phenomenon is known under the name of *anomalies*. Certain Ward identities get modified by the anomalies, and the corresponding currents are no longer conserved. In this case, the effective Lagrangian should also contain terms that reproduce the correct anomaly in the underlying theory. An example is given by the Wess-Zumino-Witten (WZW) term in the effective chiral Lagrangian, which yields the anomalous divergence of the singlet axial-vector current in QCD.

As we will see in the following, symmetries can be realized in various ways. Most simply, they can be exact or approximate. However, in many case symmetries are hidden (spontaneously broken) or anomalous. All these intricate phenomena will appear in QCD and its related effective field theories.

2.2 Euler-Heisenberg Lagrangian

Ex. 1

2.2.1 The role of symmetry

To elucidate the role of symmetries, we start with an warm-up example, where perturbative calculations can be carried out explicitly in order to verify the result. Consider Quantum Electrodynamics (QED) for momenta/energies much smaller than the electron mass. According to the decoupling theorem, the only relevant degrees of freedom in the effective theory will be photons for such energies, $E \ll m_e$, with E the photon energy and m_e denotes the electron mass. Consequently, the effective Lagrangian of the theory should ~~be constructed from the photon~~ field $\mathscr{A}_\mu$. It is important to realize that this is *not* a theory of free photons: the corresponding Lagrangian contains vertices with 4,6,... photons (an odd number is not allowed because of Furry's theorem [2]). These vertices describe interactions which in the original theory are mediated by closed electron loops, see Fig. 2.1.

In order to construct the effective Lagrangian, one writes down all possible terms, which can be built using the field $\mathscr{A}_\mu$. In the next step, the couplings in front of these terms should be matched to the underlying theory, in this case QED. Here one arrives at the central question: what is a criterion for the *possible* terms? The corresponding procedure is based on the following rules:

- Use only those fields that correspond to the relevant degrees of freedom at the given energy.
- Respect all symmetries. In our example, Lorentz invariance and the discrete C, P, T symmetries of QED should be maintained. Here, C, P and T refer to charge conjugation, parity transformations and time reversal, in order (see, e.g., Ref. [3]). However, in addition to these general symmetries, QED possesses a $U(1)$ gauge symmetry. In this section we shall demonstrate that the requirement of $U(1)$-invariance of the effective theory severely limits the number of the possible terms. This simplifies the procedure of constructing the effective Lagrangian.

Now, let us focus on the role of gauge invariance. It is straightforward to see that the integrand in Eq. (2.4) is invariant under the gauge transformations

$$\psi(x) \mapsto e^{-i\alpha(x)}\psi(x)\,, \quad \bar{\psi}(x) \mapsto \bar{\psi}(x)e^{i\alpha(x)}\,, \quad \mathscr{A}_\mu \mapsto \mathscr{A}_\mu + \frac{1}{e}\,\partial_\mu\alpha(x)\,. \tag{2.5}$$

Here, $\alpha(x)$ denotes the real-valued parameter of the gauge transformation.

Consequently, assuming that the path integral measure is also invariant with respect to the gauge transformations[1], and performing these transformations in Eq. (2.4), we easily obtain

$$\exp\left\{i\int d^4x\mathcal{L}_{\text{eff}}(\mathscr{A}_\mu)\right\} = \exp\left\{i\int d^4x\mathcal{L}_{\text{eff}}(\mathscr{A}_\mu + \partial_\mu\alpha)\right\}. \tag{2.6}$$

In the above equation, $\mathscr{A}_\mu$ is considered as an external classical field. Eq. (2.6) holds if the effective Lagrangian $\mathcal{L}_{\text{eff}}(\mathscr{A}_\mu)$ is gauge-invariant, i.e., if it only depends on the gauge-invariant field strength tensor

$$\mathcal{L}_{\text{eff}}(\mathscr{A}_\mu) = \mathcal{L}_{\text{eff}}(\mathscr{F}_{\mu\nu})\,. \tag{2.7}$$

Note that the gauge invariance naturally leads to consistent counting rules: since $\mathscr{F}_{\mu\nu}$ contains field derivatives, insertions of $\mathscr{F}_{\mu\nu}$ into the loop diagrams result in the suppression of the loop corrections at low energies.

The non-linear contributions to the Lagrangian arise first at $O(m_e^{-4})$. To this order, there are only two such terms, consistent with all symmetries:

$$\mathcal{L}_{\text{eff}} = -\frac{\alpha c_0}{4}\,\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} + \frac{\alpha^2}{m_e^4}\left\{c_1(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})^2 + c_2(\mathscr{F}_{\mu\nu}\tilde{\mathscr{F}}^{\mu\nu})^2\right\} + O(m_e^{-6})\,, \tag{2.8}$$

where $\tilde{\mathscr{F}}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\alpha\beta}$, and $\varepsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor. The overall factor m_e^{-4} appears on dimensional grounds, and the factor α^2, where $\alpha = e^2/(4\pi)$ is the electromagnetic fine-structure constant, appears because this term couples with four photons, each carrying a factor e. So, to this order, only two constants c_1, c_2 have to be determined from matching to QED. Note also the constant c_0 is ultraviolet-divergent. Note further that the first term in Eq. (2.8) combines with the free photon kinetic term and leads to a renormalization of the photon field.

Irrespective of the actual values of the constants c_1, c_2, one may investigate, e.g., the dependence of the photon-photon scattering cross section on photon energy E at $E \ll m_e$. From the explicit form of the Euler-Heisenberg Lagrangian given in

[1] At first glance, this seems self-evident, since $d\psi d\bar{\psi} \doteq \prod_x d\psi(x)d\bar{\psi}(x) = \prod_x(e^{-i\alpha(x)}d\psi(x))(e^{i\alpha(x)}d\bar{\psi}(x))$. However, a certain care is needed in performing the continuum limit, where the number of integration variables tends to infinity. One, in particular, needs to regularize the ultraviolet divergence emerging in this limit, and remove the regularization at the end of the calculations. In the given particular case, this can be done without a problem, justifying the assumption about the gauge-invariance of the fermionic measure. However, if the gauge transformation contains γ_5, the fermionic measure is, in general, no more gauge-invariant, giving rise to the so-called *anomalies*. In the following, we shall consider this issue in detail.

It can be shown that (see below)

$$-i \ln \det(D) = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s}\, e^{-is(m_e^2 - i0)} \left(e^2 ab \frac{\cosh(eas)\cos(ebs)}{\sinh(eas)\sin(ebs)} - \frac{1}{s^2} \right), \tag{2.14}$$

where the factor $i0$ is needed to make the integral convergent. Expanding in powers of a, b and using Eq. (2.13), we obtain

$$\begin{aligned} -i \ln \det(D) &= \frac{e^2}{24\pi^2} (\mathbf{E}^2 - \mathbf{B}^2) \int_0^\infty \frac{ds}{s}\, e^{-is(m_e^2 - i0)} \\ &- \left(\frac{e^4}{360\pi^2} (\mathbf{E}^2 - \mathbf{B}^2)^2 + \frac{7e^4}{360\pi^2} (\mathbf{E}\cdot\mathbf{B})^2 \right) \int_0^\infty ds\, s\, e^{-is(m_e^2 - i0)} + \cdots . \end{aligned} \tag{2.15}$$

The ultraviolet divergence at $s = 0$ in the first integral can be removed by the renormalization of the free-photon term $\sim \mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}$ in the Lagrangian. The second term is finite. Performing the integration over s in this term, we finally get

$$-i \ln \det(D) = -\frac{\alpha c_0}{4} \mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} + \frac{\alpha^2}{90 m_e^4} (\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})^2 + \frac{7\alpha^2}{360 m_e^4} (\mathscr{F}_{\mu\nu}\tilde{\mathscr{F}}^{\mu\nu})^2 + \cdots , \tag{2.16}$$

where c_0 denotes an ultraviolet-divergent constant, as discussed before. From this equation, one may directly read off the values of c_1, c_2:

$$c_1 = \frac{1}{90}, \qquad c_2 = \frac{7}{360}. \tag{2.17}$$

2.2.3 The fermion determinant in a constant field

Here, we perform the explicit calculation of the fermion determinant in a constant field. Namely, our final goal will be to derive Eq. (2.14), which was already been used to match the coefficients c_1, c_2 in the Euler-Heisenberg Lagrangian.

