Solutions to the Exercises – Dispersion Relations

Yu-Fei Wang

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[Problem 1] The Omnès function:

$$\Omega(s) = \exp\Big\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\Big\}.$$

(i) Show $\mathrm{Arg}\Omega(s) = \delta(s)$.

(ii) Assume $\delta(s > s_0) = c\pi = \text{const}$; show $\Omega(s \to \infty) \sim s^{-c}$.

(iii) Assume the phase shift of an infinitely narrow resonance, $\delta(s) = \pi\Theta(s - 4M^2)$. What is the resulting Omnès function?

[Solution 1] Using $1/(s'(s'-s)) = (1/(s'-s) - 1/s')/s$, it is convenient to rewrite the Omnès function as

$$\Omega(s) = \exp\Big\{\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \delta(s')\Big(\frac{1}{s'-s} - \frac{1}{s'}\Big)\Big\}. \tag{1}$$

(i) From Eq. (1), $\mathrm{Arg}\Omega(s) = \mathrm{Im}[\frac{1}{\pi}\int \cdots]$. Remember that we are discussing this problem in the physical region, i.e. $s > 4M_\pi^2$, consequently the singularity in the integrand $1/(s'-s)(s'=s)$ lies in the integral interval $[4M_\pi^2, \infty)$, causing the imaginary part. Specifically, one should regard that term as

$$\frac{1}{s'-s} \equiv \frac{1}{s'-s-i\epsilon} = \mathcal{P}\frac{1}{s'-s} + i\pi\delta_D(s'-s)$$

where $\epsilon \to 0^+$, δ_D being the Dirac's delta function and $\mathcal{P}$ standing for the Cauchy's principle value. Eventually

$$\mathrm{Arg}\Omega(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds'\delta(s')\Big[\pi\delta_D(s'-s)\Big] = \delta(s).$$

(ii) Under this assumption,

$$\Omega(s) = \exp\Big\{\frac{1}{\pi}\int_{4M_\pi^2}^{s_0} ds'\delta(s')\Big(\frac{1}{s'-s} - \frac{1}{s'}\Big) + \frac{1}{\pi}\int_{s_0}^{\infty} ds'(c\pi)\Big(\frac{1}{s'-s} - \frac{1}{s'}\Big)\Big\} \equiv \exp(I_1 + I_2). \tag{2}$$

When $s \to \infty$, it is easy to find that

$$\exp(I_1) \to \exp(0) = 1.$$

And for I_2

$$I_2 = \frac{1}{\pi}\int_{s_0}^{\infty} ds'(c\pi)\Big(\frac{1}{s'-s} - \frac{1}{s'}\Big) = c\ln\left(\frac{s'-s}{s'}\right)\Big|_{s_0}^{\infty} = c\ln\left(\frac{s_0}{s_0-s}\right).$$

Thus

$$\exp(I_2) = \left(\frac{s_0}{s_0-s}\right)^c,$$

so $\Omega(s \to \infty) \sim s^{-c}$.

(iii) Under this assumption,

$$\Omega(s) = \exp\Big\{\frac{1}{\pi}\int_{M^2}^{\infty} ds'\pi\Big(\frac{1}{s'-s}-\frac{1}{s'}\Big)\Big\}. \tag{3}$$

Similar to the I_2 term in Eq. (2), the integral can be easily figured out

$$\Omega(s) = \exp\Big\{\ln\Big(\frac{M^2}{M^2-s}\Big)\Big\} = \frac{M^2}{M^2-s}.$$

It is found that in the case of infinitely narrow resonance, the Omnès function is proportional to the propagator of the resonance i.e. $(s-M^2)^{-1}$; the numerator M^2 serves as a normalization constant making $\Omega(0) = 1$.

[Problem 2] For the pion charge radius r_π^V, we have

$$\langle (r_\pi^V)^2\rangle = 6\frac{dF_\pi^V(s)}{ds}|_{s=0}.$$

From Omnès representation $F_\pi^V(s) = \Omega(s)$. Deduce the sum rule

$$\langle (r_\pi^V)^2\rangle = \frac{6}{\pi}\int_{4M_\pi^2}^{\infty} dz\frac{\delta(z)}{z^2}.$$

[Solution 2] Taking an one-order derivative on the Omnès function in Eq. (1) one gets

$$\frac{d\Omega(s)}{ds} = \Omega(s)\times\Big\{\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dz\frac{\delta(z)}{(z-s)^2}\Big\}.$$

Then set $s = 0$, note that the normalization condition $\Omega(0) = 1$:

$$\frac{d\Omega(0)}{ds} = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dz\frac{\delta(z)}{z^2},$$

hence $\langle (r_\pi^V)^2\rangle = 6\frac{dF_\pi^V(s)}{ds}|_{s=0} = \frac{6}{\pi}\int_{4M_\pi^2}^{\infty} dz\frac{\delta(z)}{z^2}$.

[Problem 3] For the tensor form factor of the pion

$$\langle \pi^+(p')\pi^-(p)|\bar{q}\sigma_{\mu\nu}q|0\rangle \equiv \frac{i}{M_\pi}(p_\mu p'_\nu - p_\nu p'_\mu)B_T^{\pi,q}(s),$$

show

$$\text{Im}[B_T^{\pi,q}(s)] = \sigma t_1^*(s)B_T^{\pi,q}(s).$$

That is to say, the tensor form factor obeys exactly the same discontinuity condition as the vector form factor.

