Evaluating Feynman integrals by hypergeometric function method

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Zhang, Feng, GKZ hypergeometric systems of the three-loop vacuum Feynman integrals, *JHEP* 05 (2023) 075 [arXiv: 2303.02795].

III. Feynman integrals on Grassmannian

Feng, Zhang, Chang, Feynman integrals of Grassmannians, *PRD* 106 (2022) 116025 [arXiv: 2206.04224].

V. Summary

I. Introduction

1. Background

- Higher-order radiative corrections are more important, with the increasing precision of measurements at the future colliders: CLIC, ILC, CEPC, FCC, HL-LHC $\cdots$
- One-loop Feynman integrals are well known analytically in the time-space dimension $D = 4 - 2\varepsilon$.
  However, how to perform analytically multi-loop Feynman integrals is still a challenge.
- **Considering Feynman integrals as the generalized hypergeometric functions, one finds that the $D-$module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) $D-$module.**

I. Introduction

2. Relevant research

- Hypergeometric functions of some Feynman integrals are obtained from Mellin-Barnes representations.

  Feng, Chang, Chen, Gu, Zhang, *NPB* 927(2018)516 [arXiv:1706.08201]
  Feng, Chang, Chen, Zhang, *NPB* 940(2019)130 [arXiv:1809.00295]
  Gu, Zhang, *CPC* 43(2019)083102 [arXiv:1811.10429]
  Gu, Zhang, Feng, *IJMPA* 35(2020)2050089.

- Using GKZ hypergeometric system, we can obtain the fundamental solution systems of Feynman integrals.

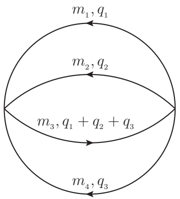
  Feng, Chang, Chen, Zhang, *NPB* 953(2020)114952, [arXiv:1912.01726]
  Feng, Zhang, Chang, *PRD* 106(2022)116025 [arXiv: 2206.04224]
  Feng, Zhang, Dong, Zhou, *EPJC* 83(2023)314 [arXiv:2209.15194].
  Zhang, Feng, *JHEP* 05(2023)075 [arXiv: 2303.02795].
  Zhang, Feng, [arXiv: 2403.13025].

I. Introduction

3. Generally strategy

- **Hypergeometric function method: Evaluating Feynman integrals as hypergeometric functions.**
- Steps: (1) we write out the GKZ hypergeometric systems satisfied by the Feynman integrals on general compact manifold or proper Grassmannian manifold $G_{k,n}$.
  (2) fundamental solution systems are constructed in neighborhoods of regular singularities of the GKZ hypergeometric systems. The combination coefficients can be determined from Feynman integrals with some special kinematic parameters.

II. 3-loop vacuum on compact manifold



- Feynman integral of the 3-loop vacuum diagram with 4 propagates is written as

$$U_4 = \left(\Lambda^2_{\rm RE}\right)^{6-\frac{3D}{2}} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \times \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)((q_1 + q_2 + q_3)^2 - m_3^2)(q_3^2 - m_4^2)} . \quad (2.1)$$

- Zhang, Feng, GKZ hypergeometric systems of the three-loop vacuum Feynman integrals, *JHEP* 05 (2023) 075 [arXiv: 2303.02795].

II. 3-loop vacuum on compact manifold

- Through Mellin-Barnes transformation

$$U_4 = \frac{\left(\Lambda_{\mathrm{RE}}^2\right)^{6-\frac{3D}{2}}}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \left[\prod_{i=1}^{3} (-m_i^2)^{s_i} \Gamma(-s_i)\Gamma(1+s_i)\right] I_q \,, \quad (2.2)$$

where

$$I_q \equiv \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \frac{1}{(q_1^2)^{1+s_1} (q_2^2)^{1+s_2} ((q_1+q_2+q_3)^2)^{1+s_3} (q_3^2 - m_4^2)} \,. \quad (2.3)$$