Subtracting a constant that does not depend on the field $\mathscr{A}_\mu$, we may define

$$\begin{aligned} \ln \det(\bar{D}) &= \ln \det(D) - \ln \det(i\gamma^\mu \partial_\mu - m_e) \\ &= \mathrm{Tr}\, \ln\Big((i\,\partial\!\!\!/ - e\,\mathscr{A}\!\!\!/ - m_e)(i\,\partial\!\!\!/ - m_e)^{-1} \Big), \end{aligned} \tag{2.18}$$

where "Tr" denotes the trace both in the coordinate-space and in the space of the Dirac indices. Using $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, where $C = i\gamma^2\gamma^0$, Eq. (2.18) can be rewritten as

$$2 \ln \det(\bar{D}) = \mathrm{Tr}\, \ln\Big(((i\,\partial\!\!\!/ - e\,\mathscr{A}\!\!\!/)^2 - m_e^2)((i\,\partial\!\!\!/)^2 - m_e^2)^{-1} \Big). \tag{2.19}$$

Further, using the relation

$$\ln \frac{\alpha}{\beta} = \int_0^\infty \frac{ds}{s} \left(e^{is(\beta + i0)} - e^{is(\alpha + i0)} \right), \tag{2.20}$$

$$C\,(i\not\partial - e\not A - m)\,CC\,(i\not\partial - m)^{-1}\,C$$

$$= (-i\not\partial^T + e\not A^T - m)\,(-i\not\partial^T - m)$$

$$= (i\not\partial - e\not A + m)\,(i\not\partial + m)$$

$|x_3\rangle$. Calculating the matrix elements in Eq. (2.30) separately, for the first one we get

$$\langle x_0x_3|e^{isH_{03}}|x_0x_3\rangle = \int \frac{dp_0dp_3dp_0'dp_3'dq_0dq_3dq_0'dq_3'}{(2\pi)^8}\, e^{ix_0(p_0-p_0')+ix_3(p_3-p_3')}$$
$$\times\ \langle p_0p_3|e^{ip_0p_3/ea}|q_0q_3\rangle\langle q_0q_3|e^{is(p_0^2-e^2a^2q_0^2)}|q_0'q_3'\rangle$$
$$\times\ \langle q_0'q_3'|e^{-ip_0p_3/ea}|p_0'p_3'\rangle\,. \tag{2.31}$$

Using the relations

$$\langle p_0p_3|e^{\pm ip_0p_3/ea}|q_0q_3\rangle = e^{\pm ip_0p_3/ea}(2\pi)^2\delta(p_0-q_0)\delta(p_3-q_3)\,,$$
$$\langle q_0q_3|e^{is(p_0^2-e^2a^2q_0^2)}|q_0'q_3'\rangle = (2\pi)\delta(q_3-q_3')\langle q_0|e^{is(p_0^2-e^2a^2q_0^2)}|q_0'\rangle\,, \tag{2.32}$$

we obtain

$$\langle x_0x_3|e^{isH_{03}}|x_0x_3\rangle = \frac{ea}{4\pi^2}\int dp_0\,\langle p_0|e^{is(p_0^2-e^2a^2x_0^2)}|p_0\rangle\,. \tag{2.33}$$

In order to calculate the matrix element in Eq. (2.33), we consider the quantum-mechanical problem of a harmonic oscillator given by the Hamiltonian

$$h_{\mathsf{osc}} = \frac{1}{2}\,p_0^2 + \frac{\omega_0^2}{2}\,x_0^2\,. \tag{2.34}$$

The eigenfunctions of this Hamiltonian are labeled by an integer $n = 0, 1, \cdots$

$$h_{\mathsf{osc}}|n\rangle = \omega_0\left(n+\frac{1}{2}\right)|n\rangle\,. \tag{2.35}$$

Consider now the matrix element

$$\frac{ea}{4\pi^2}\int dp_0\,\langle p_0|e^{2ish_{\mathsf{osc}}}|p_0\rangle = \frac{ea}{4\pi^2}\sum_{n=0}^{\infty}\int dp_0|\langle p_0|n\rangle|^2\exp\left\{2is\omega_0\left(n+\frac{1}{2}\right)\right\}$$
$$= \frac{ea}{2\pi}\sum_{n=0}^{\infty}\exp\left\{2is\omega_0\left(n+\frac{1}{2}\right)\right\}. \tag{2.36}$$

In order to recover the original matrix element in Eq. (2.33), one has to substitute $\omega_0 \to iea$. Carrying out the summation over n, we finally arrive at the following result

$$\langle x_0x_3|e^{isH_{03}}|x_0x_3\rangle = \frac{ea}{4\pi\sinh(eas)}\,. \tag{2.37}$$

Evaluating the second matrix element in Eq. (2.30) with the same method, we obtain

$$\langle x_1x_2|e^{isH_{12}}|x_1x_2\rangle = \frac{eb}{4\pi i\sin(ebs)}\,. \tag{2.38}$$

Finally, substituting Eqs. (2.37) and (2.38) into Eqs. (2.30) and (2.26), we arrive at Eq. (2.14), which was used for matching the couplings of the Euler-Heisenberg Lagrangian.

$_i a \to a$

egree of divergence ω of G reads

$$\omega(G) = 4L + \left(\sum_{\text{vertices}} \delta_v\right) - I_F - 2I_B \qquad (8\text{-}13a)$$

$$\omega(G) - 4 = 3I_F + 2I_B + \sum_{\text{vertices}} (\delta_v - 4) \qquad (8\text{-}13b)$$

lation (6-69):

$$L = I_B + I_F + 1 - V \qquad (8\text{-}14)$$

δ_v counts the number of derivatives acting on the fields at vertex to give internal propagators. If f_v (respectively b_v) denotes the number rmion (respectively boson) lines incident to the vertex v, we obviously

$$I_F = \tfrac{1}{2}\sum_v f_v \qquad \text{and} \qquad I_B = \tfrac{1}{2}\sum_v b_v \qquad (8\text{-}15)$$

internal line is counted twice in the sum over the vertices. Thus ay be rewritten

$$\omega(G) - 4 = \sum_v (\hat{\omega}_v - 4) \qquad (8\text{-}13c)$$

$$\hat{\omega}_v \equiv \delta_v + \tfrac{3}{2} f_v + b_v \qquad (8\text{-}16)$$

tation of this index attached to the vertex v is clear if we remember nal considerations of Sec. 6-2-1. A spin $\frac{1}{2}$ fermion field is given the in mass scale and a boson field the dimension 1. Therefore ω_v is tion to the dimension of the interaction monomial of the f_v internal internal bosons, and δ_v derivatives on internal fields. Alternatively, imension of the monomial of $\mathscr{L}_{\text{int}}$ attached to the vertex v, including internal fields, that is,

number of boson fields + $\frac{3}{2}$ times the total number of fermion fields

tal number of field derivatives (8-17)

d E_B stand for the number of external fermion and boson lines of it is clear from (8-13c) that

$$\omega(G) - 4 = \sum_{\text{vertices}} (\omega_v - 4) - \tfrac{3}{2} E_F - E_B - \delta \qquad (8\text{-}18)$$

e total power of external momenta factorized from the Feynman ourse, the dimension ω_v of the vertex does not include the contribution sioned coupling constant pertaining to this vertex, g_v. As already hap. 6,

$$\omega_v + [g_v] = 4 \qquad (8\text{-}19)$$

surprising to see that the convergence of Feynman integrals has to

Ex. 2

the coupling constant g_v has a positive dimension. Going to higher and higher orders in perturbation theory yields more and more powers of g and forces the Feynman integrand to vanish faster and faster at large momenta in order to maintain the total dimension fixed. Conversely, if all $\omega_v > 4$, we expect the integrals to be more and more divergent. All these concepts apply to any subdiagram as well. In general, a diagram with $\omega(G) \geq 0$ is called superficially divergent.

We are now led to a classification of field theories into three classes:

1. Nonrenormalizable theories are those containing at least an interaction monomial of degree $\omega_v > 4$. For a given Green function Eq. (8-18) shows that the superficial degree of divergence grows with the number of vertices, i.e., with the order of perturbation theory. Any function becomes divergent at a high enough order, as will be demonstrated below.
2. Renormalizable theories are the most interesting. All their interaction monomials have $\omega_v \leq 4$, and at least one of them has $\omega_v = 4$. If all monomials have $\omega_v = 4$, we see from (8-18) that all diagrams contributing to a given function have the same degree of divergence. Only a finite number of Green's functions gives rise to overall divergences.
3. Super-renormalizable theories have only vertices with $\omega_v < 4$. The degree of divergence decreases with the order of perturbation. Such theories have only a finite number of divergent diagrams.

The word "nonrenormalizable" may be misleading. It does not mean that such theories cannot be made finite but rather that the proliferation of their divergences, and hence of counterterms, make them unrealistic in the framework of perturbation theory. After renormalization they will depend on an infinite set of arbitrary parameters barring any deeper principle allowing to relate them. We shall not consider them any more. On the other hand, super-renormalizable theories form a too restricted class and are often pathological.

From the rule (8-17), it is easy to list by simple inspection all the possible renormalizable or super-renormalizable theories. We have to construct all possible interaction monomials that are Lorentz scalars, hermitian, and have a degree ω_v less or equal to 4, from derivatives, scalar fields φ, Dirac fields ψ, and vector fields A_μ (supposed to be endowed with a Stueckelberg propagator). The monomials $\bar{\psi}\psi\varphi$ or $\bar{\psi}\gamma_5\psi\varphi$, φ^4 or $(A^2)^2$, $\bar{\psi}A\!\!\!/\psi$ (or perhaps $\bar{\psi}A\!\!\!/\gamma_5\psi$), $\varphi^\dagger\partial_\mu\varphi A^\mu$, $\varphi^\dagger\varphi A^2$, as well as monomials usually appearing in the kinetic lagrangian, $\bar{\psi}\partial\!\!\!/\psi$, $(\partial\varphi)^2$, $(\partial_\mu A_\nu)^2$, $(\partial_\mu A^\mu)^2$, have $\omega_v = 4$ and exhaust the list of renormalizable monomials, up to the introduction of several species of each field and possible internal symmetries. The terms φ^3, $\bar{\psi}\psi$, $\bar{\psi}\gamma_5\psi$, φ^2, A^2 have $\omega_v = 3$ or 2, and thus lead to super-renormalizable lagrangians. Among nonrenormalizable theories, let us quote the pseudovector coupling $\bar{\psi}\gamma_\rho\gamma_5\psi\partial^\rho\varphi$, the Fermi coupling $\bar{\psi}\gamma_\rho(1-\gamma_5)\psi\bar{\psi}\gamma^\rho(1-\gamma_5)\psi$, or higher-degree monomials in φ: φ^5, φ^6, etc.

