



Solving the left-hand cut problem in lattice QCD: $T_{cc}(3875)$ from the E_{FV}

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15th Dec, 2023, 强子物理在线论坛

Base on [JHEP10\(2021\)051](#), [PoS LATTICE2022 \(2023\) 201](#) and [arXiv:2312.01930](#)

Together with V. Baru, E. Epelbaum, A. Filin, A.M. Gasparyan

Lattice QCD and finite volume energy levels

- QCD is the fundamental theory of the strong interaction

$$\mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M}_{q_f}) - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a}$$

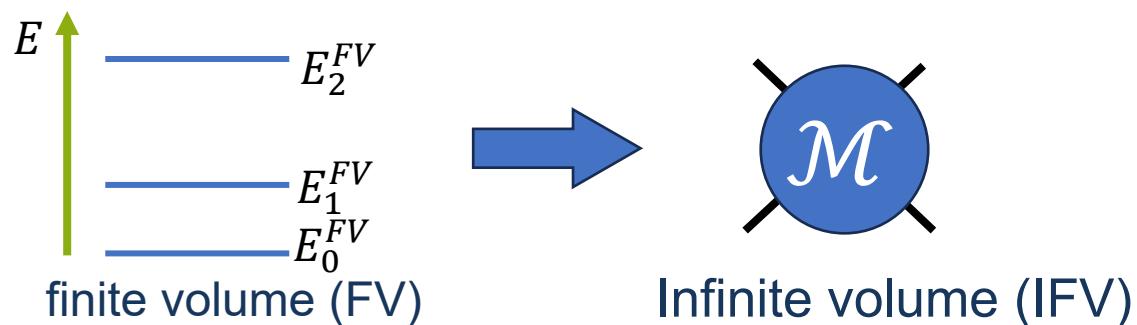
- Extract hadron-hadron interactions from QCD? Lattice QCD

► formulated on a lattice of points in space and time in a finite volume (FV)

- How to extract observables in the infinite volume (IFV) from a FV calculation?

► Energy level method: Lüscher's formula, $E^{FV} \sim \delta_l(E^{FV})$

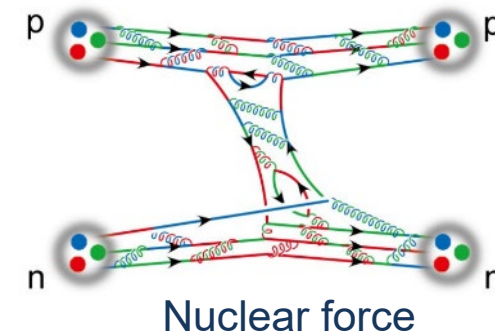
Luscher:1990ux



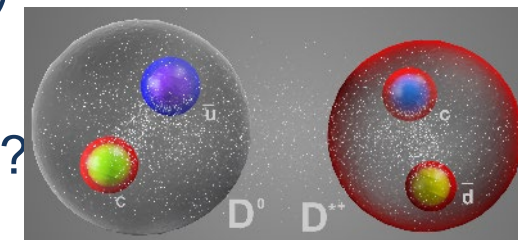
AKA: Lüscher Quantization conditions (LQCs)

► Potential (or HAL QCD) method: Bethe–Salpeter amplitude → potential

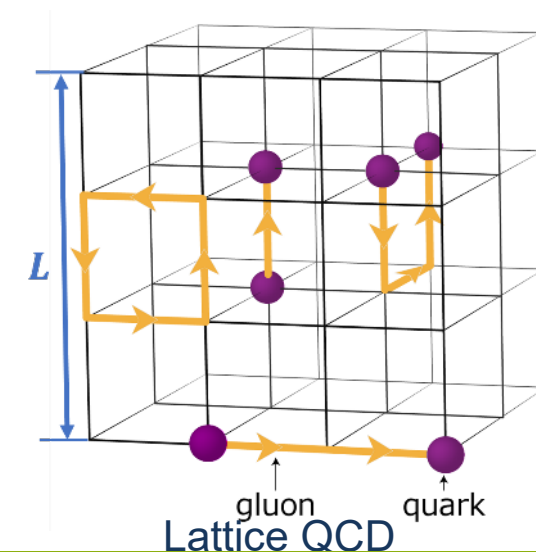
Ishii:2006ec



Nuclear force

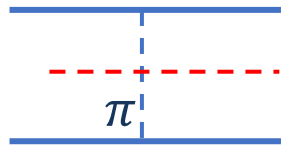


Hadronic molecules



Left-hand cut

- Left-hand cut (lhc) from the one-pion exchange interaction



Left-hand cut

$$V(\vec{p}, \vec{p}') = \frac{1}{(\vec{p}' - \vec{p})^2 + m^2}$$

- Partial wave decomposition, e.g. S-wave

$$V_{l=0}(p, p') = \int_{-1}^1 dz \frac{1}{p^2 + p'^2 - 2pp'z + m^2} = -\frac{1}{2pp'} \log \left(\frac{(p - p')^2 + m^2}{(p + p')^2 + m^2} \right)$$

- On-shell $p = p' = k$, $k^2 = 2\mu E$

$$V_{l=0}(k, k) = \int_{-1}^1 dz \frac{1}{2k^2(1 - z) + m^2}$$

$$2k^2(1 - z) + m^2 = 0 \Rightarrow z = \frac{m^2}{2k^2} + 1$$

on-shell pion

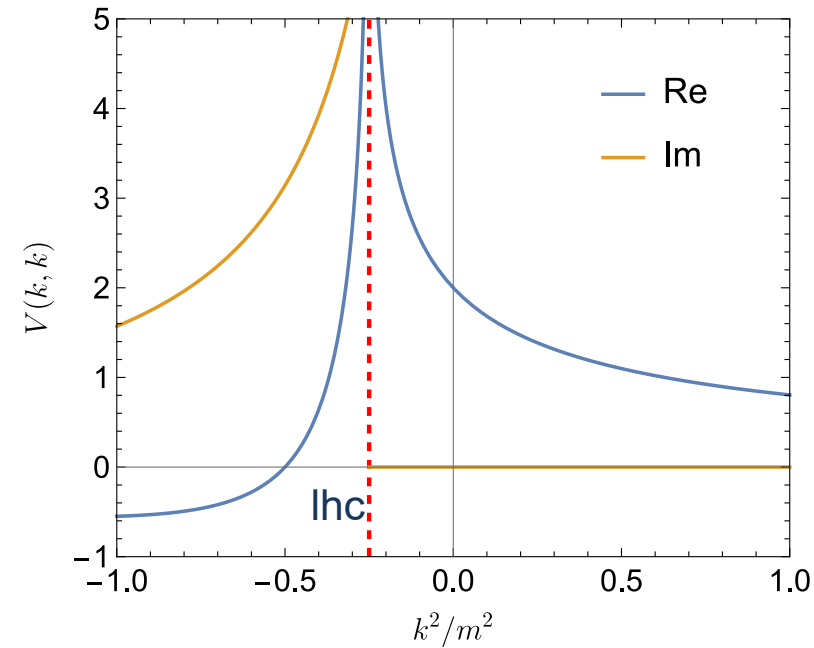
$$-1 < z < 1 \Rightarrow k^2 < -\frac{m^2}{4}$$

► Branch point: $k^2 < -\frac{m^2}{4}$

- K-matrix

$$K = V + VG^{\mathcal{P}}K$$

► The imaginary part arises



Left-hand problems for LQCs

- Lüscher formula: $\det [G_F^{-1}(L, E) - K(E)] = 0$

kinematic term

K matrix in the IFV

- Effective range expansion (ERE)

$$K^{-1}(p) = p \cot \delta(p) = \frac{1}{a} + \frac{1}{2} r p^2 + \dots$$

- ▶ Well used in lattice community
- ▶ No left-hand cut

- NN system: modified ERE



Baru:2015ira, Baru:2016evv

- DD^* at unphysical pion mass: Chiral effective field theory (ChEFT) Du:2023hlu

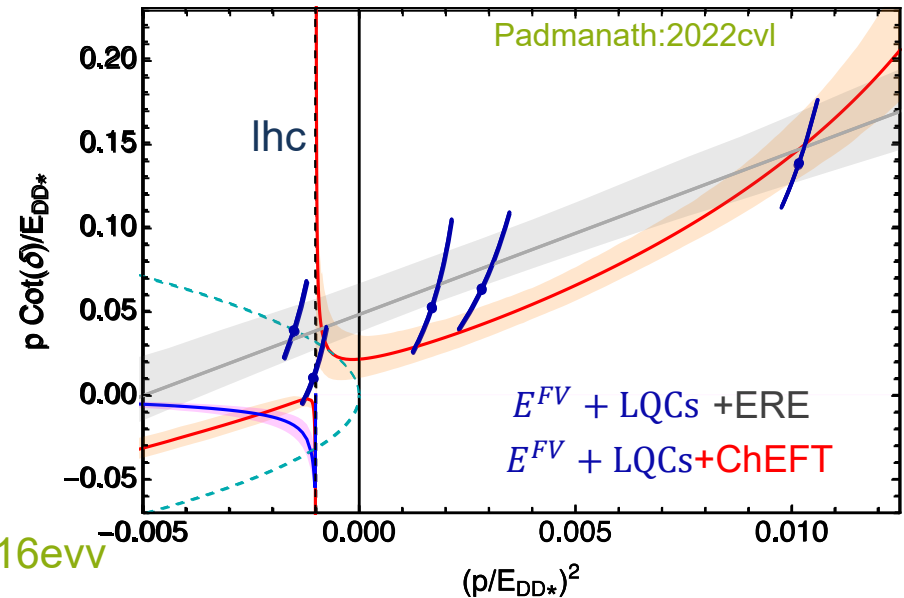
Infinite volume

- ▶ $K^{-1}(p) \neq \frac{1}{a} + \frac{1}{2} r p^2 + \dots$

Rely on the δ_l from the Lüscher formula

- ▶ $V = V_{OPE} + V_{ctc}^{LO} + V_{ctc}^{NLO}$ and Lippmann-Schwinger eq. (LSE)

- The Lüscher formula fails because of the left-hand cut



- ▶ Finite volume effect

- ▶ Heated discussions in Lattice2023 conference

<https://indico.fnal.gov/event/57249/>

Finite-volume scattering in the left-hand cut

$$\det [G_F^{-1}(L, E) - K(E)] \neq 0$$

Andre Baião Raposo

Curia II, WH2SW

13:30 - 13:50

Breakdown of Lüscher Formalism near Left Hand Cuts

real

$$\text{Im } K \neq 0 \text{ for } p^2 < -\frac{m^2}{4}$$

Md Habib E Islam

Curia II, WH2SW

13:50 - 14:10

Resolving the left-hand-cut problem in lattice studies of the doubly-charmed tetraquark

Steve Sharpe

Curia II, WH2SW

14:10 - 14:30

- Lüscher's formula
- Our formalism
- $T_{cc}(3875)^+$ state
- Summary and outlook

Lüscher's formula

- Boundary conditions in the cubic box

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \psi(\mathbf{x}_1 + \mathbf{n}_1 L, \mathbf{x}_2 + \mathbf{n}_2 L)$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}, \quad \mathbf{p}_1 = \frac{2\pi}{L} \mathbf{n}, \quad \mathbf{P} = \frac{2\pi}{L} \mathbf{d}, \quad \mathbf{n}, \mathbf{d} \in \mathbb{Z}^3$$

- ▶ 2-body rest systems: $\mathbf{d} = (0,0,0)$

- The rotation symmetry is broken: $SO(3) \rightarrow O_h$

- ▶ $\{l, m\}$ are not good quantum numbers to label states

- ▶ Partial wave mixing, for $l \neq l'$ and $m \neq m'$,

$$\langle lm | H^{FV} | l' m' \rangle \neq 0$$

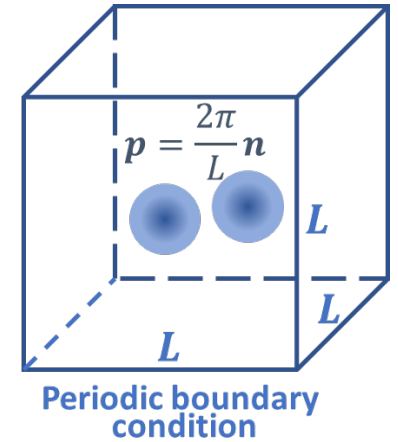
- ▶ The FV energy should be classified by irreducible representations (irreps.) of O_h

$$\{l, m\} \rightarrow \{A_1, A_2, E, T_1, T_2\}$$

- Moving system in the box $\mathbf{P} = \frac{2\pi}{L} \mathbf{d} \neq 0$

- ▶ Other point groups, D_{4h} , D_{2h} ...

- ▶ If $m_1 \neq m_2$, space inversion invariance is broken; states with different parities could mix



Detailed derivation of Lüscher's formula

$$C_L(P) = \text{Diagram 1} + \text{Diagram 2} + \dots$$

$$\text{Diagram 3} = \text{Diagram 4} + \text{Diagram 5} + \dots$$

$$\text{Diagram 6} = \text{Diagram 7} + \text{Diagram 8} \qquad \text{Diagram 9} \equiv \text{Diagram 10}$$

$$C_L(P) = C_\infty(P) + \text{Diagram 11} + \text{Diagram 12} + \dots$$

$$\text{Diagram 13} = \text{Diagram 14} + \text{Diagram 15} + \text{Diagram 16} + \dots$$

$$\text{Diagram 17} = \text{Diagram 18} + \text{Diagram 19} + \dots$$

Note: all the $\int$ should be treated in the sense of P.V.

• F cut

$$\left[\frac{1}{L^3} \sum_k -\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$

► Only the singularities are important

• Within on-shell approximation:

$$G_F + G_F K G_F + \dots = G_F (1 - K G_F)^{-1} = (G_F^{-1} - K)$$

$$C_L(P) = C_\infty(P) + iA \frac{1}{G_F^{-1}(E, L) - K(P)} A^\dagger$$

• Poles of C_L is the FV energy levels

• Lüscher's formula: $\det [G_F^{-1} - K] = 0$

Luscher:1990ux, Kim:2005gf, Polejaeva:2012ut

- Expanding it in partial wave (PW) basis

$$\det[G_F - K^{-1}] = 0, \Rightarrow \det[M_{l'l'm',lm} - \delta_{ll'}\delta_{mm'} \cot \delta_l] = 0$$

- ▶ Determinate equation of a matrix with infinite dimensions.
- ▶ Truncate at some $l_{\max}$
- Reduce to irreps. Γ_i of point group:: $\det [M_{ln,l'n'}^{(\Gamma,P)} - \delta_{ll'}\delta_{nn'} \cot \delta_l] = 0$
- Example $\mathbf{d} = (0,0,1)$, $\Gamma = A_1^+$, w_{lm} depends on E but independent on V

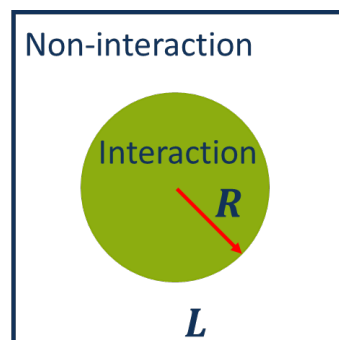
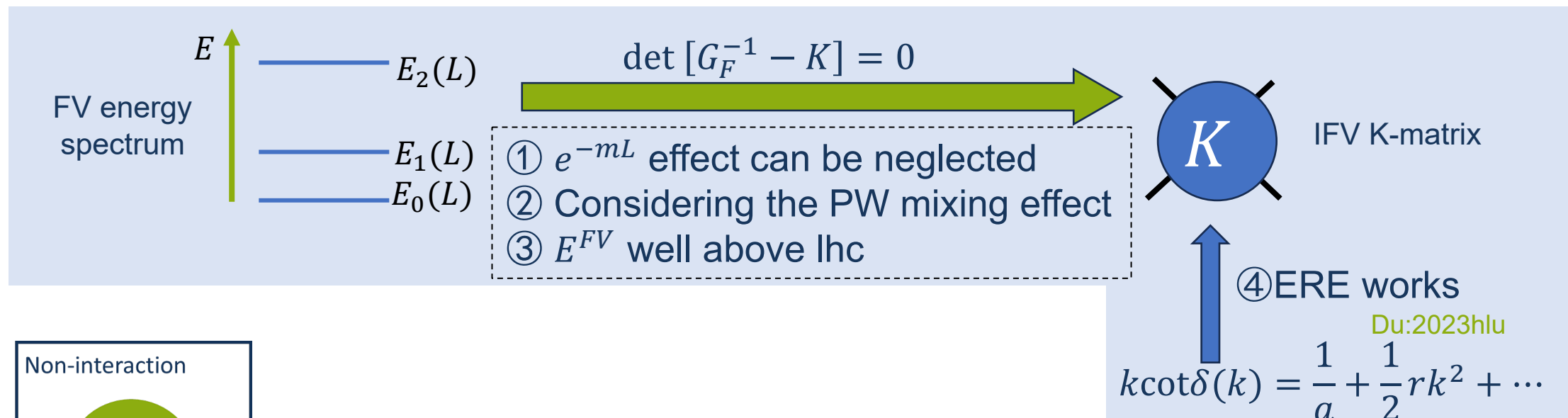
$$\det [M_{ln,l'n'}^{(\Gamma,P)} - \delta_{ll'}\delta_{nn'} \cot \delta_l] = 0, \quad M^{(A_1^+, \mathbf{d})} = \begin{bmatrix} w_{00} & -\sqrt{5}w_{20} & \cdots \\ -\sqrt{5}w_{20} & w_{00} + \frac{10}{7}w_{20} + \frac{18}{7}w_{40} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Bernard:2008ax

- Truncate at $l_{\max} = 0$, one-to-one relation: $\delta_0(E^{FV}) \sim E^{FV}$
- Truncate at $l_{\max} > 0$, no one-to-one relation
 - ▶ E.g. $\{E_1^{FV}, E_2^{FV}\} \not\Rightarrow \{\delta_S(E_1^{FV}), \delta_S(E_2^{FV}), \delta_D(E_1^{FV}), \delta_D(E_2^{FV}) \dots\}$
 - ▶ One has to parameterize the K-matrix: e.g. effective range expansions (ERE)

Luscher:1990ux, Rummukainen:1995vs, Feng:2004ua, Kim:2005gf, Fu:2011xz, Polejaeva:2012ut, Leskovec:2012gb, Gockeler:2012yj, ...

