Family tree decomposition of cosmological correlators

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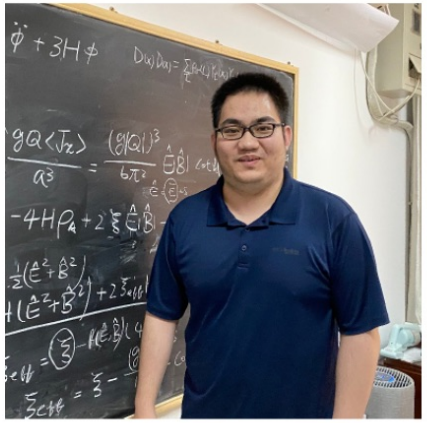
Based on

ZX, Jiaju Zang: Inflation Correlators with Multiple Massive Exchanges,

JHEP **03** (2024) 070 [arXiv: 2309.10849]

Bingchu Fan, **ZX**: Cosmological Amplitudes in Power-Law FRW Universe,
arXiv: 2403.07050

Jiaju Zang
臧家驹



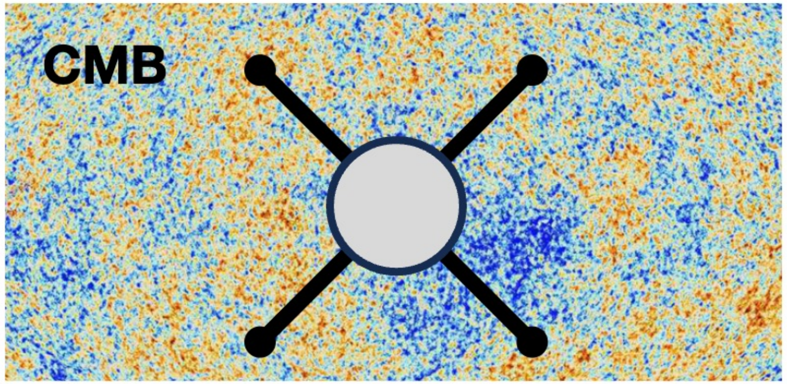
Bingchu Fan
樊秉初

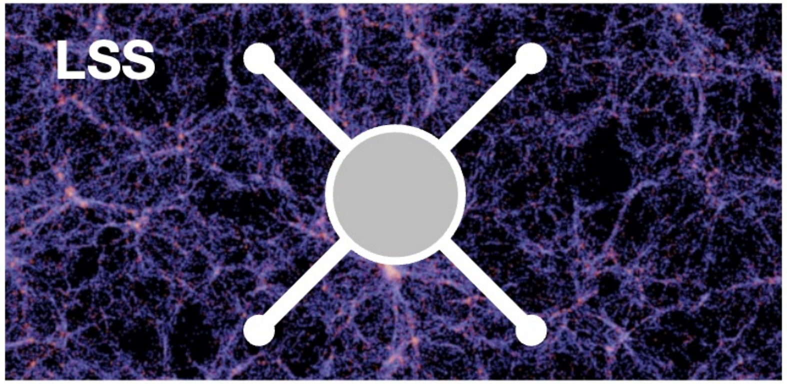
Outline

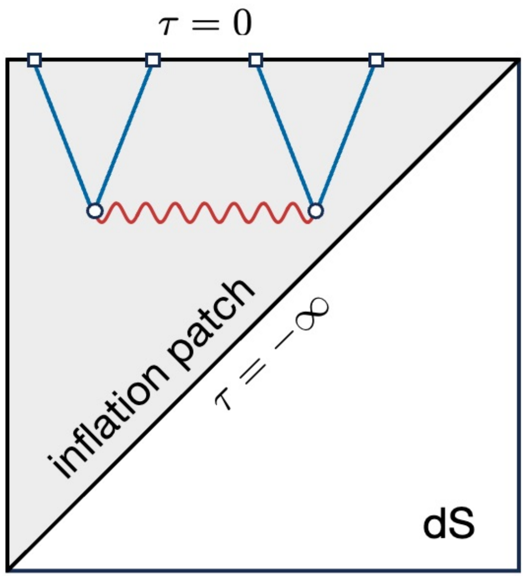
1. Background [cosmological collider | CC signal | particle pheno]
2. Cosmological correlators [structure | analytical methods | PMB rep]
3. Family tree decomposition [time int | series solution | analytical continuation]
4. Pheno application: multiple massive exchanges
5. Conformal amplitudes in FRW [general rule | energy int | inflation limit]

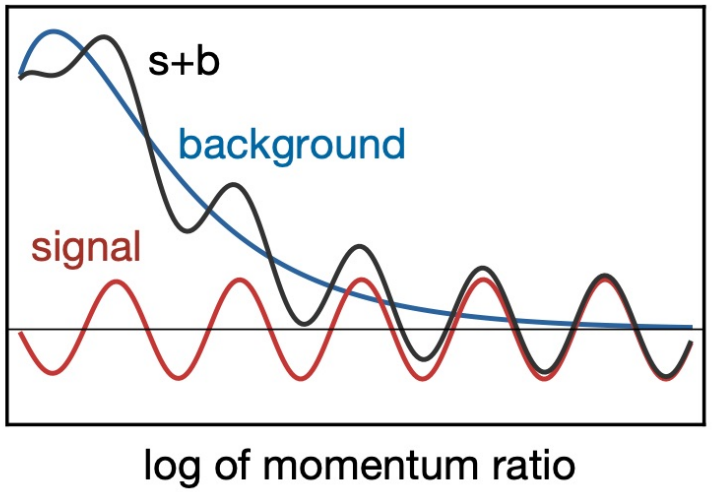
A Cosmological collider program

[Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more]









particle production

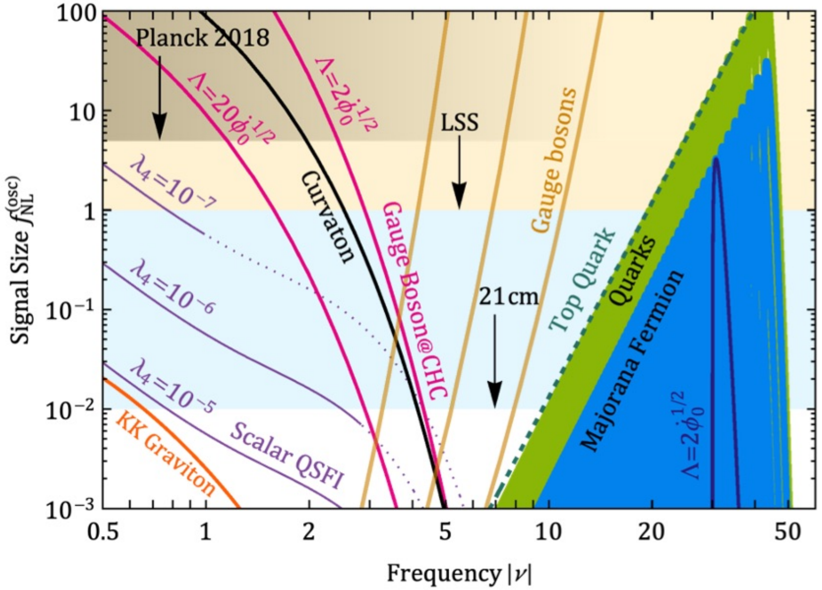
mass ~ 10^{14} GeV

superhorizon resonance

nonanalyticity

[Zhehan Qin, **ZX**, 2304.13295; 2308.14802]