The previous analysis and list have been presented for a four-dimensional space-time. For purposes of generalization, or for the needs of statistical mechanics,

VI. PROZEßE ZU HÖHEREN ORDNUNGEN (Schleifen)

• Idee: Berechne Korrekturen zu den Baumgraphen. Konzepte der Regularisierung & Renormierung treten auf.

1) GRAD DER DIVERGENZ

Um Diagramme (Prozeße) in höheren Ordnungen studieren zu können, brauchen wir das Konzept des "oberflächlichen Divergenzgrades".

Zähle:

$$\boxed{\begin{array}{lll} \text{Schleifen-Integral} & : & p^4 \\ \text{Elektron-Propagator} & : & 1/p = p^{-1} \\ \text{Photonen-Propagator} & : & 1/p^2 = p^{-2} \end{array}}$$

(1a) (1b) (1c)

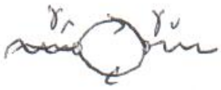
Beachte: Dies sind die sogenannten UV (Ultraviolett) Divergenzen, die von den großen Impulsen also kleinen Abständen herrühren. Später werden wir noch IR Divergenzen kennenlernen (kleine Impuls -> weiche Photonen)

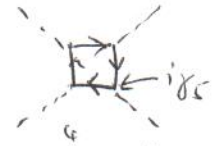
Die algebraische Summe dieser Potenzen ergibt den Divergenzgrad D eines Diagramms. Man hat

$$\boxed{\begin{array}{ll} D = 2 & \text{quadratisch divergent} \\ D = 1 & \text{linear divergent} \\ D = 0 & \text{logarithmisch divergent} \\ D < 0 & \text{konvergent} \end{array}}$$

(2a) (2b) (2c) (2d)

Beispiele: Elektron-Selbstenergie $\qquad\qquad$ $\pi\pi \to \pi\pi$





$$\sim \int d^4k \, \frac{1}{\not k - m}\,\frac{1}{\not k - m} \underset{k \gg m}{\sim} \int \frac{d^4k}{k^2} \sim k^2 \quad \Rightarrow \underline{D=2}$$

$$\sim \int d^4p \prod_{i=1}^{4} \frac{1}{\not p_i - M} \underset{p \gg M}{\sim} \int \frac{d^4p}{p^4} \sim \int \frac{dp}{p} \sim \ln p \quad \Rightarrow \underline{D=0}$$

(ln zählt wie p^0)

In der <u>QED</u> (mit der wir uns nun beschäftigen) ist die Kopplungskonstante $g = \sqrt{\alpha} = e/\sqrt{4\pi} \simeq 0.3$ dimensionslos. Dann hängt der Grad der oberfl. Divergenz nur von der Zahl der externen Beine (Teilchen) ab und ist unabh. von der internen Struktur des jeweiligen Diagrams. Dies ist offensichtlich, denn über die Energie-Impuls-Erhaltung können wir jeden Propagator über Vertices mit den äußeren Beinen in Verbindung bringen [Übung -> zeige dies, eg. IZ, S. 344/380]
In der QED gilt dann (Abl. Seite HO IIa,b):

$$\boxed{\begin{aligned} &D = 4 - \tfrac{3}{2} F_e - B_e \\ &F_e \equiv \#\ \text{der externen Fermionlinien} \\ &B_e = \#\ \text{"} \quad \text{"}\ \text{Photonlinien} \end{aligned}} \qquad (3)$$

Da die Zahl der ext. Linien ≥ 0 ist, gibt es nur eine endliche Anzahl von Typen divergenter Diagramme.
Allerdings ist der Div.grad D nicht immer zuverlässig. Manchmal ist $D<0$, aber ein Untergraph kann divergent sein. Oder aber Diagramme mit $D>0$ werden von anderen div. Diagrammen weggehoben. Das werden wir noch an Beispielen sehen.

Ableitung von Gl. (3)

Kopplungen enthalten keine Ableitungen, also ein gewöhnliches Integral:

$$I = \underbrace{\int d^d l_1 \dots \int d^d l_L}_{L \text{ mal}} \; \frac{1}{(l_\alpha^2 - m^2)^{P_i}} \; \frac{1}{(\not{l} - m_e)^{E_i}}$$

E_i: Zahl der inneren Fermionl.

P_i: Zahl der inneren Photon (Boson) linien

$\Rightarrow \quad D = dL - 2P_i - E_i$ uns interessiert nur $l_\mu \to \infty$, $l_\mu^2 \gg m_{(e)}^2$

L: Zahl der unabh. Impulsintegrationen

$=$ Zahl der internen Linien $- n + 1$ ← overall $\delta^{(4)}(P_i)$

n: Zahl der Vertices, 4er Impulserhaltung an jedem Vertex

also $L = E_i + P_i - n + 1$ (checke an einfachen Diagrammen)

 $E_i = 2$, $P_i = 0$, $n = 2$

$\Rightarrow L = 2 - 2 + 1 = 1$

also ein unabh. 4er Impuls ✓

Jeder Vertex: 2 Fermionen Beine

$\Rightarrow \quad 2n = E_e + 2E_i$ ← jedes innere Elektron nimmt an 2 Vertices teil →

dito $\quad n = P_e + 2P_i$

alles zusammen

$$D = (d-1)E_i + (d-2)P_i - d(n-1)$$
$$= d + n\left(\frac{d}{2} - 2\right) - \left(\frac{d-1}{2}\right)E_e - \left(\frac{d-2}{2}\right)P_e$$

$d = 4 \Rightarrow$

$$\boxed{\begin{aligned} D &= 4 - \tfrac{3}{2}E_e - P_e \\ &= 4 - \tfrac{3}{2}F_e - B_e \end{aligned}}$$

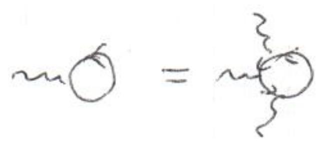
$E_e \equiv F_e$
$P_e \equiv B_e$

Oberflächlich divergente Diagramme in der QED

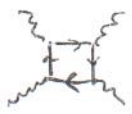
B_e	F_e	D	Kommentar
0	0	4	Vakuumgraph
1	0	3	$=0$, Furry's Theorem
2	0	2	Vakuumpolarisation (Photon Σ)
3	0	1	$=0$, Furry's Theorem
4	0	0	endlich (!)
0	2	1	Elektron Selbstenergie
1	2	0	Vertexkorrektur
(0	1	5/2	Verletzung Leptonzahlerh.)

Graphisch (Bsp)

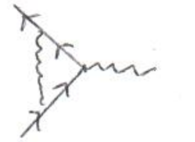
Vakuumgraph*  $\int \frac{d^4k\, d^4k}{k^2 \cdot k \cdot k} \sim k^4 \Rightarrow D=4$ ✓

Furry (s.u.)  $=0$

Vakuumpol.  $\int \frac{d^4k}{k^2} \sim k^2 \Rightarrow D=2$ ✓

Boxgraph  $\int \frac{d^4k}{k \cdot k \cdot k \cdot k} \sim \ln k \Rightarrow D=0$, tatsächlich endlich ✓

Elektr. Selbsten.  $\int \frac{d^4k}{k \cdot k^2} \sim \int dk \sim k^1 \Rightarrow D=1$ ✓

Vertexkorrektur  $\int \frac{d^4k}{k^2 \cdot k \cdot k} \sim \ln k \Rightarrow D=0$ ✓

Also 4 Klassen von div. Diagrammen, beschäftigen uns mit den 3 unterstrichenen Typen.

(*) Haben wir schon abgehandelt, siehe

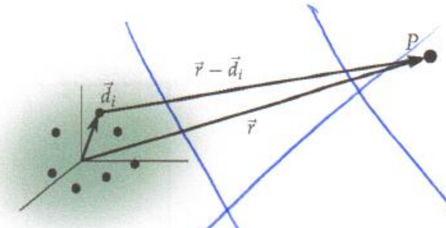
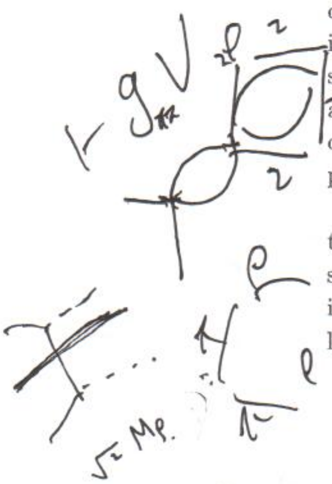


Fig. 1.1 An electric field produced by an arbitrary localized static distribution of charges. The observer is located at the point P far away from the charge distribution.



The equation (1.3) demonstrates that choosing the appropriate degrees of freedom, or variables, for describing the problem at large distances (i.e., choosing $\mathbf{r}$ instead of the individual distances $\mathbf{r}_i$), the solution of the problem is considerably simplified and can be described by a few parameters, here $Q, \mathbf{P}, Q_{\alpha\beta}$. These characterize the system as a whole rather than its individual components. Eq. (1.2) can be then considered as the *matching condition,* giving the expressions of these parameters in terms of the underlying physics at short distances.