Requirements of a practical Lüscher method



Requirement: $\frac{L}{2} \gg R$

Typically: $m_\pi L > 3$

High partial wave suppression

- Threshold effect: $T_l(p) \sim p^{2l}$
- The large scale: m_π

Taylor's textbook P197



All four requirements constrained by the $V_{1\pi}$

Our strategy

August 2021



25 Aug 第三十届HAPOF论坛, Two-particle scattering from finite-volume quantization conditions using the plane wave basis, 孟璐 博士, 2021年08月25日

Solving the lhc problem

- The lhc problems arise from taking OPE on-shell

- ▶ Keep OPE off-shell
- ▶ Modified Lüscher QCs [Raposo:2023oru](#)

时时勤拂拭，勿使惹尘埃

- In the FV, given discrete momenta and energies, why the “cut” becomes a problem?

$$\det [G_F^{-1}(L, E) - K(E)] = 0$$

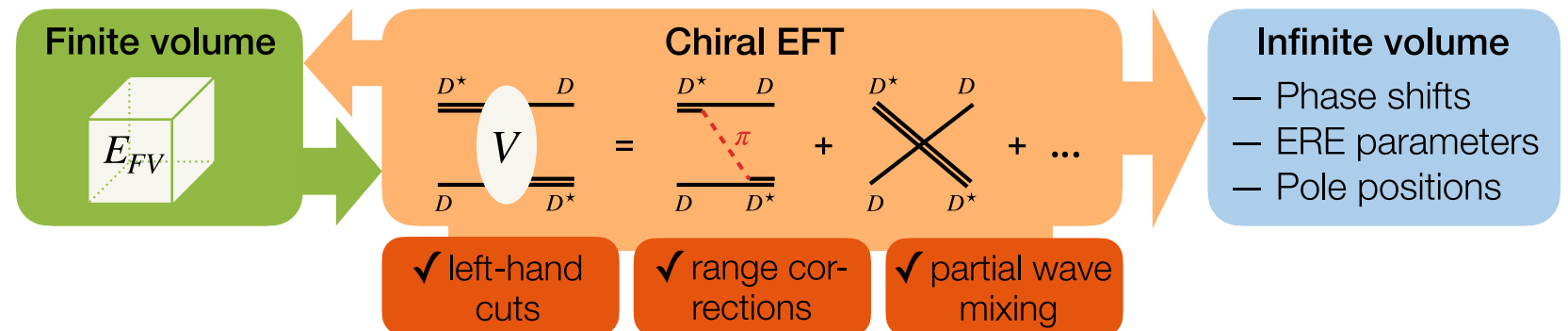
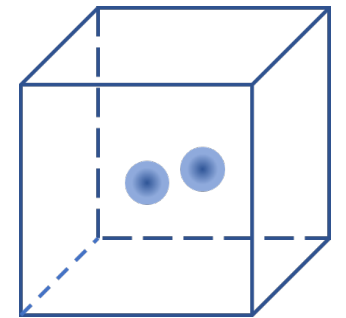
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- ▶ The on-shell K in the infinite volume in LQCs is the problem

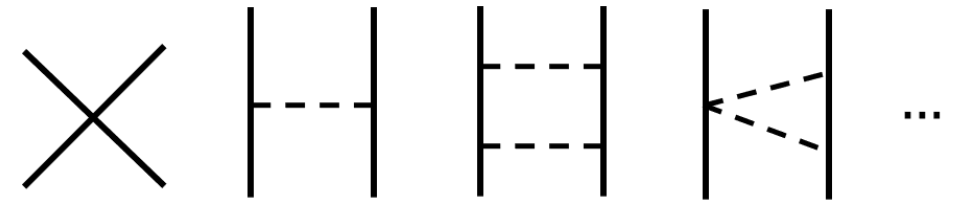
- ▶ Given a interaction, solve the Schrödinger eq. in the FV $\Rightarrow E^{FV}$
- $$\frac{\mathbf{p}^2}{2\mu} \psi(\mathbf{p}) + \int \frac{d^3 \mathbf{p}'}{(2\pi)^3} V(\mathbf{p}, \mathbf{p}') \psi(\mathbf{p}') = E \psi(\mathbf{p}), \quad \int \frac{d^3 \mathbf{p}'}{(2\pi)^3} \rightarrow \frac{1}{L^3} \sum_{\mathbf{p}_n}$$

Off-shell, for $E < 0$

- ▶ Plane wave basis with discrete momentum: cubic group symmetry
- ▶ Include the OPE explicitly



$$V(\vec{p}', \vec{p}) = V_{\text{contact}} + V_{1\pi} + V_{2\pi}$$

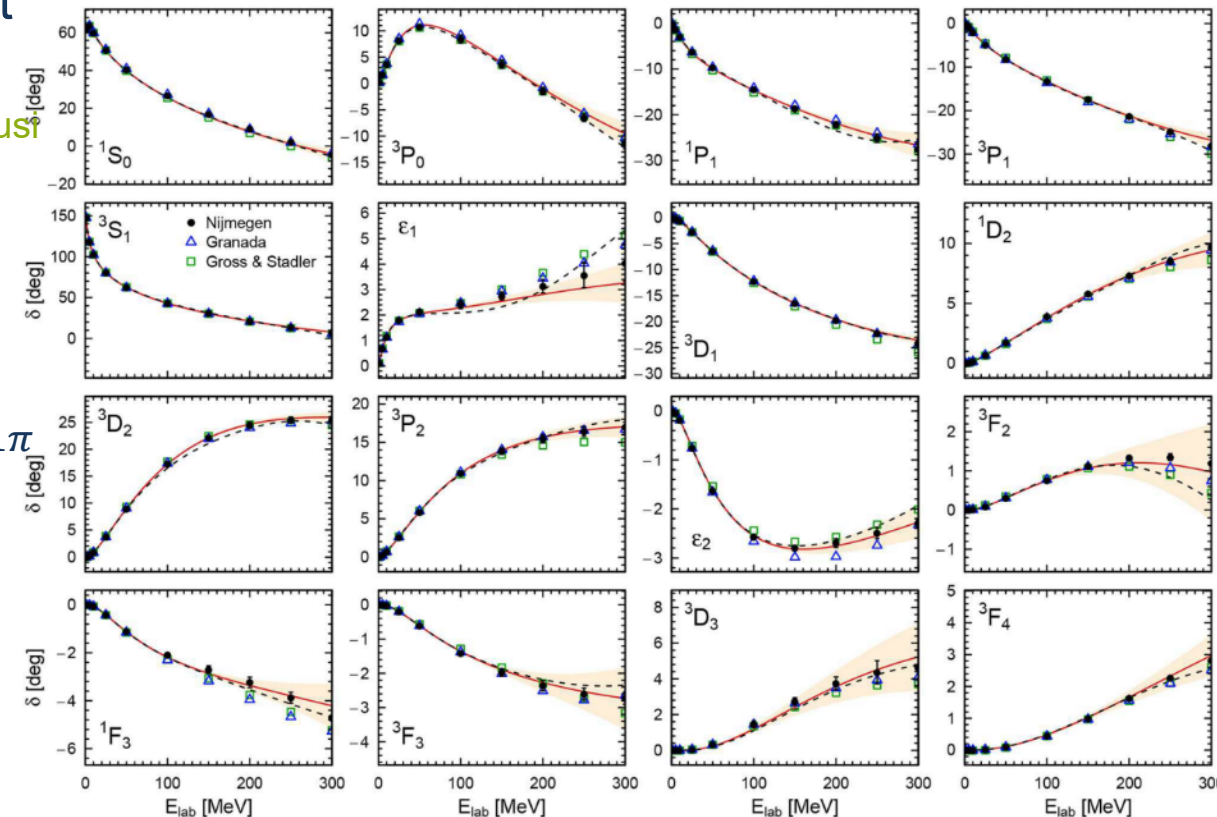


- Derived in the momentum space, E -independent
- Semilocal momentum-space regularization

Reinert:2017usi

$$V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4F_\pi^2} \left(\frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_\pi^2} + C(m_\pi) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) e^{-\frac{q^2 + m_\pi^2}{\Lambda^2}}$$

- Benefit from the known long-range interaction $V_{1\pi}$
- Low energy constants (LECs) for short-range interaction (contact interaction)
 - fitting lattice QCD data



Our formalism: plane wave basis expansion

- $|\mathbf{p}_n, \boldsymbol{\eta}\rangle$: $\mathbf{p}_n$ discrete momentum, $\boldsymbol{\eta}$: polarization vector for $S = 1$

$$\hat{D}(g)|\mathbf{p}, \boldsymbol{\eta}\rangle = |g\mathbf{p}, g\boldsymbol{\eta}\rangle, \hat{P}|\mathbf{p}, \boldsymbol{\eta}\rangle = |-\mathbf{p}, \boldsymbol{\eta}\rangle, \langle \mathbf{p}_{n'}, \boldsymbol{\eta}'^\dagger | \hat{D}(g) | \mathbf{p}_n, \boldsymbol{\eta} \rangle = \delta_{n'n} (\boldsymbol{\eta}'^\dagger \cdot g\boldsymbol{\eta})$$

- $\{|\mathbf{p}_n, \boldsymbol{\eta}\rangle\}$ form the representation space of corresponding point group
- For non-relativistic systems, Lippmann-Schwinger equation (LSE)
 - ▶ matrix equation $\mathbb{T} = \mathbb{V} + \mathbb{V}\mathbb{G}\mathbb{T}$
 - ▶ Finite volume levels $\Rightarrow$ Eigenvalue problem

$$\det(\mathbb{G}^{-1} - \mathbb{V}) = 0 \rightarrow \det(\mathbb{H} - E\mathbb{I}) = 0,$$

- ▶ Reduce the $\mathbb{H}$ according to irreducible representations (irreps) of the point group

$$\mathbb{H} \Rightarrow \text{diag}\{\mathbb{H}_{\Gamma_i}, \mathbb{H}_{\Gamma_j}, \dots\} \Rightarrow \mathbb{H}_{\Gamma} \mathbf{v} = E_{\Gamma} \mathbf{v}$$

- For moving systems, elongated boxes, particles with arbitrary spin...
- dim of the $\mathbb{H}_{\Gamma}$: cubic function

$$\dim \sim \left(\frac{\Lambda_{\text{UV}}}{2\pi/L} \right)^3 \sim \mathcal{O}(1000), \quad \Lambda_{\text{UV}} \approx 4\text{GeV}$$

Similar approaches: Momentum lattice ([Doring:2011ip](#)), Hamiltonian EFT([Wu:2014vma](#), [Liu:2015ktc](#))...

Towards a practical approach: eigenvector continuation

- Plane wave basis+Eigenvector continuation

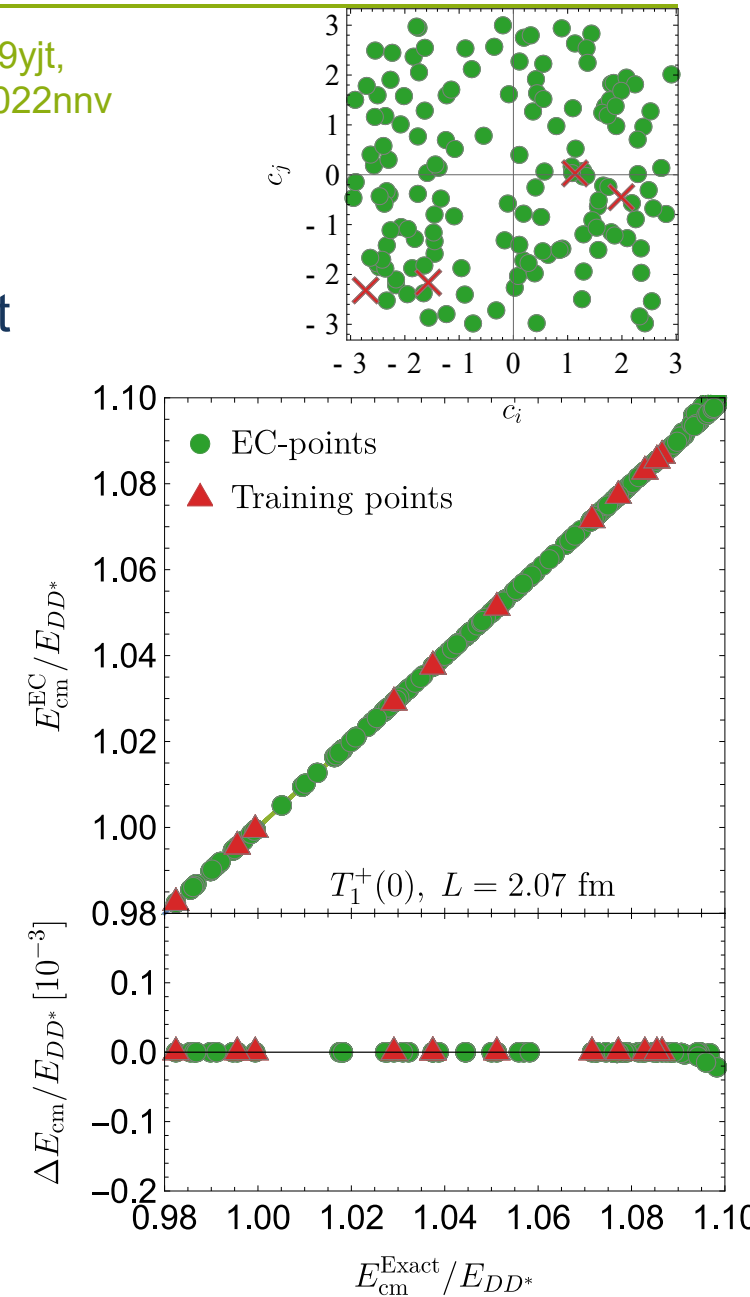
► Eigenvector continuation (EC) with subspace learning

Frame:2017fah, Demol:2019yjt,
Furnstahl:2020abp, Yapa:2022nnv

- To fit or quantify uncertainty: solve eigenvalue problem with different $\{c_i\}$ repeatedly
- EC basis: eigenvectors from a selection of parameter sets $\{c_i\}_1, \{c_i\}_2, \dots$ (training point)
- Naturalness of LEC in EFT (~ 1) makes the EC more reliable
- dim is linear function

$$\dim^{EC} = \frac{p_{max}}{2\pi/L} \sim \mathcal{O}(10), \quad p_{max} \approx 0.6 \text{ GeV}$$

- The subspace learning is the one-time cost
- Make the calculation fast and accurate