II. 3-loop vacuum on compact manifold

- Mellin-Barnes representation of the Feynman integral:

$$U_4 = \frac{-im_4^4}{(2\pi i)^3(4\pi)^6}\Big(\frac{4\pi\Lambda_{\rm RE}^2}{m_4^2}\Big)^{6-\frac{3D}{2}}\int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \Big[\prod_{i=1}^{3}\Big(\frac{m_i^2}{m_4^2}\Big)^{s_i}\Gamma(-s_i)\Big]$$
$$\times\Big[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-s_i)\Big]\Gamma(3-D+\sum_{i=1}^{3}s_i)\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{3}s_i)\ . \quad (2.4)$$

- It is well known that negative integers and zero are simple poles of the function $\Gamma(z)$. As all s_i contours are closed to the right in corresponding complex planes, one finds that the analytic expression of the the three-loop vacuum integral can be written as the linear combination of generalized hypergeometric functions.

II. 3-loop vacuum on compact manifold

$$U_4 \ni \frac{im_4^4}{(4\pi)^6}\left(\frac{4\pi\Lambda_{\mathrm{RE}}^2}{m_4^2}\right)^{6-\frac{3D}{2}} \frac{\pi^3}{\sin^3\frac{\pi D}{2}} T_4(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x})\ , \tag{2.5}$$

$$T_4(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x}) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} A_{n_1n_2n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}\ , \tag{2.6}$$

$$A_{n_1n_2n_3} = \frac{\Gamma(a_1+\sum_{i=1}^{3} n_i)\Gamma(a_2+\sum_{i=1}^{3} n_i)}{n_1!n_2!n_3!\Gamma(b_1+n_1)\Gamma(b_2+n_2)\Gamma(b_3+n_3)}\ . \tag{2.7}$$

where $x_i = m_i^2/m_4^2$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with

$$a_1 = 3-D,\ a_2 = 4-\frac{3D}{2},\ b_1 = b_2 = b_3 = 2-\frac{D}{2}\ . \tag{2.8}$$

II. 3-loop vacuum on compact manifold

- We can define auxiliary function

$$\Phi_4(\mathbf{a},\ \mathbf{b}\,\big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_3}T_4(\mathbf{a},\ \mathbf{b}\,\big|\,\mathbf{x})\ . \tag{2.9}$$

Through Miller's transformation,

$$\begin{aligned}
&\vartheta_{u_j}\Phi_4(\mathbf{a},\ \mathbf{b}\,\big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j\Phi_4(\mathbf{a},\ \mathbf{b}\,\big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\\
&\vartheta_{v_k}\Phi_4(\mathbf{a},\ \mathbf{b}\,\big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k-1)\Phi_4(\mathbf{a},\ \mathbf{b}\,\big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,
\end{aligned} \tag{2.10}$$

which naturally induces the notion of GKZ hypergeometric system. Euler operators: $\vartheta_{x_k} = x_k\partial_{x_k}$.

II. 3-loop vacuum on compact manifold

- Through the transformation

$$z_j = \frac{1}{u_j}, \quad z_{2+k} = v_k, \; z_{5+k} = \frac{x_k}{u_1 u_2 v_k}, \tag{2.11}$$

we have GKZ hypergeometric system for the integral

$$\mathbf{A_4} \cdot \vec{\vartheta}_4 \Phi_4 = \mathbf{B_4} \Phi_4 \; , \tag{2.12}$$

$$\mathbf{A_4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix} ,$$

$$\vec{\vartheta}_4^{\,T} = (\vartheta_{z_1}, \; \cdots \; , \; \vartheta_{z_8}) \; ,$$

$$\mathbf{B_4}^T = (-a_1, \; -a_2, \; b_1 - 1, \; b_2 - 1, \; b_3 - 1) \; . \tag{2.13}$$

II. 3-loop vacuum on compact manifold

- Defining the combined variables

$$y_1 = \frac{z_3 z_6}{z_1 z_2} , \quad y_2 = \frac{z_4 z_7}{z_1 z_2} , \quad y_3 = \frac{z_5 z_8}{z_1 z_2} , \tag{2.14}$$

we write the solutions as

$$\Phi_4(\mathbf{z}) = \Big(\prod_{i=1}^{8} z_i^{\alpha_i} \Big) \varphi_4(y_1, \; y_2, \; y_3) \; . \tag{2.15}$$