Phenomenological motivations



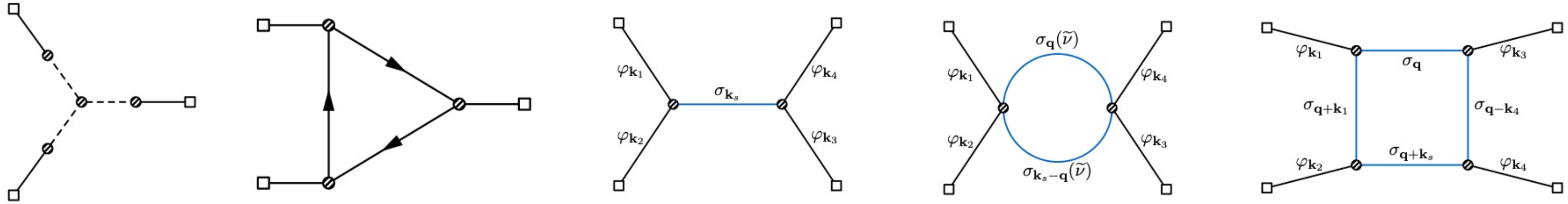
[Lian-Tao Wang, **ZX**, 1910.12876]

Over the years, many particle models identified in SM/BSM, with large signals

Many types of diagrams (tree + loop) involved

Understanding the amplitudes!

- efficient numerical implementation
- analytical structure



Data are coming in!

Constraints from CMB data (~ 2σ) [2404.07203]

Searching for Cosmological Collider in the Planck CMB Data

Wuhyun Sohn[1], Dong-Gang Wang[2], James R. Fergusson[2], and E. P. S. Shellard[2]

Constraints from LSS data [2404.01894]

BOSS Constraints on Massive Particles during Inflation: The Cosmological Collider in Action

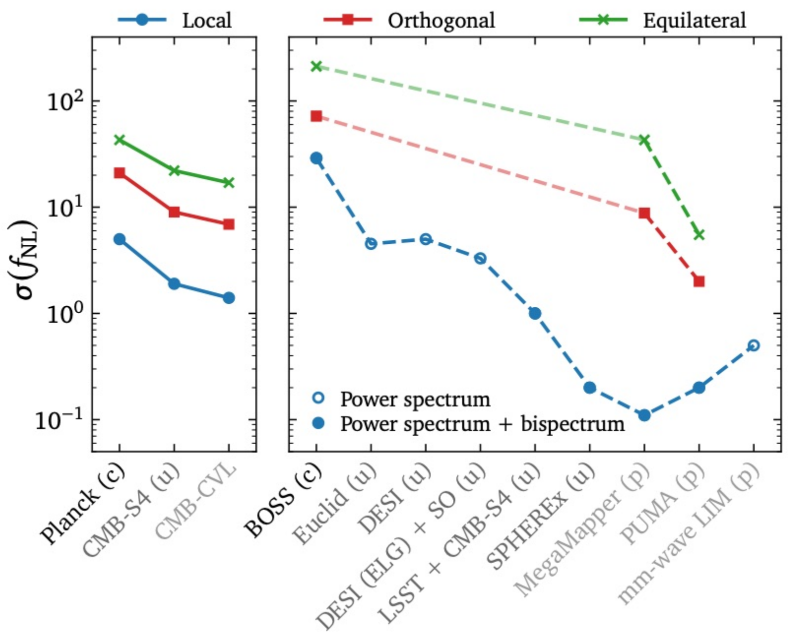
Giovanni Cabass,[1,*] Oliver H.E. Philcox,[2,3,†] Mikhail M. Ivanov,[4,‡] Kazuyuki Akitsu,[5] Shi-Fan Chen,[6] Marko Simonović,[7,8] and Matias Zaldarriaga[6]

inflaton self-interactions. This is made possible through an improvement in Cosmological Bootstrap techniques and the combination of perturbation theory and halo occupation distribution models for galaxy clustering. Our work sets the standard for inflationary spectroscopy with cosmological observations, providing the ultimate link between physics on the largest and smallest scales.

~ 2 orders in near future

~ 4 ultimately with 21cm

[Snowmass 2021: 2203.08128]



Cosmological correlators: General structure

[See Chen, Wang, **ZX**, 1703.10166 for a review]

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{q} \times (-\tau)^p \times e^{\mathrm{i}E\tau} \times \mathrm{H}_{\mathrm{i}\widetilde{\nu}}\Big[-K(\boldsymbol{q},\boldsymbol{k})\tau\Big] \times \theta(\tau_i - \tau_j)$$

vertex int loop int ext line bulk line

Complications and strategies

Special functions in propagators

☐ cosmo bootstrap [Arkani-Hamed et al. 1811.00024 etc] ☐ AdS + Mellin [Sleight 1907.01143 etc]

☐ **partial Mellin-Barnes** [Qin, **ZX**, 2205.01692, 2208.13790 etc]

Loop (momentum) integral

☐ spectral decomposition [**ZX**, Zhang, arXiv:2211.03810] ☐ dispersion, star-mesh [ongoing]

Nested time integral

☐ diff eq [Arkani-Hamed et al. 2312.05303] ☐ **family tree** [this talk]

Partial Mellin-Barnes representation

[Qin, **ZX**, 2205.01692, 2208.13790]

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}\tau \int \mathrm{d}^d \boldsymbol{q} \times (-\tau)^p \times e^{\mathrm{i}E\tau} \times \mathrm{H}_{\mathrm{i}\widetilde{\nu}}\Big[-K(\boldsymbol{q},\boldsymbol{k})\tau\Big] \times \theta(\tau_i - \tau_j)$$

Partial Mellin-Barnes: All massive lines in Mellin space [expanded in dilatation eigenmodes]

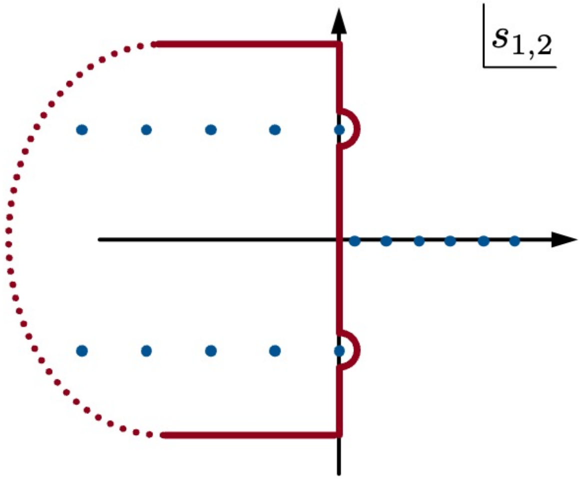
$$\mathrm{H}_\nu^{(j)}(az) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{(az/2)^{-2s}}{\pi} e^{(-1)^{j+1}(2s-\nu-1)\pi\mathrm{i}/2} \Gamma\Big[s-\frac{\nu}{2}, s+\frac{\nu}{2}\Big]$$