The above separation of scales is encountered in any field of physics. In this chapter, we consider the application of this idea in quantum field theory and demonstrate, how the physics at the heavy scales (at short distances) can be consistently integrated out from the theory, leading to an effective theory, which contains the light degrees of freedom only.

Ex. 3

1.2 Warm up: effective theory for scattering on the potential well

1.2.1 Effective range expansion

Before addressing effective field theories, we would like to start from a more familiar example and consider constructing an effective theory for the quantum-mechanical scattering on a short-ranged potential. This allows to explain many fundamental concepts and notions of the effective field theories in an intuitive and transparent fashion[1]. Namely, we shall consider a spherical potential well, depicted in Fig. 1.2. The potential of the well is given by:

$$U(r) = \begin{cases} -U_0 & \text{for} \quad r \le b\,, \\ 0 & \text{for} \quad r > b\,. \end{cases} \tag{1.4}$$

[1] A similar problem has been addressed, for example, in the beautiful lectures given by G.P. Lepage [1], which we strongly recommend for further reading.

$\frac{\rho\rho}{\pi\pi}\ \frac{2}{7}$

d27

$\boxed{d72 = d27/g_\rho(s) \times g_a(s)}$

$\frac{g_{\pi\pi}}{g_{\rho\rho}}$ $g_{\pi\pi}$ $\sqrt{g_2}$

In the following, we restrict ourselves to S-wave scattering with $\ell = 0$ and thus drop the index ℓ. The S-wave scattering phase is given by:

$$\tan\delta(k) = \frac{k\tan(Kb) - K\tan(kb)}{K + k\tan(kb)\tan(Kb)}\,. \tag{1.9}$$

Using this expression, one may write down the *effective-range expansion* for the phase shift:

$$k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}\,rk^2 + v_4k^4 + O(k^6)\,. \tag{1.10}$$

Here, a is the *scattering length*, r is called *effective range*, and the higher coefficients $v_4, v_6, \ldots$ are known under the name *shape parameters*. Generally, $a, r, v_4, v_6, \ldots$ are referred to as *effective-range parameters*. Note that in our sign convention, a positive value of a corresponds to attraction. The explicit expressions for these parameters are obtained by Taylor-expanding Eq. (1.9). It is convenient to express the results in terms of b and the dimensionless parameter $x = b\sqrt{U_0}$:

$$\begin{aligned} a &= bf_0(x)\,,\\ r &= bf_2(x)\,,\\ v_{2n} &= b^{2n-1}f_{2n}(x)\,. \end{aligned} \tag{1.11}$$

Below, we display the first two coefficients explicitly:

$$\begin{aligned} f_0(x) &= 1 - \frac{\tan x}{x}\,,\\ f_2(x) &= \frac{3\tan x - 3x + 3x\tan^2 x - 6x^2\tan x + 2x^3}{3x(x-\tan x)^2}\,, \end{aligned} \tag{1.12}$$

and so on.

Next, let us consider the limit $x \to \frac{\pi}{2} + \pi n$. One can easily convince oneself that $f_2(x), f_4(x), \ldots$ stay finite in this limit. With the first coefficient, matters are however different. As seen from Eq. (1.12), $f_0(x) \to \infty$ as $\tan x \to \infty$. Thus, we have two distinct possibilities:

1. All effective-range parameters are of *natural size,* which is determined by the interaction range b. Namely, $a \sim b$, $r \sim b$, $v_{2n} \sim b^{2n-1}$.
2. We have an *unnaturally large scattering length,* namely, $a \gg b$. All other parameters are of natural size, i.e., we still have $r \sim b$, $v_{2n} \sim b^{2n-1}$.

Note also that the convergence of the effective range expansion is in both cases controlled by the parameter b. In other words, the effective range expansion converges when $kb \ll 1$. Physically, this means that for the large distances $r \sim 1/k \gg b$, the scattering on the short-ranged potential, irrespective of its shape, can be parameterized by the first few coefficients in this expansion (like any static charge distribution in classical electrodynamics at large distances can be characterized by

$C \int \frac{q^2 dq}{q^2-k^2} R(q,k) + C = R(p,k)$

One may also introduce the scattering R-matrix, which obeys the the Lippmann-Schwinger equation

$$R(p,k) = V(p,k) + \frac{2}{\pi} \fint \frac{q^2 dq}{q^2 - k^2} V(p,q) R(q,k)\,. \tag{1.18}$$

Here, unlike Eq. (1.15), the integral is equipped with the principal value prescription (the bar across the integral sign denotes the principal-value integral). The T- and R-matrices are related. On shell, this relation takes the form

$$T(k) = \frac{R(k)}{1 - ikR(k)}\,, \qquad R(k) = \frac{1}{k} \tan\delta(k)\,. \tag{1.19}$$

In the following, we prefer to work with the R-matrix. The Fourier transform of the potential in Eq. (1.14) is given by

$$V(p,k) = C_0\,. \tag{1.20}$$

$R = C_0 + C_0 I_2 R$

The solution of the Lippmann-Schwinger equation (1.18) is given by:

$R = \frac{C_0}{1 - C_0 I_2}$

$$R(k) = \frac{C_0}{1 - C_0 I_2(k^2)}\,, \qquad I_2(k^2) = \frac{2}{\pi} \fint \frac{q^2 dq}{q^2 - k^2}\,. \tag{1.21}$$

Here, we encounter the problem of an ultraviolet (UV) divergence: since the δ-function potential is singular at short distances, the integral I_2 diverges at the upper limit and should be regularized. The most straightforward way to do this is to introduce a momentum cutoff Λ. Then, the integral is equal to

$$I_2(k^2) = \frac{2}{\pi} \fint^{\Lambda} \frac{q^2 dq}{q^2 - k^2} = \frac{2}{\pi}\Lambda + O(1/\Lambda)\,. \tag{1.22}$$

In the following, the terms of order $1/\Lambda$ are always neglected and never displayed explicitly. The R-matrix at leading order, which is given by Eq. (1.21), turns out to be constant. The matching condition then reads:

$$R(0) = -a\,, \qquad C_0 = -\frac{a}{1 - aI_2(0)}\,, \tag{1.23}$$

It follows from the above equation that C_0 should be Λ-dependent, in order to ensure that the observable (the scattering length a) does not depend on the cutoff.

In order to illustrate the matching, let us do a simple numerical exercise. We arbitrarily choose the parameters of the square well as $b = 1$ and $x = b\sqrt{U_0} = \pi/4$. In Fig. 1.3, we display the exact phase shift, given by Eq. (1.9), as well as the phase shift, obtained in the effective field theory with a zero-range potential given in Eq. (1.14), where the parameter C_0 is determined from the matching to the exact scattering length. The parameter Λ is set equal to $1/b$ (i.e., to the inverse of the short-range scale of the model). It is seen that, up to the momenta $k^2/\Lambda^2 \leq 0.5$, the phase shift is reproduced in the effective theory rather well.

We are not going to stop here, however, and ask ourselves whether it is possible to *systematically* improve the description of the phase shift. To this end, note that, using the leading-order potential (with no derivatives), it is possible to adjust only respect to Λ. Recalling the discussion in the previous section, one may expect that the beta-functions in cutoff regularization at different orders are merely different by a quantity of order one.

1.2.7 What did we learn from this example?

- At low momenta ($k \ll 1/b$) a scattering process can be characterized by a small set of effective range expansion parameters.
- The interaction range is implicitly encoded in the size of the effective range expansion parameters. Namely, if the scattering length is of natural size, then we have $a \sim b$, $r \sim b$, $v_4 \sim b^3$ and so on. In case of an unnaturally large scattering length, only the first of these relations is not valid.
- An unnaturally large scattering length is related to the formation of a near-threshold bound state (or, of a virtual state, in general).
- One may construct a low-energy effective theory, approximating the square well potential by a series of the δ-function potential and derivatives thereof. The couplings in front of these potentials are adjustable parameters and are used to reproduce the effective-range expansion parameters order by order. This procedure goes under the name of matching.
- Albeit the matching conditions may look differently in different regularizations, the resulting scattering amplitude, expressed in terms of the effective range parameters, is the same in all regularizations up to the terms of higher orders.
- Last but not least, it is interesting to mention that the matching fixes not only the scattering amplitude at small momenta, but the spectrum of the shallow bound states as well. To see this, it suffices to note that, according to Eq. (1.19), the poles of the T-matrix (corresponding to the bound states) emerge for purely imaginary values of k, corresponding to the solution of the equation

$$R^{-1}(k) - ik = -\frac{1}{a} + \frac{1}{2}rk^2 + \cdots - ik = 0\,. \tag{1.39}$$

  If the effective theory reproduces the values of $a, r, \ldots$, then the solution of this equation will also be reproduced up to higher-order terms.
- Theoretical uncertainties can and should be estimated.

