcal{Z}(i\kappa, L) =& \frac{1}{L} \sum_{k=-\infty}^{\infty} \frac{1}{[k^2+\kappa^2]} \\
=& \int_{-\infty}^{\infty} dk \frac{1}{L} \sum_{n=-\infty}^{\infty} \delta(\frac{2\pi n}{L} - k) \frac{1}{k^2+\kappa^2} \\
=& \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{inkL}}{k^2+\kappa^2} \\
=& \sum_{n>0} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{inkL}}{k^2+\kappa^2} + \sum_{n<0} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{inkL}}{k^2+\kappa^2} + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{k^2+\kappa^2} \\
=& \sum_{n>0} \frac{ie^{-n\kappa L}}{2i\kappa} + \sum_{n>0} (-i) \frac{e^{-n\kappa L}}{2(-i\kappa)} + \frac{1}{2\kappa} \\
=& \frac{1}{\kappa} (\frac{1}{2} + \sum_{n>0} e^{-n\kappa L}) \\
=& \frac{1}{\kappa} (-\frac{1}{2} + \frac{1}{1-e^{-\kappa L}}) \\
=& \frac{1}{2\kappa} \frac{1+e^{-\kappa L}}{1-e^{-\kappa L}} = \frac{1}{2\kappa} \coth\left(\frac{\kappa L}{2}\right).
\end{aligned}
\tag{7.22}
$$

Continuate $\mathcal{Z}(i\kappa, L)$ to complex plane, i.e., $\kappa = -ip$, we have

$$\mathcal{Z}(p, L) = -\frac{1}{2p}\cot\frac{pL}{2}. \tag{7.23}$$

Now we solve Lüsche equation,

$$\cot\delta = 2p\mathcal{Z}(p, L) = -\cot\frac{pL}{2} \tag{7.24}$$

yielding to

$$\delta + \frac{pL}{2} = n\pi \tag{7.25}$$

then

$$p = \frac{1}{L}(2n\pi - \delta). \tag{7.26}$$

8 Twisted boundary conditions

Suppose, a shallow bound state is made of a particle and an antiparticle. In the NREFT, these are described by separate fields ϕ and ϕ_a, respectively.
In the lectures, we considered the derivation of the finite-volume shift of the bound-state energy on the torus, if the fields are subject to the periodic boundary conditions:

$$\phi(\mathbf{x} + \mathbf{e}_i L, t) = \phi(\mathbf{x}, t), \quad \phi_a(\mathbf{x} + \mathbf{e}_i L, t) = \phi_a(\mathbf{x}, t)$$

where $\mathbf{e}_i$, $i = 1, 2, 3$ are unit vectors in the direction of spatial axis. However, sometimes, it is useful to impose more general boundary conditions:

$$\phi(\mathbf{x} + \mathbf{e}_i L, t) = e^{i\theta_i}\phi(\mathbf{x}, t), \quad \phi_a(\mathbf{x} + \mathbf{e}_i L, t) = e^{-i\theta_i}\phi_a(\mathbf{x}, t).$$

Here, $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^{\mathrm{T}}$ is the twisting angle, with $0 \le \theta_i < 2\pi$.

a) How does the Fourier-transform of a field look like? How do you interpret the vector $\boldsymbol{\theta}$?

Solution

We have

$$\phi(\mathbf{x},t) = \frac{1}{L^3}\sum_{\mathbf{k}} \tilde{\phi}(\mathbf{k},t)e^{i\mathbf{kx}}. \tag{8.1}$$

The twisting boundary condition asks for

$$\frac{1}{L^3}\sum_{\mathbf{k}} \tilde{\phi}(\mathbf{k},t)e^{i\mathbf{k}(\mathbf{x}+\mathbf{e}_i L)} = \frac{1}{L^3}\sum_{\mathbf{k}} \tilde{\phi}(\mathbf{k},t)e^{i\mathbf{kx}+i\theta_i} \tag{8.2}$$

yielding to

$$k_i L = 2\pi n_i + \theta_i \tag{8.3}$$

or say

$$\mathbf{k} = \frac{2\pi\mathbf{n} + \boldsymbol{\theta}_i}{L}. \tag{8.4}$$

Since $0 \leq \theta_i < 2\pi$, the twisting angle is the derivation of momentum from integer. By the same token, for the antiparticle,

$$\mathbf{k} = \frac{2\pi\mathbf{n} - \boldsymbol{\theta}_i}{L}. \tag{8.5}$$

b) Derive the energy shift of the two-body state in case of a non-zero $\boldsymbol{\theta}$.