Time and loop integrals factorize in the Mellin integrand

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \Big[\int \mathrm{d}^d \boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\Big] \times \Big[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j)\Big]$$

loop int — nested time int

Mellin integrand meromorphic (IR finite): Sum of residues



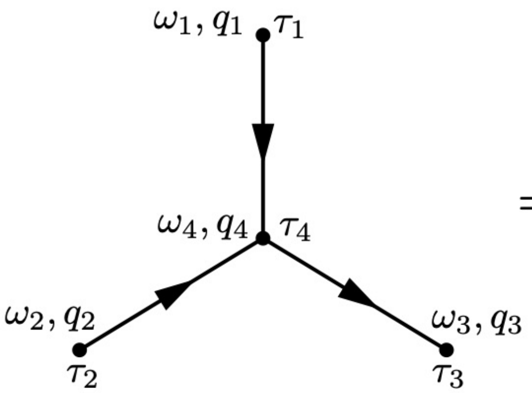
Family tree decomposition

[**ZX**, Zang, 2309.10849]

The most general time integral: $$(-\mathrm{i})^N \int_{-\infty}^{0} \prod_{\ell=1}^{N} \left[\mathrm{d}\tau_\ell\,(-\tau_\ell)^{q_\ell-1} e^{\mathrm{i}\omega_\ell\tau_\ell}\right] \prod \theta(\tau_j - \tau_i)$$

They naturally acquire graphic representations [NOT original Feynman diagrams]

For example:



$$= (-\mathrm{i})^4 \int \prod_{\ell=1}^{4} \left[\mathrm{d}\tau_\ell\,(-\tau_\ell)^{q_\ell-1} e^{\mathrm{i}\omega_\ell\tau_\ell}\right] \theta(\tau_4 - \tau_1)\theta(\tau_4 - \tau_2)\theta(\tau_3 - \tau_4)$$

Family tree decomposition

Complications all from theta functions

Irremovabile, but can flip directions, at the expense of additional factorized graphs

$$\theta(\tau_1 - \tau_2) + \theta(\tau_2 - \tau_1) = 1$$

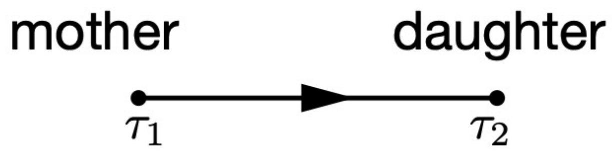
Family tree decomposition:

We always flip the directions such that all nested graphs are partially ordered

Partial order:

A mother can have any number of daughters

but a daughter must have only one mother



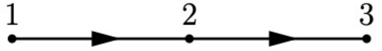
Every resulting nested graph can be interpreted as a maternal family tree

A useful notation for family trees: $\left[12(34\cdots)(5\cdots)\right]$

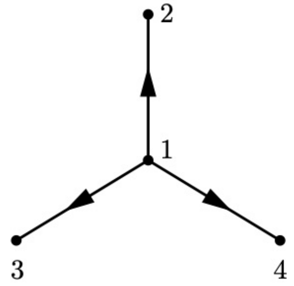
sisters

mother-daughter

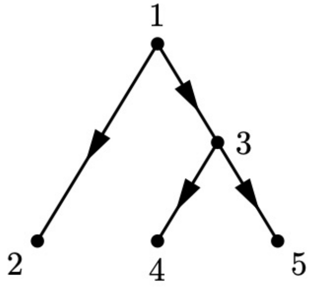
Examples:



$$[123] = (-\mathrm{i})^3 \int \prod_{i=1}^{3} \left[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \right] \theta_{32}\theta_{21}$$



$$[1(2)(3)(4)] = (-\mathrm{i})^4 \int \prod_{i=1}^{4} \left[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \right] \theta_{41}\theta_{31}\theta_{21}$$

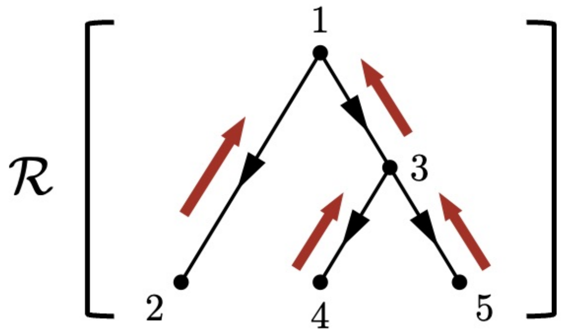


$$\left[1(2)\big(3(4)(5)\big)\right] = (-\mathrm{i})^5 \int \prod_{i=1}^{5} \left[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \right] \theta_{43}\theta_{53}\theta_{31}\theta_{21}$$

$$\theta_{ij} \equiv \theta(\tau_i - \tau_j)$$

Computing the family tree

Expand-and-integrate strategy works, but more streamlined with MB reps



→

$$\int_{-\infty}^{\tau_1} \mathrm{d}\tau_2\,(-\tau_2)^{q_2-1} e^{\mathrm{i}\omega_2\tau_2} = (-\tau_1)^{q_2}\mathrm{E}_{1-q_2}(-\mathrm{i}\omega_2\tau_1)$$

(exp int)

→

$$\mathrm{E}_p(z) = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\Gamma(s) z^{-s}}{s+p-1}$$

MB rep

→ next layer: again powers and exp → go through all layers → finish MB int

Mellin integrals finished by the residue theorem, with a series expansion:

$$[\mathscr{P}(\widehat{1}2\cdots N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}} \sum_{n_2,\cdots,n_N=0}^{\infty} \Gamma(q_{1\cdots N} + n_{2\cdots N}) \prod_{j=2}^{N} \frac{(-\omega_j/\omega_1)^{n_j}}{(\widetilde{q}_j + \widetilde{n}_j) n_j!}$$

earliest site   sum of all q's on Site j and her descendants $(q_{12\cdots} \equiv q_1 + q_2 + \cdots)$

All family trees are multivariate hypergeometric series

For simple family trees, the series sum to named hypergeometric functions [all dressed]

$$[1] = \frac{-\mathrm{i}}{(\mathrm{i}\omega_1)^{q_1}}\Gamma[q_1]$$ Euler Gamma function

$$[12] = \frac{-1}{(\mathrm{i}\omega_1)^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right]$$ Gauss hypergeometric functiona