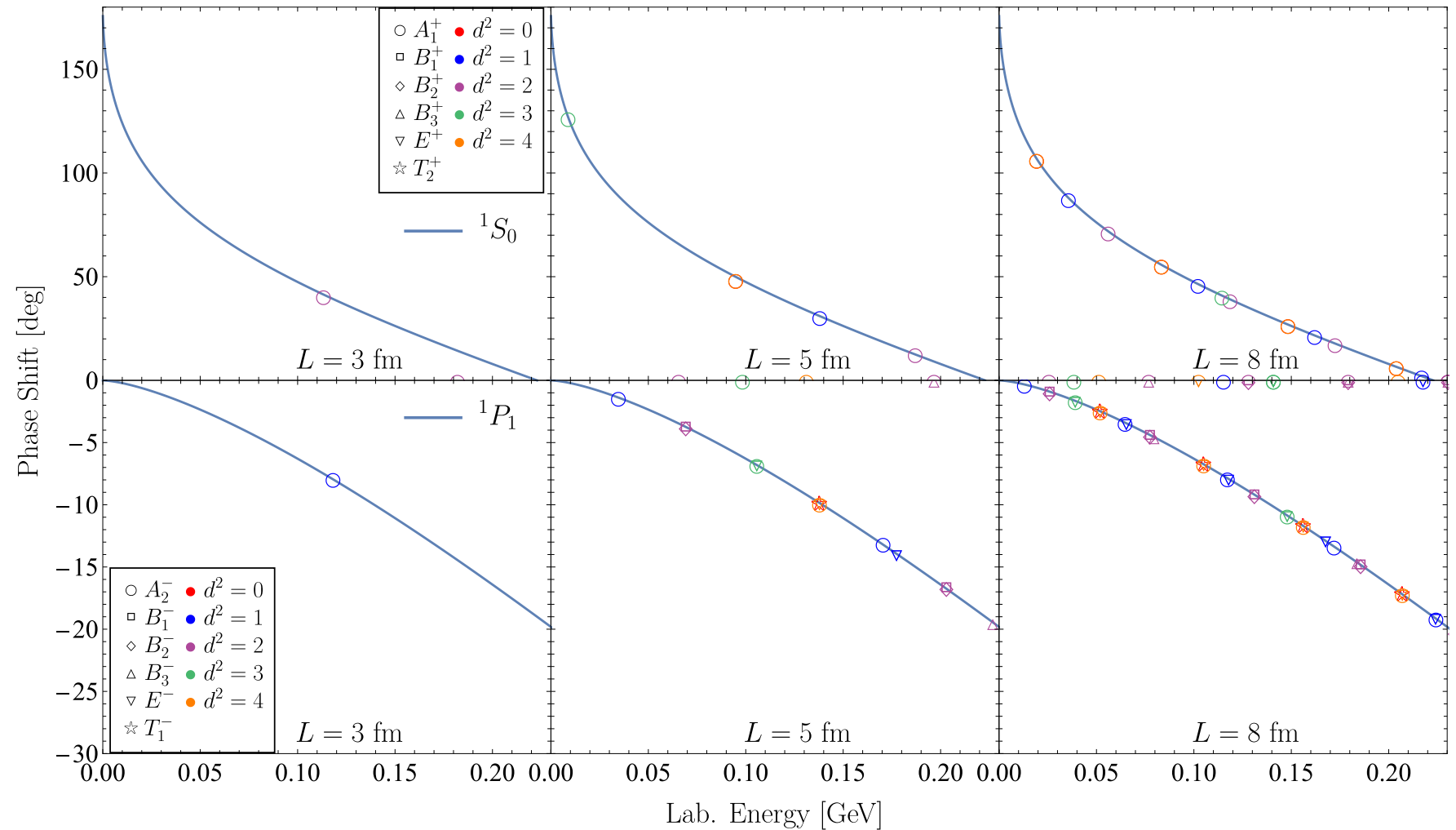
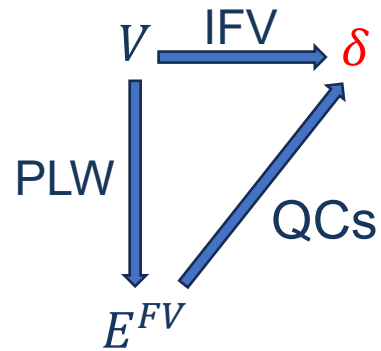


January 2022

21 Jan 第四十届HAPOF论坛, New results on eigenvector continuation and quantum mechanical bootstrap, 柏栋 副研究员, 2022年01月21日

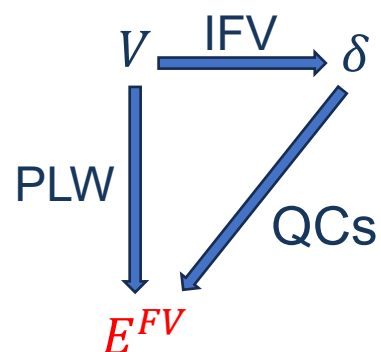
Benchmark: contact interaction

- Contact interaction: $V(\mathbf{p}, \mathbf{p}') = C_S + C_1 \mathbf{q}^2 + C_2 \mathbf{k}^2$
- Only contribute to S-wave and P-wave

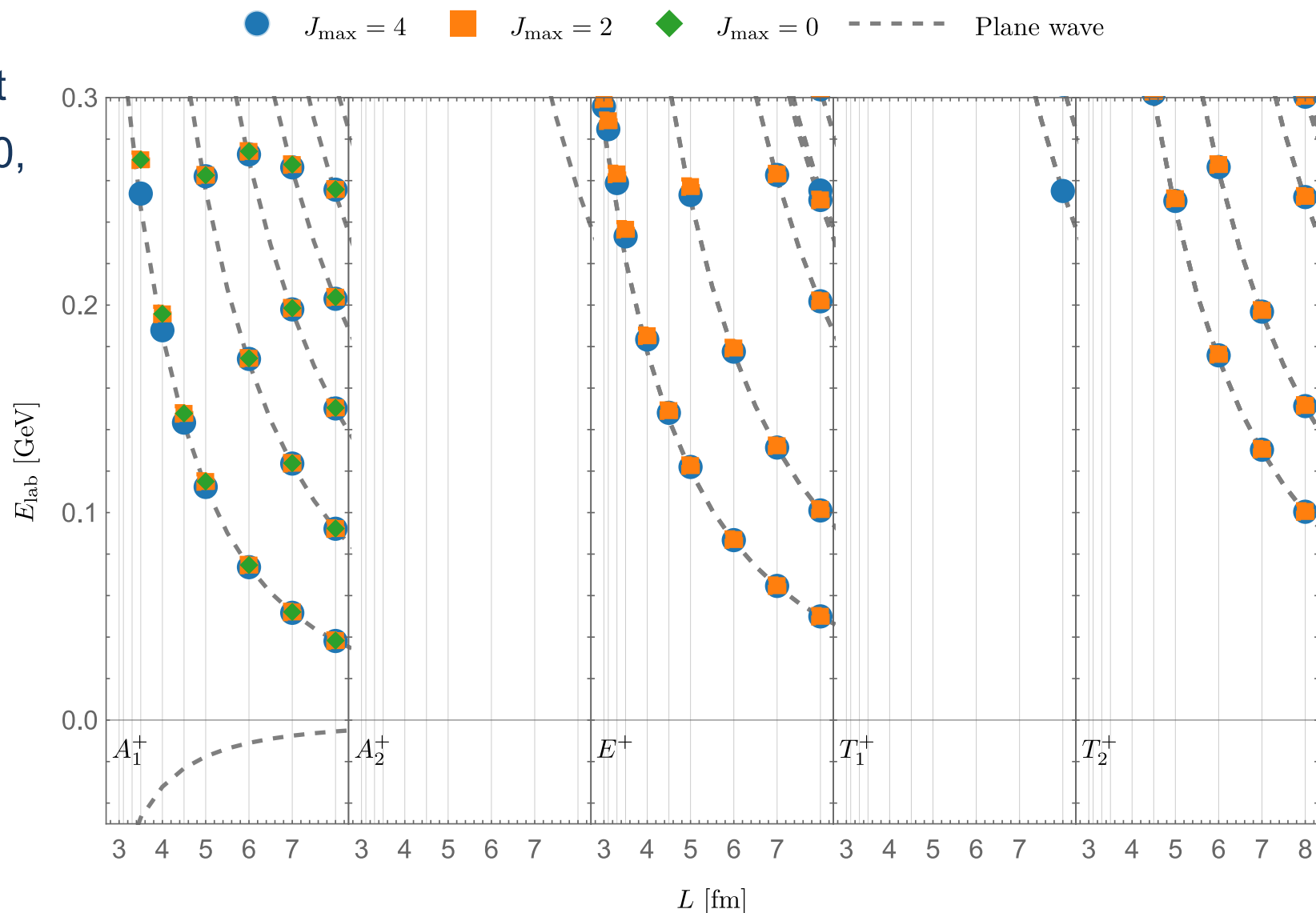


Benchmark: chiral EFT

- ChEFT nuclear force: NNLO
- $S=0$, $d = (0,0,0)$, even parity
- QCs with partial mixing effect
- $L=\{3.0, 3.1, 3.3, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0\}$ fm

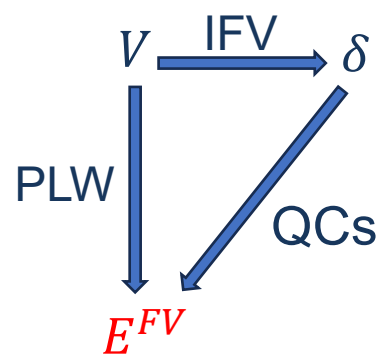


- The discrepancy
 - ▶ Small box
 - ▶ Small $J_{\max}$ truncation

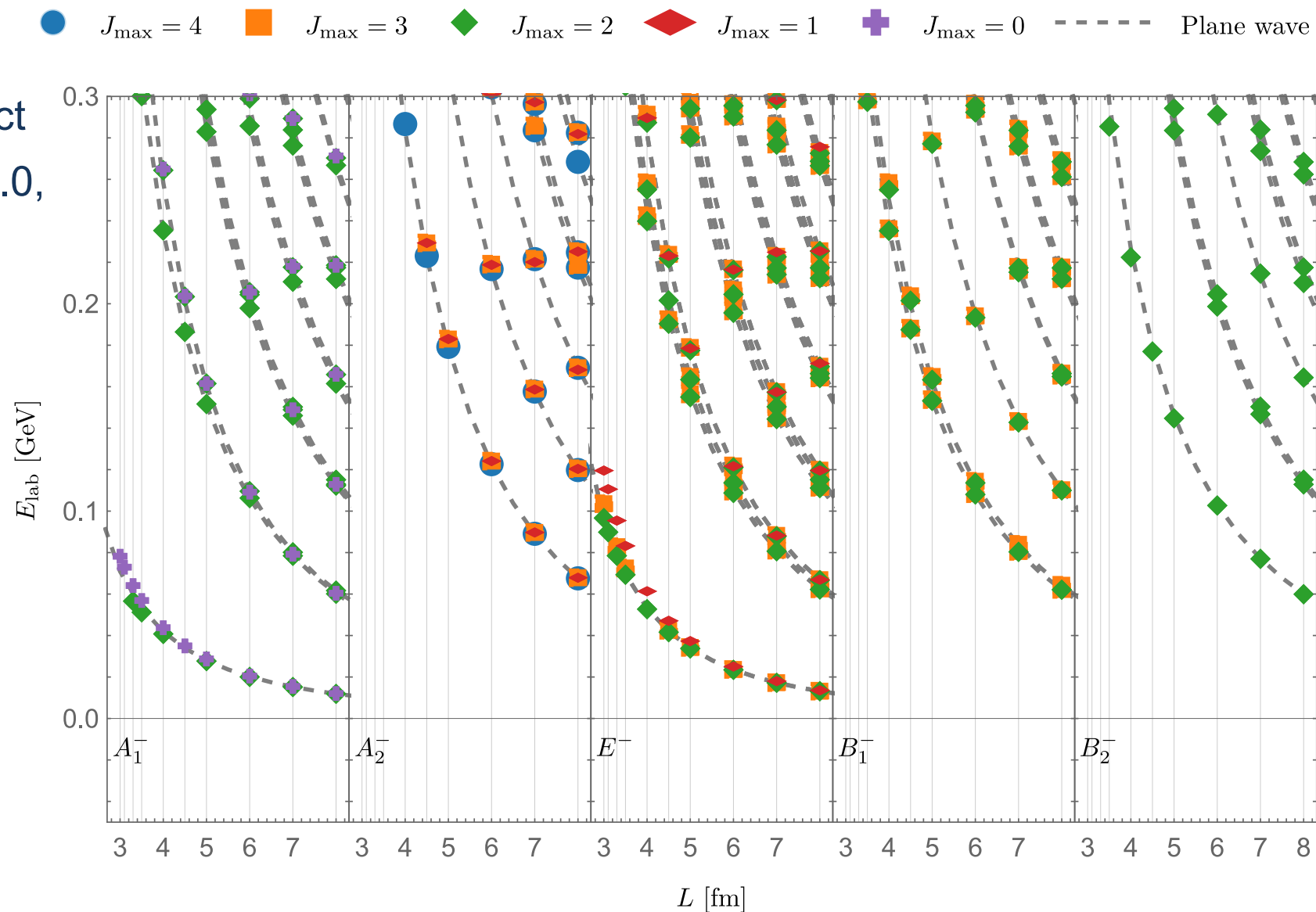


Benchmark: chiral EFT

- ChEFT nuclear force: NNLO
- $S=1$, $d=(0,0,1)$, odd parity
- QCs with partial mixing effect
- $L=\{3.0, 3.1, 3.3, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0\}$ fm

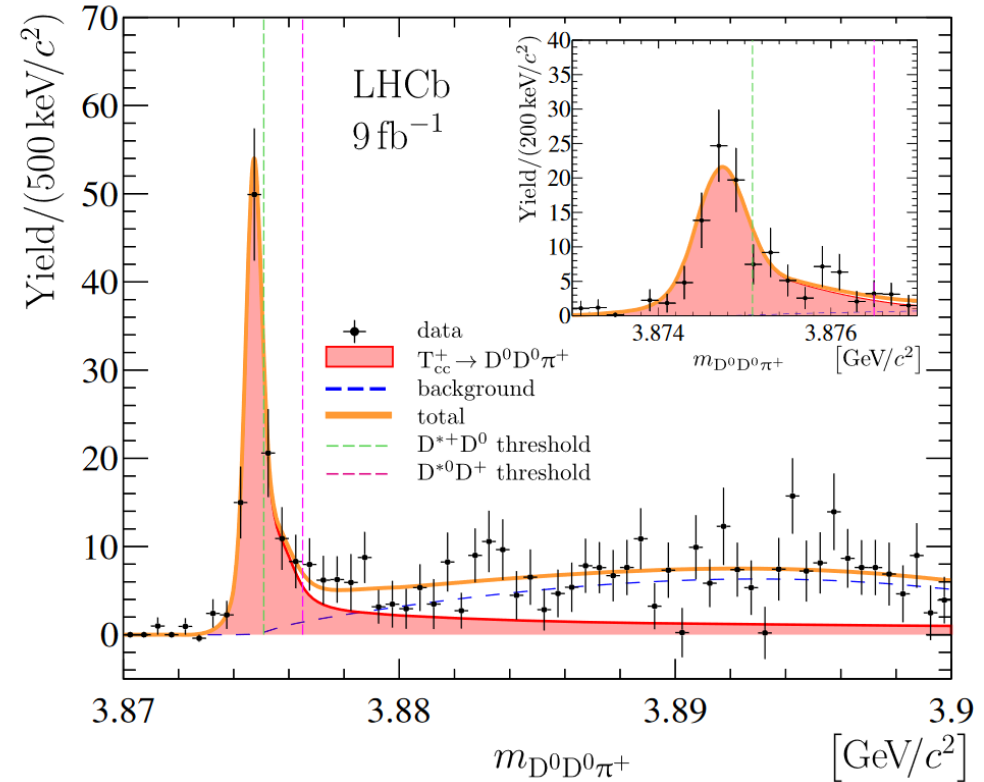
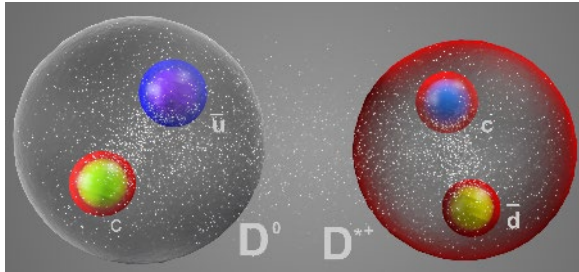


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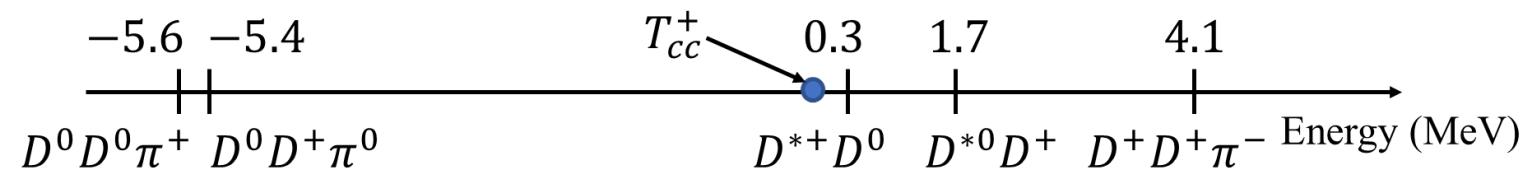


$T_{cc}(3875)^+$ **state**

- $T_{cc}(3875)^+$ was observed in 3-body final states: $D^0 D^0 \pi^+$
- Very close to $D^0 D^{*+}$ thresholds: $\delta m_U \approx -360 \text{ keV}$, $\Gamma \approx 48 \text{ keV}$
- Exotic hadrons: minimal quark content: $cc\bar{u}\bar{d}$
- Good candidates of $D^0 D^{*+}$ molecule



- 3-body dynamics could be important



LHCb:2021vvq, LHCb:2021auc, Du:2021zzh, Meng:2021jnw...