Here $\vec{\alpha}^{\,T} = (\alpha_1, \; \alpha_2, \; \cdots \; , \; \alpha_8)$ denotes a sequence of complex number such that

$$\mathbf{A_4} \cdot \vec{\alpha} = \mathbf{B_4} \; , \tag{2.16}$$

namely,

$$\alpha_1 + \alpha_6 + \alpha_7 + \alpha_8 = -a_1 \; , \quad \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 = -a_2 \; ,$$
$$\alpha_3 - \alpha_6 = b_1 - 1 \; , \quad \alpha_4 - \alpha_7 = b_2 - 1 \; , \quad \alpha_5 - \alpha_8 = b_3 - 1. \tag{2.17}$$

II. 3-loop vacuum integrals

- Correspondingly the dual matrix $\tilde{\mathbf{A}}_4$ of $\mathbf{A}_4$ is

$$\tilde{\mathbf{A}}_4 = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.18)$$

  The row vectors of the matrix $\tilde{\mathbf{A}}_4$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

- We denote the submatrix composed of the first, third, and fourth column vectors of the dual matrix of Eq. (2.18) as $\tilde{\mathbf{A}}_{134}$, i.e.

$$\tilde{\mathbf{A}}_{134} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}. \qquad (2.19)$$

II. 3-loop vacuum on compact manifold

- Obviously $\det \tilde{\mathbf{A}}_{134} = -1 \neq 0$, and

$$\begin{aligned}\mathbf{B}_{134} &= \tilde{\mathbf{A}}_{134}^{-1} \cdot \tilde{\mathbf{A}}_{4} \\ &= \left(\begin{array}{cccccccc} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right) . \end{aligned} \quad (2.20)$$

Taking 3 row vectors of the matrix $\mathbf{B}_{134}$ as the basis of integer lattice, one constructs the GKZ hypergeometric series solutions in parameter space through choosing the sets of column indices $I_i \subset [1, 8]$ $(i = 1, \cdots, 8)$ which are consistent with the basis of integer lattice $\mathbf{B}_{134}$.

II. 3-loop vacuum on compact manifold

- We take the set of column indices $I_1 = [2,5,6,7,8]$, i.e. the implement $J_1 = [1,8] \setminus I_1 = [1,3,4]$. The choice on the set of indices implies the exponent numbers $\alpha_1 = \alpha_3 = \alpha_4 = 0$. Through Eq. (2.17), one can have

$$\begin{aligned}&\alpha_2 = a_1 - a_2,\ \alpha_5 = b_1 + b_2 + b_3 - a_1 - 3,\\&\alpha_6 = 1 - b_1,\ \alpha_7 = 1 - b_2,\ \alpha_8 = b_1 + b_2 - a_1 - 2\,.\end{aligned} \quad (2.21)$$

Combined with Eq. (2.8), we can have

$$\alpha_2 = \frac{D}{2} - 1,\ \alpha_5 = -\frac{D}{2},\ \alpha_6 = \frac{D}{2} - 1,\ \alpha_7 = \frac{D}{2} - 1,\ \alpha_8 = -1\,. \quad (2.22)$$

II. 3-loop vacuum on compact manifold

- According the basis of integer lattice $\mathbf{B}_{134}$, the hypergeometric series solution can be written as

$$\Phi^{(1)}_{[134]}(\alpha,z)=y_1^{\frac{D}{2}-1}y_2^{\frac{D}{2}-1}y_3^{-1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}c^{(1)}_{[134]}(\alpha,\mathbf{n})\Big(\frac{1}{y_3}\Big)^{n_1}\Big(\frac{y_1}{y_3}\Big)^{n_2}\Big(\frac{y_2}{y_3}\Big)^{n_3},$$

$$c^{(1)}_{[134]}(\alpha,\mathbf{n})=\frac{\Gamma(\frac{D}{2}+n_1+n_2+n_3)\Gamma(1+n_1+n_2+n_3)}{n_1!n_2!n_3!\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)}\,. \quad (2.23)$$

Here, the convergent region is

$$\Xi_{[134]}=\{(y_1,\ y_2,\ y_3)\Big|1<|y_3|,\ |y_1|<|y_3|,\ |y_2|<|y_3|\}\,, \quad (2.24)$$

which shows that $\Phi^{(1)}_{[134]}(\alpha,z)$ is in neighborhood of regular singularity ∞.