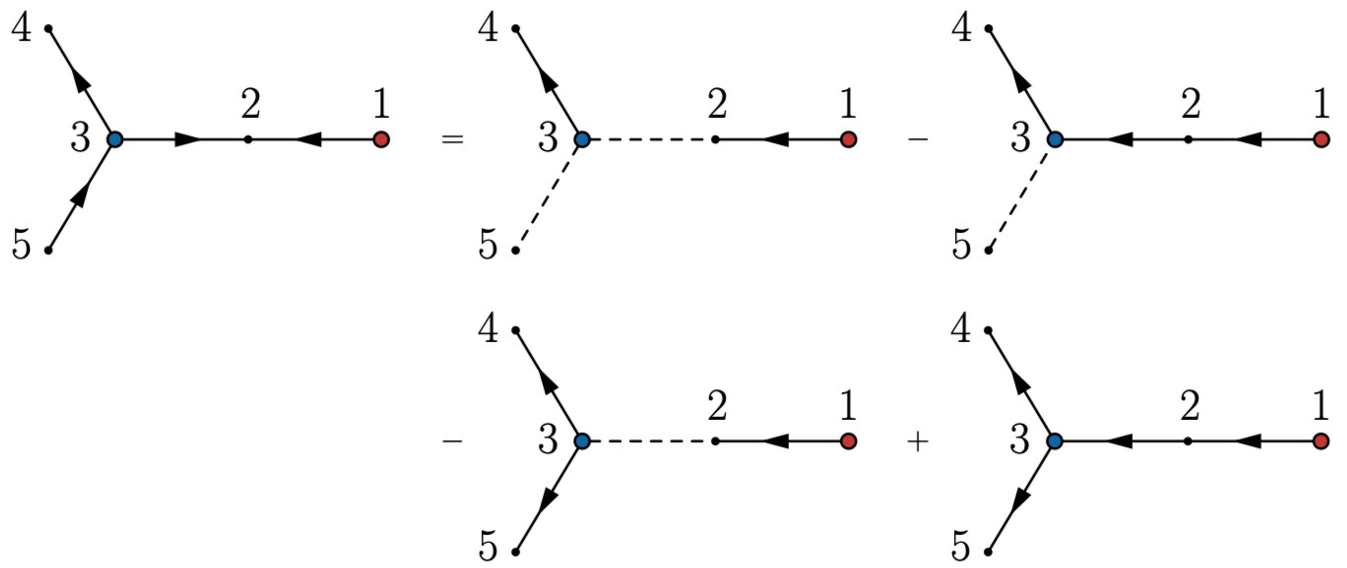
$$[2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix} q_1, q_3 \\ q_1+1, q_3+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$ Appell function

$$[123] = \frac{\mathrm{i}}{(\mathrm{i}\omega_1)^{q_{123}}}\ {}^{2+1}\mathcal{F}_{1+1}\left[\begin{matrix} q_{123}, q_{23} \\ q_{23}+1 \end{matrix}\middle|\begin{matrix} \text{-}\ , q_3 \\ \text{-}\ , q_3+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right]$$ Kampé de Fériet function

$$[1(2)\cdots(N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}}\mathcal{F}_A\left[q_{1\cdots N}\middle|\begin{matrix} q_2, \cdots, q_N \\ q_2+1, \cdots, q_N+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, \cdots, -\frac{\omega_N}{\omega_1}\right]$$ Lauricella function

… while more complicated family trees are not yet named

Example of family tree decomposition



Choose Site 1 as the earliest

1->2 good 2->3 flip

3->4 good 4->5 flip

$$\int \prod_{\ell=1}^{N} \left[\mathrm{d}\tau_\ell \, (-\tau_\ell)^{q_\ell - 1} e^{\mathrm{i}\omega_\ell \tau_\ell} \right] \theta(\tau_2 - \tau_1)\theta(\tau_2 - \tau_3)\theta(\tau_4 - \tau_3)(\tau_3 - \tau_5)$$

$$= [12]\,[34]\,[5] - [1234]\,[5] - [12]\,[3(4)(5)] + [123(4)(5)]$$

Functional identities & analytical continuation

The family tree decomposition yields many functional identities for family trees:

$$[12] = [12] \qquad \frac{1}{\omega_1^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right] = \frac{\Gamma[q_2]}{\omega_{12}^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} 1, q_{12} \\ q_2+1 \end{matrix}\middle| \frac{\omega_2}{\omega_{12}}\right]$$

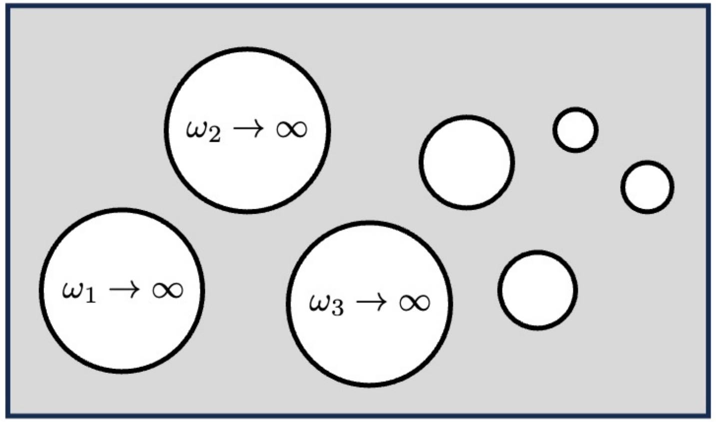
$$[12] + [21] = [1]\,[2] \qquad \frac{1}{\omega_1^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right] + \frac{1}{\omega_2^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_1, q_{12} \\ q_1+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}\right] = \frac{\Gamma[q_1, q_2]}{\omega_1^{q_1}\omega_2^{q_2}}$$

$$[123] + [2(1)(3)] = [1]\,[23] \qquad \frac{1}{\omega_1^{q_{123}}}\ {}_{2+1}\mathcal{F}_{1+1}\left[\begin{matrix} q_{123}, q_{23} \\ q_{23}+1 \end{matrix}\middle|\begin{matrix} \text{-}\,, q_3 \\ \text{-}\,, q_3+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right] + \frac{1}{\omega_2^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix} q_1, q_3 \\ q_1+1, q_3+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$
$$= \frac{\Gamma[q_1]}{\omega_1^{q_1}\omega_2^{q_{23}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_3, q_{32} \\ q_3+1 \end{matrix}\middle| -\frac{\omega_3}{\omega_2}\right]$$

More importantly, when series do not close, the identities amount to analytical continuation beyond the region of convergence

Unfortunately, conv. regions do not overlap

Numerical interpolation works. Better strategy?

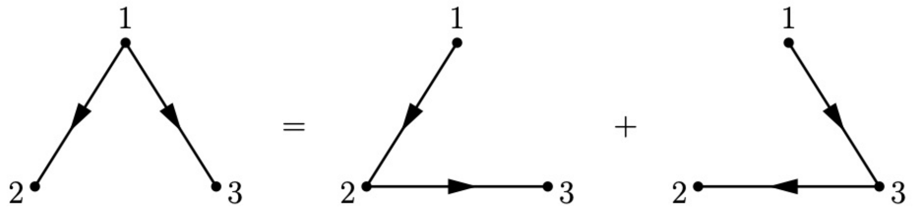


Minimal set of functions: family chains

birthday rule: Compare the birthdays of all family members and sum over all possibilities

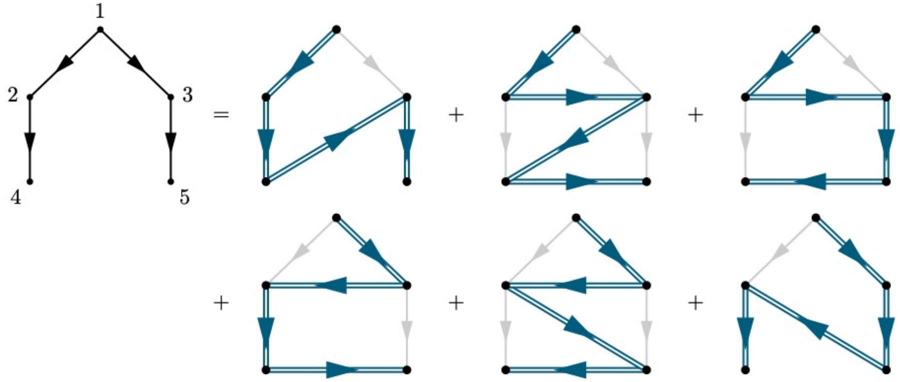
Formally, take shuffle products recursively among all subfamilies

$$\theta_{21}\theta_{31}(\theta_{32}+\theta_{23}) = \theta_{32}\theta_{21} + \theta_{23}\theta_{31}$$