Padmanath:2022cvi

● LQCD: $m_\pi \approx 280$ MeV, $m_D \approx 1927$ MeV, $m_{D^*} \approx 2049$ MeV, $L \approx 2.07, 2.76$ fm, $a \approx 0.086$ fm

● Some quick estimations

▶ $m_{\text{eff}}^2 = m_\pi^2 - (m_{D^*} - m_D)^2 > 0, m_{\text{eff}} \approx 252$ MeV

▶ $m_{\text{eff}}L = 2.6, 3.5$

▶ $p_{\text{lhc}}^2 \approx -\left(\frac{m_{\text{eff}}}{2}\right)^2 = -(126 \text{ MeV})^2$

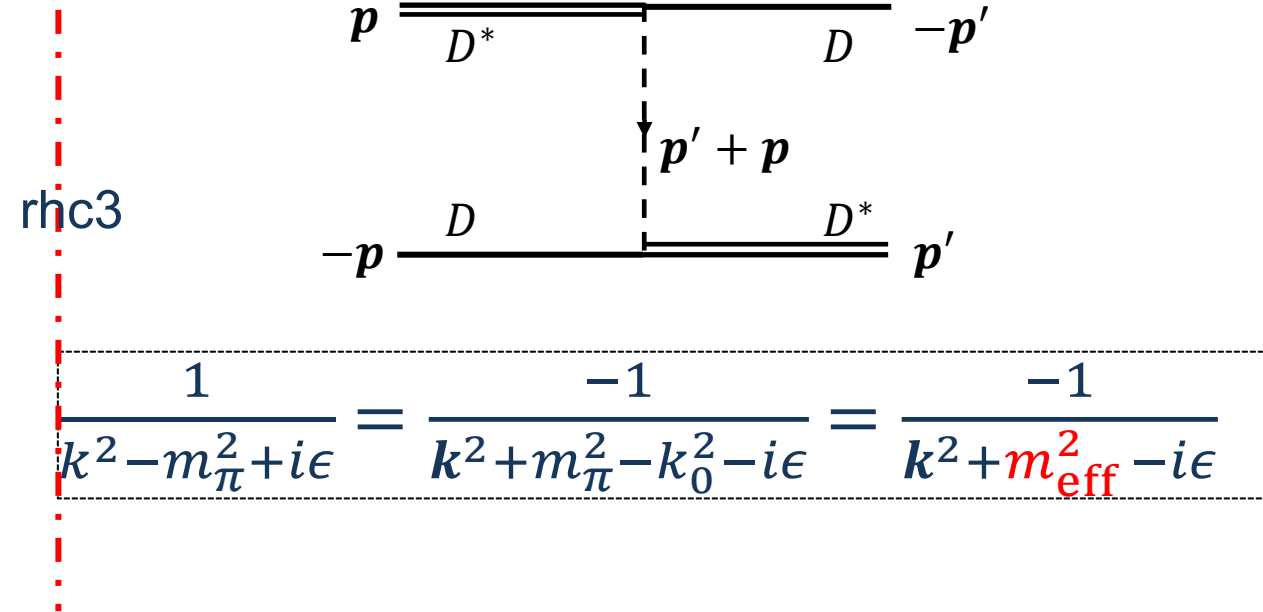
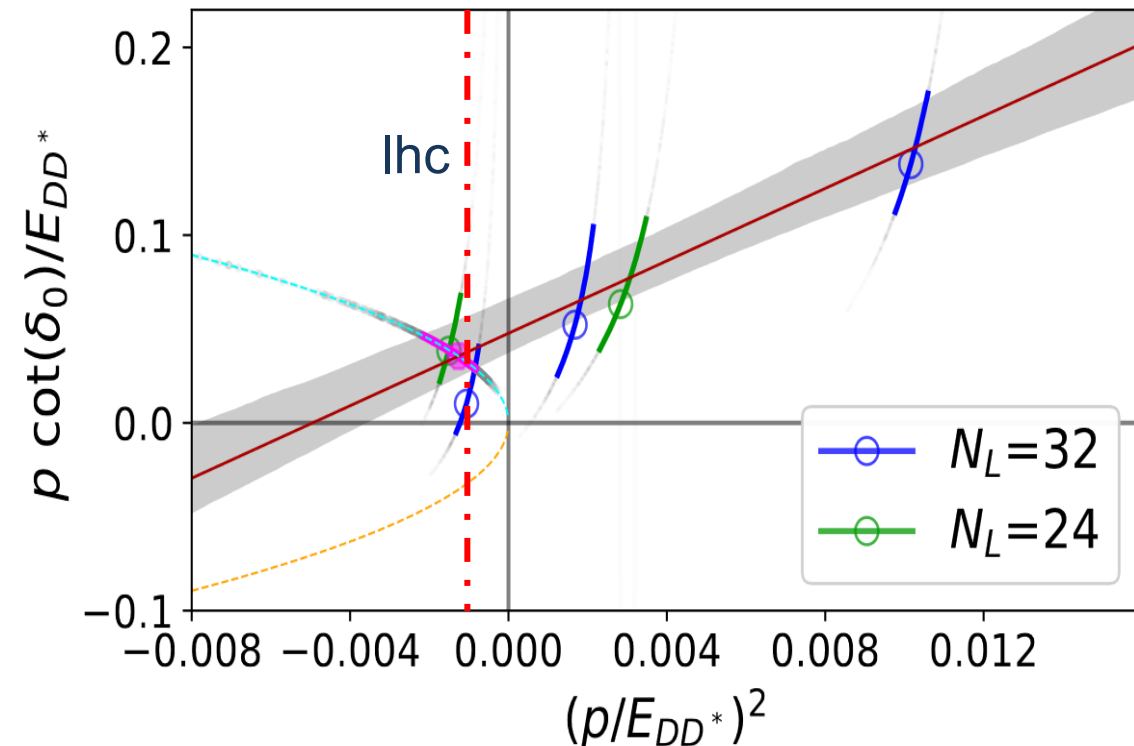
▶ $p_{\text{rhc3}}^2 \approx 2\mu_{DD^*}(2m_D + m_\pi - m_D - m_{D^*}) \approx (560 \text{ MeV})^2$

- ① e^{-mL} effect can be neglected

② Considering the PW mixing effect

③ E^{FV} well above lhc

④ ERE works in IFV Du:2023hlu



Other lattice results: Cheung:2017tnt, Junnarkar:2018twb,
Chen:2022vpo, Lyu:2023xro

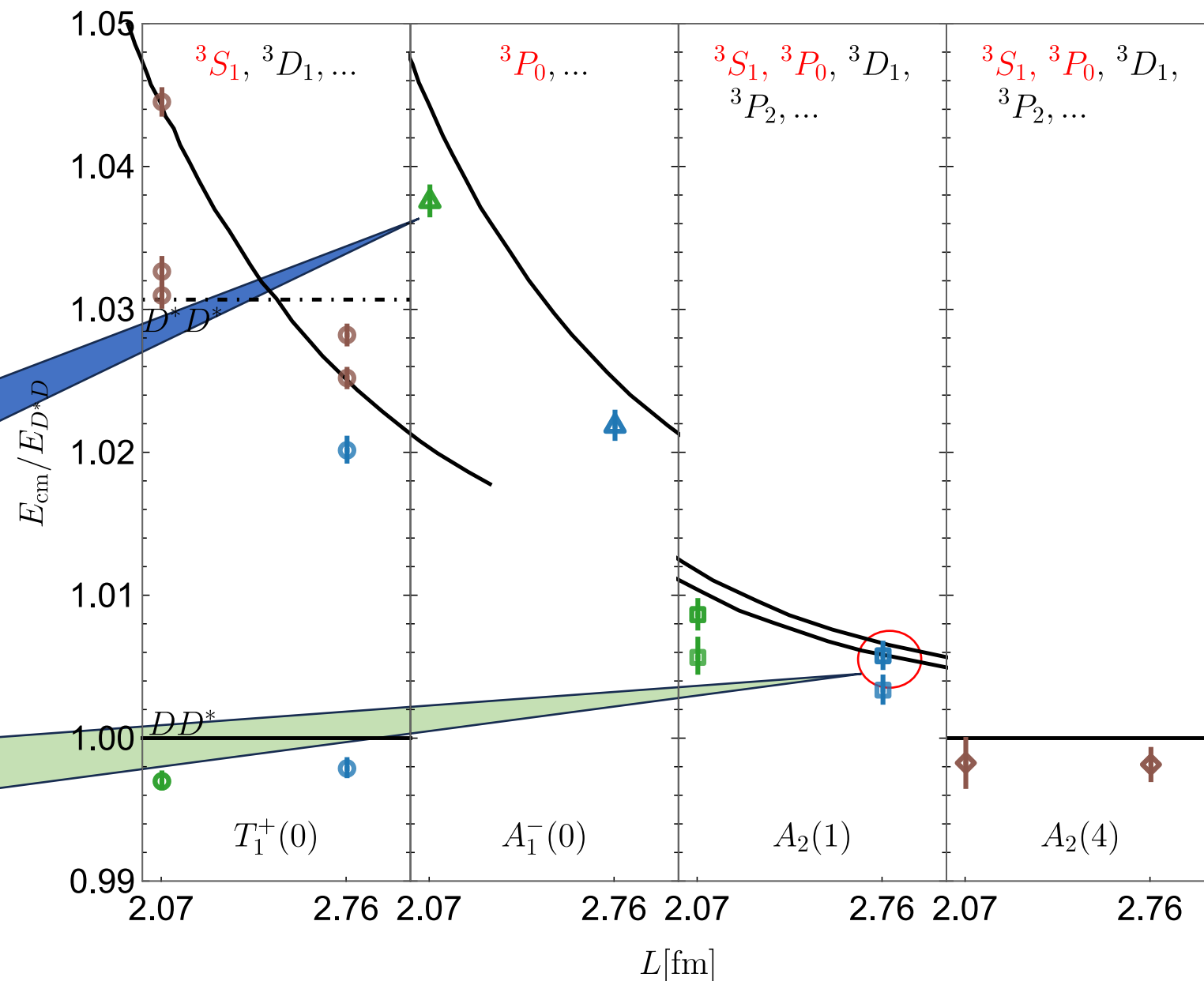
Lattice FV energy levels

- Green and blue points as input
- 9 inputs in total

Highest input: $p \approx 0.56$ GeV

- relativistic effect
- large cutoff $\Lambda \geq 0.7$

Overlap with noninteracting energy levels: $\cot \theta = \infty$;
Singular in $\text{pcot } \delta$ figures



- The energy levels of $A_1^-(0)$ is high, relativistic formalism

$$T(\mathbf{p}, \mathbf{p}') = V(\mathbf{p}, \mathbf{p}') + \int \frac{d^3 \mathbf{q}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}) \frac{1}{2w_1 w_2} \frac{(w_1 + w_2)}{P_0^2 - (w_1 + w_2)^2 + i\epsilon} T(\mathbf{q}, \mathbf{p}')$$

$$w_i = \sqrt{m_i^2 + \mathbf{q}^2}$$

$$G = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(P - q)^2 - m_1^2 + i\epsilon} \frac{1}{q^2 - m_2^2 + i\epsilon}$$

- Replace integral into summation to get $\mathbb{T} = \mathbb{V} + \mathcal{J} \mathbb{V} \cdot \mathbb{G} \cdot \mathbb{T}$

$$\int \frac{d^3 \mathbf{q}}{(2\pi)^3} \rightarrow \mathcal{J} \int \frac{d^3 \mathbf{q}_{box}}{(2\pi)^3} \rightarrow \mathcal{J} \sum_{\mathbf{n}} \frac{1}{L^3}$$

$\mathcal{J}$: is the Jacobian determinant of the Lorentz boost

Li:2021mob

- Get the poles

$$\det(\mathbb{H} - \lambda \mathbb{I}) = 0 \rightarrow \mathbb{H} \mathbf{v} = \lambda \mathbf{v},$$

- Contact terms to NLO

$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$	$\mathcal{O}_6$
$\epsilon'^{\dagger} \cdot \epsilon$	$\mathbf{q}^2 \epsilon'^{\dagger} \cdot \epsilon$	$4 \mathbf{k}^2 \epsilon'^{\dagger} \cdot \epsilon$	$4 \mathbf{k} \cdot \epsilon'^{\dagger} \mathbf{k} \cdot \epsilon$	$\mathbf{q} \cdot \epsilon'^{\dagger} \mathbf{q} \cdot \epsilon$	$(\epsilon'^{\dagger} \times \epsilon) \cdot (\mathbf{q} \times 2 \mathbf{k})$

$$\mathbf{q} = \mathbf{p}' - \mathbf{p},$$

$$\mathbf{k} = \frac{\mathbf{p}' + \mathbf{p}}{2}$$

$$\mathcal{O}_{^3S_1}^1 = \mathcal{O}_1$$

$$\mathcal{O}_{^3S_1}^2 = \mathcal{O}_2 + \mathcal{O}_3$$

$$\mathcal{O}_{^3S_1-^3D_1} = -\mathcal{O}_2 - \mathcal{O}_3 + 3\mathcal{O}_4 + 3\mathcal{O}_5$$

$$\mathcal{O}_{^3P_0} = -(\mathcal{O}_4 - \mathcal{O}_5) + \mathcal{O}_6$$

$$\mathcal{O}_{^3P_1} = -\frac{3}{2}(\mathcal{O}_2 - \mathcal{O}_3) + \frac{3}{2}(\mathcal{O}_4 - \mathcal{O}_5) + \frac{3}{2}\mathcal{O}_6$$

$$\mathcal{O}_{^3P_2} = -\frac{3}{2}(\mathcal{O}_2 - \mathcal{O}_3) - \frac{1}{2}(\mathcal{O}_4 - \mathcal{O}_5) - \frac{5}{2}\mathcal{O}_6$$

- In present calculation: LO and NLO 3S_1 contact terms, NLO 3P_0
 - ▶ 3S_1 - 3D_1 transition term and 3P_2 are included to estimate the systemic uncertainties
- Separable regulator: $e^{-\frac{p^n + p'^n}{\Lambda^n}}$, $n = 2, 4, 6$

One-pion-exchange interaction

- Pion propagator: static approximation

$$D \approx -\mathbf{k}^2 - [m_\pi^2 - (M_{D^*} - M_D)^2] + i\epsilon$$

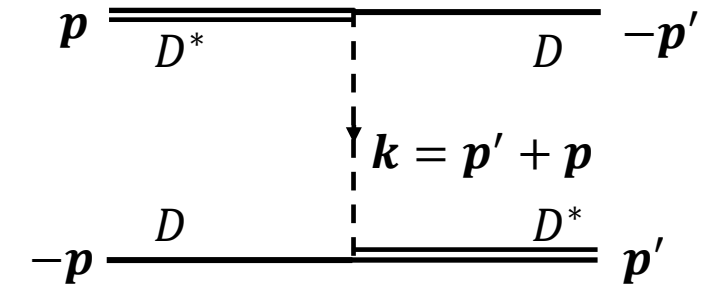
- Semilocal momentum-space regularization

$$\mathcal{V}(k) = -\frac{g^2}{4F_\pi} \left[\frac{\mathbf{k} \cdot \boldsymbol{\epsilon}'^* \mathbf{k} \cdot \boldsymbol{\epsilon}}{\mathbf{k}^2 + u^2} + C_{sub} \boldsymbol{\epsilon}'^* \cdot \boldsymbol{\epsilon} \right] e^{-\frac{\mathbf{k}^2 + u^2}{\Lambda^2}}$$

$$C_{sub} = -\frac{\Lambda(\Lambda^2 - 2u^2) + 2\sqrt{\pi}u^3 e^{\frac{u^2}{\Lambda^2}} \text{erfc}(\frac{u}{\Lambda})}{3\Lambda^3}$$

- ▶ The regulator will not change the long-range behavior
- ▶ The short-range part of OPE is subtracted: $V_{\epsilon\cdot\epsilon}(r=0) = 0$