II. 3-loop vacuum on compact manifold

- In a similar way, we can obtain other seven hypergeometric solutions which are consistent with the basis of integer lattice $\mathbf{B}_{134}$, and the convergent region is also $\Xi_{[134]}$.
- The above eight hypergeometric series solutions $\Phi^{(i)}_{[134]}(\alpha, z)$ whose convergent region is $\Xi_{[134]}$ can constitute a fundamental solution system.
- Multiplying one of the row vectors of the matrix $\mathbf{B}_{134}$ by -1, the induced integer matrix can also be chosen as a basis of the integer lattice space of certain hypergeometric series.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned} \mathbf{B}_{\tilde{1}34} &= \mathrm{diag}(-1, 1, 1) \cdot \mathbf{B}_{134} \\ &= \begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix} , \end{aligned} \tag{2.25}$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[\tilde{1}34]}(\alpha, z)$ $(i = 1, \cdots, 8)$ similarly. The convergent region is

$$\Xi_{[\tilde{1}34]} = \{(y_1,\ y_2,\ y_3) \big| |y_1| < 1,\ |y_2| < 1,\ |y_3| < 1\} , \tag{2.26}$$

which shows that $\Phi^{(i)}_{[\tilde{1}34]}(\alpha, z)$ are in neighborhood of regular singularity 0 and can constitute a fundamental solution system.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{1\tilde{3}4} = \mathrm{diag}(1,-1,1)\cdot\mathbf{B}_{134}$$
$$= \left(\begin{array}{cccccccc} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right), \quad (2.27)$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[1\tilde{3}4]}(\alpha, z)$ $(i = 1, \cdots, 8)$ similarly. The convergent region is

$$\Xi_{[1\tilde{3}4]} = \{(y_1,\ y_2,\ y_3)\Big| 1 < |y_1|,\ |y_2| < |y_1|,\ |y_3| < |y_1|\}, \quad (2.28)$$

which shows that $\Phi^{(i)}_{[1\tilde{3}4]}(\alpha, z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{13\tilde{4}} = \mathrm{diag}(1,1,-1)\cdot\mathbf{B}_{134} = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}, \quad (2.29)$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[13\tilde{4}]}(\alpha, z)$ $(i = 1, \cdots, 8)$ similarly. The convergent region is

$$\Xi_{[1\tilde{3}4]} = \{(y_1,\ y_2,\ y_3)\Big| 1 < |y_2|,\ |y_1| < |y_2|,\ |y_3| < |y_2|\}, \quad (2.30)$$

which shows that $\Phi^{(i)}_{[13\tilde{4}]}(\alpha, z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

II. 3-loop vacuum integrals

- **3-loop vacuum with 4 massive propagates**
  CPU i9-13th, 64GB: FeynGKZ $\sim$ 1 s, FIESTA $\sim$ 50 s

```
SumLim = 15;
ParameterSub = {Dϵ → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1,
   a2 → 1, a3 → 1, a4 → 1, m4 → 0.01, m1 → 0.02, m2 → 10 , m3 → 0.04};
NumericalSum[SeriesSolution, ParameterSub, SumLim];
Numerical result = -7.5628 × 10^8
Time Taken 1.16336 seconds
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = -7.56285 × 10^8
Time Taken 53.2344 seconds
```

- **3-loop vacuum with 5 massive propagates**
  CPU i9-13th, 64GB: FeynGKZ $\sim$ 120 s, FIESTA $\sim$ 600 s

```
SumLim = 15;
ParameterSub = {Dϵ → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 7 / 8,
   a4 → 3 / 4, a5 → 1, m1 → 0.1, m2 → 5, m3 → 0.3, m4 → 0.3, m5 → 100};
NumericalSum[SeriesSolution, ParameterSub, SumLim, RunInParallel → True];
In[46]:=
Numerical result = 1.42136 × 10^7
In[46]:=
Time Taken 120.792 seconds
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = 1.42136 × 10^7
Time Taken 623.497 seconds
```