Example:

$$\begin{aligned}[1(24)(35)] &= \{1(24) \sqcup\!\sqcup (35)\} \\ &= \{12435\} + \{12345\} + \{12354\} \\ &+ \{13245\} + \{13254\} + \{13524\}\end{aligned}$$



Family trees over-complete: further decomposable to chains; tree topology erased

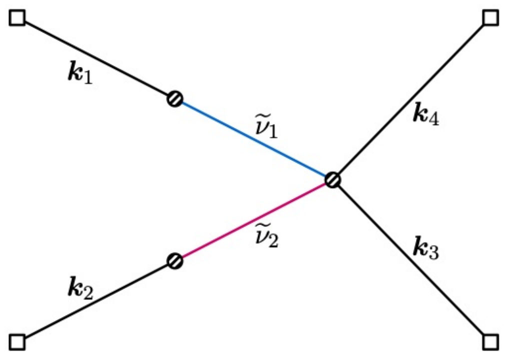
Family chains: standard iterated integrals; Hopf algebra; symbology?

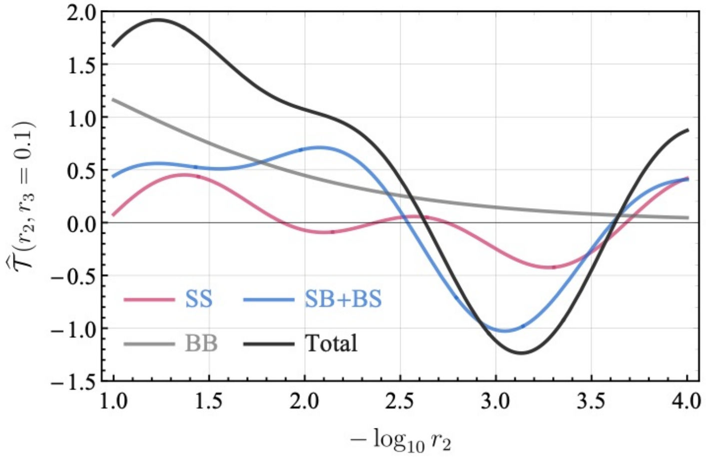
Multiple massive exchange in dS

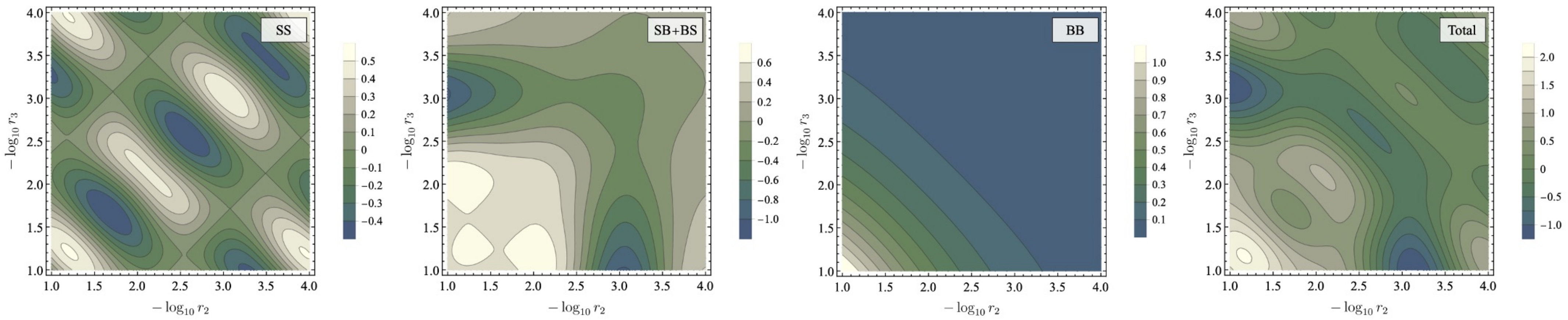
[**ZX**, Zang, 2309.10849]

Partial MB + family tree yields analytical expressions and enables fast numerical implementation of multiple massive exchanges

[**ZX**, Zang, 2309.10849]







Conformal amplitudes in power-law FRW

[Fan, **ZX**, 2403.07050]

An interesting toy model: conformal scalar with non-conformal self-interactions

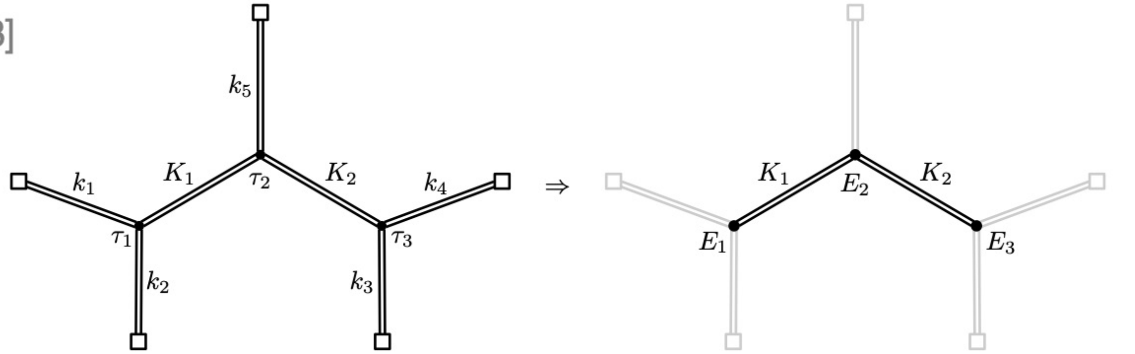
$$S[\phi_c] = -\int \mathrm{d}^{d+1}x\sqrt{-g}\left[\frac{1}{2}(\partial_\mu\phi_c)^2 + \frac{1}{2}\xi R\phi_c^2 + \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi_c^n\right] \qquad \xi \equiv (d-1)/(4d)$$

Conformal to Mink massless scalar + time-dep int: $\varphi(k,\tau) = \frac{1}{\sqrt{2k}}e^{-\mathrm{i}k\tau}$

Many activities in recent years

- ☐ Cosmo polytopes [Arkani-Hamed et al. 1709.02813 etc]
- ☐ Symbol recursion [Hillman 1912.09450]
- ☐ Twisted cohomology [De & Pokraka 2308.03753]
- ☐ Diff eqs [Arkani-Hamed et al. 2312.05303]

--- We get the full analytical results.



$$E_1 = k_{12}, \quad E_2 = k_5, \quad E_3 = k_{34}$$

Tree conformal amplitudes ~ sum & products of family trees

[finally, computing tree graphs like in flat space!]