Ex. 4

1.3 Integrating out a heavy scale: A model at tree level

1.3.1 Matching at tree level

After this warm-up example considered in the previous section, let us proceed with the construction of an effective theory in a simple field-theoretical model. This

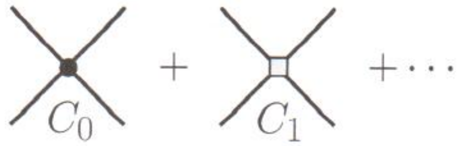


Fig. 1.6 The tree-level scattering amplitude for the process $\phi\phi \to \phi\phi$ in the effective theory described by the Lagrangian given in Eq. (1.43). This amplitude can be obtained from the amplitude shown in Fig. 1.5 by contracting all heavy lines to a point.

a Taylor series:

$$T_{\text{tree}} = \frac{3g^2}{M^2} + \frac{g^2}{M^4}(s+t+u) + \frac{g^2}{M^6}(s^2+t^2+u^2) + \cdots \tag{1.42}$$

At low energies, each subsequent term in this expansion is suppressed by a factor E^2/M^2 with respect to the previous one, where E is the characteristic energy of the light particles.

Our aim is to find a Lagrangian that contains only ϕ-fields, and which reproduces the expansion of the amplitude in Eq. (1.42). In general, such an *effective* Lagrangian must contain an infinite tower of quartic terms in the field ϕ. In analogy with Eq. (1.24) one may try to use the Lagrangian of the following form:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - C_0\phi^4 - C_1\phi^2\Box\phi^2 - C_2\phi^2\Box^2\phi^2 + \cdots, \tag{1.43}$$

with $\Box = \partial^\mu\partial_\mu = \partial\partial$. Note that at tree level the mass parameters in both the underlying and effective Lagrangians are equal. As we shall see below, this is no more the case at one loop.

The tree-level amplitude, obtained from this Lagrangian, takes the form

$$T_{\text{tree}}^{\text{eff}} = -24C_0 + 8C_1(s+t+u) - 8C_2(s^2+t^2+u^2) + \cdots. \tag{1.44}$$

This amplitude is shown in Fig. 1.6. Demanding $T_{\text{tree}}^{\text{eff}} = T_{\text{tree}}$ leads to *matching conditions* which enable one to express the couplings of the effective theory $C_0, C_1, C_2, \ldots$ in terms of the parameters of the underlying theory g, m and M.

1.3.2 Equations of motion

The matching condition is imposed on *observables,* i.e., in our case, on the scattering amplitude defined on shell, $p_i^2 = m^2$. As it is known, the Mandelstam variables on shell obey the constraint

$$s + t + u = 4m^2, \tag{1.45}$$

(13) Examples:

a) The SM without Higgs

$$SU(2)_L \times U(1)_Y \to U(1)_{em}$$

technicolor, ... (Higgs as composite)

$\Rightarrow$ local symmetry is broken $\Rightarrow$ Goldstone's are not physical particles but they become the longitudinal component of the V-bosons ($V = W, Z$)

$\Rightarrow$ $V_L V_L \to V_L V_L \equiv \pi\pi \to \pi\pi$

(would have been studied beautifully at the SSC)

b) The SM at $E < 1$ GeV

As argued: Relevant d.o.f's not q's and g's, but π, K, η (+ other hadrons). The corresponding EFT is called <u>CHIRAL PERTURBATION THEORY</u> (CHPT)

(see sections 3, 4, ..)

Ex. 5

(14) <u>APPLICATIONS OF EFT</u>

- If the fundamental theory is not known, use EFT to parametrize the effects of new physics.

<u>Ex</u>: Physics beyond the SM

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \frac{1}{\Lambda}\mathcal{L}_5 + \frac{1}{\Lambda^2}\mathcal{L}_6 + \dots$$

There is only one operator of dimension 5 which obeys the symmetries of the SM:

$$\mathcal{L}_5 = c_5\ \varepsilon^{ij} \left(\bar{\ell}_L^{(c)}\right)_i \Phi_{Hj}\ \varepsilon^{kl} (\ell_L)_k \Phi_{Hl} \qquad (i,j,k,l=1,2)$$

lepton Higgs doublet(s)

$\Rightarrow$ (Majorana) neutrino mass $m_\nu \sim \frac{c_5}{2}\frac{v^2}{\Lambda}$

with $v = \langle \Phi_H \rangle = 250$ GeV

Say $m_\nu < 10$ eV $\Rightarrow \Lambda > 10^{10}$ TeV , $c_5 \simeq O(1)$

$$\mathcal{L}_6 = \frac{c'}{\Lambda^2}\left(\Phi_H^\dagger \Phi_H\right) \vec{W}_{\mu\nu}\vec{W}^{\mu\nu} + \frac{c''}{\Lambda^2}\, \bar{e}\gamma_\mu(1+\gamma_5)\mu\ \bar{s}\gamma^\mu(1+\gamma_5)d + \text{h.c.} + \ldots$$

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} = 1 - c'\frac{v^2}{\Lambda^2} \qquad\qquad \frac{\Gamma(K_L^0 \to \mu^+ e^-)}{\Gamma(K_L^0 \to \mu^+\nu_\mu)} = \left(\frac{c''}{V_{us}G_F\Lambda^2}\right)^2$$

$\Rightarrow \Lambda > 4.0$ TeV $\Rightarrow \Lambda > 250$ TeV

(should be updated w/ m_t !!) $c', c'' \sim O(1)$

$\vdots$

- EVEN IF THE FUNDAMENTAL TH'Y IS KNOWN, EFT CAN BE USEFUL. EITHER THE FULL TH'Y IS NOT NEEDED ($\exists$ QED), OR CAN NOT BE TREATED BY RELIABLE METHODS IN THE REGION OF INTEREST (QCD @ LOW ENERGIES)

$\swarrow$

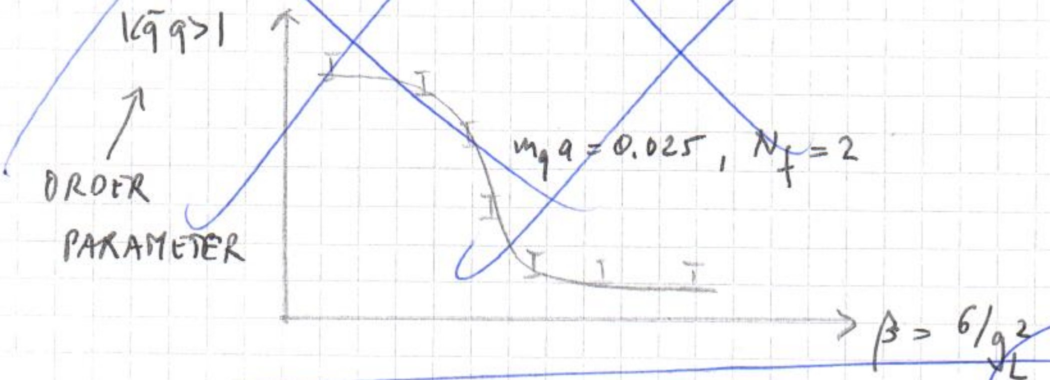
IN THE CONFINEMENT REGIME OF QCD, CHPT TAKES OVER.

[Remark: I always leave aside Lattice gauge theory.]

6) ~~IN VECTOR-LIKE THEORIES (LIKE QCD), $SU(N_f)_V$ CANNOT BE SPONTANEOUSLY BROKEN (for $\theta_{QCD} = 0$)~~

~~Vafa + Witten, NP B234 (1984) 173~~

7) ~~EVIDENCE FROM LATTICE QCD~~



Ex. 6

IDENTIFYING THE ORDER PARAMETER:

- In the chiral limit, we have

$$\boxed{\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle \neq 0}$$

  - broken by mass terms
  - intuitively clear $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{u}_L u_R|0\rangle + \langle 0|\bar{u}_R u_L|0\rangle$

    couples L/R quark fields

- FORMAL PROOF:

$$\langle 0|[Q_V^a, \bar{q}\lambda^b q]|0\rangle = 0 \qquad a,b = 1,\ldots,8$$

since $Q_V^a|0\rangle = 0$

62

Now use: $[q^\dagger_\alpha(\vec{x},t), q_\beta(\vec{y},t)]_+ = \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y})$

$\rightsquigarrow [\bar{q}_\alpha (\frac{\lambda^a}{2})_{\alpha\beta}\,\gamma_0\, q_\beta, \bar{q}_\gamma (\lambda^b)_{\gamma\delta} q_\delta]$ omit $\int d^3x$, will be killed later by $\delta^{(3)}(..)$

$= [q^\dagger_\alpha q_\beta, q^\dagger_\gamma q_\delta](\lambda^a \cancel{\gamma_0}^{\,1\!\!1})_{\alpha\beta}(\lambda^b\gamma^0)_{\gamma\delta}\cdot\frac{1}{2}$ $\bar{q} = q^\dagger\gamma^0$

$= (-q^\dagger_\alpha q^\dagger_\gamma \cancel{\{q_\beta, q_\delta\}} + q^\dagger_\alpha\{q_\beta, q^\dagger_\gamma\}q_\delta - q^\dagger_\gamma\{q^\dagger_\alpha, q_\delta\}q_\beta$

$+ \cancel{\{q^\dagger_\alpha, q^\dagger_\gamma\}} q_\beta q_\delta)(\lambda^a)_{\alpha\beta}(\lambda^b\gamma^0)_{\gamma\delta}\cdot\frac{1}{2}$ $\left[[q^{(\dagger)}_\alpha, q^{(\dagger)}_\beta]_\pm = 0 \right.$

omit $\delta^{(3)}$ $\rightsquigarrow$ $= (q^\dagger_\alpha \delta_{\beta\gamma} q_\delta - q^\dagger_\gamma \delta_{\alpha\delta} q_\beta)(\lambda^a)_{\alpha\beta}(\lambda^b\gamma^0)_{\gamma\delta}\cdot\frac{1}{2}$