Reinert:2017usi



- F_π and $g_{D^*D\pi}$ at $m_\pi = 280$ MeV are determined by lattice QCD data, physical values by either linear extrapolation or chiral extrapolation

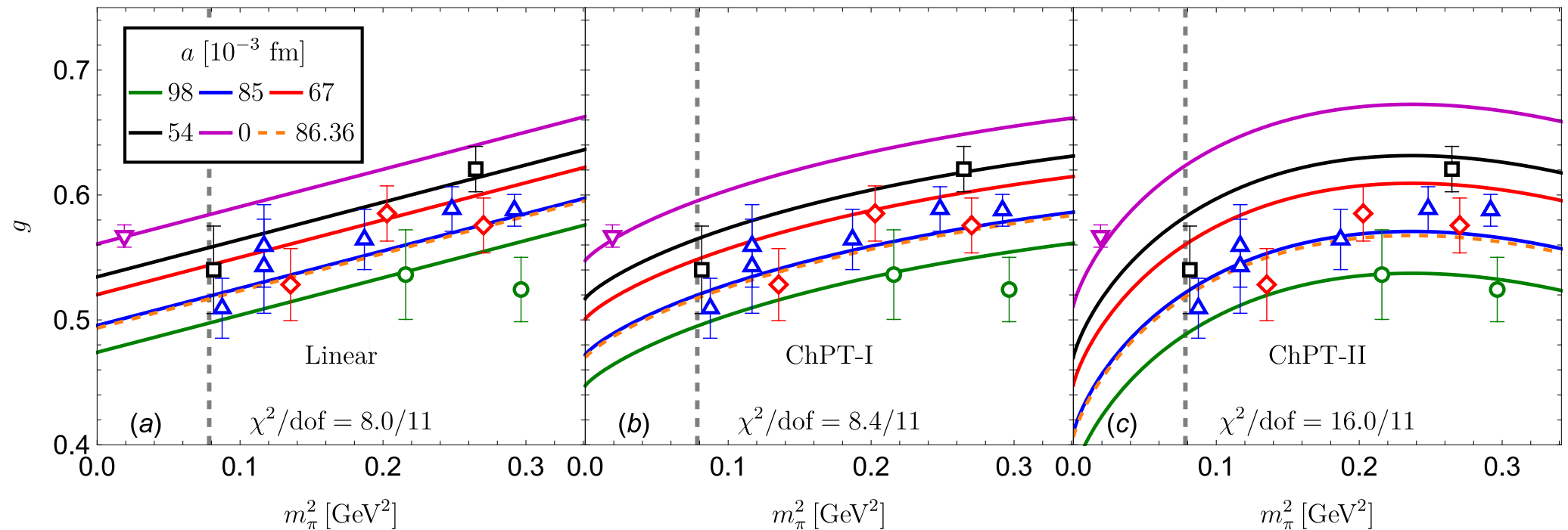
- $F_{ph} = 92.1$ MeV, $F_0 = 85$ MeV, chiral extrapolation, $\xi = m/m^{ph}$

$$f_\pi(\xi) = f_\pi^{ph} \left[1 + \left(1 - \frac{f_0}{f_\pi^{ph}} \right) (\xi^2 - 1) - \frac{(m_\pi^{ph})^2}{8\pi^2 f_0^2} \xi^2 \log \xi \right]$$

Du:2023hlu, Becirevic:2012pf

- Three extrapolations give the consistent results

- ▶ The g is slightly smaller than the value in Ref. [Du:2023hlu]
- ▶ $g = 0.517 \pm 0.015$ for $a = 0.086$ fm



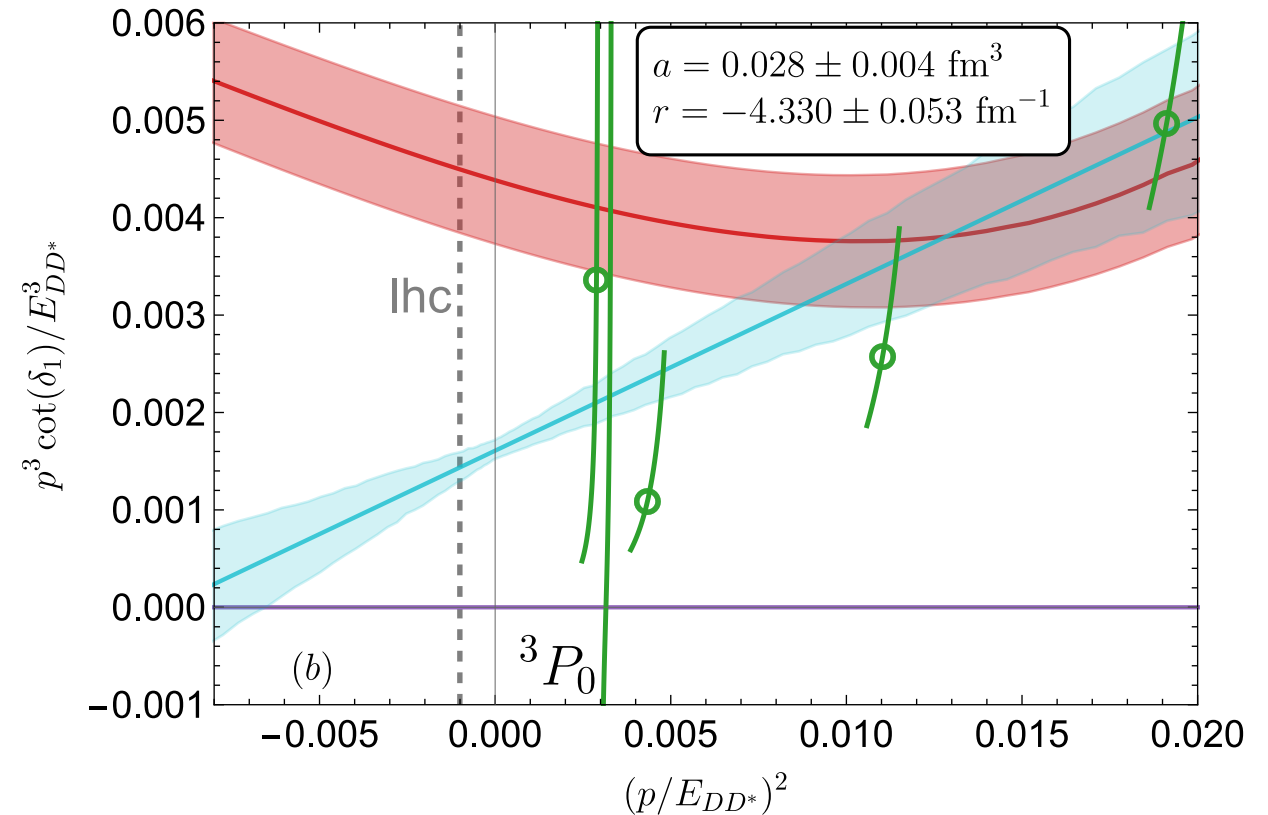
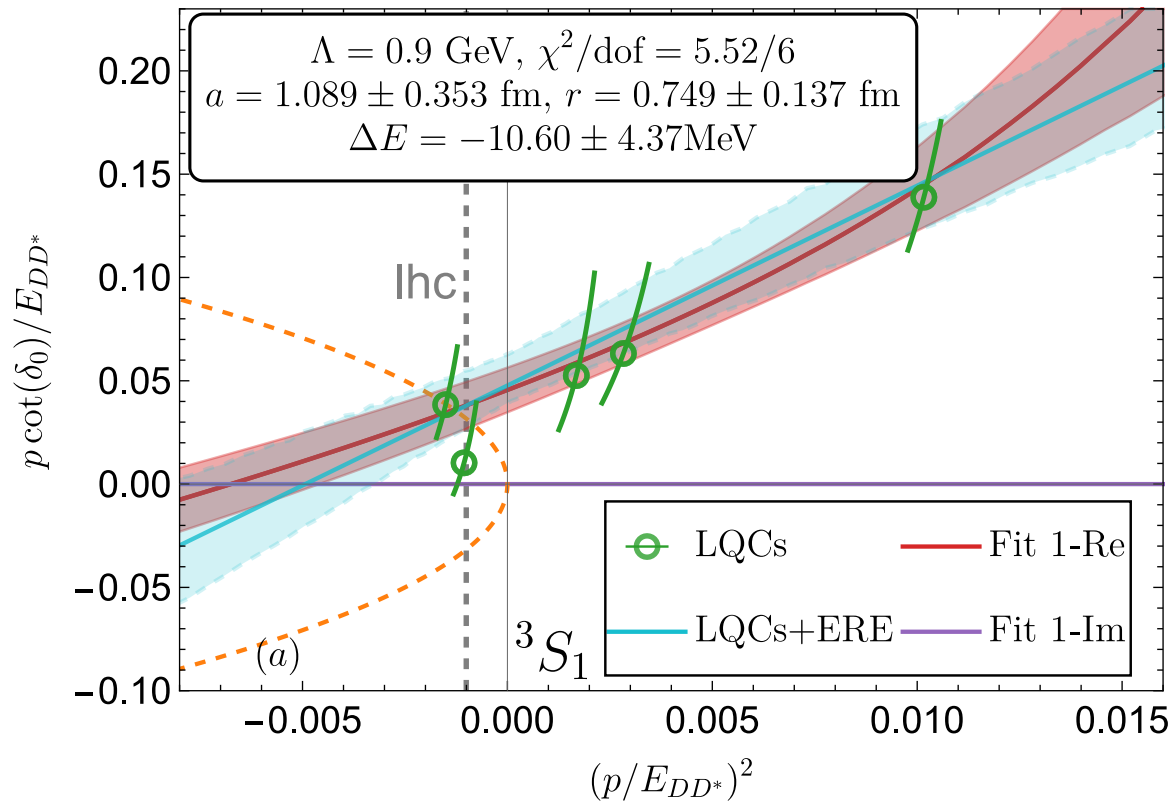
● Results using Lüscher's QCs Padmanath:2022cvi

$$\chi^2/\text{dof}=3.7/5, E_{\text{pole}}^{^3S_1} = -9.9_{-7.2}^{+3.6} \text{ MeV}$$

$$a_{^3S_1} = 1.04(29)\text{fm}, \quad r_{^3S_1} = 0.96_{-0.20}^{+0.18}\text{fm}$$

$$a_{^3P_0} = 0.076_{-0.009}^{+0.008}\text{fm}^3, \quad r_{^3P_0} = 6.9(2.1)\text{fm}^{-1}$$

● Our results



Results using Lüscher's QCs

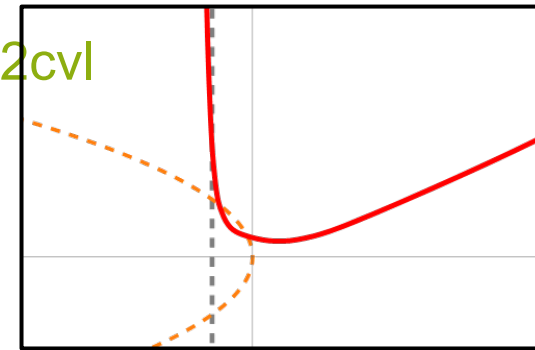
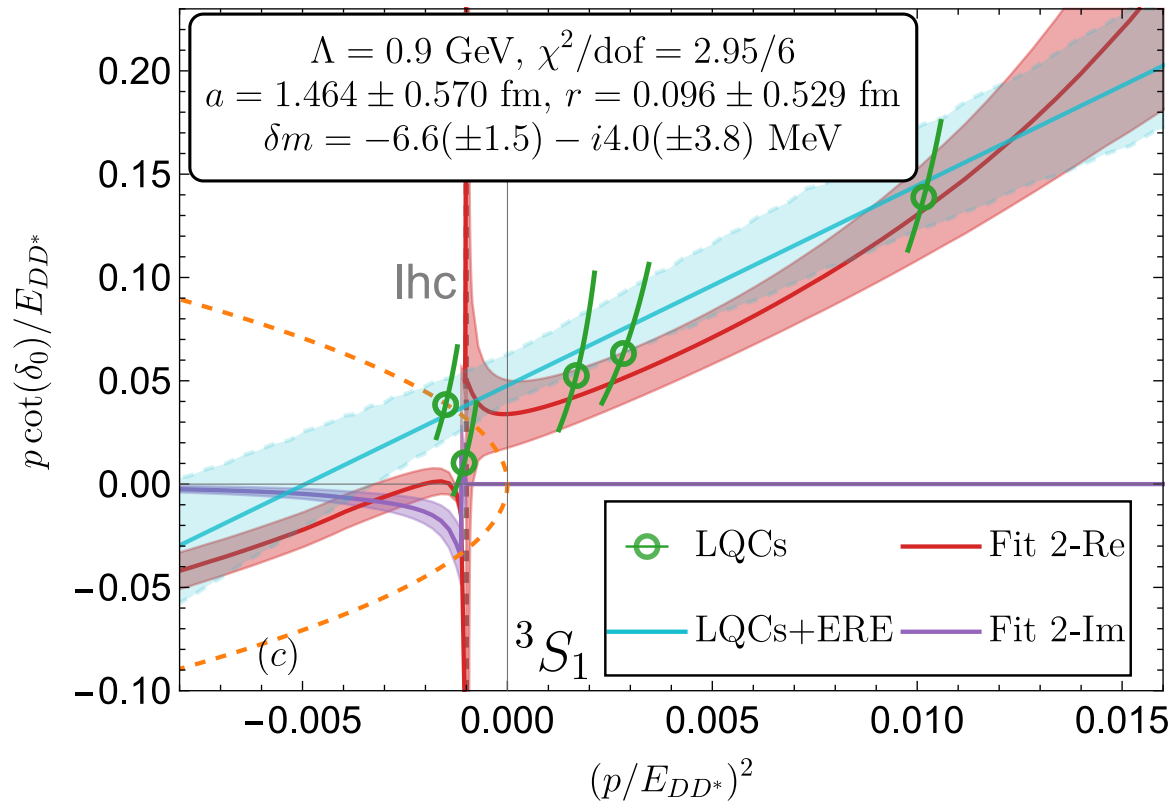
Padmanath:2022cvl

$$\chi^2/\text{dof}=3.7/5, E_{\text{pole}}^{^3S_1} = -9.9_{-7.2}^{+3.6} \text{ MeV}$$

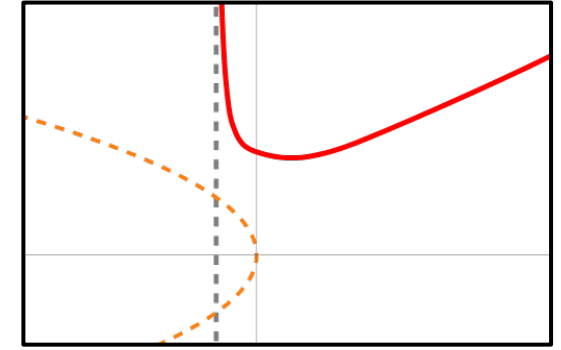
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Our results