4-loop vacuum, arXiv: 2403.13025

- **4-loop vacuum with 6 propagates Type A: 3 massive**
  CPU i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 1500 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2×0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.01, m2 → 0.1 , m6 → 10};
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = 1.2624×10^{11}

Time Taken 0.059701 seconds

```
SumLim = 15;
ParameterSub = {De → 4 - 2×0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.01, m2 → 0.1 , m6 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = 1.26241×10^{11}

Time Taken 1525.73 seconds

- **4-loop vacuum with 7 propagates Type B: 3 massive**
  CPU i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 6000 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2×0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 9 / 10,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1, m7 → 10 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = 1.43756×10^{8}

Time Taken 0.094408 seconds

```
SumLim = 15;
ParameterSub = {De → 4 - 2×0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 9 / 10,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1, m7 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = 1.43754×10^{8}

Time Taken 6370.08 seconds

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2 = 0$, $m_2^2 \neq 0$

- Adopting Feynman parametric representation

$$A_{1SE}(p^2, 0, m_2^2) = \left(\Lambda^2_{\rm RE}\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2((q+p)^2 - m_2^2)}$$

$$= \frac{i\Gamma(2-\frac{D}{2})\left(\Lambda^2_{\rm RE}\right)^{2-D/2}}{(-)^{2-D/2}(4\pi)^{D/2}} \int_0^1 dt_1 dt_3 \frac{t_1^{D/2-2}\delta(t_1+t_3-1)}{(t_3p^2 - m_2^2)^{2-D/2}}$$

$$= \frac{i\Gamma(2-\frac{D}{2})\left(\Lambda^2_{\rm RE}\right)^{2-D/2}}{(-)^{2-D/2}(4\pi)^{D/2}} \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D}\delta(t_1+t_2+t_3)}{(t_3p^2 + t_2m_2^2)^{2-D/2}} \quad (3.1)$$

  with homogeneous coordinate $t_2 = -1$, volume element of projective space $\omega_3(t) = t_1 dt_2 dt_3 - t_2 dt_1 dt_3 + t_3 dt_1 dt_2$.

- Feng, Zhang, Chang, Feynman integrals of Grassmannians, *PRD* 106 (2022) 116025 [arXiv: 2206.04224].

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2 = 0$, $m_2^2 \neq 0$

- The integral

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} ,$$

can be embedded in the subvariety of the Grassmannian $G_{3,5}$, with splitting local coordinates as

$$A^{(1)} = \left(\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{array} \right) . \quad (3.2)$$

- first row: t_1, second row: t_2, third row: t_3, first column: $t_1^{D/2-2}$, second column: t_2^{2-D}, third column: $t_3^0 = 1$, fourth column represents the function $\delta(t_1 + t_2 + t_3)$, fifth column: $(z_{1,5} t_1 + z_{2,5} t_2 + z_{3,5} t_3)^{D/2-2} = (t_2 m_2^2 + t_3 p^2)^{D/2-2}$.

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2 = 0$, $m_2^2 \neq 0$

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} ,$$

satisfies the following GKZ-system

$$\Big\{\vartheta_{1,1} + \vartheta_{1,4}\Big\}A_{1SE} = -A_{1SE} \;,\; \Big\{\vartheta_{2,2} + \vartheta_{2,4} + \vartheta_{2,5}\Big\}A_{1SE} = -A_{1SE} \;, \tag{3.3}$$
$$\Big\{\vartheta_{3,3} + \vartheta_{3,4} + \vartheta_{3,5}\Big\}A_{1SE} = -A_{1SE} \;,\; \vartheta_{1,1}A_{1SE} = (\frac{D}{2} - 2)A_{1SE} \;,\; \vartheta_{2,2}A_{1SE} = (2 - D)A_{1SE} \;,$$
$$\vartheta_{3,3}A_{1SE} = 0 \;,\; \Big\{\vartheta_{1,4} + \vartheta_{2,4} + \vartheta_{3,4}\Big\}A_{1SE} = -A_{1SE} \;,\; \Big\{\vartheta_{2,5} + \vartheta_{3,5}\Big\}A_{1SE} = (\frac{D}{2} - 2)A_{1SE} \;.$$