$= \frac{1}{2}q^\dagger_\alpha(\lambda^a)_{\alpha\gamma}(\lambda^b\gamma^0)_{\gamma\delta}q_\delta - \frac{1}{2}q^\dagger_\gamma(\lambda^b\gamma^0)_{\gamma\delta}(\lambda^a)_{\delta\beta}q_\beta$

$= q^\dagger\gamma^0[\lambda^a, \lambda^b]q/2$

$= i\bar{q}f^{abc}\lambda^c q$ $\rightsquigarrow$ $\underline{[Q^a_V, \bar{q}\lambda^b q] = i f^{abc}\,\bar{q}\lambda^c q}$

In the vacuum: $\langle 0|\bar{q}_\alpha q_\beta|0\rangle(\lambda^c)_{\alpha\beta}f^{abc} = 0$

$\Rightarrow$ all non-diagonal elements are zero:

$\underline{\langle 0|\bar{u}s|0\rangle = \langle 0|\bar{d}s|0\rangle = \ldots 0}$

Choose $a=1, b=2$: $f^{123} = 1$

$(\bar{u}\,\bar{d}\,\bar{s})\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\begin{pmatrix}u\\d\\s\end{pmatrix} = 0 \Rightarrow \langle 0|\bar{u}u - \bar{d}d|0\rangle = 0 \Rightarrow \langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle$

$a=6, b=7$: $[\lambda_6, \lambda_7] = 2i\underbrace{f_{678}}_{\sqrt{3}}\lambda_8 + 2i\underbrace{f_{673}}_{-1}\lambda_3 \rightsquigarrow$

$$\bar{q}\,(\sqrt{3}\lambda_8 - \lambda_3)\,q = (\bar{u}\,\bar{d}\,\bar{s})\begin{pmatrix}0 & & \\ & 2 & \\ & & -2\end{pmatrix}\begin{pmatrix}u\\d\\s\end{pmatrix} = 2(\bar{d}d - \bar{s}s) = 0$$

$$\Rightarrow \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle$$

$$\Rightarrow \langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle$$

Now calculate (as before):

$$[Q_A^1, \bar{q}\gamma_5\lambda^1 q] = -(\bar{u}u + \bar{d}d)$$

Since $Q_A^a|0\rangle \neq 0$

$$\underline{\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle \neq 0}$$

SUMMARY:

THE VACUUM IS NON-TRIVIAL!

$\boxed{\text{THE CHIRAL SYMMETRY IN QCD UNDERGOES SSB. FOR } SU(N)_L \times SU(N)_R \text{, WE HAVE } N^2-1 = 3, 8\ (N=2,3) \text{ MASSLESS PSEUDOSCALAR GOLDSTONES*. THE ORDER PARAMETER OF THE S}\chi\text{SB IS } \langle 0|\bar{q}q|0\rangle \neq 0\ (q=u,d,s) \text{ SINCE } \langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R|0\rangle + \langle 0|\bar{q}_R q_L|0\rangle.}$

* π, K, η are not exactly massless ⇒ approximate chiral sym.
↓
related to the quark masses (see later)

(4) DISCRETE SYMMETRIES

* QCD is a hermitean and local QFT ⇒ PCT INVARIANCE

* In the absence of a θ-term: QCD conserves P, C and T SEPARATELY!

(4) "HANDS-ON" RENORMALIZATION OF M_π

Ex. 7

- From previous (p.), $SU(2)$: $U = \exp\{i\phi/F\}$

$$\mathcal{L}^{(2)} = \frac{1}{2}\left[\partial_\mu\phi\cdot\partial^\mu\phi - M^2\phi\cdot\phi\right] + \frac{M^2}{24}(\phi\cdot\phi)^2$$

$$+\frac{1}{6F^2}\left[(\phi\cdot\partial^\mu\phi)(\phi\cdot\partial_\mu\phi) - (\phi\cdot\phi)(\partial^\mu\phi\cdot\partial_\mu\phi)\right] + \dots$$

- Self-energy of the pion:

 $\equiv I(M^2) = \mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M^2} = \frac{\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)(M^2)^{\frac{d}{2}-1}$

a) $\sim \frac{M^2}{24}(\phi\cdot\phi)^2$

contractions $\phi_a\phi_a\phi_b\phi_b$ 3, 1, 1

$$\left.\right\} \quad \frac{M^2}{24F^2}(3+1+1)\phi\cdot\phi \times 2 \times I(M^2) = \frac{5M^2}{12F^2}I(M^2)\,\phi\cdot\phi$$

b) $\sim \frac{1}{6F^2}\left[(\phi\cdot\partial^\mu\phi)(\phi\cdot\partial_\mu\phi) - (\phi\cdot\phi)(\quad)\right]$

$\phi_i\partial^\mu\phi_k\,\phi_j\partial_\mu\phi_\ell\,\delta_{ik}\delta_{j\ell}$

a $\phi_i\partial^\mu\phi_k$ does not contribute to self-energy

$\rightarrow$ only possible contractions

$\phi_i\partial^\mu\phi_k\,\phi_j\partial_\mu\phi_\ell\,\delta_{ik}\delta_{j\ell}$

$\frac{g^\mu_\mu}{d}I(M^2)$

$\Rightarrow \delta_{ik}\delta_{j\ell}(\delta_{ij}\partial^\mu\phi_k\partial_\mu\phi_\ell + \delta_{k\ell}\frac{M^2}{4}\phi_j\phi_i)I(M^2)$

$\phi_i\phi_j\partial^\mu\phi_k\partial_\mu\phi_\ell\,\delta_{ij}\delta_{k\ell}$

$= \partial^\mu\phi_k\partial_\mu\phi_\ell\,\delta_{ij}\,I(M^2)$

$+\phi_i\phi_j\,\delta_{k\ell}\,M^2 I(M^2)$

We use:

$$\delta_{ij}\, I_{\mu\nu}(M^2) = \langle 0|\partial_\mu \phi_j(x)\,\partial_\nu \phi_k(x)|0\rangle$$

$$I_{\mu\nu}(M^2) = \mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\, k_\mu k_\nu \frac{i}{k^2-M^2} = g_{\mu\nu}\frac{M^2}{d} I(M^2)$$

c) contact terms from $\mathcal{L}^{(4)}$ ($\tilde{L}_i$ to denote $SU(2)$)

$\tilde{L}_{1,2,3} \sim \varphi^4 \Rightarrow$ do not contribute

$\tilde{L}_4\, Tr(D_\mu U D^\mu U^+)\, Tr(\chi U^+ + U\chi^+)$ $\qquad \chi = 2MB = M_\pi^2$

$\tilde{L}_4\, Tr(\tau^a\tau^b \partial_\mu\phi^a\partial^\mu\phi^b)\, M_\pi^2\, Tr(2\cdot\mathbb{1}) \Rightarrow 16 L_4 \underbrace{\frac{M^2}{F^2}\frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi}$

$\tilde{L}_5\, Tr(\tau^a\tau^b\partial_\mu\phi^a\partial^\mu\phi^b\mathbb{1})\frac{M^2}{F^2} \Rightarrow 8L_5$ --

two such terms

$\tilde{L}_6\left[Tr(\chi^+U+U^+\chi)\right]^2 \to -M^2\, Tr(\mathbb{1})\frac{1}{F^2}\, Tr(\tau^a\tau^b\phi^a\phi^b)\cdot 2$

$$= -32\frac{M^2}{F^2} L_6\, \frac{1}{2}\phi\cdot\phi$$

$\tilde{L}_8\, Tr(\chi^+U\chi^+U + U^+\chi U^+\chi) \to \frac{M^2}{F^2} Tr(\mathbb{1} i\phi^a \mathbb{1} i\phi^b - \frac{1}{2}\phi^2\cdot\mathbb{1}^3\cdot 2)$

$$= \frac{16}{2} L_8 \frac{M^2}{F^2}\phi\cdot\phi$$

$\Rightarrow$

$$\mathcal{L}^{(2)}_{eff} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}M^2\phi\cdot\phi + \frac{5M^2}{12F^2} I(M^2)\,\phi\cdot\phi$$

$$+\frac{1}{6F^2}(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl})\, I(M^2)\,(\delta_{ij}\,\partial^\mu\phi_k\,\partial_\mu\phi_l + \delta_{kl} M^2\phi_i\phi_j)$$

$$+\frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi\,\frac{M^2}{F^2}(16\tilde{L}_4 + 8\tilde{L}_5) - \frac{1}{2}M^2\phi\cdot\phi\,\frac{M^2}{F^2}(32\tilde{L}_6 + 16\tilde{L}_8)$$

use: $\Big(\underbrace{\delta_{ik}\delta_{jl}\delta_{ij}}_{\delta_{kl}} - \underbrace{\delta_{ij}\delta_{kl}\delta_{ij}}_{3\delta_{kl}}\Big)\partial^\mu\phi_k\partial_\mu\phi_l = -2\,\partial^\mu\phi\cdot\partial^\mu\phi$

same for $\phi_i\phi_j = -2\phi\cdot\phi$

(38)

$$\leadsto \mathcal{L}_{eff}^{(2)} = \frac{1}{2}\partial^\mu\phi\cdot\partial_\mu\phi\left[1+\frac{M^2}{F^2}(16\tilde{L}_4+8\tilde{L}_5)-\frac{2}{3F^2}I(M^2)\right]$$

$$-\frac{1}{2}M^2\phi\cdot\phi\left[1+(32\tilde{L}_6+16\tilde{L}_8)\frac{M^2}{F^2}-\frac{1}{6F^2}I(M^2)\right]$$