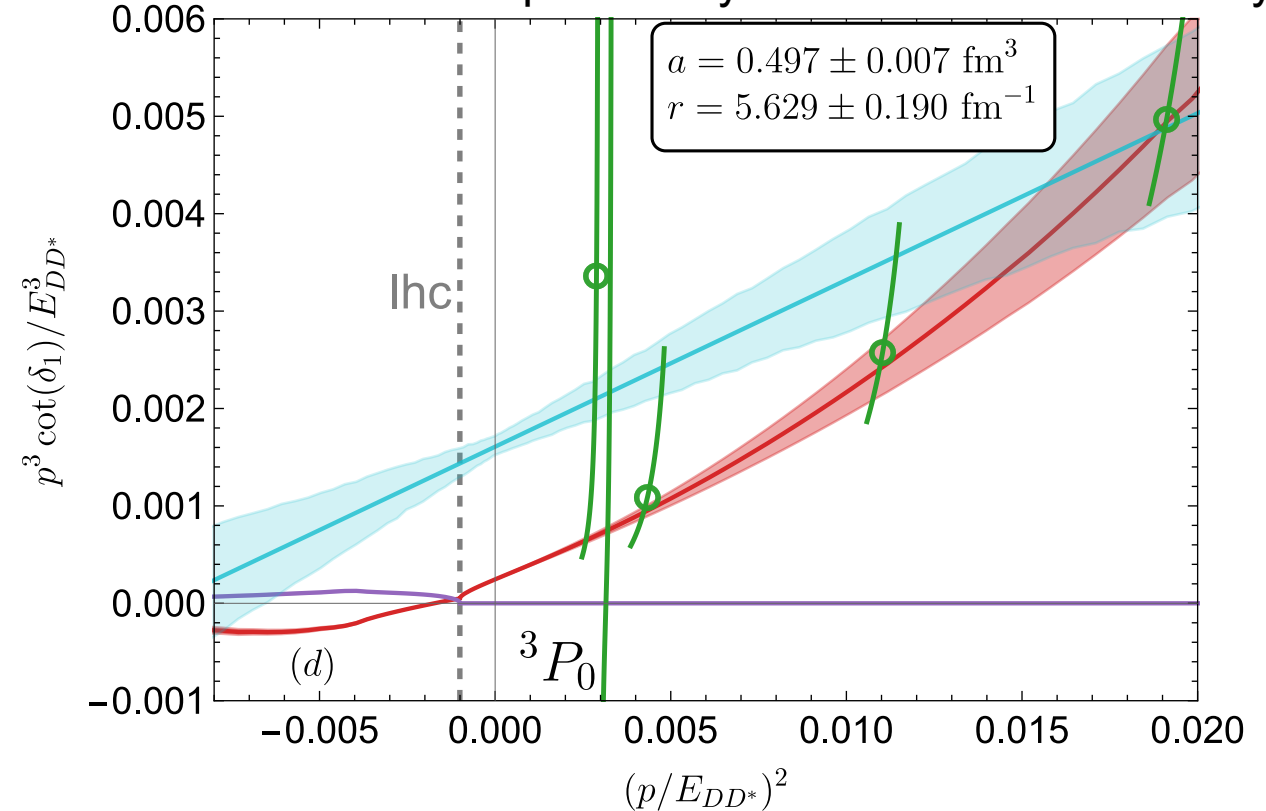


two virtual states



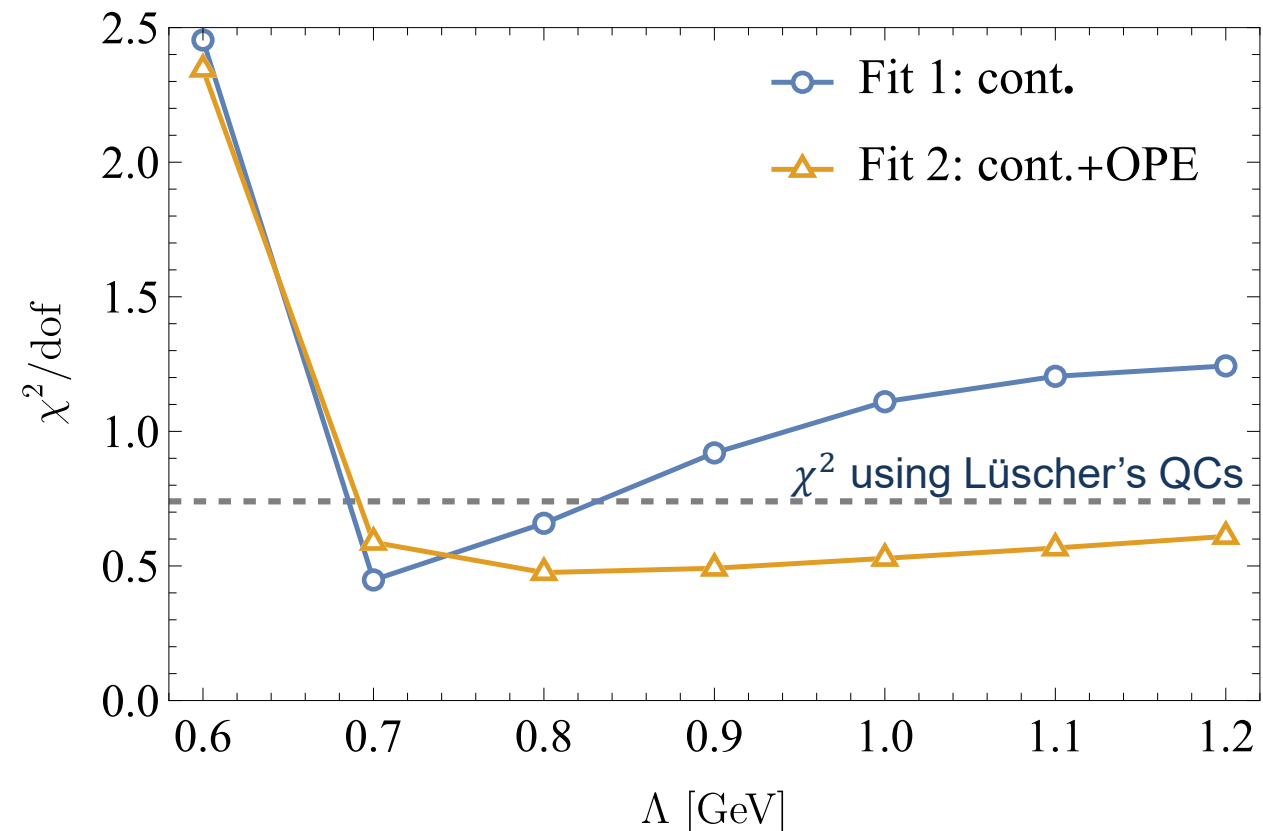
Resonance

Resonance with 85% probability within the 1σ uncertainty

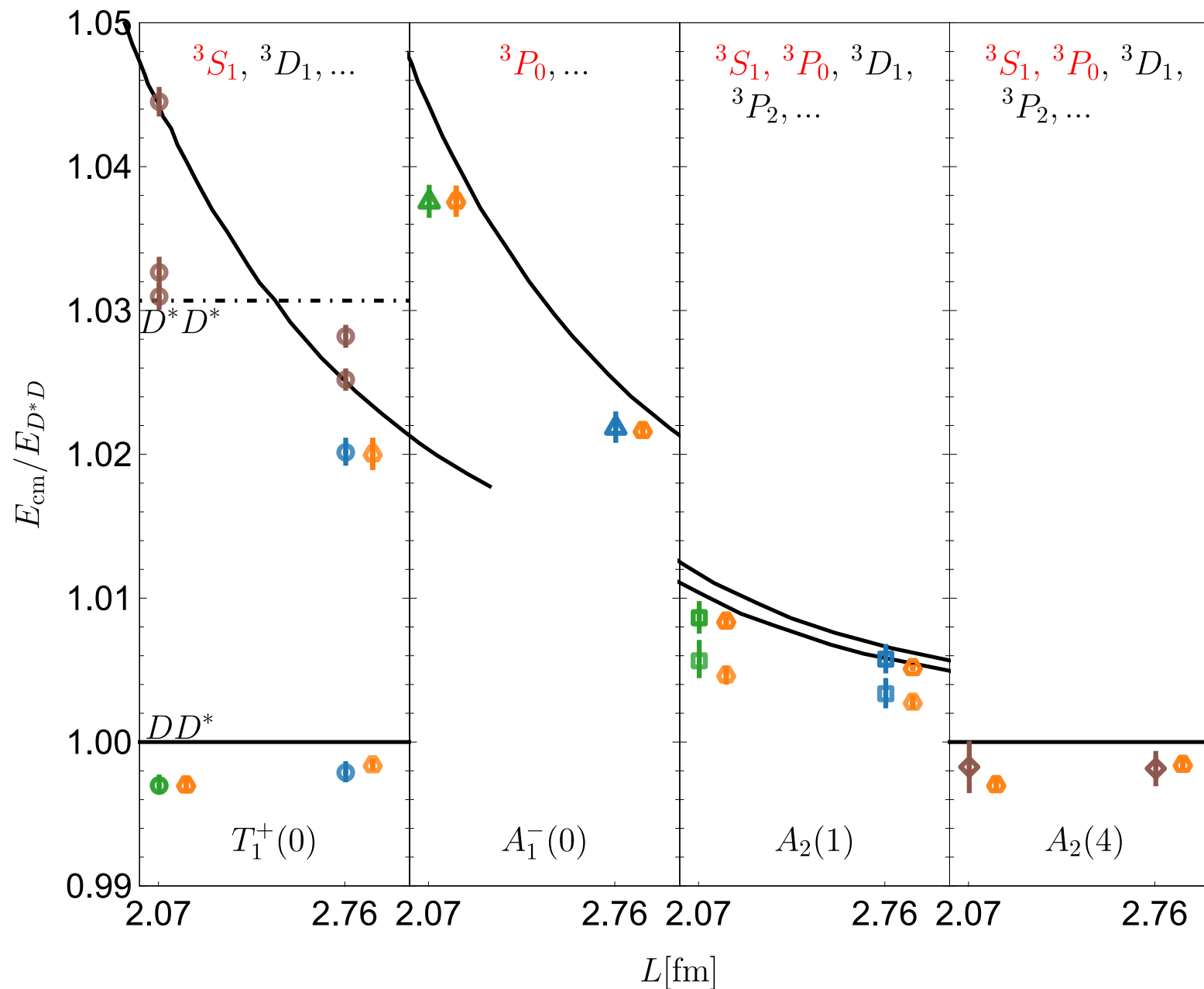


Cutoff dependence of χ^2

- 3 LECs: LO and NLO 3S_1 contact terms, NLO 3P_0
- In V_{ctc} fit, the P-wave dominate states control Λ -dependence of the χ^2
 - ▶ The shape of the of $k^3 \cot \delta_1$ is determined by regulator and cutoff
 - ▶ Sensitive to Λ
- The $V_{\text{ctc}} + V_{1\pi}$ fit is stable with Λ
- The $V_{\text{ctc}} + V_{1\pi}$ fit is even better than QCs

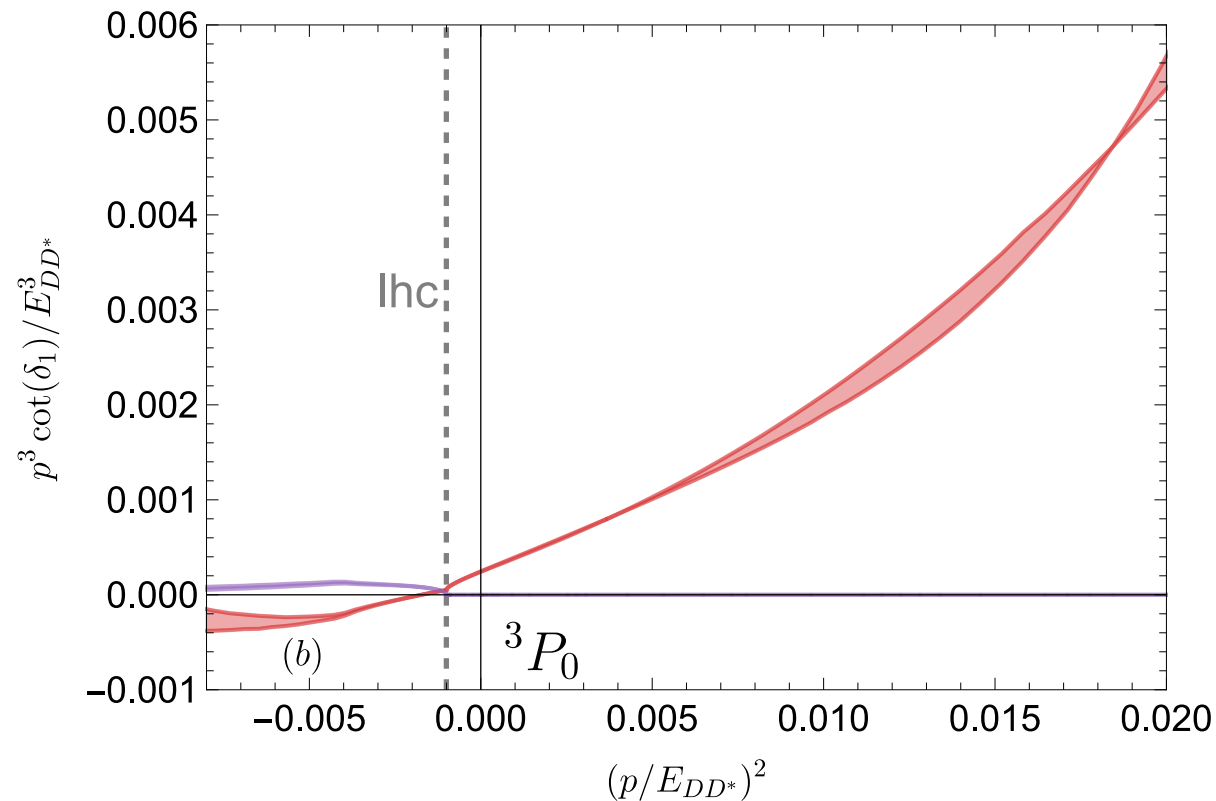
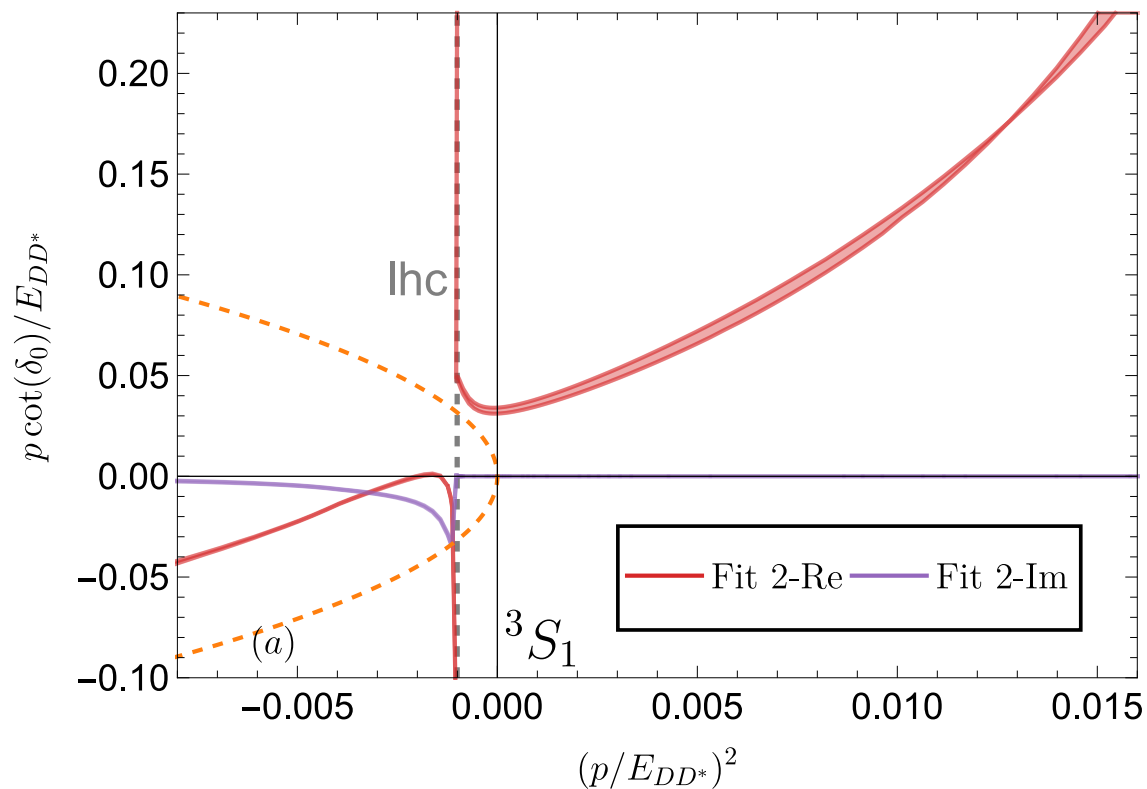


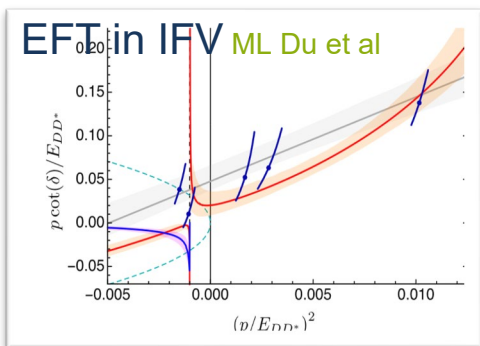
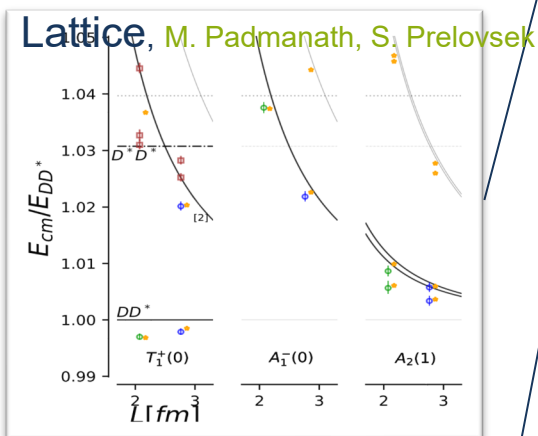
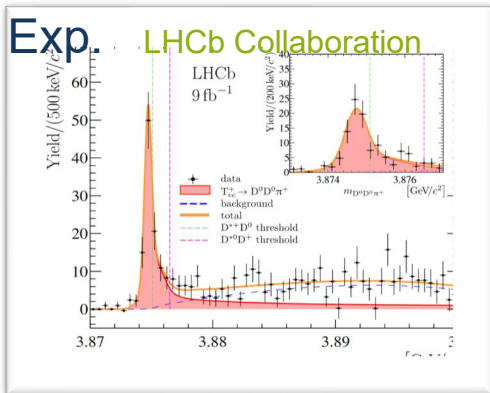
Energy levels from the best fit