Solution

We have LS in the infinite volume and finite volume,

$$\begin{aligned} T =& V + VG_0T, \\ T_L =& V + VG_0^L T_L \end{aligned} \tag{8.6}$$

where

$$\begin{aligned} G_0 =& \int \frac{d^4q}{(2\pi)^4 i}\frac{1}{m+\frac{q^2}{2m}-q_0-i\epsilon}\frac{1}{m+\frac{q^2}{2m}-E+q_0-i\epsilon} \\ =& m\int \frac{d^3q}{(2\pi)^3}\frac{1}{q^2-p^2-i\epsilon}, \quad p^2 = m(E-2m) \\ =& \frac{m}{4\pi}ip \end{aligned} \tag{8.7}$$

and

$$
\begin{aligned}
G_0^L =& \int \frac{dq^0}{(2\pi)i} \frac{1}{L^3} \sum_{\mathbf{q}=\frac{2\pi\mathbf{n}+\boldsymbol{\theta}}{L}} \frac{1}{m+\frac{q^2}{2m}-q_0-i\epsilon} \frac{1}{m+\frac{q^2}{2m}-E+q_0-i\epsilon} \\
=& \frac{m}{L^3} \sum_{\mathbf{q}=\frac{2\pi\mathbf{n}+\boldsymbol{\theta}}{L}} \frac{1}{q^2-p^2} \\
=& \frac{m}{4\pi} \mathcal{Z}_{\boldsymbol{\theta}}(p,L).
\end{aligned} \tag{8.8}
$$

We define twisting Lüsche zeta function,

$$
\mathcal{Z}_{\boldsymbol{\theta}}(p,L) = \frac{4\pi}{L^3} \sum_{\mathbf{q}=\frac{2\pi\mathbf{n}+\boldsymbol{\theta}}{L}} \frac{1}{q^2-p^2}. \tag{8.9}
$$

The infinite volume T-matrix reads,

$$
T = \frac{4\pi}{m} \frac{1}{\frac{4\pi}{m}V^{-1}-ip} = \frac{4\pi}{m} \frac{1}{p\cot\delta - ip} \tag{8.10}
$$

while the finite volume T is

$$
T_L = \frac{4\pi}{m} \frac{1}{\frac{4\pi}{m}V^{-1}-\mathcal{Z}_{\boldsymbol{\theta}}(p,L)}. \tag{8.11}
$$

The quantization condition or say Lüsche equation is therefore,

$$
p\cot\delta = \mathcal{Z}_{\boldsymbol{\theta}}(p,L). \tag{8.12}
$$

For the shallow bound state, $p \to i\kappa$. In the infinite volume,

$$
p\cot\delta\big|_{p=i\kappa_B} = -\kappa_B \tag{8.13}
$$

and finite volume,

$$
\begin{aligned}
\mathcal{Z}_{\boldsymbol{\theta}}(i\kappa,L) =& \frac{4\pi}{L^3} \sum_{\mathbf{q}=\frac{2\pi\mathbf{n}+\boldsymbol{\theta}}{L}} \frac{1}{q^2+\kappa^2} \\
=& \frac{1}{\pi L} \sum_{\mathbf{n}} \frac{1}{(\mathbf{n}+\frac{\boldsymbol{\theta}}{2\pi})^2+\nu^2} \\
=& 4\pi \sum_{\mathbf{n}} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{nk}L} \frac{1}{(\mathbf{k}+\frac{\boldsymbol{\theta}}{L})^2+\kappa^2} \\
=& 4\pi \sum_{\mathbf{n}} e^{-i\mathbf{n}\boldsymbol{\theta}} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{nk}L} \frac{1}{k^2+\kappa^2} \\
=& 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+\kappa^2} + 4\pi \sum_{\mathbf{n}\neq\mathbf{0}} e^{-i\mathbf{n}\boldsymbol{\theta}} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{nk}L} \frac{1}{k^2+\kappa^2}.
\end{aligned}
$$

$$= -\kappa + \frac{2}{L}\sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{n} e^{-i\mathbf{n}\boldsymbol{\theta} - n\kappa L}. \tag{8.14}$$

We consider that

$$p\cot\delta|_{p=i\kappa} \simeq p\cot\delta|_{p=i\kappa_B} = -\kappa_B,$$
$$\frac{2}{L}\sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{n} e^{-i\mathbf{n}\boldsymbol{\theta} - n\kappa L} \simeq \frac{2}{L}\sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{n} e^{-i\mathbf{n}\boldsymbol{\theta} - n\kappa_B L}, \tag{8.15}$$

then quantization equation

$$p\cot\delta|_{p=i\kappa} = \mathcal{Z}_{\boldsymbol{\theta}}(i\kappa, L) \tag{8.16}$$

gives that

$$\kappa - \kappa_B = \frac{2}{L}\sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{n} e^{-i\mathbf{n}\boldsymbol{\theta} - n\kappa_B L}. \tag{8.17}$$

Taking $\boldsymbol{\theta} = (0, 0, \theta)$ and $|\mathbf{n}| = 1$, we have

$$\kappa - \kappa_B = \frac{4}{L}(2 + \cos\theta) e^{-\kappa_B L}. \tag{8.18}$$