• Now: $\boxed{\phi_r = Z_\pi^{-1/2}\phi}$ renormalized pion field

$$\leadsto Z_\pi = 1-\frac{8M^2}{F^2}(2\tilde{L}_4+\tilde{L}_5)+\frac{2}{3F^2}I(M^2)$$

$$I(M^2) = \frac{M^2}{16\pi^2}\left\{\frac{2}{d-4}-1-\ln(4\pi)+\ln\frac{M^2}{\mu^2}\right\}$$

$$\leadsto \boxed{Z_\pi = 1-\frac{8M^2}{F^2}(2\tilde{L}_4+\tilde{L}_5)+\frac{M^2}{24\pi^2F^2}\left(\frac{2}{d-4}+\gamma-1-\ln 4\pi+\ln\frac{M^2}{\mu^2}\right)}$$

$$\bullet\ M_\pi^2 = M^2\left[1+(32\tilde{L}_6+16\tilde{L}_8)\frac{M^2}{F^2}-\frac{1}{6F^2}I(M^2)\right]\left[1-\frac{8M^2}{F^2}(2\tilde{L}_4+\tilde{L}_5)+\frac{2}{3F^2}I(M^2)\right]$$

$$= M^2\left[1-\frac{8M^2}{F^2}(2\tilde{L}_4+\tilde{L}_5-4\tilde{L}_6-2\tilde{L}_8)+\frac{1}{2F^2}I(M^2)\right]$$

use: $$2\tilde{L}_4+\tilde{L}_5-4\tilde{L}_6-2\tilde{L}_8 = 2\tilde{L}_4^r+\tilde{L}_5^r-4\tilde{L}_6^r-2\tilde{L}_8^r+\frac{1}{32\pi^2}(2\gamma_4+\gamma_5-4\gamma_6-2\gamma_8)*$$

$$*\left[\frac{2}{d-4}-\ln(4\pi)+\gamma-1\right]$$

SU(2): $2\gamma_4+\gamma_5-4\gamma_6-2\gamma_8 = \frac{1}{8}$

$$\leadsto \boxed{M_\pi^2 = M^2\left[1-\frac{8M^2}{F^2}(2\tilde{L}_4^r+\tilde{L}_5^r-4\tilde{L}_6^r-2\tilde{L}_8^r)+\frac{M^2}{32\pi^2F^2}\ln\frac{M^2}{\mu^2}\right]} \quad (*)$$

lowest order in the quark mass expansion

$$M_\pi^2 = 2\hat{m}B(1+O(M)) = M^2(1+O(M))$$
$$F_\pi^2 = F(1+O(M))$$

* Scale-dependence of the $L_i^r(\mu)$

$$L_i = L_i^r(\mu_1) + \frac{\gamma_i}{(4\pi)^2}\frac{\mu_1^\varepsilon}{\varepsilon} = L_i^r(\mu_2) + \frac{\gamma_i}{(4\pi)^2}\frac{\mu_2^\varepsilon}{\varepsilon} \qquad [\varepsilon = 4-d$$

$$\mu^\varepsilon = \exp\{\varepsilon \ln\mu\} \simeq \varepsilon \ln\mu$$

$$\Rightarrow \quad \boxed{L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\gamma_i}{(4\pi)^2}\ln\left(\frac{\mu_1}{\mu_2}\right)}$$

. slow variation between $\mu = M_\eta \cdots \mu = 1$ GeV

e.g.: $L_1^r(\mu = M_\eta) = 0.4\cdot 10^{-3} + \frac{3/32}{16\pi^2}\ln\left(\frac{M_\rho}{M_\eta}\right) \simeq 0.6\cdot 10^{-3}$

* Good check: <u>OBSERVABLES ARE SCALE-INDEPENDENT</u>

$\downarrow$

choice of μ shuffles individual contributions between loop and counterterms!

* Relation to SU(2): $\tilde{L}_i$ in SU(2), L_i in SU(3)

$$2\tilde{L}_1^r + \tilde{L}_3^r = 2L_1^r + L_3^r - \frac{1}{4}\ell_K \quad ; \quad \tilde{L}_2^r = L_2^r - \frac{1}{4}\ell_K$$

$$2\tilde{L}_4^r + \tilde{L}_5^r = 2L_4^r + L_5^r - \frac{3}{2}\ell_K \quad ; \quad \tilde{L}_9^r = L_9^r - \ell_K$$

$$2\tilde{L}_6^r + \tilde{L}_8^r = 2L_6^r + L_8^r - \frac{3}{4}\ell_K - \frac{1}{12}\ell_\eta \; ; \quad \tilde{L}_{10}^r = L_{10}^r + \ell_K$$

$$\tilde{L}_4^r - \tilde{L}_6^r - 9\tilde{L}_7^r - 3\tilde{L}_8^r = L_4^r - L_6^r - 9L_7^r - 3L_8^r + \frac{3}{2}\ell_K + \frac{F_\pi^2}{24M_\eta^2} + \frac{5}{1152\pi^2}\ln\left(\frac{M_\eta^2}{\mu^2}\right)$$

with $\ell_{K,\eta} \equiv [\ln(M_{K,\eta}^2/\mu^2) + 1]/384\pi^2$

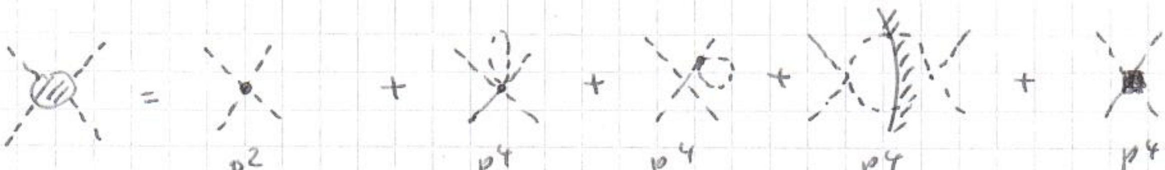
note: In <u>SU(2)</u> one can define <u>scale-independent</u> $\bar{\ell}_i$ $(i = 1,\ldots,7)$

- All are finite, agree with GL, NPB 250 (1985) 465 ✓
- Strictly, this is the η_8 (some mixing)
- Numerically: $$\frac{F_K}{F_\pi} = 1 - 0.01 + \frac{4(M_K^2 - M_\pi^2)}{F_\pi^2} L_5^r(M_\eta^2)$$ $$\frac{F_{\eta_8}}{F_\pi} = \left(\frac{F_K}{F_\pi}\right)^{4/3} + 0.02$$
- Exp ($K^+ \to \mu^+ \nu_\mu$): $$\underline{\frac{F_K}{F_\pi} = 1.23 \pm 0.02} \Rightarrow \underline{L_5^r(M_\eta) = (2.3 \pm 0.2) \cdot 10^{-3}}$$

Ex. 8

APPLICATION 3: $\pi\pi$ SCATTERING TO ONE LOOP

- Don't need detailed calculation anymore!



p^2 done ✓ $\quad$ p^4 trivial ✓ $\quad$ p^4 wf ✓ $\quad$ p^4 imag. parts → done ✓ $\quad$ p^4 cts

$$\boxed{A(s,t,u) = \frac{s - M^2}{F^2} + B(s,t,u) + C(s,t,u)} + O(p^6)$$

unitarity $\qquad$ polynomial correction

- Structure of $B(s,t,u)$

$$B(s,t,u) = \frac{1}{F^4}\Big\{ P_1(s^2, M^4)\,\bar{J}(s) + P_2(t^2, ut, M^2 t, M^2 u, M^4)\,\bar{J}(t) + P_2(t \leftrightarrow u, M^4)\,\bar{J}(u) \Big\}$$

polynom in s^2, M^4 $\qquad$ crossing

"fundamental bubble" done on p. ___

- First derived in the chiral limit by Lehmann, simply based on analyticity (derivation on p. 58a):

$$A(s,t,u) = \frac{s}{F^2} + \frac{1}{96\pi^2 F^4}\left\{ 3s^2 \ln\left(\frac{\mu_1^2}{-s}\right) + t(t-u)\ln\left(\frac{\mu_2^2}{-t}\right) + u(u-t)\ln\left(\frac{\mu_2^2}{-t}\right)\right\}$$

$\Rightarrow$ the two scales are equivalent to two LECs!