Cutoff dependence of phase shift

$$\Lambda = 0.7 - 1.2 \text{ GeV}$$





2108.02709

2109.01038

2202.10110

2303.09441

Lattice2023

2311.18793

2312.01930

Summary

Lattice2021

- A proof-of-principle of an alternative method of Lüscher's formalism
- LSE or BSE in **plane wave** expansion+projection operator technique reduction to irreps
 - ⇒ Including partial wave mixing effect naturally, avoid complications of PW expansion
 - ⇒ Rest and moving two-particle systems, spinless, equal mass
- Non-relativistic example: spin-triplet NN
 - ⇒ S-wave dominant states: LF works well for $L \gtrsim 5$ fm
 - ⇒ P-wave dominant states: OPE→large PW mixing effect regardless the box size
 - ⇒ EFT-based approach in the plane wave basis: $V_{1\pi} + V_{1h} \xrightarrow[\text{data}]{\text{fit}} V_{1\pi} + V_{\text{cont}}^{(0)} + V_{\text{cont}}^{(2)} \dots$
 - ⇒ Advantages: 1) insensitive to PW mixing artifact; 2) small box (long-range interaction)

g the plane wave basis

EFT in FV

Finite-volume scattering on the left-hand cut

Andre Baião Raposo

Curia II, WH2SW

13:30 - 13:50

Breakdown of Lüscher Formalism near Left Hand Cuts

Md Habib E Islam

Curia II, WH2SW

13:50 - 14:10

Resolving the left-hand-cut problem in lattice studies of the doubly-charmed tetraquark

Steve Sharpe

Curia II, WH2SW

14:10 - 14:30

Finite-volume scattering on the left-hand cut

André Baião Raposo, Maxwell T. Hansen

Nov 30, 2023

59 pages

e-Print: [2311.18793 \[hep-lat\]](#)

Modified Lüscher QCs
framework and toy models

Solving the left-hand cut problem in lattice QCD: $T_{cc}(3875)^+$ from finite volume energy levels

Lu Meng, Vadim Baru, Evgeny Epelbaum, Arseniy A. Filin, Ashot M. Gasparyan

Dec 4, 2023

13 pages

e-Print: [2312.01930 \[hep-lat\]](#)

EFT in FV

Application in Tcc

Two approaches

● Hansen's approach Raposo:2023oru

► FV: lattice data fix $\bar{\mathcal{K}}^{os}$

$$\det_{\mathbf{k}^* \ell m} \left[S(P_j, L)^{-1} + \xi^\dagger \bar{\mathcal{K}}^{os}(P_j) \xi + 2g^2 \mathcal{T}(P_j) \right] = 0$$

► IFV: solve a integral equation

$$\mathcal{M}^{\text{aux}}(P, p, p') = \mathcal{K}^{\mathcal{T}}(P, p, p') - \frac{1}{2} \int \frac{d^3 \mathbf{k}^*}{(2\pi)^3} \frac{\mathcal{M}^{\text{aux}}(P, p, k) H(\mathbf{k}^*) \mathcal{K}^{\mathcal{T}}(P, k, p')}{4\omega_N(\mathbf{k}^*) [(k_{os}^*)^2 - (\mathbf{k}^*)^2 + i\epsilon]},$$

$$\mathcal{K}^{\mathcal{T}}(P, p, p') = \bar{\mathcal{K}}^{os}(P, p, p') + 2g^2 \mathcal{T}(P, p, p'),$$

$$S_{\mathbf{k}^* \ell m, \mathbf{k}'^* \ell' m'}(P, L) = \frac{1}{2L^3} \frac{4\pi Y_{\ell m}(\hat{\mathbf{k}}^*) Y_{\ell' m'}^*(\hat{\mathbf{k}}^*) \delta_{\mathbf{k}^* \mathbf{k}'^*} |\mathbf{k}^*|^{\ell+\ell'} e^{-\alpha[(\mathbf{k}^*)^2 - (k_{os}^*)^2]}}{4\omega_N(\mathbf{k}^*) [(k_{os}^*)^2 - (\mathbf{k}^*)^2]},$$

$$\mathcal{T}_{\mathbf{k}^* \ell m, \mathbf{k}'^* \ell' m'}(P) = -\frac{1}{4\pi |\mathbf{k}^*|^\ell |\mathbf{k}'^*|^{\ell'}} \int d\Omega_{\hat{\mathbf{k}}^*} d\Omega_{\hat{\mathbf{k}}'^*} Y_{\ell m}(\hat{\mathbf{k}}^*) Y_{\ell' m'}^*(\hat{\mathbf{k}}'^*) \times \frac{1}{(p' - p)^2 - M_\pi^2}$$

$$\omega_N(\mathbf{p})^2 = m_N^2 + \mathbf{p}^2, \quad p = (\omega_N(\mathbf{k}^*), \mathbf{k}^*), \quad p' = (\omega_N(\mathbf{k}'^*), \mathbf{k}'^*)$$

$\xi = 1$

Model-independent?

You have to choose a parameterization of $\bar{\mathcal{K}}^{os}$: ERE

To some how, the ERE is equivalent to the contact EFT

● Our approach

► FV: lattice data fix contact terms

$$\det[\mathbb{G}^{-1}(E) - \mathbb{V}] = 0.$$

► IFV: solve a integral equation

$$T(\mathbf{p}, \mathbf{p}', E) = V(\mathbf{p}, \mathbf{p}') + \int \frac{d^3 \mathbf{q}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}) G(\mathbf{q}, E) T(\mathbf{q}, \mathbf{p}', E).$$

$$V_{EFT} = \begin{array}{c} D^* \quad D \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ D \quad D^* \end{array} \begin{array}{c} \pi \\ \text{---} \end{array} + \begin{array}{c} D^* \quad D \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ D \quad D^* \end{array} + \dots$$

$$\mathbb{G}(E) = \frac{\mathcal{J}(\mathbf{q}_n)}{L^3} G(\mathbf{q}_n, E) \delta_{\mathbf{n}', \mathbf{n}}, \quad \mathbb{V} = V(\mathbf{q}_n, \mathbf{q}_{n'})$$

$$\begin{aligned} G(\mathbf{q}, E) &= i \int \frac{dq^0}{2\pi} \frac{1}{(P - q)^2 - m_1^2 + i\epsilon} \frac{1}{q^2 - m_2^2 + i\epsilon} \\ &= \frac{1}{4\omega_1 \omega_2} \left(\frac{1}{E - \omega_1 - \omega_2} - \frac{1}{E + \omega_1 + \omega_2} \right) \\ &= \frac{1}{2\omega_1 \omega_2} \frac{(\omega_1 + \omega_2)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon}, \end{aligned}$$

Summary and Outlook

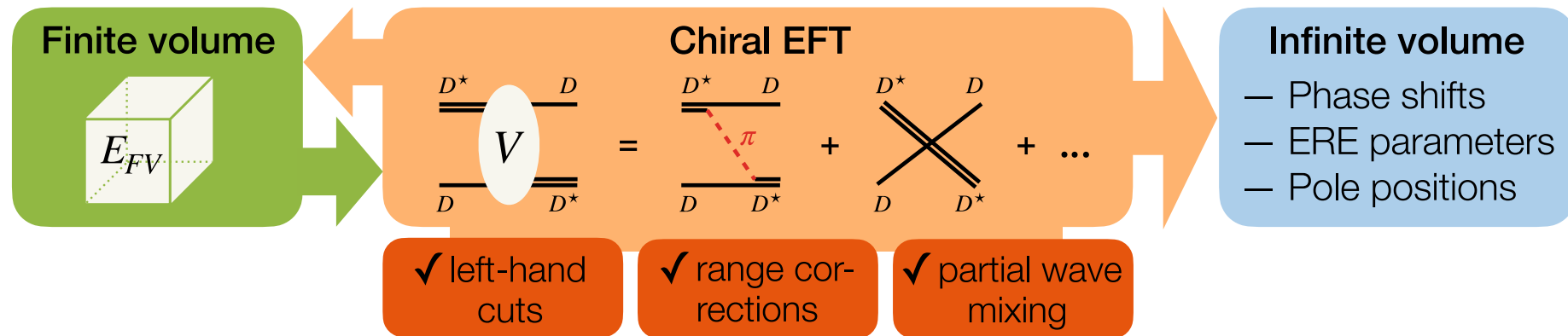
● Validation of Lüscher's formula

- ① e^{-mL} effect can be neglected
- ② Considering the PW mixing effect
- ③ E^{FV} well above lhc
- ④ ERE works in IFV Du:2023hlu

← Invalidate



● Our formalism



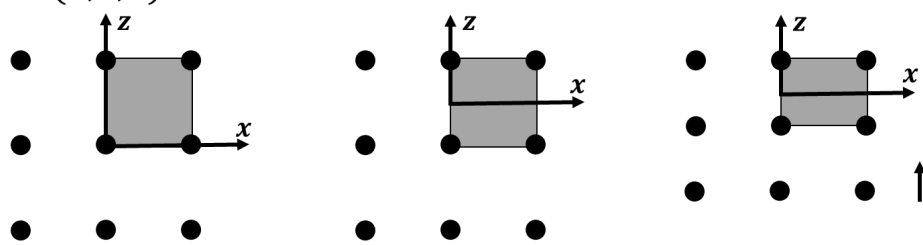
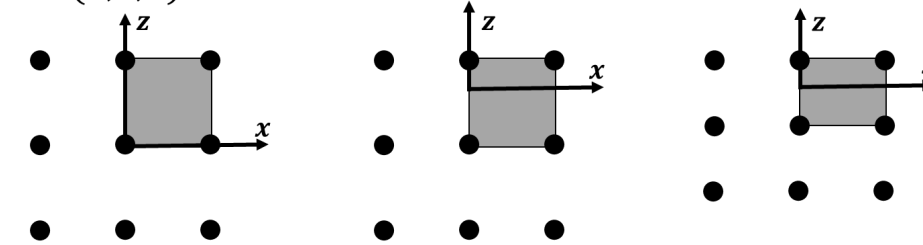
● T_{cc} lattice data

- ▶ Better fit than QCs
- ▶ The possible partial wave mixing effect
- ▶ E^{FV} below the left-hand cut

Thanks for your attentions!

Back up

Moving systems

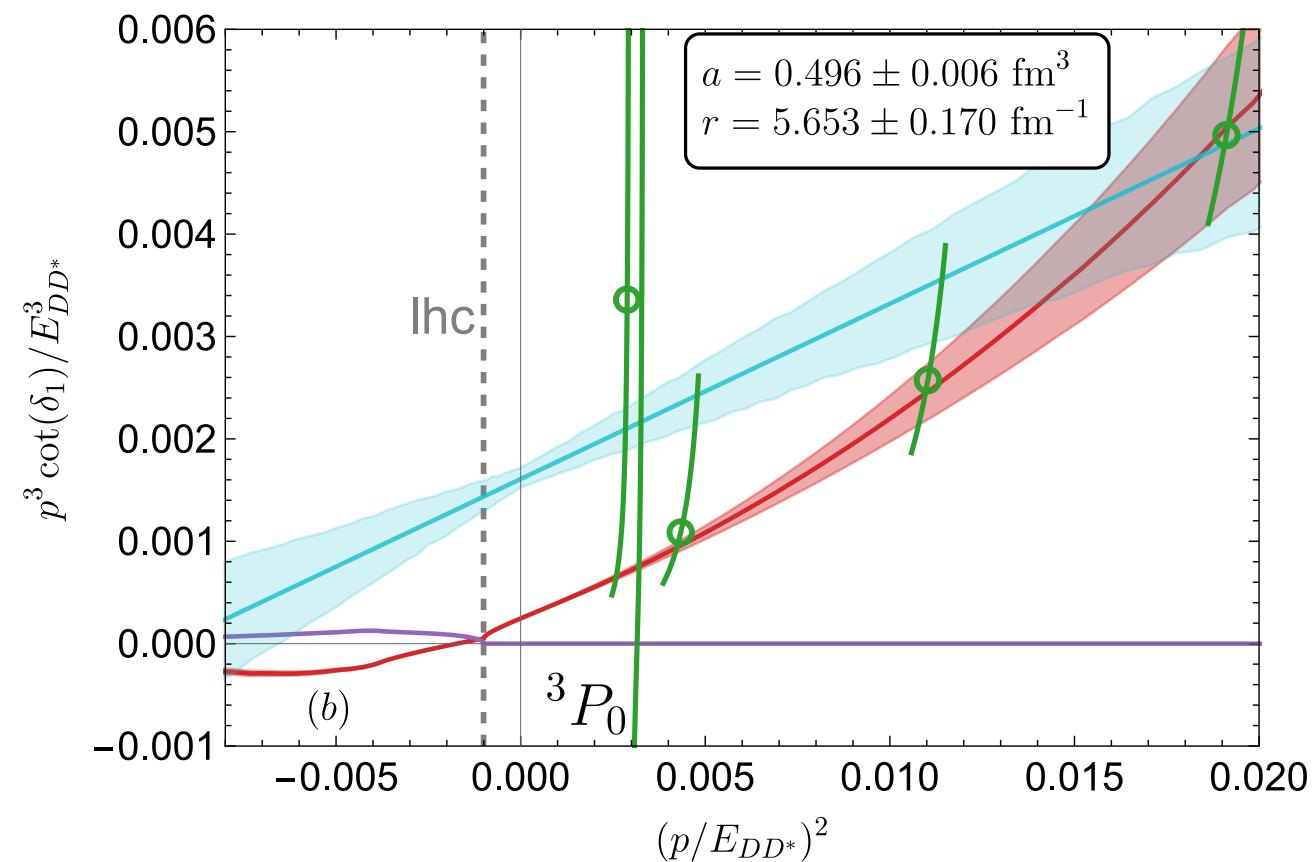
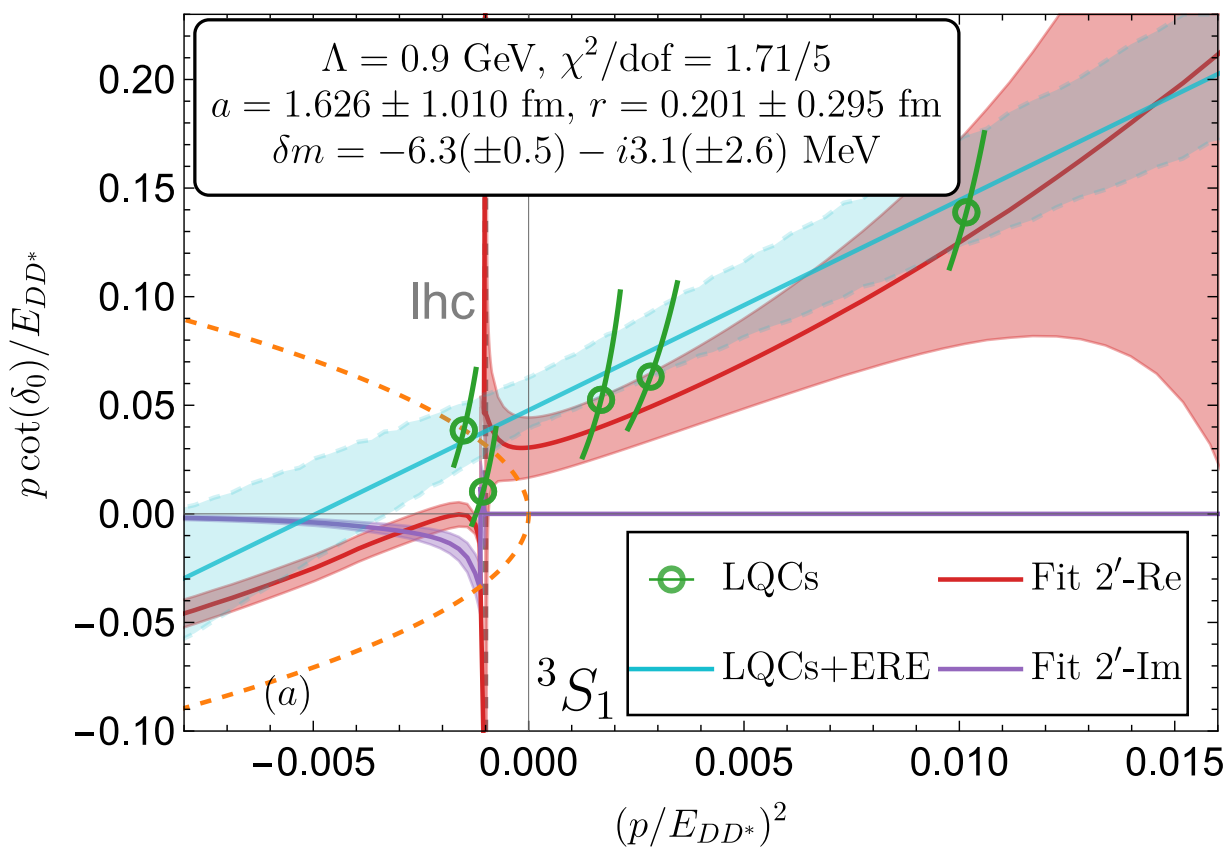
$m_1 = m_2, \quad A = 1$			$m_1 \neq m_2, \quad A = 1 + \frac{m_1^2 - m_2^2}{E^*}$		
$n \in Z$	$n - \frac{1}{2}d$	$\gamma^{-1} \left(n_{\parallel} - \frac{d}{2} \right) + n_{\perp}$	$n \in Z$	$n - \frac{A}{2}d$	$\gamma^{-1} \left(n_{\parallel} - \frac{A}{2}d \right) + n_{\perp}$
$d = (0,0,1)$ 			$d = (0,0,1)$ 		