Rule for wavefunctions:

[correlators similar]

1. Fix a partial order

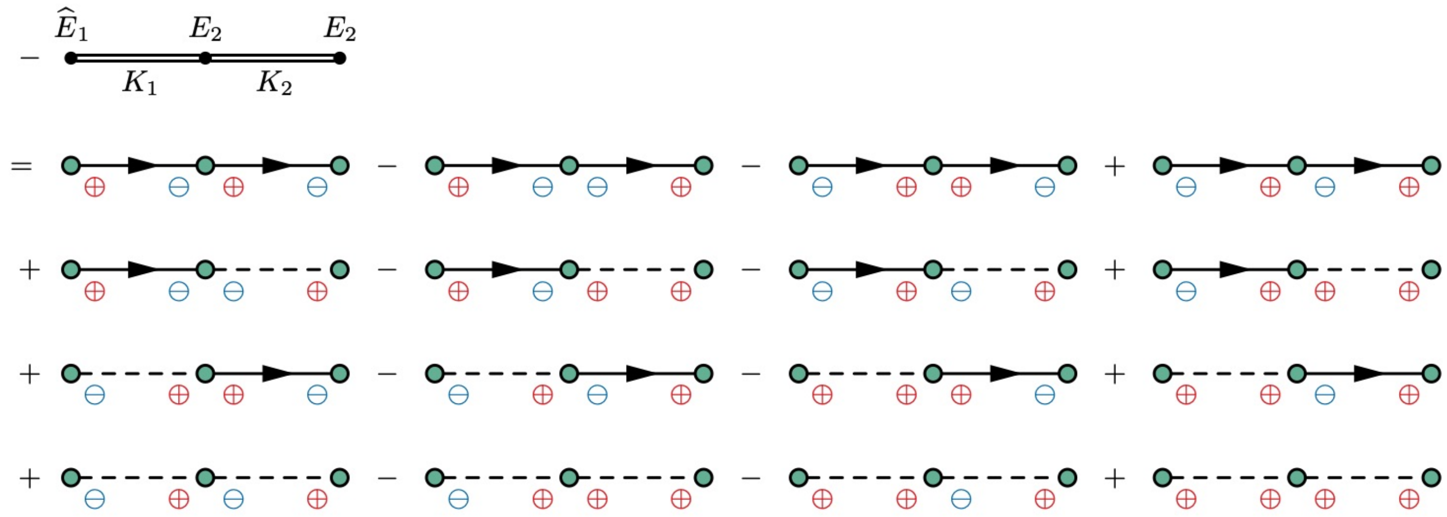
2. Write the uncut tree

[E_K | bar | sign index]

3. Cut! [bar ↔ unbar |

remove later index]

Example: 3-site chain



$$-\widetilde{\psi}_{3\text{-chain}}(\widehat{1}) = \sum_{a,b=\pm} ab\left\{[1_{1^a}2_{\bar{1}^a2^b}3_{\bar{2}^b}] + [1_{1^a}2_{\bar{1}^a\bar{2}^b}][3_2] + [1_{\bar{1}^a}][2_{12^b}3_{\bar{2}^b}] + [1_{\bar{1}^a}][2_{1\bar{2}^b}][3_2]\right\}$$

$$2_{\bar{1}^a2^b} \equiv E_2 - aK_1 + bK_2$$

Rich structures

FRW conformal amplitudes => Twisted energy integrals
integrand being fractions (cosmological polytope) [Arkani-Hamed et al. 1709.02813, 2312.05303]

$$\mathcal{T} \sim \int_0^\infty \frac{\mathrm{d}\epsilon_1 \cdots \mathrm{d}\epsilon_V}{(\epsilon_1 \cdots \epsilon_V)^q} \times \{\text{energy integrand}\}$$

Easily recovered from the family trees: not only the integrand, but the whole integral!

$$[1 \cdots N] = (-\mathrm{i})^N \int_0^\infty \prod_{i=1}^N \left[\frac{\mathrm{d}\epsilon_i \, (\mathrm{i}\epsilon_i)^{-q_i}}{\Gamma[1-q_i]} \right] \frac{1}{\mathcal{E}_1 \mathcal{E}_{12} \cdots \mathcal{E}_{1\cdots N}} \equiv \{1 \cdots N\}$$

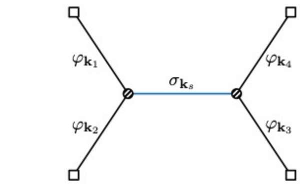
Inflation limit (q->0): all family trees reduce to polylogarithms. For instance:

$$\widetilde{\psi}_{\text{2-chain}} = \mathrm{Li}_2 \frac{E_2 - K}{E_{12}} + \mathrm{Li}_2 \frac{E_1 - K}{E_{12}} + \log \frac{E_1 + K}{E_{12}} \log \frac{E_2 + K}{E_{12}} - \frac{\pi^2}{6}$$

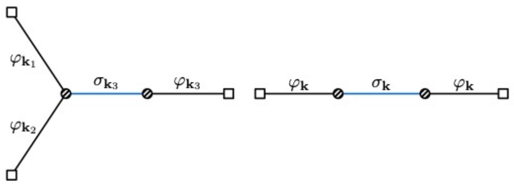
The family tree decomposition provides a new way to derive the symbol

Concluding remarks

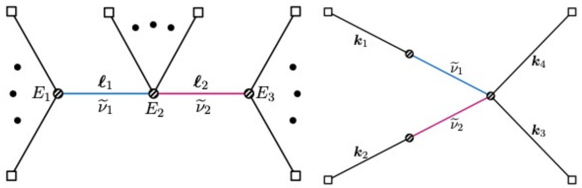
Analytical progress of cosmological correlators from our group since 2022:



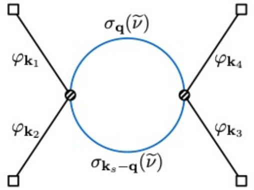
boost-less graphs
PMB / bootstrap
[2205.01692; 2208.13790]



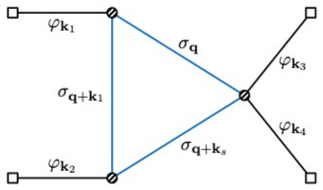
Closed-form formula
improved bootstrap
[2301.07047]

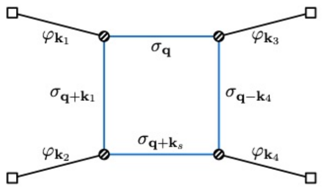


Multiple massive exchange
family-tree decomposition
[2309.10849]

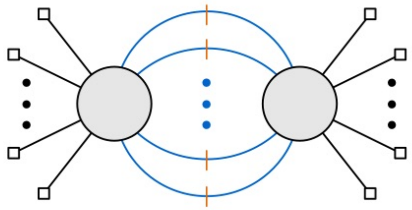


1-loop bubble graphs
spectral decomposition
[2211.03810]





1-loop signal
PMB, bootstrap
[2304.13295]



All-loop signal
factorization theorem
[2308.14802]

Analytical results enable fast numerical implementation:

1-loop: Brute-force numerical [O(10^5) CPU hours] vs. Analytical [O(10s) on a laptop]

[Lian-Tao Wang, **ZX**, Yi-Ming Zhong, 2109.14635] [**ZX**, Hongyu Zhang, 2211.03810]

Yet still a lot more to be understood. Far from done!