Exponent matrix:

$$\begin{pmatrix} \frac{D}{2} - 2 & 0 & 0 & 1 - \frac{D}{2} & 0 \\ 0 & 2 - D & 0 & \alpha_{2,4} & \alpha_{2,5} \\ 0 & 0 & 0 & \alpha_{3,4} & \alpha_{3,5} \end{pmatrix} , \tag{3.4}$$

$$\alpha_{2,4} + \alpha_{2,5} = D - 3,\; \alpha_{3,4} + \alpha_{3,5} = -1,\; \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2,\; \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2.$$

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2 = 0$, $m_2^2 \neq 0$

- Dual space of the GKZ-system: 3×5 matrix $(0_{3\times 3} \big| E_3^{(1)})$ with

$$E_3^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} . \qquad (3.5)$$

- Integer lattice $(0_{3\times 3} \big| nE_3^{(1)})$ $(n \geq 0)$ is compatible with two choices of the exponents. Through

$$\alpha_{2,4} + \alpha_{2,5} = D - 3,\ \alpha_{3,4} + \alpha_{3,5} = -1,\ \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2,\ \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2,$$

the first choice is written as

$$\alpha_{2,4} = 0,\ \alpha_{2,5} = D - 3,\ \alpha_{3,4} = \frac{D}{2} - 2,\ \alpha_{3,5} = 1 - \frac{D}{2} . \qquad (3.6)$$

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2=0$, $m_2^2\neq 0$

$$\text{Splitting local coordinates: } A^{(1)}=\left(\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{array}\right).$$

$$\text{Integer lattice } (0_{3\times 3}\big|nE_3^{(1)}):\quad nE_3^{(1)}=\left(\begin{array}{cc} 0 & 0 \\ n & -n \\ -n & n \end{array}\right).$$

$$\text{Exponent matrix: } \left(\begin{array}{ccccc} \frac{D}{2}-2 & 0 & 0 & 1-\frac{D}{2} & 0 \\ 0 & 2-D & 0 & 0 & D-3 \\ 0 & 0 & 0 & \frac{D}{2}-2 & 1-\frac{D}{2} \end{array}\right).$$

Hypergeometric function:

$$\psi^{(1)}_{\{1,2,3\}}\sim (m_2^2)^{\alpha_{2,5}}(p^2)^{\alpha_{3,5}}\sum_{n=0}^{\infty}\frac{\Gamma(-\alpha_{2,5}+n)\Gamma(-\alpha_{3,4}+n)}{\Gamma(1+\alpha_{2,4}+n)\Gamma(1+\alpha_{3,5}+n)}(m_2^2)^{-n}(p^2)^n$$

$$\sim (m_2^2)^{D-3}(p^2)^{1-D/2}\sum_{n=0}^{\infty}\frac{\Gamma(3-D+n)}{n!}\left(\frac{p^2}{m_2^2}\right)^n . \tag{3.7}$$

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2 = 0$, $m_2^2 \neq 0$

- For integer lattice $(0_{3\times 3} \big| nE_3^{(1)})$ $(n \geq 0)$, second choice is written as

$$\alpha_{3,5} = 0,\ \alpha_{2,4} = \frac{D}{2} - 1,\ \alpha_{2,5} = \frac{D}{2} - 2,\ \alpha_{3,4} = -1\ . \qquad (3.8)$$