- STRUCTURE OF $G(s,t,u)$

general crossing symmetry polynomial of order p^4:

$$\boxed{G(s,t,u) = \underbrace{c_1 s^2 + c_2 (t-u)^2}_{\text{TWO LECs PRESENT IN THE CH. LIMIT}} + \underbrace{c_3 s M^2 + c_4 M^4}_{\text{TWO LECs RELATED TO SHIFTS IN } F \to F_\pi,\ M \to M_\pi}}$$

For explicit forms see: GL, Ann. Phys. (NY) 158 (1984) 142
GM, Phys. Lett.
BKM, Nucl. Phys. B ___ (19 SU(3) $+\pi K$

- DEFINITION OF PHASE SHIFTS

- Ambiguity
$$\boxed{\delta_\ell^I = \mathrm{Re}\, T_\ell^{I(2)} + \mathrm{Re}\, T_\ell^{I(4)} + O(q^6)}$$

or $\delta_\ell^I = \tan^{-1}(\mathrm{Re}\, T_\ell^I) + O(q^6)$

* upper one more commonly used
* only differs in the terms of $O(q^6)$

$\pi\pi$ TO $O(q^4)$ FROM UNITARITY & ANALYTICITY

- Massless case $\quad s+t+u=0$, $A(s,t,u) = s/F^2$ to $O(p^2)$

- only symmetric terms under $t \leftrightarrow u$: $s^2 = (t+u)^2$, tu

$$\rightsquigarrow \boxed{A = \frac{s}{F^2} + G_1 s^2 + G_2 tu + s^2 \log s + \ldots + O(s^3)}$$

from the loops

- Derivation:

$$\boxed{\operatorname{Im} T = |T|^2}$$

$$A = \frac{s}{F^2} + \frac{i}{16\pi^2 F^4}\left\{\frac{1}{2}s^2\,\Theta(s) + \frac{1}{6}t(t-u)\,\Theta(t) + \frac{1}{6}u(u-t)\,\Theta(u)\right\} \quad (*)$$

s,t,u – channel cut

use analyticity, which function produces this $\operatorname{Im} A$?

$$\Rightarrow \quad A = \frac{s}{F^2} + \frac{1}{16\pi^2 F^4}\left\{-\frac{1}{2}s^2\log(-s/s_0) - \frac{1}{6}t(t-u)\log(-t/t_0) - \frac{1}{6}u(u-t)\log(-u/t_0)\right\}$$

same const. because of Bose!

* s_0, t_0 can not be determined from analyticity

$\Rightarrow$ introduce to scale-dependent parameters $G_1(\mu^2), G_2(\mu^2)$

$$\rightsquigarrow \boxed{\begin{aligned} A = {} & \frac{s}{F^2} + G_1(\mu^2)s^2 + G_2(\mu^2)\,tu + \\ & + \frac{1}{16\pi^2F^4}\left\{-\frac{1}{2}s^2\log\left(-\frac{s}{\mu^2}\right) - \frac{1}{6}t(t-u)\log\left(\frac{-t}{\mu^2}\right) - \frac{1}{6}u(u-t)\log\left(-\frac{u}{\mu^2}\right)\right\} \\ G_1(\mu^2) = {} & \frac{1}{32\pi^2F^4}\left(\log\left(\frac{s_0}{\mu^2}\right) + \frac{1}{3}\log\left(\frac{t_0}{\mu^2}\right)\right) \\ G_2(\mu^2) = {} & -\frac{1}{24\pi^2F^4}\log\left(\frac{t_0}{\mu^2}\right) \end{aligned}}$$

(58b)

DERIVATION OF (*):

$$T^I = 32\pi \sum_\ell{}'(2l+1)\, t_{\ell I}(s)\, P_\ell(\cos\theta) \quad , \cos\theta = x \,, \quad s+t+u=0$$

$$\uparrow$$

$$\mathrm{Im}\, t_{\ell I} = |t_{\ell I}|^2 \quad \text{elastic unitarity}$$

From before: $t_{00} = \frac{s}{16\pi F^2}$, $t_{11} = \frac{s}{96\pi F^2} = \frac{1}{6} t_{00}$, $t_{20} = -\frac{s}{32\pi F^2} = -\frac{1}{2} t_{00}$ [others vanish to $O(p^2)$!

$$t = -\frac{s}{2}(1-x)\,, \quad u = -\frac{s}{2}(1+x) \Rightarrow t-u = sx \Rightarrow \boxed{x = \frac{t-u}{s}}$$

$$\mathrm{Im}\, T^0 = 32\pi \cdot 1 \cdot \underbrace{\frac{1}{(16\pi F^2)^2} s^2}_{|t_{00}|^2} \cdot \underset{P_0(x)}{1} = \frac{32\pi}{(16\pi F^2)^2} s^2 = \frac{1}{16\pi F^4} \cdot 2s^2$$

$$\mathrm{Im}\, T^1 = \frac{32\pi}{(16\pi F^2)^2} \cdot 3 \cdot \frac{s^2}{6\cdot 6} \underset{x}{P_1(x)} = \frac{1}{16\pi F^4} \frac{6}{6^2} s^2 \frac{t-u}{s} = \frac{1}{16\pi F^4} \frac{1}{6} s(t-u)$$

$$\mathrm{Im}\, T^2 = \left(-\frac{1}{2}\right)^2 \mathrm{Im}\, T^0 = \quad = \frac{1}{16\pi F^4} \cdot \frac{1}{2} s^2$$

so that:

$$T^0 = 3A(s,t,u) + T^2$$

$$T^1 = A(t,u,s) - A(u,s,t)$$

$$T^2 = A(t,u,s) + A(u,s,t)$$

$$\underline{\mathrm{Im}\, A(s,t,u)} = \frac{1}{3} \mathrm{Im}\, (T^0 - T^2)$$

$$= \frac{1}{16\pi F^4} \frac{1}{3} \underbrace{\left(2s^2 - \frac{1}{2}s^2\right)}_{\frac{3}{2}s^2} = \underline{\frac{1}{16\pi F^4} \frac{s^2}{2}\, \theta(s)}$$

$s>0$ in s-channel!

$$\mathrm{Im}\, T^1 = \frac{1}{16\pi F^4} \frac{1}{6} \underbrace{-(t+u)}_{=s}(t-u)$$

$$\left(\frac{1}{16\pi F^4}\right)\left(-\frac{1}{6}\right)\left[t(t-u) + u(t-u)\right]$$

$$\left(\frac{1}{16\pi F^4}\right)\left(-\frac{1}{6}\right)\left[t(t-u) - u(u-t)\right]$$

$t>0$ in t-channel

$$= \mathrm{Im}\, A(t,u,s) - \mathrm{Im}\, A(u,s,t) \Rightarrow \underline{\mathrm{Im}\, A(t,u,s) = \frac{1}{16\pi F^4}\left(-\frac{1}{6}\right) t(t-u)\, \theta(t)}$$

$$\underline{\mathrm{Im}\, A(u,s,t) = \frac{1}{16\pi F^4} \cdot \left(-\frac{1}{6}\right) u(u-t)\, \theta(u)}$$

$u>0$ in u-channel

$$\sqrt{6}\,U_\eta \to \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\eta&0\\0&\eta\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}\eta&0\\0&\eta\end{pmatrix} \to \eta \to \eta$$

$$U = \begin{pmatrix}a&b\\-b^*&a^*\end{pmatrix},\quad U^* = \begin{pmatrix}a^*&b^*\\-b&a\end{pmatrix},\quad U^\dagger = \begin{pmatrix}a^*&-b\\b^*&a\end{pmatrix},\quad U^\dagger = (U^*)^T$$

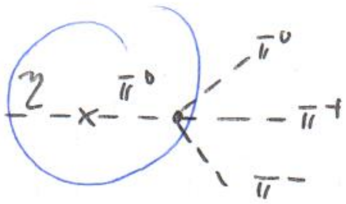
$$\chi U^\dagger + \chi^\dagger U \to \hat{m}\begin{pmatrix}1&0\\0&1\end{pmatrix}(U+U^\dagger) \sim \mathbb{1}\,(U + (U^*)^T)$$

$$(i\tau^2)U^T(-i\tau^2) = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}a&-b^*\\b&a^*\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}a^*&-b\\b^*&a\end{pmatrix} = U^\dagger$$

$$(i\tau^2)U^*(-i\tau^2) = (\quad)\begin{pmatrix}-b&a\\-a^*&-b^*\end{pmatrix}(\quad) = \begin{pmatrix}a&b\\-b^*&a^*\end{pmatrix} = U$$

$$\Rightarrow U + U^\dagger \to U^\dagger + U \quad \checkmark$$

- $m_u \neq m_d$:



Ex. 9

Diagonalize:
$$\mathcal{L}^{(2)} = \tfrac{1}{2}\left(\partial_\mu\pi^0\partial^\mu\pi^0 - M_{\pi^0}^2\pi^{0\,2}\right) + \tfrac{1}{2}\left(\partial_\mu\eta\,\partial^\mu\eta - M_\eta^2\eta^2\right)$$
$$+\, a\,\eta\,\pi^0(\pi^{0\,2} + 2\pi^+\pi^-) + b\left(\partial_\mu(\pi^0\eta)\partial^\mu(\pi^+\pi^-) + \ldots\right)$$

with $a, b \sim (m_d - m_u)$ (work this out in more detail!)

- Amplitude:
$$\boxed{\mathcal{M}(\eta \to \pi^0\pi^+\pi^-) = \frac{1}{F_\pi^2}\frac{\delta m^2}{\sqrt{3}}\left(1 - \frac{2E_0}{M_\eta}\right)}$$

Some details on page 20a

- $\delta m^2 = M_{K^+}^2 - M_{K^0}^2 - M_{\pi^+}^2 + M_{\pi^0}^2 \sim m_d - m_u$ ✓
- E_0 = energy of the π^0
- $\eta = \eta_8$ (mixing neglected)

$$\Rightarrow \boxed{\Gamma(\eta \to \pi^+\pi^-\pi^0) = 66\ \text{eV}} \qquad \text{exp}: (310 \pm 20)\,\text{eV}$$

- LARGE DISCREPANCY → higher orders in ... ?