Space inversion invariance is broken

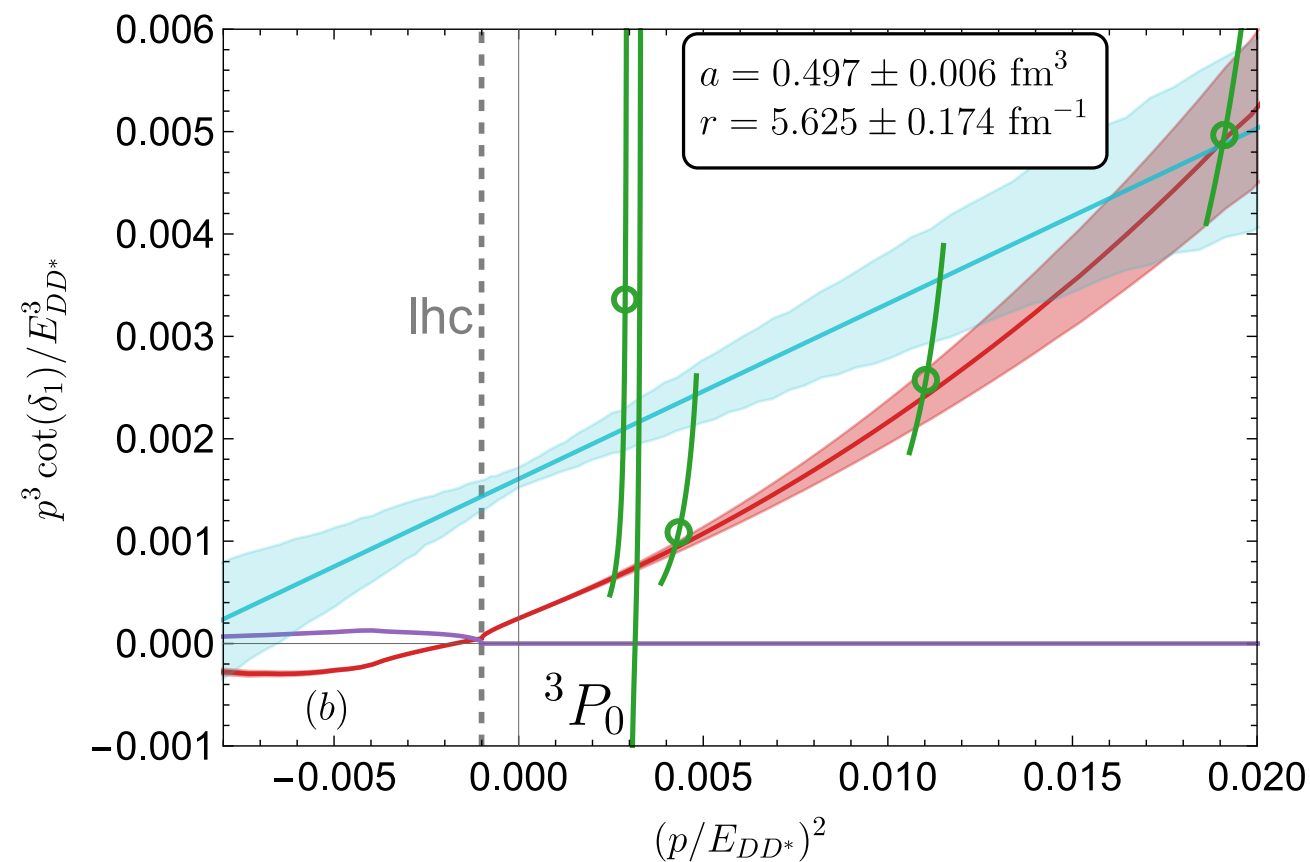
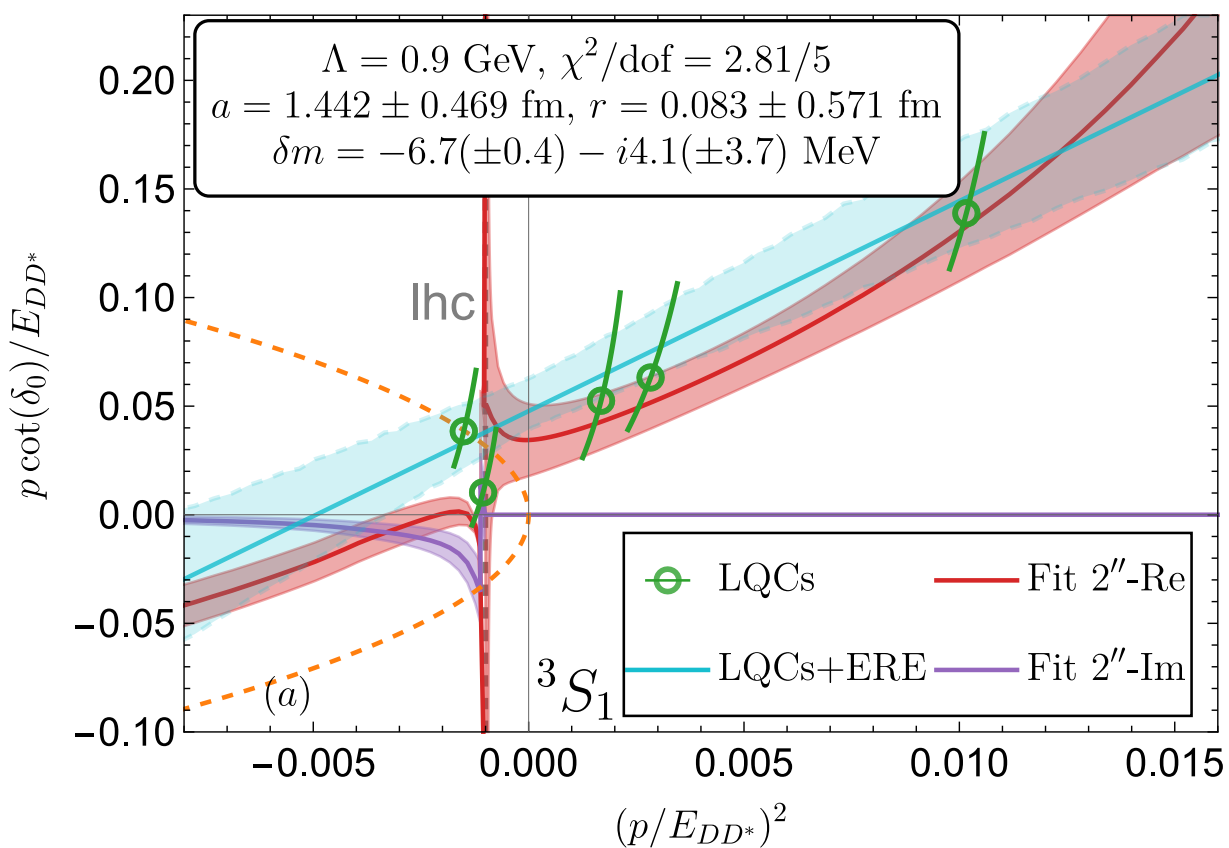
- Moving system in the box $\mathbf{P} = \frac{2\pi}{L} \mathbf{d} \neq 0$
 - ▶ For LQCD, changing box size is expensive
 - ▶ Calculate E^{FV} of moving two-body systems in a box
- Box frame (BF) $\mathbf{p}$ and center of mass frame (CMF) $\mathbf{p}^*$
 - ▶ BF: $\mathbf{p} = \frac{2\pi}{L} \mathbf{n}$; CMF: $\mathbf{p}^* = \gamma^{-1} \left(\mathbf{p}_{\parallel} - \frac{A}{2} \mathbf{P} \right) + \mathbf{p}_{\perp}$
 - ▶ For moving systems with $m_1 \neq m_2$, states with different parities could mix
- $\mathbf{d} = (0,0,1)$, D_{4h} group for $m_1 = m_2$, C_{4v} group for $m_1 \neq m_2$
- $\mathbf{d} = (1,1,0)$, ...

Rummukainen:1995vs, Leskovec:2012gb

Including SD transition terms



Including 3P2 term



- Lippmann-Schwinger equation in the finite volume

Luscher:1990ux,Polejaeva:2012ut

$$T^L(\mathbf{p}, \mathbf{q}; z) = V(\mathbf{p}, \mathbf{q}) + \int \frac{d^3 \mathbf{k}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}) G_0^L(\mathbf{k}; z) T(\mathbf{k}; z)$$

$$G_0^L(\mathbf{k}, z) = \left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{p} \in \frac{2\pi}{L} \mathbf{n}} \frac{2\mu \delta^3(\mathbf{p} - \mathbf{k})}{q_0^2 - \mathbf{p}^2} = \text{P.V.} \cdot \frac{2\mu}{q_0^2 - \mathbf{k}^2} + G_F(\mathbf{k}, z) = G_K(\mathbf{k}, z) + G_F(\mathbf{k}, z)$$

with $z = m_1 + m_2 + \frac{q_0^2}{2\mu}$

- The “=” relation is valid up to the exponentially suppressed terms in L
- K matrix in the infinite volume: $K = V + V G_K K$

$$T^L = V + V(G_K + G_F)T^L = K + K G_F T^L$$

- E^{FV} corresponding to poles of T^L : interaction-independent form

$$\det[1 - K G_F] = 0, \text{ or } \det[G_F - K^{-1}] = 0$$

Detailed derivation of Lüscher's formula

$$C_L(P) = \text{O} \text{---} \Sigma \text{---} \text{O}^\dagger + \text{O} \text{---} \Sigma \text{---} V \text{---} \Sigma \text{---} \text{O}^\dagger + \dots$$

$$V = \text{X} + \text{---} + \dots$$

$$\Sigma = \int + \Sigma - \int$$

$$\Sigma - \int \equiv F$$

$$\left[\frac{1}{L^3} \sum_k - \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$

$$C_L(P) = C_\infty(P) + A \text{---} F \text{---} A' + A \text{---} F \text{---} K \text{---} F \text{---} A' + \dots$$

Within on-shell approximation:

$$F + FKF + \dots = F(1 - KF)^{-1} = (F^{-1} - K)$$

$$A = \text{O} + \text{O} \text{---} \int \text{---} V + \text{O} \text{---} \int \text{---} V \text{---} \int \text{---} V + \dots$$

$$K = V + V \text{---} \int \text{---} V + \dots$$

Note: all the $\int$ should be treated in the sense of P.V.

- Expanding it in partial wave basis

$$\det[G_F - K^{-1}] = 0, \Rightarrow \det[M_{l'm',lm} - \delta_{ll'}\delta_{mm'}\cot\delta_l] = 0$$

- ▶ Determinate equation of a matrix with infinite dimensions.
- ▶ Truncate at some $l_{\max}$

- Reduce to irreps. Γ_i of point group

$$\det[F_{l'm',lm}] = 0 \Rightarrow \begin{vmatrix} F_{\Gamma_1} & & \\ & F_{\Gamma_2} & \\ & & \ddots \end{vmatrix} = 0, \Rightarrow \det[F_{\Gamma_i}] = 0$$

- Obtain the basis of the irreps. $|l, m\rangle \rightarrow |\Gamma, l, \alpha\rangle$ (Projection operator technique)

Bernard:2008ax

- Lüscher quantization conditions: $\det \left[M_{ln,l'n'}^{(\Gamma,P)} - \delta_{ll'}\delta_{nn'}\cot\delta_l \right] = 0$

Luscher:1990ux,Rummukainen:1995vs,Feng:2004ua,Kim:2005gf,Fu:2011xz,Polejaeva:2012ut,Leskovec:2012gb,Gockeler:2012yj,...

- Example $\mathbf{d} = (0,0,1)$, $\Gamma = A_1^+$, w_{lm} depends on E but independent on V

$$\det \left[M_{ln,l'n'}^{(\Gamma,\mathbf{P})} - \delta_{ll'} \delta_{nn'} \cot \delta_l \right] = 0, \quad M^{(A_1^+,\mathbf{d})} = \begin{bmatrix} w_{00} & -\sqrt{5}w_{20} & \cdots \\ -\sqrt{5}w_{20} & w_{00} + \frac{10}{7}w_{20} + \frac{18}{7}w_{40} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- Truncate at $l_{max} = 0$, one-to-one relation: $\delta_0(E^{FV}) \sim E^{FV}$
- Truncate at $l_{max} > 0$, no one-to-one relation
 - ▶ E.g. $\{E_1^{FV}, E_2^{FV}\} \neq \{\delta_S(E_1^{FV}), \delta_S(E_2^{FV}), \delta_D(E_1^{FV}), \delta_D(E_2^{FV})\}$
 - ▶ One has to parameterize the K-matrix
 - ▶ Effective range expansion
- Lüscher's formula: quantization conditions in partial wave basis
- Why not quantization conditions in plane wave basis + Hamiltonian method?

- **Seven patterns** of representation space $\{n_1, n_2, n_3\}_{dim}$ for O_h group

$$\Rightarrow \{0, 0, 0\}_{1 \times 3}, \{0, 0, a\}_{6 \times 3}, \{0, a, a\}_{12 \times 3}, \{0, a, b\}_{24 \times 3} \dots$$

- Reduce to irreducible representations (irreps): projection operator

e.g. textbook by M.Dresselhaus et.al

$$\hat{P}_{\alpha\beta}^{\Gamma_a} \equiv \sum_{g_i \in G} \frac{N(\Gamma_a)}{n_G} R_{\alpha\beta}^{\Gamma_a}(g_i)^* \hat{D}(g_i), \quad \hat{P}_{\alpha\alpha'}^{\Gamma_a} |\psi\rangle = a_{\alpha'}^{\Gamma_a} |\Gamma_a, \alpha\rangle.$$

- An example: $\{0, 0, a\}_{6 \times 3} = 2T_1^+ \oplus T_2^+ \oplus A_1^- \oplus E_1^- \oplus T_1^- \oplus T_2^-$
- For moving systems, elongated boxes, particles with arbitrary spin...

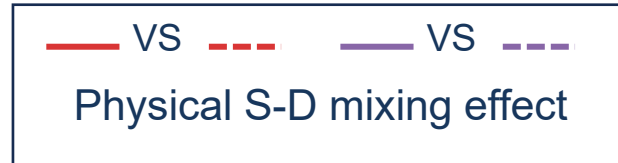
Symmetric group (character table) $\xrightarrow{\hat{P}^\Gamma}$ unitary irrep matrices $\xrightarrow{\hat{P}_{\alpha\beta}^\Gamma}$ rep space $|p_n\rangle \rightarrow$ irreps

- dim of the $\mathbb{H}_\Gamma$: cubic function of L^{-1}

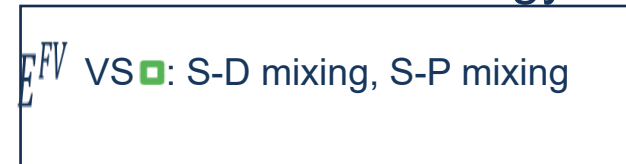
$$\dim \sim \left(\frac{\Lambda_{UV}}{2\pi/L} \right)^3 \times \frac{1}{10} \sim \mathcal{O}(1000)$$

Test consistency with Lüscher's formula

- Using the LECs of our best fitting
- V_{full} and S-wave-projected V_S
- Obtain IFV T-matrix



- Obtain the FV energy levels



- Using Lüscher's QCs to get δ
► Single partial wave QCs