- Adopting integer lattice and the corresponding exponents matrices, we obtain hypergeometric function as

$$\begin{aligned}
&\psi^{(2)}_{\{1,2,3\}}(p^2,\ 0,\ m_2^2) \\
&\sim (m_2^2)^{\alpha_{2,5}}(p^2)^{\alpha_{3,5}} \sum_{n=0}^{\infty} \frac{\Gamma(-\alpha_{2,5}+n)\Gamma(-\alpha_{3,4}+n)}{\Gamma(1+\alpha_{2,4}+n)\Gamma(1+\alpha_{3,5}+n)} (m_2^2)^{-n}(p^2)^n \\
&\sim (m_2^2)^{D/2-2} \sum_{n=0}^{\infty} \frac{\Gamma(2-\frac{D}{2}+n)}{\Gamma(\frac{D}{2}+n)} \left(\frac{p^2}{m_2^2}\right)^n . \qquad (3.9)
\end{aligned}$$

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2 = 0$, $m_2^2 \neq 0$

- For integer lattice $(0_{3\times 3}\Big| - nE_3^{(1)})$ $(n \geq 0)$, two possibilities:

$$\alpha_{2,5} = 0,\ \alpha_{2,4} = D-3,\ \alpha_{3,4} = 1-\frac{D}{2},\ \alpha_{3,5} = \frac{D}{2}-2\ ; \quad (3.10)$$

$$\alpha_{3,4} = 0,\ \alpha_{2,4} = \frac{D}{2}-2,\ \alpha_{2,5} = \frac{D}{2}-1,\ \alpha_{3,5} = -1\ . \quad (3.11)$$

- Adopting integer lattice and exponents matrices, we obtain two linear independent hypergeometric functions as

$$\psi^{(3)}_{\{1,2,3\}}(p^2,\ 0,\ m_2^2) \sim (p^2)^{D/2-2}\sum_{n=0}^{\infty}\frac{\Gamma(3-D+n)}{n!}\Big(\frac{m_2^2}{p^2}\Big)^n\ ; \quad (3.12)$$

$$\psi^{(4)}_{\{1,2,3\}}(p^2,\ 0,\ m_2^2) \sim \frac{(m_2^2)^{D/2-1}}{p^2}\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{D}{2}+n)}{\Gamma(\frac{D}{2}+n)}\Big(\frac{m_2^2}{p^2}\Big)^n\ . \quad (3.13)$$

III. Feynman integrals on Grassmannian

1-loop self-energy: $m_1^2 = 0$, $m_2^2 \neq 0$

- $\det(A^{(1)}_{\{1,2,3\}}) = 1$, $G_{3,5}$ splitting local coordinates

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}. \tag{3.14}$$

Convergent regions: $|1/x| \leq 1$, $|x| \leq 1$ $(x = m_2^2/p^2)$. Neighborhoods: $x = \infty,\ 0$.

- $\det(A^{(1)}_{\{1,2,5\}}) = p^2$, $G_{3,5}$ splitting local coordinates

$$\left(A^{(1)}_{\{1,2,5\}}\right)^{-1} \cdot A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -\frac{m_2^2}{p^2} & 1 - \frac{m_2^2}{p^2} & 0 \\ 0 & 0 & \frac{1}{p^2} & \frac{1}{p^2} & 1 \end{pmatrix}. \tag{3.15}$$

Convergent regions: $|1/(1-1/x)| \leq 1$, $|1-1/x| \leq 1$. Neighborhoods: $x = 0,\ 1$.

- $\det(A^{(1)}_{\{1,3,5\}}) = -m_2^2$, $\det(A^{(1)}_{\{2,3,4\}}) = 1$, $\det(A^{(1)}_{\{2,4,5\}}) = -p^2$, $\det(A^{(1)}_{\{3,4,5\}}) = m_2^2$.
- 24 fundamental solutions

IV. Summary

- **Using Mellin-Barnes representation and Miller's transformation, we derive GKZ hypergeometric systems of Feynman integrals on compact manifold. In the neighborhoods of origin 0 including infinity ∞, we can obtain analytical hypergeometric solutions through GKZ-systems. One can see that evaluating Feynman integrals by hypergeometric function method is efficiency .**

- **Feynman integrals also can be taken as functions on the subvarieties of Grassmannians through homogenizing the parametric representation. The GKZ-systems can be obtained in splitting local coordinates. Fundamental solution systems can be obtained in neighborhoods of all possible regular singularities.**

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THANKS!