[Solution 3] Fig. 1 shows the discontinuity structure of $B_T^{\pi,q}(s)$. Similar to the vector form factor shown in the lecture, the discontinuity condition can be written as:

$$\frac{i}{M_\pi}P_{\mu\nu}\text{disc}\,[B_T^{\pi,q}(s)] = \frac{i\sigma}{32\pi^2}\int d\Omega_l T_{\pi\pi}^*(s,z_l)\Big[\frac{i}{M_\pi}Q_{\mu\nu}B_T^{\pi,q}(s)\Big], \tag{4}$$

where $P_{\mu\nu} \equiv p_\mu p'_\nu - p_\nu p'_\mu$, $Q_{\mu\nu} \equiv q_\mu l_\nu - q_\nu l_\mu$ with $q \equiv p + p' - l$. Note that the pre-factor $\frac{i\sigma}{32\pi^2}$ ($\sigma \equiv [(s-4M_\pi^2)/s]^{1/2}$) in Eq. (4) stems from applying Cutkosky rule and integrating out the modulus of the loop momentum l; since there is a dependence of the angles in the scattering amplitude $T_{\pi\pi}$

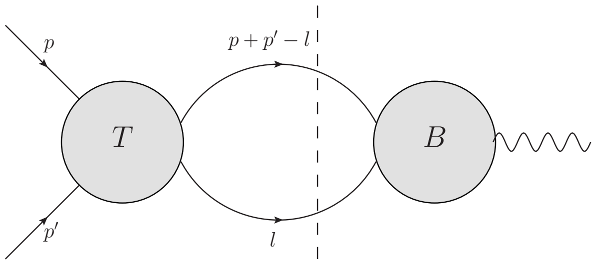

Figure 1: Schematic diagram of the discontinuity structure in the tensor form factor. The vertical dash line stands for the application of Cutkosky rule.

($z_l = \cos\theta$ with θ being the scattering angle), the integral on solid angle Ω_l remains. Due to the anti-symmetric Lorentz structure,

$$\int d\Omega_l T^*_{\pi\pi} Q_{\mu\nu} \propto P_{\mu\nu}$$

must holds. The calculation can be carried out in the centre-of-mass frame, whereas under Cutkosky rule all the momentums in Fig. 1 are on-shell. Specifically,

$$\begin{aligned} p^\mu &= \left(\frac{\sqrt{s}}{2}, 0, 0, k\right), \; p'^\mu = \left(\frac{\sqrt{s}}{2}, 0, 0, -k\right), \\ q^\mu &= \left(\frac{\sqrt{s}}{2}, k\sin\theta, 0, k\cos\theta\right), \; l^\mu = \left(\frac{\sqrt{s}}{2}, -k\sin\theta, 0, -k\cos\theta\right), \end{aligned}$$

with $s \equiv (p+p')^2$ and $k = \sqrt{s}\sigma/2$. The scalar products can be found:

$$\begin{aligned} p^2 = p'^2 = l^2 = q^2 = M_\pi^2, \; p\cdot p' = q\cdot l = \frac{1}{2}(s - 2M_\pi^2), \\ p\cdot l = p'\cdot q = \frac{s}{4}(1+\sigma^2 z_l), \; p'\cdot l = p\cdot q = \frac{s}{4}(1-\sigma^2 z_l), \end{aligned}$$

yielding

$$P_{\mu\nu}P^{\mu\nu} = -\frac{s}{2}(s-4M_\pi^2), \; Q_{\mu\nu}P^{\mu\nu} = -\frac{s}{2}(s-4M_\pi^2)z_l.$$

Multiplied by $P^{\mu\nu}$, after some simplifications, Eq. (4) becomes

$$\mathrm{disc}\,[B_T^{\pi,q}(s)] = \frac{i\sigma}{32\pi^2}\int d\Omega_l z_l T^*_{\pi\pi}(s,z_l) B_T^{\pi,q}(s).$$

Using $\mathrm{disc}[\cdots] = 2i\mathrm{Im}[\cdots]$, and $\int d\Omega_l = 2\pi\int_{-1}^1 dz_l$ since the integrand depends only on z_l, one obtains

$$\mathrm{Im}\,[B_T^{\pi,q}(s)] = \frac{\sigma}{32\pi}\int_{-1}^1 dz_l z_l T^*_{\pi\pi}(s,z_l) B_T^{\pi,q}(s).$$

The integral $\frac{1}{32\pi}\int_{-1}^1 dz_l z_l T^*_{\pi\pi}(s,z_l)$ is nothing but the P-wave projection of the scattering amplitude, namely $t_1^*(s)$. Finally

$$\mathrm{Im}[B_T^{\pi,q}(s)] = \sigma t_1^*(s) B_T^{\pi,q}(s).$$

[Problem 4] Check that you arrive at the same result for $t_{\rm II}$, starting from

$$S_{\rm II} = 1 - 2i\sigma t_{\rm II}$$

and using $S_{\rm II} = \frac{1}{S_{\rm I}}$.

[Solution 4] Seen that

$$S_{\mathrm{II}} = \frac{1}{S_{\mathrm{I}}} = 1 - 2i\sigma t_{\mathrm{II}},$$

that is to say

$$t_{\mathrm{II}} = \frac{i}{2S_{\mathrm{I}}} - \frac{i}{2} = \frac{i(1 - S_{\mathrm{I}})}{2S_{\mathrm{I}}}.$$

By definition $S_{\mathrm{I}} = 1 + 2i\sigma t_{\mathrm{I}}$, thus

$$t_{\mathrm{II}} = \frac{t_{\mathrm{I}}}{S_{\mathrm{I}}},$$

which is the same result as in the lecture note.