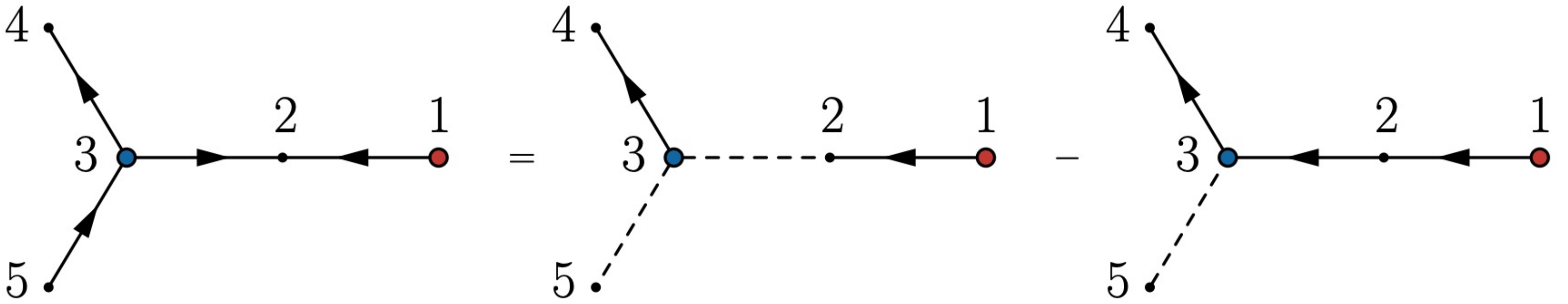
Concluding remarks

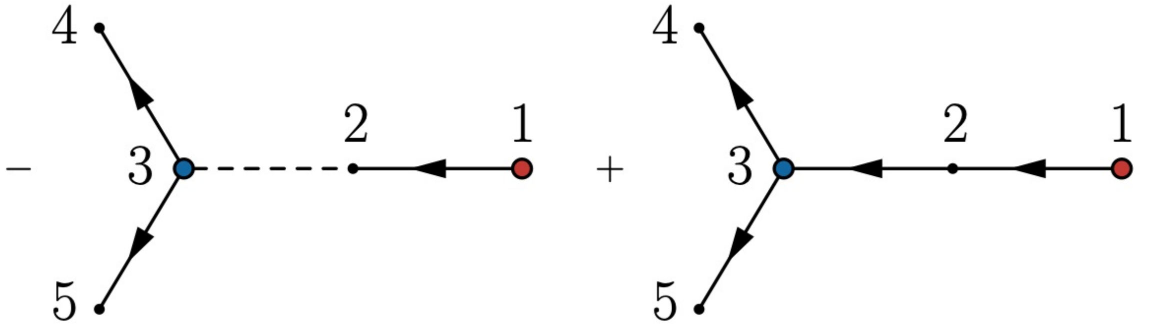
- Massive tree diagrams essentially solved [PMB + family tree];
  Simple loops (relevant to pheno) should be doable as well [ongoing]
- Many structures seen in conformal amp should persist in general IR finite amp
  [Singularity structure | cut structure]
- In particular, redundant info in amplitudes: Full amp recoverable from a subset of data:
- From the uncut part [cosmological bootstrap]:
  Initial data given in early-time (flat-space) limit; evolved with bootstrap eqs
- From the cut part [dispersion]:
  Recover the amplitude from the branch cut [late time | signal | factorizable] [ongoing]
- Rich mathematical structure ↔ deep physics of QFT in dS

Thank you!

Back up

Example: a 5-fold int





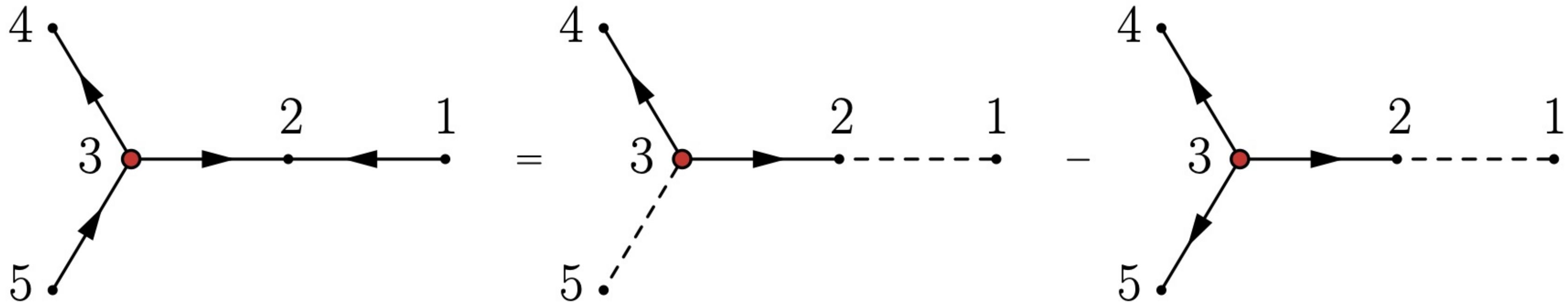
Choose Site 1 as the earliest

1->2 good   2->3 flip

3->4 good   4->5 flip

[Also need to decide “locally” earliest site in all nested subgraph, in this case Site 3]

Example: a 5-fold int

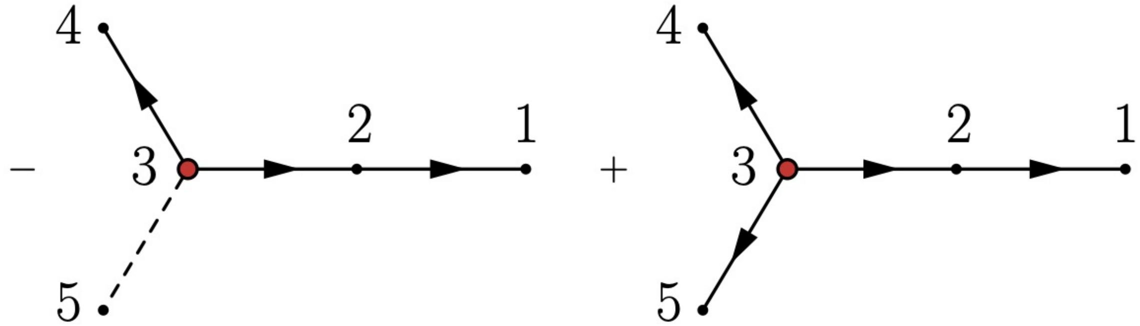


We can as well choose a differerent site as the earliest

Say Site 3:

3->4 good 3->2 good

3->5 flip 2->1 flip



For a tree graph: choosing an earliest site fixes the partial order

Total-energy series

$$\int_{-\infty}^{\tau_1} \mathrm{d}\tau_2\,(-\tau_2)^{q_2-1} e^{\mathrm{i}\omega_2\tau_2} = (-\tau_1)^{q_2}\mathrm{E}_{1-q_2}(-\mathrm{i}\omega_2\tau_1)$$

→

$$\mathrm{E}_p(z) = e^{-z}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\Gamma\begin{bmatrix} p+s, 1+s, -s \\ p \end{bmatrix} z^{-s-1}$$

The energy of a site is collected by her mother

→ A new series expanded in reciprocal of total energy:

$$[\mathscr{P}(\widehat{1}2\cdots N)] = \frac{-\mathrm{i}^N}{(\mathrm{i}\omega_{1\cdots N})^{q_{1\cdots N}}}\sum_{n_2,\cdots,n_N=0}^{\infty}\Gamma(q_{1\cdots N}+n_{2\cdots N})\prod_{j=2}^{N}\frac{(-\widetilde{\omega}_j/\omega_{1\cdots N})^{n_j}}{(-\widetilde{q}_j-\widetilde{n}_j)_{1+n_j}}$$

MB reps very flexible, opening up many possibilities to expand family trees:

Mix of single- & total-energy rep, vanishing total-energy limit, etc [progress underway]

family tree vs energy integral

FRW conformal amplitudes => Twisted integrals of flat amplitudes

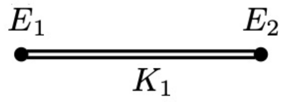
[Arkani-Hamed et al. 2312.05303]

$$\mathcal{T} \sim \int_0^\infty \frac{\mathrm{d}\epsilon_1 \cdots \mathrm{d}\epsilon_V}{(\epsilon_1 \cdots \epsilon_V)^q} \times \{\text{energy integrand}\}$$

The energy integrand constructable recursively (cosmological polytope)

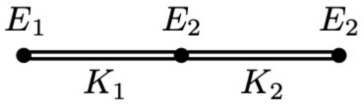
[Arkani-Hamed et al. 1709.02813]

Example: 2-site wavefunction



$$\frac{2K_1}{\mathcal{E}_{12}(\mathcal{E}_1 + K_1)(\mathcal{E}_2 + K_1)} \qquad \mathcal{E}_i \equiv E_i + \epsilon_i$$

3-site wavefunction



$$\frac{4K_1K_2}{\mathcal{E}_{123}(\mathcal{E}_1 + K_1)(\mathcal{E}_2 + K_{12})(\mathcal{E}_3 + K_2)}\left[\frac{1}{\mathcal{E}_{12} + K_2} + \frac{1}{\mathcal{E}_{23} + K_1}\right]$$

Time and energy integrals essentially related by Fourier transform:

$$(-\tau)^{q-1} = \frac{\mathrm{i}^{1-q}}{\Gamma(1-q)} \int_0^\infty \mathrm{d}\epsilon\, e^{\mathrm{i}\epsilon\tau}\, \epsilon^{-q}$$

Energy integrand easily recovered from family tree integrals for chain diagrams:

$$\begin{aligned}
[1\cdots N] &= (-\mathrm{i})^N \int_{-\infty}^{0} \prod_{i=1}^{N} \left[\mathrm{d}\tau_i\, (-\tau_i)^{q_i-1} e^{\mathrm{i}\omega_i\tau_i}\right] \theta_{N,N-1}\cdots\theta_{21} \\
&= \int_0^\infty \prod_{i=1}^{N} \left[\frac{\mathrm{d}\epsilon_i\, (\mathrm{i}\epsilon_i)^{-q_i}}{\Gamma[1-q_i]} \int_{-\infty}^{0} \mathrm{d}\tau_i\, e^{\mathrm{i}\mathcal{E}_i\tau_i}\right] \theta_{N,N-1}\cdots\theta_{21} \\
&= (-\mathrm{i})^N \int_0^\infty \prod_{i=1}^{N} \left[\frac{\mathrm{d}\epsilon_i\, (\mathrm{i}\epsilon_i)^{-q_i}}{\Gamma[1-q_i]}\right] \frac{1}{\mathcal{E}_1\mathcal{E}_{12}\cdots\mathcal{E}_{1\cdots N}} \equiv \{1\cdots N\}
\end{aligned}$$

Energy integrand of a chain is 1 / product of successive sums of external energies

Inflationary limit

Interesting to consider the special case of ϕ^3 theory in dS limit (all q = 0)

Boundary of IR safe region: A family tree of V sites contains q = 0 poles up to deg V
All poles cancel out in amplitudes, finite terms being polylogs

Example 2-site wavefunction: $\widetilde{\psi}_{\text{2-chain}} = [1_1 2_{\bar{1}}] - [1_{\bar{1}} 2_1] + [1_{\bar{1}}][2_1] - [1_1][2_1]$

$$[1_1 2_{\bar{1}}]_{q_1=q_2=q} = \frac{-1}{[\mathrm{i}(E_1+K)]^{2q}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma[n+2q]}{n+q} \left(\frac{E_2-K}{E_1+K}\right)^n$$

$$= \underbrace{-\frac{1}{2q^2} + \frac{\gamma_E + \log[\mathrm{i}(E_1+K)]}{q}}_{\text{divergent terms}} \underbrace{- \mathrm{Li}_2 \frac{K-E_2}{K+E_1} - \Big(\log[\mathrm{i}(E_1+K)] + \gamma_E\Big)^2 - \frac{\pi^2}{6}}_{\text{finite terms}} + \mathcal{O}(q)$$

Final answer: $\widetilde{\psi}_{\text{2-chain}} = \mathrm{Li}_2 \frac{E_2-K}{E_{12}} + \mathrm{Li}_2 \frac{E_1-K}{E_{12}} + \log \frac{E_1+K}{E_{12}} \log \frac{E_2+K}{E_{12}} - \frac{\pi^2}{6}$

Question: a new algorithm for all family-trees/amplitudes? [See also Hillman 1912.09450]

Inflationary limit

More sites: integrated polylogs could be tedious, but easy to get the symbol

A 3-site example: $[2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}} \sum_{n_1,n_3=0}^{\infty} \frac{(-1)^{n_{13}}\Gamma[n_{13}+q_{123}]}{(n_1+q_1)(n_3+q_3)} \frac{u_1^{n_1}}{n_1!} \frac{u_3^{n_3}}{n_3!} \qquad u_i \equiv \omega_i/\omega_2$

Inflationary limit: $\lim_{q\to 0} [2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{3q}} \left\{ \frac{\Gamma[3q]}{q^2} + \frac{\mathrm{Li}_2(-u_1)+\mathrm{Li}_2(-u_3)}{q} + \mathbf{L}_3(u_1,u_3) \right\} + \mathcal{O}(q)$

$$\mathbf{L}_3(u_1,u_3) \equiv \sum_{n_1,n_3=1}^{\infty} \frac{(-1)^{n_{13}}\Gamma[n_{13}]}{n_1 n_3} \frac{u_1^{n_1}}{n_1!} \frac{u_3^{n_3}}{n_3!}$$

$\mathbf{L}_3$ is a weight-3 polylog: $\frac{\partial}{\partial \log u_1} \frac{\partial}{\partial \log u_3} \mathbf{L}_3(u_1,u_3) = \log(1+u_1) + \log(1+u_3) - \log(1+u_{13})$

$$\mathcal{S}(\mathbf{L}_3) = \frac{(1+u_1)(1+u_3)}{1+u_{13}} \otimes u_1 \otimes u_3 + \frac{(1+u_1)(1+u_3)}{1+u_{13}} \otimes u_3 \otimes u_1 + \frac{1+u_{13}}{1+u_1} \otimes (1+u_1) \otimes u_1 + \frac{1+u_{13}}{1+u_3} \otimes (1+u_3) \otimes u_3$$